Journal of Convex Analysis
Volume 28 (2021), No. 2, 471–494

Sharp Trudinger Type Inequalities for Measure-Valued Lagrangeans

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Dedicated to Umberto Mosco on the occasion of his 80th birthday.

Received: March 26, 2020
Accepted: May 17, 2020

The aim of the paper is to prove weighted John-Nirenberg and sharp Trudinger type inequalities for measure-valued (α, p)-Lagrangeans.

Keywords: Embedding theorems, metric spaces, doubling measure, Lagrangeans, weight.

2010 Mathematics Subject Classification: 35J60, 35J65, 35B30, 35J87

1. Introduction

The aim of the paper is to prove weighted John-Nirenberg and sharp Trudinger type inequalities for measure-valued (α, p)-Lagrangeans.

We recall that in the classical setting of Sobolev spaces $W^{k,p}(\mathbb{R}^n)$ with $kp = n$, $k \in \mathbb{N}$, N. S. Trudinger proved the continuous embedding

$$W_0^{k,p}(\Omega) \subseteq exp\, L^{\frac{n}{n-1}}(\Omega) \tag{1}$$

where $exp\, L^{\frac{n}{n-1}}(\Omega)$ denotes the exponential type Orlicz space endowed with the norm

$$||u||_{\exp L^{\frac{n}{n-1}}(\Omega)} = \inf\{\lambda > 0 : \int_\Omega \exp\Big(\Big(\frac{|u(x)|}{\lambda}\Big)^{\frac{n}{n-1}}\Big)\, dx\ \leq 1\}$$

and Ω is an open subset of $\mathbb{R}^n$ with finite measure (see [29] and [18]).

Similar results have been obtained independently by V. I. Yudovich in [30] and S. I. Pokhozhaev in [26]. The previous integrability result was improved by R. S. Strichartz who pointed out that the power $\frac{n}{n-1}$ can be replaced by $\frac{p}{p-1}$ if u belongs to the Bessel space $W_0^{\frac{n}{p},p}(\Omega)$ (see [27]). Later on, K. Hansson in [11] and H. Brézis and S. Wainger in [2] established the so called *sharp* Trudinger inequality. More precisely, they proved the continuous embedding

$$W_0^{1,n}(\Omega) \subseteq BW_n(\Omega) \tag{2}$$

where
$$||u||_{BW_n(\Omega)} = \Big(\int_0^{|\Omega|} u^*(t)^n (\ln(\frac{|\Omega|}{t}))^{-n} \frac{dt}{t}\Big)^{\frac{1}{n}}$$

and $u^*(t)$ denotes the non-increasing rearrangement of u (see, for example, [18]).

ISSN 0944-6532 / $ 2.50

The *sharp* Trudinger inequality implies the Trudinger embedding mentioned before but it is stronger as the inclusion $BW_n(\Omega) \subset \exp L^{\frac{n}{n-1}}(\Omega)$ is proper (see *e.g.* [11] Section 3).

Functional inequalities of the previous mentioned type can be also obtained by using the capacitary estimates of Maz'ya in [20]. For a more exhaustive discussion on this topic in the classical setting of domains of $\mathbb{R}^n$ we refer to the interesting survey by L. Tartar [28].

There is a vast literature concerning this type of functional inequalities also in the setting of abstract metric and quasi-metric spaces (see [10] and the references therein). Similar results to those of the present paper have been obtained by P. Hajłasz, P. Koskela [10], B. Franchi, C. Pérez, R. L. Wheeden [8], J. Malý J., L. Pick [18], [19] P. MacManus, C. Pérez [15], [16], E. Mbakop in [21]. In particular, Mbakop in [21] establishes Trudinger type inequalities for $(1, p)$-Lagrangeans.

The main tools of the previous works are dyadic sets, covering techniques, classical good $\lambda-$inequality, maximal operators. The approach of our paper is based on Riesz potential estimates in the spirit of [17] and [4].

The measure-valued Lagrangeans on homogeneous spaces have been introduced in the paper [17] of J. Malý and U. Mosco in order to describe the dynamics of very general self-similar structures and, in particular, of the most common fractals. In fact, in the fractal case, the gradient and hence the associated Lagrangean cannot be defined point-wise because the structure is not differentiable: a way to overcome this difficulty consists in defining the Lagrangean as a measure on the space, by stating the gradient-like behavior in a suitable form. Self-similar variational fractals, introduced by U. Mosco in [23], provide examples of $(1, 2)$-Lagrangeans. For $p \neq 2$, we can consider the p-Lagrangeans corresponding to the nonlinear energy forms on Koch curve type fractals (see [3], [5]), on Sierpinski gaskets and on larger class of fractals called weakly completely symmetric fractals (see [12]). In [17], Sobolev inequalities and embedding in Lorentz spaces for measure-valued (α, p)-Lagrangeans on homogeneous spaces X have been proved by using an approach based on Riesz potential estimates when $\alpha p < \nu$, where ν is the dimension of the homogeneous space X. In [4], using similar techniques, three different situations $\alpha = 1$, $p < \nu$, $\alpha = 1$, $p = \nu$ and $\alpha > 0$, $\alpha p > \nu$, have been faced. Weighted Sobolev inequalities and embedding in weighted Lorentz spaces have been proved in the first case, John-Nirenberg type inequalities are established for the second one and Morrey type inequalities have been obtained in the last one.

In the present paper, we consider the case (α, p)-Lagrangeans for any positive α with $\alpha p = \nu$ on a quasi metric space $X = (X, d, m)$ where m is a non negative Borel measure satisfying a *doubling condition* in a given ball (see assumption (L1)). We prove embeddings in weighted Lorentz spaces (see Proposition 4.4) and *weighted* John-Nirenberg type inequalities, for weights belonging to the Muckenhoupt class $A_\infty(m)$, (see Theorem 4.1). Moreover, we establish *sharp weighted* Trudinger inequalities assuming that the space X is connected (see Theorem 5.3). We note that inequality (37) can be compared to embedding (2).

In fact we have

$$\int_0^{|\Omega|} u^*(t)^p \big(\ln(\frac{|\Omega|}{t})\big)^{-p} \frac{dt}{t} = \frac{p}{p-1} \int_0^{\infty} \Big(\ln(\frac{|\Omega|}{|\{x \in \Omega : |u(x)| > y\}|})\Big)^{1-p} y^{p-1} dy$$

(see Theorem 2.1 in [6]). As it is proved by H. Brézis and S. Wainger (in the classical setting [2]) *sharp* Trudinger inequalities imply the exponential Trudinger embedding (see Theorem 5.5). Actually estimate (37) is stronger than estimate (48): in fact, the integral in (48) may be finite in cases when the integral in (37) diverges (see, *e.g.* Section 3 in [11]). Moreover, we recall that if we do not assume the space X to be connected then Trudinger type inequalities may not hold true. We refer to *e.g.* [15] for discussion and examples.

