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Some New Results about Mosco Convergence

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Dedicated to Umberto Mosco on the occasion of his 80th birthday.

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We consider the problem $\min_{v \in C} J(v)$, where J is the standard integral functional

$$J(v) = \int_\Omega j(x, \nabla v) - \int_\Omega f(x)\, v(x),$$

defined in the Sobolev space $\wp$. We study the convergence of the minima u if we perturb the convex set C in accordance with the Mosco convergence.

Keywords: Mosco convergence, minimization, integral functionals, continuous dependence, real analysis methods.

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1. Introduction

This paper is related to my last two years as student at the "Università di Roma" and is dedicated to Umberto Mosco, one of my last two teachers.

The main objective of the papers [19] and [20] by Umberto Mosco is the study of convergence properties of solutions of variational inequalities (they are related to the minima of functionals on convex subsets of a Banach space). The author asks under what condition on the convex the solutions converge and he introduced a definition of convergence for sequences of convex sets in Banach spaces.

Since these papers, this condition is known as Mosco convergence.

Moreover the main result of [21] states that the Mosco convergence of quadratic forms is equivalent to strong convergence of the corresponding semigroups.

There are many papers devoted to Mosco convergence; here we only recall [1], [2], [3], [5], [9], [13], [14], [15], [16], [17], [18], [22], [23], [24].

2. Setting

In line with [20] this paper deals with the stability of the minima on convex subsets of integral functionals defined as

$$J(v) = \int_\Omega j(x, \nabla v) - \int_\Omega f(x)\, v(x), \tag{1}$$

under the standard assumptions below.

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Here Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 1$, and $j\colon \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function, that is, measurable with respect to x in Ω, for every $\xi \in \mathbb{R}^N$, and continuous with respect to ξ in $\mathbb{R}^N$, for almost every x in Ω.

We assume that there exist two real positive constants α, β such that for almost every x in Ω, for every ξ we have

$$\alpha|\xi|^p \leq j(x,\xi), \quad j(x,\xi) \leq \beta|\xi|^p, \tag{2}$$

with

$$p > 1. \tag{3}$$

With respect to the function f we assume that

$$f \in L^m(\Omega), \qquad \text{with } m \geq (p^*)', \tag{4}$$

where p^* is the Sobolev conjugate ($p^* = \frac{pN}{N-p}$, $1 \leq p < N$) and $(p^*)'$ is the Hölder conjugate of p^*. Thus $J(v)$ is well defined in $\wp$. We also assume

$$j(x,\xi) \text{ is strictly convex with respect to } \xi, \tag{5}$$

for a.e. $x \in \Omega$; so that J is a strictly convex functional on $\wp$. Then J is strongly continuous and weakly lower semicontinuous; thus the minimization of J on $\wp$ or on a convex subset C is a classical problem.

3. Mosco convergence

We recall the definition of convergence for convex subsets due to Mosco (see [20]).

Definition 3.1. Let X be a Banach space and C_n, C_0 closed, convex subsets of X. The sequence $\{C_n\}$ *Mosco-converges* to C_0 if

(i) C_0 is the set of all $v_0 \in X$ such that $\|v_0 - v_n\|_X \to 0$, with $v_n \in C_n$;

(ii) for every subsequence $\{v_{n_i}\}$, with $v_{n_i} \in C_{n_i}$, weakly convergent to v_0, we have $v_0 \in C_0$.

Proposition 3.2. *The following examples of Mosco-convergence are proved in* [20].

(A) *Let $\{C_n\}$ be a decreasing sequence of closed convex subset of the Banach space X. Then $\{C_n\}$ Mosco-converges to C_0, where $C_0 = \cap C_n$.*

(B) *Let $\{C_n\}$ be an increasing sequence of closed convex subset of the Banach space X. Then $\{C_n\}$ Mosco-converges to C_0, where C_0 is the closure of $(\cup\, C_n)$.*

The obstacle type convex sets are important cases and we recall the following results.

Proposition 3.3. (Obstacle type convex sets) *Let ψ_n be a measurable function such that*

$$C_n = K(\psi_n) = \{v \in \wp : \, v \geq \psi_n\} \tag{6}$$

is not empty. $K(\psi_n)$ Mosco-converges to $K(\psi_0)$, provided that one of the following assertions holds:

(a) *the sequence $\{\psi_n\}$ is weakly convergent in $\wp$ to ψ_0, with $\psi_n \leq \psi_0$;*

(b) *the sequence $\{\psi_n\}$ strongly converges in $\wp$ to ψ_0 (see Lemma* 1.7 *of* [20]*);*

(c) *the sequence $\{\psi_n\}$ is weakly convergent in $\wp$ to ψ_0, $\psi_n \geq \psi_0$ and $A(\psi_n) \leq T$, in the sense of distributions, where $A(v) = -\mathrm{div}(a(x, \nabla v))$ is a nonlinear differential operator (Leray-Lions operator (see below)) and T belongs to the dual of $\wp$;*

(d) *the sequence $\{\psi_n\}$ strongly converges in $L^\infty(\Omega)$ to ψ_0 and there exists $\Psi \in \wp$, such that $\Psi \geq \psi_n$, $\forall\, n$ (see [11]);*

(e) *the sequence $\{\psi_n\}$ weakly converges in $W_0^{1,q}(\Omega)$ to ψ_0, with $q > p$, (see* [11]*).*

We recall that a nonlinear differential operator of the type $A(v) = -\mathrm{div}\,(a(x, \nabla v))$ is a Leray-Lions operator if the function $a(x, \xi)$ is a Carathéodory function of the form $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ such that, for almost every $x \in \Omega$, for every $\xi \in \mathbb{R}^N$ and η in $\mathbb{R}^N$, with $\xi \neq \eta$,

(i) $a(x, \xi) \cdot \xi \geq \alpha\, |\xi|^p\,, \quad |a(x, \xi)| \leq \beta\, |\xi|^{p-1}\,,$ *where α and β are positive constants;*

(ii) $\big(a(x, \xi) - a(x, \eta)\big) \cdot (\xi - \eta) > 0\,.$

Proof. (a):

- If $v_n \geq \psi_n$, with v_n weakly convergent in $\wp$ to v_0, then $v_0 \geq \psi_0$.
- If $v_0 \geq \psi_0$, then $v_n = v_0$, $\forall\, n$, is such that $v_n = v_0 \geq \psi_0 \geq \psi_n$ and v_n strongly converges to v_0.

(c): Indeed we have that, thanks to the non-negativity of $\psi_n - \psi_0$,

$$0 \leq \langle A(\psi_n) - A(\psi_0), \psi_n - \psi_0 \rangle \leq \langle T - A(\psi_0), \psi_n - \psi_0 \rangle \to 0$$

and the convergence $\langle A(\psi_n) - A(\psi_0), \psi_n - \psi_0 \rangle \to 0$ implies the strong convergence of ψ_n to ψ_0 in $\wp$ (see [12]).

Remark 3.4. In Proposition 3.3, we do not have results concerning the case of a sequence $\{\psi_n\}$ only decreasing and weakly convergent in $\wp$.

Nevertheless, in $L^p(\Omega)$ it is easy to prove that if the sequence $\{\psi_n\}$ is decreasing and weakly convergent in $L^p(\Omega)$ to ψ_0, then ψ_n strongly converges to ψ_0 in $L^p(\Omega)$, since

$$\begin{aligned} \int_\Omega |\psi_n - \psi_0|^p &= \int_\Omega |\psi_n - \psi_0|^{p-2}(\psi_n - \psi_0)(\psi_n - \psi_0) \\ &\leq \int_\Omega |\psi_1 - \psi_0|^{p-2}(\psi_1 - \psi_0)(\psi_n - \psi_0) \to 0. \end{aligned}$$

4. Stability

We consider the minimization problems

$$u_n \in C_n : J(u_n) \leq J(v),\ \forall\, v \in C_n, \tag{7}$$

$$u_0 \in C_0 : J(u_0) \leq J(v),\ \forall\, v \in C_0. \tag{8}$$

The standard minimization theory of integral functionals ensures the existence and uniqueness of u_n, u_0.