To our knowledge *sharp weighted* Trudinger inequalities for (α, p)-Lagrangeans on a quasi-metric space are new. Moreover we think that our approach is rather flexible and it is also adapt to study measure-valued Lagrangeans scaled according to refined power laws and more general weights introduced in [25] and in [22] to treat the Lagrangeans for fluctuating fractals and for scale irregular mixtures (see [24] and [1]). We note that U. Mosco and E. Mbakop (see [25] and [22]) study the case $\alpha p < \nu$ and, to our knowledge, the limiting case $\alpha p = \nu$ has not yet been addressed for this type of (α, p)-Lagrangeans.

The plan of the paper is the following. In the second section we define the measure-valued (α, p)-Lagrangeans on homogeneous spaces under the assumption of the weak chain rule which is more general than the one required in [17] (see Remark 2.1). In the third section we introduce the Riesz type potentials and we give a quantitative estimate in term of parameters (see Theorem 3.1). In Section 4 we deal with John-Nirenberg inequalities (see Theorem 4.1). First we establish *weak inequalities* for measure-valued (α, p)-Lagrangeans (12) (see Proposition 4.2). Inequality (12) is new in this framework and it can be seen as an improvement of inequality (10) in [17] as it allow us to treat the case $\alpha p = \nu$. Then we use the *weak chain rule condition* (see (L4)) to prove the weighted John-Nirenberg inequality (10). This inequality can be seen as an improvement of John-Nirenberg inequalities in [4] as it allow us to consider the case $\alpha \neq 1$. We note that condition (L4) plays a role similar to that of the *truncation properties* (see *e.g.* [10] and [20]) which are commonly used for obtaining strong type inequalities from weak type ones. In Section 5 we prove *sharp* weighted Trudinger type inequalities (see Theorem 5.3) and exponential weighted Trudinger embeddings (see Theorem 5.5).

2. (α, p)-Lagrangians on a homogeneous space X

We recall the definition of (α, p)-Lagrangeans on a homogeneous space X (as in [17]). We suppose that

(L1) $X = (X, d, m)$ is a locally compact quasi-metric measure space with a non-negative Borel measure satisfying the doubling condition. More precisely, X is a locally compact topological space endowed with a quasi-distance $d\colon X \times X \to \mathbb{R}^+$ such that $d(x, y) > 0$ if and only if $x \neq y$; $d(x, y) = d(y, x)$; for all $x, y, z \in X$ with $c_T \geq 1$ we have $d(x, z) \leq c_T(d(x, y) + d(y, z))$.

The (quasi-) balls $B(x,r) = \{y : y \in X,\ d(x,y) < r\}$, $x \in X$, $r > 0$ form a basis of open neighborhoods in X (see e.g. Theorem 2 in [14]). The measure m is a non-negative Borel measure on X satisfying the doubling condition within a given ball $B_0^{**} = B(z, R_0)$, $z \in X$, $R_0 > 0$, that is, there exists a constant $\nu > 0$ such that, for every $\sigma \in (0,1)$, there exists a constant $c_D \geq 1$ (possibly depending on σ) such that

$$0 < \left(\frac{r}{s}\right)^{\nu} m(B(x,s)) \leq c_D m(B(x,r)) \tag{3}$$

for every $x \in B(z, \sigma R_0)$ and for every $0 < r \leq s$, $B(x,s) \subseteq B(z, R_0)$.

We denote by $C(X)$ the space of continuous functions on X and by $\mathcal{M}^+(X)$ the space of non-negative Radon measures on X. For every ball $B = B(x,r)$ we set

$$|B| = m(B), \qquad \tau B = B(x, \tau r),\ \tau > 0$$

and from now on we denote by u_B the mean of the function u on the ball B with respect to the measure m, that is,

$$u_B = \frac{1}{|B|}\int_B u\, dm.$$

Moreover we suppose that

(L2) $\mathcal{L}^{(p)} : \mathcal{C} \to \mathcal{M}^+(X)$ is a map which associates with each function u from a given subspace $\mathcal{C} \subset C(X)$ a measure $\mathcal{L}^{(p)}[u] \in \mathcal{M}^+(X)$. We denote by $(X, \mathcal{L}^{(p)})$ the homogeneous space X with a Lagrangean $\mathcal{L}^{(p)}$.

We suppose that, for a given $p \geq 1$ and $\alpha > 0$, the following family of *Poincaré inequalities* holds:

(L3) there exist constants $\tau \geq 1$ and $c_P > 0$ such that we have

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u - u_B|\, dm \leq c_P r^{\alpha} \left(\frac{1}{|B(x,\tau r)|}\int_{B(x,\tau r)} d\mathcal{L}^{(p)}[u]\right)^{1/p} \tag{4}$$

for every $u \in \mathcal{C}$ and for every $B(x,r) \subseteq B(z, R_0)$.

To establish strong type estimates (see Theorems 4.1, 5.3 and 5.5) we will require that the *weak chain rule* holds as in [25]:

(L4) for every function $u \in \mathcal{C}$ and every non-negative real-valued function $g \in C^1(\mathbb{R})$, with g' bounded on $\mathbb{R}$, we have $g(u) \in \mathcal{C}$ and the following inequality holds in measure-valued sense

$$\mathcal{L}^{(p)}[g(u)] \leq \max |g'(u)|^p \chi_{\{g' \neq 0\}} \mathcal{L}^{(p)}[u].$$

Remark 2.1. The property (L4) is weaker that the one assumed in [17]: we remark that this weak chain rule property is satisfied also by Lagrangeans of fractional order α, possibly of non-local Besov type (see [25]). □

The measure $\mathcal{L}^{(p)}[u]$ satisfying the previous property is called the (α, p)-Lagrangian. We note that from condition (3) we derive the following *eccentric doubling condition.*

Proposition 2.2. *For any $\sigma \in (0,1)$, let $R \leq \frac{R_0}{c_T(c_T+\sigma)}$. Then there exists a constant $c_E \geq 1$, such that the eccentric doubling condition holds, that is, for every $x \in B(z, \sigma R)$ and $0 < r \leq R$ we have*

$$c_E |B(x,r)| \geq \left(\frac{r}{R}\right)^{\nu} |B(z,R)|. \tag{5}$$

Proof. We observe that if $x \in B(z, \sigma R)$, then

$$B(z,(1-\sigma)R) \subseteq B(x, c_T R) \subseteq B(z, R_0)$$

hence condition (5) follows from (3):

$$|B(x,r)| \geq c_D^{-1}(\frac{r}{c_T R})^\nu |B(x, c_T R)| \geq c_D^{-1}(\frac{r}{c_T R})^\nu |B(z,(1-\sigma)R)|$$
$$\geq c_D^{-2}(\frac{r}{c_T R})^\nu (\frac{(1-\sigma)R}{R})^\nu |B(z,R)|$$

and so $c_E = c_D^2 (\frac{c_T}{1-\sigma})^\nu$. □

From now on, we use both the inequalities (3) and (5) hence our setting will be the fixed ball $B = B(z,R)$ where R is as in Proposition 2.2.