The main result of this paper concerns the strong convergence in $\wp$ of the sequence $\{u_n\}$ to u_0 on the assumption that the sequence $\{C_n\}$ Mosco-converges to C_0. We point out that our approach follows the classic style of the "direct method of the calculus of variations" and we do not use the differentiability of J (and of $j(x, \xi)$).

Theorem 4.1. *The sequence $\{u_n\}$ strongly converges in $\wp$ to u_0.*

The proof of the theorem requires some results proved in Lemma 4.2, Lemma 4.3, Lemma 4.5 below.

Lemma 4.2. *The sequence $\{u_n\}$ weakly converges in $\wp$ to u_0.*

Proof. Let $v_0 \in C_0$; there exists a sequence $\{v_n\}$, with $v_n \in C_n$, strongly converging to v_0. The continuity of J implies that $J(v_n)$ converges to $J(v_0)$, so that we have (for n large enough)

$$\alpha\|u_n\|^p_{\wp} - C_1\|u_n\|_{\wp} \leq J(u_n) \leq J(v_n) \leq J(v_0) + 1.$$

Thus the sequence $\{u_n\}$ is bounded; then there exist a subsequence $\{u_{n_i}\}$ and u^* such that $\{u_{n_i}\}$ weakly converges to u^*. Moreover the Mosco-convergence implies that $u^* \in C_0$.

Now we work on the subsequence n_i. Once more, let $v_0 \in C_0$; there exists a sequence $\{v_{n_i}\}$, with $v_{n_i} \in C_{n_i}$, strongly converging to v_0. The continuity of J implies that $J(v_{n_i}) \to J(v_0)$, so that in the following inequality (minimality of u_{n_i})

$$u_{n_i} \in C_{n_i} : J(u_{n_i}) \leq J(v_{n_i}),\ \forall\, v_{n_i} \in C_{n_i} \tag{9}$$

we can pass to the limit (thanks to the Mosco convergence and to the weak-lower semicontinuity of J) and we obtain

$$u^* \in C_0 : J(u^*) \leq J(v_0),\ \forall\, v_0 \in C_0. \tag{10}$$

The uniqueness of the minimum u_0, gives $u^* = u_0$ and the weak convergence of the full sequence $\{u_n\}$ to u^*. □

Lemma 4.3. *Let $\{w_n\}$ be a sequence strongly convergent to u_0, with $w_n \in C_n$ (the sequence exists thanks to the Mosco convergence assumption). Then the sequence $\{u_n\}$ satisfies*

$$0 \leq \Big[\frac{1}{2}\, J(u_n) + \frac{1}{2}\, J(w_n) - J\Big(\frac{u_n + w_n}{2}\Big)\Big] \to 0, \tag{11}$$

Proof. Since u_n minimize J on C_n, in (9) we take as test function $v_n = w_n$, we have

$$u_n \in C_n : J(u_n) \leq J(w_n),\ \forall\, w_n \in C_n,$$

which implies (once more recall the strong continuity and the weak lower semicontinuity of J)

$$J(u_0) \leq \liminf\, J(u_n) \leq \lim\, J(w_n) = J(u_0)$$

that is

$$\lim\, J(u_n) = J(u_0). \tag{12}$$

Note that $\dfrac{u_n + w_n}{2} \in C_n$, so that (recall (12) and the convexity of J) we have

$$J(u_0) \leq \liminf J\Big(\frac{u_n + w_n}{2}\Big) \leq \lim \frac{1}{2}\, J(u_n) + \lim \frac{1}{2}\, J(w_n) = J(u_0),$$

which implies (11). □

In the proof of Lemma 4.5 we will use the following measure theory lemma (proved in [10]).

Lemma 4.4. *Let (X, μ) be a measurable space, μ a positive measure satisfying $\mu(X) < +\infty$, and $\gamma\colon X \longrightarrow [0, +\infty]$ a measurable function with the property $\mu(\{x \in X : \gamma(x) = 0\}) = 0$.*
Then, for every $\epsilon > 0$ there exists $\delta > 0$ such that the statement

$$\textit{A measurable subset of } X, \quad \int_A \gamma(x)\, d\mu < \delta,$$

implies $\mu(A) < \epsilon$.