3. Riesz type potentials

We introduce the *Riesz type potentials* as in [17]. Let σ such that $0 < c_T \sigma < 1$. Let $B = B(z,R)$ be the fixed ball where $R \leq \frac{R_0}{c_T(c_T+\sigma)}$. We define the kernel for x, $y \in B$, $\beta > 0$,

$$I_\beta^B(x,y) = \sum_{k=1}^\infty \frac{r_k^\beta}{|B_k(x)|} \chi_{B_k(x)}(y) = \sum_{k=1}^\infty \frac{r_k^\beta}{|B_k(x)|} \chi_{B_k(y)}(x)$$

where $B_k(x) = B(x, r_k)$, with $r_k = 2^{1-k}\theta R$, for $k = 1, 2, \ldots$ with

$$0 < \theta \leq \frac{1 - c_T \sigma}{2c_T^2}.$$

We note that if the measure of the balls depends only on the radius then this Riesz kernel is symmetric. In general the symmetry fails, but for doubling measures the ratio

$$I_\beta^B(x,y)/I_\beta^B(y,x)$$

is bounded. In [17] the following Theorem 3.1 is proved: here we recall the proof because we are interested in the exact expression of the right hand side of estimate (6) in term of the parameter β.

Theorem 3.1. *Suppose that* (L1) *holds. Let $G \subseteq B(z, \sigma R)$ be a measurable set. Let ε_0 be a positive number and let β belong to the interval* $(0, \nu - \varepsilon_0)$ *then for every $y \in B(z,R)$ we have*

$$\int_G I_\beta^B(x,y)dm(x) \leq c\frac{1}{\beta} R^\beta \left(\frac{|G|}{|B|}\right)^{\beta/\nu} \tag{6}$$

for every $0 < \beta \leq \nu - \varepsilon_0$, where c is a constant depending only on c_E, c_T, c_D, ν, ε_0 and θ.

Proof. If $x \in B(z, \sigma R)$ and $\chi_{B_k(x)}(y) = 1$, then $x \in B(y, r_k)$ and, by definition of θ, r_k and σ,

$$B(y, r_k) \subset B(x, 2c_T r_k) \subset B(z,R):$$

in fact, if $\zeta \in B(y, r_k)$, then

$$d(\zeta, x) \leq c_T(d(\zeta,y) + d(y,x)) \leq c_T(r_k + r_k),$$

if $\zeta \in B(x, 2c_T r_k)$ then

$$d(\zeta, z) \le c_T(d(\zeta, x) + d(x, z)) \le c_T(2c_T r_k + \sigma R)$$

and $c_T(2c_T r_k + \sigma R) \le 2c_T^2 2^{1-k} \frac{1-c_T\sigma}{2c_T^2} R + c_T\sigma R \le R - c_T\sigma R + c_T\sigma R = R.$

Hence the doubling condition (3) implies

$$|B(y, r_k)| \le |B(x, 2c_T r_k)| \le c_D(2c_T)^\nu |B(x, r_k)|$$

and
$$\int_G \frac{r_k^\beta}{|B_k(x)|} \chi_{B_k(y)}(x) d\, m(x) \le c_D(2c_T)^\nu\, r_k^\beta. \tag{7}$$

By (5) we have
$$\int_G \frac{r_k^\beta}{|B_k(x)|} \chi_{B_k(y)}(x) dm(x) \le c_E R^\nu r_k^{\beta-\nu} \frac{|G|}{|B|}. \tag{8}$$

We choose K such that $\dfrac{r_{K+1}}{r_1} < \left(\dfrac{|G|}{|B|}\right)^{1/\nu} \le \dfrac{r_K}{r_1}$ and using (7), (8) we obtain

$$\begin{aligned}
\int_G I_\beta^B(x, y) dm(x) &= \sum_{k=1}^\infty \int_G \frac{r_k^\beta}{|B_k(x)|} \chi_{B_k(y)}(x) dm(x) = \\
&= \left(\sum_{1\le k<K} + \sum_{k\ge K} \right) \int_G \frac{r_k^\beta}{|B_k(x)|} \chi_{B_k(y)}(x) dm(x) \le \\
&\le \frac{c_E}{2^{\nu-\beta}-1} R^\nu r_K^{\beta-\nu} \frac{|G|}{|B|} + c_D(2c_T)^\nu \frac{2^\beta}{2^\beta-1} r_K^\beta \le \\
&\le \frac{c_E}{2^{\nu-\beta}-1}\, \theta^{\beta-\nu} R^\beta \left(\frac{|G|}{|B|}\right)^{\beta/\nu} + c_D(2c_T)^\nu \frac{2^{2\beta}}{2^\beta-1} \theta^\beta R^\beta \left(\frac{|G|}{|B|}\right)^{\beta/\nu} = \\
&= c(\beta)\, R^\beta \left(\frac{|G|}{|B|}\right)^{\beta/\nu}.
\end{aligned}$$

As $\lim_{\beta\to0^+} \beta c(\beta) = \dfrac{c_D(2c_T)^\nu}{\ln 2}$ we deduce that

$$\int_G I_\beta^B(x, y) dm(x) \le c\frac{1}{\beta} R^\beta \left(\frac{|G|}{|B|}\right)^{\beta/\nu}$$

for every $0 < \beta \le \nu - \varepsilon_0$, with c depending only on c_E, c_T, c_D, θ, ε_0 and ν. □

4. John-Nirenberg type inequalities

In this section we state John-Nirenberg type inequalities in the case $\alpha p = \nu$ also with a weight w belonging to *Muckenhoupt class* $A_\infty(m)$ (see [4] for the case $\alpha = 1$). We recall that a weight w (that is, a nonnegative locally integrable function) belongs to *Muckenhoupt class* $A_\infty(m)$ if and only if there exist positive constants C_{A_∞} and $0 < s \le 1$ such that

$$w(E) \le C_{A_\infty} \left(\frac{m(E)}{m(B)}\right)^s w(B) \tag{9}$$

for every ball B and every measurable set $E \subseteq B$ where $w(E) := \int_E w dm$ (see [9]).

Theorem 4.1. *Suppose that* (L1), (L2), (L3), (L4) *hold with* $\alpha p = \nu$, $p > 1$, $\nu > 1$, *and* $\tau \geq 1$, $0 < c_T\,\sigma\,\tau < 1$. *Let* $w \in A_\infty(m)$, *then for every* $u \in \mathcal{C}$ *we have*

$$\inf_{C\in\mathbb{R}} \frac{1}{|B(z,\sigma R)|}\int_{B(z,\sigma R)} \exp\left(\frac{|u-C|}{c_1 R^{\frac{\nu}{p}}\left(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[u]\right)^{1/p}}\right) w\,dm \leq c_2 \quad (10)$$

with the constants c_1 *and* c_2 *depending only on* c_E, c_T, c_D, c_P, σ, τ, ν, p, s *and on the constant* C_{A_∞} *of the weight* w.