Now we prove the following convergence lemma by means of real analysis methods. The approach follows the techniques of [10].

Lemma 4.5. *The sequence $\{u_n\}$ fulfills the following convergence result:*

$$\nabla u_n(x) \ \textit{ converges in measure to } \ \nabla u_0(x), \tag{13}$$

Proof. The convergence (11) implies

$$0 \le \int_\Omega \Big[\frac{1}{2}\, j(x, \nabla u_n) + \frac{1}{2}\, j(x, \nabla w_n) - j\Big(x, \frac{\nabla u_n + \nabla w_n}{2}\Big)\Big] \to 0, \tag{14}$$

since the linear part of J disappears in (11).

For all $\delta > 0$, there exists $\bar\nu(\delta) > 0$ such that $n > \bar\nu(\delta)$ implies

$$\int_\Omega \Big\{\frac{1}{2}\, j(x, \nabla w_n) + \frac{1}{2}\, j(x, \nabla u_n) - j\Big(x, \frac{\nabla w_n + \nabla u_n}{2}\Big)\Big\} < \delta. \tag{15}$$

Let $\lambda > 0$ and $\epsilon > 0$. We want to prove that there exists $\nu(\epsilon, \lambda)$ such that

$$m,\, n > \nu(\epsilon, \lambda) \ \Rightarrow \ \text{measure}\, \{|\nabla u_n - \nabla w_n| > \lambda\} < \epsilon.$$

First of all, there exists $k > 0$ such that

$$\text{measure}\, \{|\nabla u_n| > k\} < \epsilon, \ \text{measure}\, \{|\nabla w_n| > k\} < \epsilon,$$

uniformly w.r.t. n (thanks to the $W_0^{1,p}(\Omega)$ boundedness of $\{u_n\}$). We define

$$\begin{cases} \mathcal{K} = \{(\xi, \eta) \in \mathbb{R}^{2N} : |\xi| \le k,\ |\eta| \le k,\ |\xi - \eta| \ge \lambda\}, \\ \gamma(x) = \min\limits_{(\xi,\eta) \in \mathcal{K}} \Big[\frac{1}{2}\ j(x, \xi) + \frac{1}{2}\ j(x, \eta) - j\Big(x, \frac{\xi + \eta}{2}\Big)\Big], \end{cases}$$

$$A_{n,m} = \{|\nabla u_n| \le\ k\} \cap \{|\nabla w_n| \le\ k\} \cap \{|\nabla u_n - \nabla w_n| \ge \lambda\}.$$

Thanks to (15), one has, for $n, m > \bar\nu(\delta)$,

$$\int_{A_{n,m}} \gamma(x) \le \int_\Omega \Big\{\frac{1}{2}\, j(x, \nabla w_n) + \frac{1}{2}\, j(x, \nabla u_n) - j\Big(x, \frac{\nabla w_n + \nabla u_n}{2}\Big)\Big\} < \delta.$$

Now, thanks to Lemma 4.4 (here we use the strict convexity of the function $j(x,\xi)$ with respect to ξ), we choose δ such that

$$\int_A \gamma(x) \le \delta \ \text{ implies } \ \text{measure}\,(A) < \epsilon.$$

Thus
$$\text{measure}\,(A_{n,m}) < \epsilon \ \text{ if } \ n,\, m > \bar\nu = \bar\nu(\delta). \tag{16}$$

On the other hand, we have

$$\{|\nabla u_n - \nabla w_n| \ge \lambda\} \subset \{|\nabla u_n| > k\} \cup \{|\nabla w_n| > k\} \cup A_{n,m}.$$

Then for $n, m > \nu = \max(\bar\nu, \tilde\nu)$ one has

$$\text{measure}\,\{|\nabla u_n - \nabla w_n| \ge \lambda\} \le 3\epsilon.$$

Thus we have proved (13). □

Proof of Theorem 4.1. The convergence (13) implies

$$j(x, \nabla u_n(x)) \ \text{ converges in measure to } \ j(x, \nabla u_0(x)), \tag{17}$$

Now we go back to (7), with test function w_n, and we have

$$\int_\Omega j(x,\nabla u_n) - \int_\Omega f(x)\, u_n(x) \le \int_\Omega j(x,\nabla w_n) - \int_\Omega f(x)\, w_n(x),$$

which implies, since $\lim \int_\Omega f(x)\, u_n(x) = \int_\Omega f(x)\, u_0(x) = \lim \int_\Omega f(x)\, w_n(x)$,

$$\int_\Omega j(x,\nabla u_0) \le \liminf \int_\Omega j(x,\nabla u_n) \le \lim \int_\Omega j(x,\nabla w_n) = \int_\Omega j(x,\nabla u_0),$$

and then
$$\int_\Omega j(x,\nabla u_n) \to \int_\Omega j(x,\nabla u_0). \tag{18}$$

Now (17) and (18), jointly with a classical result from real analysis result, say that

$$j(x, \nabla u_n(x)) \ \text{ converges in } L^1(\Omega) \text{ to } \ j(x, \nabla u_0(x)). \tag{19}$$

Indeed (recall that $j(x,\xi) \ge 0$)

$$\int_\Omega |j(x,\nabla u_n) - j(x,\nabla u_0)|$$
$$= \int_\Omega [j(x,\nabla u_n) - j(x,\nabla u_0)] + 2\int_{\{0 \le j(x,\nabla u_n) \le j(x,\nabla u_0)\}} [j(x,\nabla u_0) - j(x,\nabla u_n)].$$

and in the last integral we pass to the limit thanks to the Lebesgue theorem. Then the inequality $\alpha|\nabla u_n|^p \le \ j(x, \nabla u_n(x))$ and the Vitali theorem give that

$$\nabla u_n(x) \ \text{ converges in } L^p(\Omega) \text{ to } \ \nabla u_0(x). \qquad \square \tag{20}$$

Remark 4.6. If $J(v) = \int_\Omega j(x, v(x), \nabla v) - \int_\Omega f(x)\, v(x)$, our proof is almost the same (see again [10]), excluding the last sentence of Lemma 4.2, which becomes (because of the nonuniqueness): u^* is a minimum of J on C_0.

Remark 4.7. In the simple case $j(x,\xi) = A(x)|\xi|^p$, $A(x) \geq \alpha > 0$, as consequence of 14 and Clarkson inequality (here we only discuss the case $p \geq 2$), we have

$$\begin{aligned}\alpha \int_\Omega \Big|\frac{\nabla u_n - \nabla w_n}{2}\Big|^p &\leq \int_\Omega A(x)\Big|\frac{\nabla u_n - \nabla w_n}{2}\Big|^p \\ &\leq \int_\Omega A(x)\Big[\frac{1}{2}|\nabla u_n|^p + \frac{1}{2}|\nabla w_n|^p - \Big|\frac{\nabla u_n + \nabla w_n}{2}\Big|^p\Big] \to 0,\end{aligned}$$

so that we do not need Lemma 4.5.

Remark 4.8. If we assume the differentiability of $j(x,\xi)$, with $a(x,\xi) = j_\xi(x,\xi)$, then u_n is the solution of the variational inequality

$$u_n \in C_n : \int_\Omega a(x, \nabla u_n)\nabla(u_n - v) \geq \int_\Omega f(x)(u_n - v), \quad \forall\, v \in C_n. \tag{21}$$

Thus the statement of Theorem 4.1 reads: *the sequence of the solutions u_n of the above variational inequality on C_n strongly converges in $\wp$ to the solution of variational inequality on C_0.*

This result was proved in [20] (Corollary of Theorem A) if we also assume that

$$\int_\Omega [a(x, \nabla w) - a(x, \nabla v)]\nabla(w - v) \geq \|w - v\|\, \gamma(\|w - v\|),$$

where γ is a real continuous strictly increasing function satisfying $\gamma(0) = 0$ and $\gamma(+\infty) = +\infty$.

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