To prove Theorem 4.1 we firstly show the embedding into weighted Lorentz spaces then we establish Sobolev type inequalities for any exponent k and we take into account the particular form of the constant appearing in the Sobolev type inequalities (in the spirit of [10]).

Proposition 4.2. *Under assumptions* (L1), (L2), (L3) *with* $\alpha p = \nu$, $p > 1$, $\nu > 1$, $\tau \geq 1$, *and* $0 < c_T\,\tau\,\sigma < 1$ *the following weak type inequality holds: for every* $u \in \mathcal{C}$ *such that*

$$\frac{1}{|B(z,R/\tau)|}\int_{B(z,R/\tau)} u\,dm \leq \frac{1}{2}, \quad (11)$$

we have that

$$\Big|B(z,\sigma R)\cap\{u\geq 1\}\Big|^{\delta(p-1)} \leq c(C^*(\delta))^p R^\nu |B(z,R)|^{\delta(p-1)}\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[u] \quad (12)$$

with $0 < \delta \leq \frac{1}{p}$, *where*

$$C^*(\delta) = \left(\frac{1}{1-(\frac{1}{2})^{\frac{\delta\frac{\nu}{p}(p-1)}{1-\delta}}}\right)^{1-\delta} \quad (13)$$

and c *is a constant depending only on* c_E, c_T, c_D, ν, c_P, τ, σ, *and* p.

Proof. We can assume $\int_{B(z,R)} d\mathcal{L}^{(p)}[u] < \infty$. Set

$$G = \{x \in B(z,\sigma R) :\ u(x) \geq 1\}$$

where $0 < c_T\,\tau\,\sigma < 1$. We consider a point $x \in G$ and then define $B = B(z,R)$, $\hat{B} = B(z,R/\tau)$, $B_k = B(x,r_k)$, $\hat{B}_k = B(x,r_k/\tau)$, $r_k = 2^{1-k}\theta R$, $k = 1,2,\ldots$ with

$$\theta = \frac{1-c_T\sigma\tau}{2c_T^2}. \quad (14)$$

We note that, for every $k = 1,2,\ldots,$

$$\hat{B}(x,r_k) \subseteq B(x,\frac{2c_T\theta R}{\tau}) \subseteq B(z,\frac{R}{\tau}) :$$

in fact, $r_k \leq r_1 \leq 2c_T\theta R$ and if $\zeta \in B(x,\frac{2c_T\theta R}{\tau})$, then

$$d(\zeta,z) \leq c_T(d(\zeta,x)+d(z,x)) \leq c_T(\frac{1-c_T\sigma\tau}{\tau c_T}+\sigma)R.$$

Let

$$I_\beta^B\mathcal{L}^{(p)}[u](x) = \int_B I_\beta^B(x,y)d\mathcal{L}^{(p)}[u](y). \quad (15)$$

By using (11) and the continuity of u we obtain

$$\begin{aligned}\frac{1}{2} &\leq |u(x) - u_{\hat{B}}| \leq |\sum_{k=1}^{\infty}(u_{\hat{B}_{k+1}} - u_{\hat{B}_k}) + u_{\hat{B}_1} - u_{\hat{B}}| \\ &\leq \left(\sum_{k=1}^{\infty}|u_{\hat{B}_{k+1}} - u_{\hat{B}_k}| + |u_{\hat{B}_1} - u_{\hat{B}}|\right).\end{aligned}$$

Hence assumption (L1) and its consequence (5) imply

$$\begin{aligned}\frac{1}{2} \leq (u(x) - u_{\hat{B}}) &\leq \sum_{k=1}^{\infty} \frac{1}{|\hat{B}_{k+1}|}\int_{\hat{B}_{k+1}} |u - u_{\hat{B}_k}|dm + \frac{1}{|\hat{B}_1|}\int_{\hat{B}_1} |u - u_{\hat{B}}|dm \\ &\leq \sum_{k=1}^{\infty} \frac{|\hat{B}_k|}{|\hat{B}_{k+1}|}\frac{1}{|\hat{B}_k|}\int_{\hat{B}_k} |u - u_{\hat{B}_k}|dm \ + \frac{|\hat{B}|}{|\hat{B}_1|}\frac{1}{|\hat{B}|}\int_{\hat{B}} |u - u_{\hat{B}}|dm \\ &\leq \sum_{k=1}^{\infty} c_D 2^{\nu}\frac{1}{|\hat{B}_k|}\int_{\hat{B}_k} |u - u_{\hat{B}_k}|dm \ + \ c_E\theta^{-\nu}\frac{1}{|\hat{B}|}\int_{\hat{B}} |u - u_{\hat{B}}|dm.\end{aligned}$$

By assumption (L3), we obtain

$$\begin{aligned}\frac{1}{2} \leq |u(x) - u_{\hat{B}}| \leq \sum_{k=1}^{\infty} 2^{\nu}\, c_D\, c_P(\frac{r_k}{\tau})^{\frac{\nu}{p}}\left(\frac{1}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{1/p} \\ + \theta^{-\nu}\, c_E\, c_P(\frac{R}{\tau})^{\frac{\nu}{p}}\left(\frac{1}{|B|}\int_{B} d\mathcal{L}^{(p)}[u]\right)^{1/p}.\end{aligned}$$

We evaluate the first term

$$\begin{aligned}&\sum_{k=1}^{\infty}(r_k)^{\frac{\nu}{p}}\left(\frac{1}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{1/p} = \sum_{k=1}^{\infty} r_k^{\frac{\nu}{p}}\left(\frac{1}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta}\left(\frac{1}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{\delta} \\ &\leq \sum_{k=1}^{\infty} r_k^{\frac{\nu}{p}(1-\delta)}\left(\frac{|B|}{|B_k|}\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta}\left(r_k^{\frac{\nu}{p}}\frac{1}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{\delta} \\ &\leq c_E^{1/p-\delta}\sum_{k=1}^{\infty} r_k^{\frac{\nu}{p}(1-\delta)} r_k^{-\nu(1/p-\delta)} R^{\nu(1/p-\delta)}\left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta}\left(r_k^{\frac{\nu}{p}}\frac{1}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{\delta} \\ &= c_E^{1/p-\delta} R^{\frac{\nu}{p}(1-p\delta)}\left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta}\sum_{k=1}^{\infty} r_k^{\nu\delta(1-1/p)}\left(\frac{r_k^{\nu/p}}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{\delta}.\end{aligned}$$

We apply the Hölder inequality to the sum in the last line of the previous inequality and we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} r_k^{\nu\delta(1-1/p)}\left(\frac{r_k^{\nu/p}}{|B_k|}\int_{B_k} d\mathcal{L}^{(p)}[u]\right)^{\delta} &\leq \Big(\sum_{k=1}^{\infty} r_k^{\nu\delta(1-1/p)\frac{1}{1-\delta}}\Big)^{1-\delta}\cdot\Big(I^B_{\frac{\nu}{p}}\mathcal{L}^{(p)}[u](x)\Big)^{\delta} \\ &= (\theta R)^{\nu\delta(1-1/p)} C^*(\delta)\cdot\Big(I^B_{\frac{\nu}{p}}\,\mathcal{L}^{(p)}\,[u](x)\Big)^{\delta}\end{aligned}$$

where the constant $C^*(\delta)$ is the value given in (13).

Then we have

$$1 \le 2|u(x) - u_{\hat{B}}| \le \frac{2c_P}{\tau^{\frac{\nu}{p}}}\Big\{\theta^{-\nu}\, c_E\, R^{\frac{\nu}{p}} \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}\,[u]\right)^{1/p} +$$

$$+ C^*(\delta) 2^\nu\, c_D\, c_E^{1/p-\delta} \theta^{\nu\delta(1-1/p)} R^{\frac{\nu}{p}(1-\delta)} \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta} \left(I^B_{\frac{\nu}{p}}\, \mathcal{L}^{(p)}\,[u](x)\right)^\delta \Big\}$$

$$:= X + Y \tag{16}$$

Now we discuss two cases.

If the first addendum X in (16) exceeds the second one Y, then we have

$$1 \le 2\,X = \frac{4c_P}{\tau^{\frac{\nu}{p}}}\theta^{-\nu}\, c_E\, R^{\frac{\nu}{p}} \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p}$$

and also $\qquad \dfrac{|G|^{\delta(p-1)}}{|B|^{\delta(p-1)}} \le 1 \le (2\,X)^p = \dfrac{(4c_P)^p}{\tau^\nu}\theta^{-\nu p}\, c_E^p\, R^\nu \left(\dfrac{1}{|B|}\displaystyle\int_B d\mathcal{L}^{(p)}[u]\right)$

and then we have proved (12). If the second addendum Y in (16) exceeds the first one X, then we have

$$1 \le 2\,Y = \frac{4c_P}{\tau^{\frac{\nu}{p}}} C^*(\delta) 2^\nu\, c_D\, c_E^{1/p-\delta} \theta^{\nu\delta(1-1/p)} R^{\frac{\nu}{p}(1-\delta)} \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta} \left(I^B_{\frac{\nu}{p}} \mathcal{L}^{(p)}[u](x)\right)^\delta.$$

We apply Hölder inequality and we obtain:

$$|G| = \int_G d\,m \le \frac{4c_P}{\tau^{\frac{\nu}{p}}} C^*(\delta) 2^\nu\, c_D\, c_E^{1/p-\delta} \theta^{\nu\delta(1-1/p)} R^{\frac{\nu}{p}(1-\delta)} \cdot$$

$$\cdot \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta} \int_G \left(I^B_{\frac{\nu}{p}} \mathcal{L}^{(p)}[u](x)\right)^\delta d\,m$$

$$\le \frac{4c_P}{\tau^{\frac{\nu}{p}}} C^*(\delta) 2^\nu\, c_D\, c_E^{1/p-\delta} \theta^{\nu\delta(1-1/p)} R^{\frac{\nu}{p}(1-\delta)} \cdot$$

$$\cdot \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta} \left(\int_G I^B_{\frac{\nu}{p}} \mathcal{L}^{(p)}[u](x) d\,m\right)^\delta |G|^{1-\delta}.$$

By using Fubini's theorem and Theorem 3.1 with $\beta = \frac{\nu}{p}$ we obtain

$$|G| \le \frac{4c_P}{\tau^{\frac{\nu}{p}}} C^*(\delta) 2^\nu\, c_D\, c_E^{1/p-\delta} \theta^{\nu\delta(1-1/p)} R^{\frac{\nu}{p}(1-\delta)} \cdot$$

$$\cdot \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta} \left(\int_G I^B_{\frac{\nu}{p}} \mathcal{L}^{(p)}[u](x) d\,m\right)^\delta |G|^{1-\delta}$$

$$= \frac{4c_p}{\tau^{\frac{\nu}{p}}} C^*(\delta) 2^\nu\, c_D\, c_E^{1/p-\delta} \theta^{\nu\delta(1-1/p)} R^{\frac{\nu}{p}(1-\delta)} \cdot$$

$$\cdot \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta} \left(\int_B \int_G I^B_{\frac{\nu}{p}} d\,m(x)) d\mathcal{L}^{(p)}[u](y)\right)^\delta |G|^{1-\delta}$$

$$\le \frac{4c_P}{\tau^{\frac{\nu}{p}}} C^*(\delta) 2^\nu\, c_D\, c_E^{1/p-\delta} \theta^{\nu\delta(1-1/p)} R^{\frac{\nu}{p}(1-\delta)} \cdot$$

$$\cdot \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p-\delta} \left(\int_B c\frac{p}{\nu} R^{\frac{\nu}{p}} \left(\frac{|G|}{|B|}\right)^{1/p} d\mathcal{L}^{(p)}[u](y)\right)^\delta |G|^{1-\delta},$$

where c is a constant depending only on c_E, c_T, c_D, ν, p, τ and σ.

Hence $|G|^{1-1+\delta-\frac{\delta}{p}} \leq cC^*(\delta)c_E^{1/p-\delta}\,\theta^{\nu\delta(1-1/p)}(\dfrac{p}{\nu})^{\delta}|B|^{\delta-\frac{\delta}{p}}R^{\frac{\nu}{p}}\left(\dfrac{1}{|B|}\displaystyle\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p}$,

where $C^*(\delta)$ is the constant in (13) and c is a constant depending only on c_E, c_T, c_D, ν, c_P, p, τ and σ. As $\theta^{\nu\delta(1-1/p)} \leq 1$, $c_E^{1/p-\delta} \leq c_E^{1/p}$ and $(\frac{p}{\nu})^{\delta} \leq \max\{1, (\frac{p}{\nu})^{1/p}\}$, we have proved inequality (12). $\square$

Now as in [4] we show that the following embedding into weighted Lorentz spaces holds. We note that condition $(L4)$ plays a role similar to that of the *truncation properties* (see *e.g.* [10] and [20]) which are commonly used for obtaining strong type inequalities from weak type ones.

Proposition 4.3. *Suppose that* (L1), (L2), (L3), (L4) *hold with* $\alpha p = \nu$, $p > 1$, $\nu > 1$, *and* $\tau \geq 1$, $0 < c_T\sigma\,\tau < 1$. *Let* $w \in A_\infty(m)$. *Then, for every* $u \in \mathcal{C}$, *there exists a median* u_M *such that, for every* $0 < \delta \leq \frac{1}{p}$,

$$\int_0^\infty t^{p-1}(w(\{B(z,\sigma R)\cap|u-u_M|>t\}))^{\frac{(p-1)\delta}{s}}dt \leq$$
$$\leq c\,(C^*(\delta))^p\left(\frac{c_D\,C_{A_\infty}^{1/s}}{\sigma^\nu}\right)^{(p-1)\delta}R^\nu w(B(z,\sigma R))^{\frac{(p-1)\delta}{s}}\frac{1}{|B|}\int_{B(z,R)} d\mathcal{L}^{(p)}[u] \tag{17}$$

where the constant c *depends only on* c_E, c_T, c_D, ν, p, c_P, σ, *and* τ.

Proof. Since $w \in A_\infty(m)$ (see (9)) we deduce from inequalities (12) and (3) that for every $v \in \mathcal{C}$ such that $\frac{1}{|B(z,R/\tau)|}\int_{B(z,R/\tau)} v\,dm \leq \frac{1}{2}$ we have

$$\Big(w(B(z,\sigma R)\cap\{v\geq 1\})\Big)^{\frac{\delta(p-1)}{s}} \leq$$
$$\leq c\left(\frac{c_D\,C_{A_\infty}^{1/s}}{\sigma^\nu}\right)^{(p-1)\delta}(C^*(\delta))^pR^\nu w(B(z,\sigma R))^{\frac{(p-1)\delta}{s}}\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v], \tag{18}$$

with the constant c depending only on c_E, c_T, c_D, ν, p, c_P, σ, and τ. We choose u_M in a such way that

$$|B(z,R/\tau)\cap\{u\geq u_M\}| \geq \frac{1}{2}|B(z,R/\tau)|, \tag{19}$$

and
$$|B(z,R/\tau)\cap\{u\leq u_M\}| \geq \frac{1}{2}|B(z,R/\tau)|. \tag{20}$$

Without loss of generality we assume that $u_M = 0$. Let $\Psi \in C^1(\mathbb{R})$ be such that $\Psi = 0$ on $(-\infty, \frac{1}{2}]$, $\Psi = 1$ on $[1,+\infty)$, $0 < \Psi < 1$ on $(\frac{1}{2},1)$, $0 < \Psi' \leq 4$, on $(\frac{1}{2},1)$. For the given function u and a positive number t we set

$$v_t^+(x) = \Psi\left(\frac{u(x)}{t}\right) \text{ for } u\geq 0, \text{ and } v_t^-(x) = \Psi\left(\frac{-u(x)}{t}\right) \text{ for } u<0.$$

Since $v_t^+ \leq 1$, we have (see (20))

$$\frac{1}{|B(z,R/\tau)|}\int_{B(z,R/\tau)} v_t^+\,dm \leq \frac{|B(z,R/\tau)\cap\{v_t^+>0\}|}{|B(z,R/\tau)|} \leq$$
$$\leq \frac{|B(z,R/\tau)\cap\{u>0\}|}{|B(z,R/\tau)|} \leq \frac{1}{2}.$$

Applying (18) to v_t^+ we obtain

$$(w(B(z,\sigma R)\cap\{u>t\}))^{\frac{\delta(p-1)}{s}} \le w(B(z,\sigma R)\cap\{v_t^+\ge 1\})^{\frac{\delta(p-1)}{s}} \le$$

$$\le c\left(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\right)^{(p-1)\delta}(C^*(\delta))^p R^\nu w(B(z,\sigma R))^{\frac{\delta(p-1)}{s}}\,\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+].$$

In a similar way, since $v_t^-\le 1$, we have (see (19))

$$\frac{1}{|B(z,R/\tau)|}\int_{B(z,R/\tau)} v_t^-\,dt \le \frac{1}{2}$$

and applying (18) to v_t^- we obtain

$$(w(B(z,\sigma R)\cap\{-u>t\}))^{\frac{\delta(p-1)}{s}} \le (w(B(z,\sigma R)\cap\{v_t^-\ge 1\}))^{\frac{\delta(p-1)}{s}} \le$$

$$\le c\left(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\right)^{(p-1)\delta}(C^*(\delta))^p R^\nu w(B(z,\sigma R))^{\frac{\delta(p-1)}{s}}\,\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-].$$

Then, we obtain that

$$(w(B(z,\sigma R)\cap\{|u|>t\}))^{\frac{\delta(p-1)}{s}} \le c\left(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\right)^{(p-1)\delta}(C^*(\delta))^p R^\nu w(B(z,\sigma R))^{\frac{\delta(p-1)}{s}}\cdot$$
$$\cdot\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-] + \frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big). \tag{21}$$

By property $(L4)$, we have that

$$\mathcal{L}[v_t^+] \le 4^p t^{-p}\mathcal{L}[u]\chi_{\{t/2<u<t\}} \tag{22}$$

and

$$\mathcal{L}[v_t^-] \le 4^p t^{-p}\mathcal{L}[u]\chi_{\{t/2<-u<t\}}. \tag{23}$$

We obtain by using (22) and (23)

$$(w(B(z,\sigma R)\cap\{|u|>t\}))^{\frac{\delta(p-1)}{s}} \le 4c\,\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{(p-1)\delta}(C^*(\delta))^p R^\nu w(B(z,\sigma R))^{\frac{\delta(p-1)}{s}}\, t^{-p}\cdot$$
$$\cdot\frac{1}{|B|}\left(\int_{B\cap\{t/2<u<t\}} d\mathcal{L}^{(p)}[u] + \int_{B\cap\{-t<u<-t/2\}} d\mathcal{L}^{(p)}[u]\right). \tag{24}$$

We define
$$F^+(t) = \int_{B\cap\{0<u<t\}} d\mathcal{L}^{(p)}[u], \qquad t>0$$

and
$$F^-(t) = \int_{B\cap\{-t<u<0\}} d\mathcal{L}^{(p)}[u], \qquad t>0.$$

From (24) we obtain

$$(w(B(z,\sigma R)\cap\{|u|>t\}))^{\frac{\delta(p-1)}{s}} \le 4c\left(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\right)^{(p-1)\delta}(C^*(\delta))^p R^\nu w(B(z,\sigma R))^{\frac{\delta(p-1)}{s}}\; t^{-p}\cdot$$
$$\cdot\frac{1}{|B|}\Big(F^+(t)-F^+(t/2)+F^-(t)-F^-(t/2)\Big), \tag{25}$$

then
$$\int_0^\infty t^{p-1}(w(B(z,\sigma R)\cap\{|u|>t\}))^{\frac{\delta(p-1)}{s}}\,dt \le$$
$$\le 4c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{(p-1)\delta}(C^*(\delta))^p R^\nu w(B(z,\sigma R))^{\frac{\delta(p-1)}{s}}\cdot$$
$$\cdot\frac{1}{|B|}\int_0^\infty\Big(F^+(t)-F^+(t/2)+F^-(t)-F^-(t/2)\Big)\frac{dt}{t}.$$

Since
$$\int_0^\infty (F^+(t)-F^+(t/2))\frac{dt}{t}=\lim_{T\to\infty}\Big(\int_{2/T}^{2T}F^+(t)\frac{dt}{t}-\int_{2/T}^{2T}F^+(t/2)\frac{dt}{t}\Big)=$$
$$=\lim_{T\to\infty}\Big(\int_{2/T}^{2T}F^+(t)\frac{dt}{t}-\int_{1/T}^{T}F^+(t)\frac{dt}{t}\Big)=$$
$$=\lim_{T\to\infty}\Big(\int_{T}^{2T}F^+(t)\frac{dt}{t}-\int_{1/T}^{2/T}F^+(t)\frac{dt}{t}\Big)\le \ln 2\,F^+(\infty)$$

and, in a similar way, $\int_0^\infty (F^-(t)-F^-(t/2))\frac{dt}{t}\le \ln 2F^-(\infty)$ we obtain
$$\int_0^\infty t^{p-1}(w(B(z,\sigma R)\cap\{|u|>t\}))^{\frac{\delta(p-1)}{s}}\,dt \le$$
$$\le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{(p-1)\delta}(C^*(\delta))^p R^\nu w(B(z,\sigma R))^{\frac{\delta(p-1)}{s}}\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]$$

where the constant c depends only on c_E, c_T, c_D, ν, p, c_P, σ, and τ. □

We proceed now to prove the following weighted Sobolev type inequalities for every $k\ge\overline{k}=\max(p,\frac{sp^2}{p-1})$.

Proposition 4.4. *Suppose that* (L1), (L2), (L3), (L4) *hold with* $\alpha p=\nu$, $p>1$, $\nu>1$, *and* $\tau\ge 1$, $0<c_T\sigma\tau<1$. *Let* $w\in A_\infty(m)$, *then we have for every* $k\ge\overline{k}=\max(p,\frac{sp^2}{p-1})$
$$\left(\frac{1}{|B(z,\sigma R)|}\int\limits_{B(z,\sigma R)}|u-u_M|^k\,w\,dm\right)^{1/k}\le c\,k^{1-\frac{sp}{k(p-1)}}R^{\frac{\nu}{p}}\left(\frac{1}{|B(z,R)|}\int\limits_{B(z,R)}d\mathcal{L}^{(p)}[u]\right)^{1/p}\tag{26}$$

where u_M *is the constant appearing in Proposition* 4.3 *and* c *is a constant depending only on* c_T, ν, p, c_P, c_D, c_E, σ, s, τ *and on the constant* C_{A_∞} *of the weight* w.

Proof. From Proposition 4.3 we deduce that the function $u-u_M$ belongs to the weighted Lorentz space $L_w^{(k^*,p)}(B(z,\sigma R))$ with $k^*=\frac{ps}{(p-1)\delta}$, and (see [6])
$$||u-u_M||^p_{L_w^{(k^*,p)}}\le\tag{27}$$
$$\le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{(p-1)\delta}k^*\,(C^*(\delta))^p\,R^\nu w(B(z,\sigma R))^{\frac{(p-1)\delta}{s}}\frac{1}{|B|}\int_{B(z,R)}d\mathcal{L}^{(p)}[u].$$

In fact, we have (see Theorem 2.1 in [6])

$$||u||^p_{L^{(k^*,p)}_w} := \int_0^\infty (u^*_w(t))^p\, t^{(\frac{p}{k^*}-1)}\, dt = k^* \int_0^\infty (w(\{x \in \Omega : |u(x)| > y\}))^{\frac{p}{k^*}} y^{p-1} dy$$

where u^*_w denotes the decreasing rearrangement of u with respect to the weight w. We note that, if $\delta \le \frac{s}{p-1}$, then $k^* \ge p$ and by embedding properties of Lorentz spaces (see [31])

$$||u - u_M||_{L^{(k^*,k^*)}_w} \le \left(\frac{p}{k^*}\right)^{\frac{1}{p}-\frac{1}{k^*}} \; ||u - u_M||_{L^{(k^*,p)}_w}. \tag{28}$$

Hence
$$\Big(\int_{B(z,\sigma R)} |u - u_M|^{k^*} w\, d\, m\Big)^{\frac{1}{k^*}} \le c\Big(\frac{p}{k^*}\Big)^{\frac{1}{p}-\frac{1}{k^*}} \; (k^*)^{\frac{1}{p}} C^*(\delta)\,\cdot$$
$$\cdot \Big(\frac{c_D\, C^{1/s}_{A_\infty}}{\sigma^\nu}\Big)^{\frac{(p-1)\delta}{p}} R^{\frac{\nu}{p}} (w(B(z,\sigma R))^{\frac{1}{k^*}} \Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[u]\Big)^{\frac{1}{p}}. \tag{29}$$

For every $k \ge \bar{k}$, we choose $\delta = \delta_k = \frac{ps}{(p-1)k}$, and we note that $(k^*)^{\frac{1}{k^*}} \le e^{\frac{1}{e}}$ and

$$\lim_{k\to\infty} k^{-1+\frac{sp}{k(p-1)}} C^*(\delta_k) = \frac{1}{s\nu \ln 2}.$$

Then, from (29) (for every $k \ge \bar{k}$) we derive

$$\left(\frac{1}{|B(z,\sigma R)|}\int\limits_{B(z,\sigma R)} |u - u_M|^k\, w\, dm\right)^{1/k} \le c k^{1-\frac{sp}{k(p-1)}} R^{\frac{\nu}{p}} \left(\frac{1}{|B(z,R)|}\int\limits_{B(z,R)} d\mathcal{L}^{(p)}[u]\right)^{1/p} \tag{30}$$

where c is a constant depending only on c_T, ν, p, c_P, c_D, c_E, σ, s, τ and on the constant C_{A_∞} of the weight w. This completes the proof of Proposition 4.4. $\square$

We are now in position to prove Theorem 4.1

Proof. We remark that, when $k < \bar{k}$, we have the estimate

$$\Big(\frac{1}{|B(z,\sigma R)|}\int_{B(z,\sigma R)} |u - u_M|^k\, w\, dm\Big)^{1/k} \le \Big(\frac{1}{|B(z,\sigma R)|}\int_{B(z,\sigma R)} |u - u_M|^{\bar{k}}\, w\, dm\Big)^{\frac{1}{\bar{k}}}.$$

We expand the exponential in a power series and using the previous estimates to bound the sum of the series:

$$\frac{1}{|B(z,\sigma R)|}\int_{B(z,\sigma R)} \exp\left(\frac{|u - u_M|}{c\, e^{\frac{p}{p-1}}\, R^{\frac{\nu}{p}} \left(\frac{1}{|B(z,R)|}\int_{B(z,R)|} d\mathcal{L}^{(p)}[u]\right)^{1/p}}\right) w\, dm =$$

$$= \sum_{k\ge 0} \frac{1}{|B(z,\sigma R)|}\int_{B(z,\sigma R)} \frac{1}{k!}\left(\frac{|u - u_M|}{c\ e^{\frac{p}{p-1}} R^{\frac{\nu}{p}} \left(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[u]\right)^{1/p}}\right)^k w\, dm =$$

$$= \Big(\sum_{0\le k<[\bar{k}]} + \sum_{k\ge[\bar{k}]} \Big) \frac{1}{|B(z,\sigma R)|} \cdot$$

$$\cdot \int_{B(z,\sigma R)} \frac{1}{k!} \left(\frac{|u-u_M|}{c\, e^{\frac{p}{p-1}} R^{\frac{\nu}{p}} \left(\frac{1}{|B(z,R)|} \int_{B(z,R)} d\mathcal{L}^{(p)}[u] \right)^{1/p}} \right)^k w\, dm \le$$

$$\le c + \sum_{k\ge[\bar{k}]} \frac{k^{k(1-\frac{sp}{k(p-1)})}}{e^{\frac{pk}{p-1}}\, k!} \le c.$$

The convergence of the series follows from Stirling's formula $k! \approx k^k e^{-k}\sqrt{2\pi k}$. □

5. Sharp Trudinger type inequalities

In this section we prove *sharp Trudinger type inequalities* for measure-valued Lagrangeans assuming that the space X is connected (Theorem 5.3). We remark that if the space X is not connected the sharp Trudinger inequality fails (see, for instance, [15] for discussion and examples).

To prove Theorem 5.3 we firstly establish *weak inequalities* (similar to inequalities (12) in Proposition 4.2), and then we use the weak chain rule condition ($L4$) as in the proof of Proposition 4.3.

We recall that each connected doubling space satisfies a chain condition (see e.g. [10] or [16]). This means that for every point x and for any numbers r, R such that $0 < r < R < 1/4\,\mathrm{diam}(X)$ and $X \setminus B(x,R) \neq \emptyset$ we can pick a finite chain of balls that joins $B(x, r/2)$ to $X \setminus B(x, 2R)$ (see [10], p. 29).

According to the approach of [10] (see Section 6 of [10]), in the following Proposition we adapt the chain condition to our setting.

Proposition 5.1. *In the previous assumptions and notations, for every $\tau \ge 1$, there exist constants N, M and L such that, for every point $x \in B(z,\sigma R)$ and for any numbers r, R^* such that $0 < r < R^* \le \frac{R\theta}{4c_T\tau}$ and $B(z,R) \setminus B(x,R^*) \neq \emptyset$, we can pick a finite chain of balls B^*_k, $k = 0, \cdot, \cdot, \cdot, h$ ($h \in \mathbb{N}$) that joins $B(x, r/2)$ to $B(z,R) \setminus B(x, 2R^*)$. The chain has the following properties:*

(1) *$B^*_0 \subset B(z, R/\tau) \setminus B(x,R^*)$ and $B^*_h \subseteq B(x,r)$;*

(2) *$N^{-1} diam(B^*_i) \le dist(x, B^*_i) \le N\, diam(B^*_i)$ for $i = 0, 1, 2. \dots, h$;*

(3) *there is a ball $\tilde{B}_i \subseteq B^*_i \cap B^*_{i+1}$ such that $B^*_i \cup B^*_{i+1} \subseteq M\tilde{B}_i \subseteq B(z,R_0)$ for $i = 0, 1, 2. \dots, h-1$;*

(4) *no point of $B(z,R)$ belongs to more than L balls τB^*_i.*

Proof. For every integer j between 1 and $M^* = 5 + [\lg_2 \frac{c_T^2 R^*}{r}]$, let $A_j = A_j(x)$ denote the annulus $A_j(x) = \{y : \frac{4c_T R^*}{2^j} \le d(y,x) < \frac{4c_T R^*}{2^{j-1}}\}$. For every positive ε, as m is doubling, we can cover each annulus $\bar{A}_j$ by at most N^* (open) balls of radii equal to $\varepsilon\frac{4c_T R^*}{2^j}$ with centers in the annulus A_j where N^* is independent of x and j. The balls B corresponding to the annuli A_j, $j = 1, \cdots, M^*$ together with $B(x, r/2)$, $B(z,R) \setminus \overline{B(x, 2R^*)}$ form an open cover of $B(z,R)$.

As $B(z,R)$ is connected and contains a point inside $B(x,r/2)$ and another point outside $B(x,2R^*)$ we can pick a chain of balls that joins $B(x,r/2)$ to $B(z,R)\setminus B(x,2R^*)$. The required chain $B_i^*,\ i=0,.....h$, is then obtained as the collection of the balls $2B$ from the balls B different from $B(x,r/2)$ in this chain. When ε is sufficiently small, depending only on τ and c_T, the balls $2\tau B$ with B corresponding to A_j and the balls $2\tau B'$ with B' corresponding to A_i do not intersect, provided $|j-i|\geq 2$.

It is easy to verify that if $\varepsilon \leq \frac{1}{4c_T}$ then the ball B_0^* does not intersect the ball $B(x,R^*)$; moreover if $R^*\leq \frac{R\theta}{4\tau c_T}$ then the ball B_0^* is contained in the ball $B(z,R/\tau)$.

To complete the proof of condition (1) we note that the choice of $M^*=5+[\lg_2 \frac{c_T^2R^*}{r}]$ implies that the ball B_h^* is contained the ball $B(x,r)$.

We also point out that the constant N in condition (2) depends only on c_T and ε as we can choose $N=\frac{c_T(1+\varepsilon)}{2\varepsilon}$. □

From Theorem 3.1, we obtain the following *weak type inequalities.* The spirit of the proof is the same of the proof of Theorem 4 in [17] concerning the case $\alpha p<\nu$. In order to treat the case $\alpha p=\nu$, we follow some ideas from [10].

Theorem 5.2. *Suppose that* (L1), (L2) *and* (L3) *hold with* $\alpha p=\nu$, $p>1$, $\nu>1$, $\tau\geq 1$, *and* $0<c_T\tau\sigma<1$. *Assume that* $B(z,R_0)$ *is connected. Let* $u\in\mathcal{C}$, *with*

$$\frac{1}{|B(z,R/\tau)|}\int_{B(z,R/\tau)} u\,dm \leq \frac{1}{2}. \tag{31}$$

Let ε_0 *be a positive number then, for* $q>\max(p,\frac{\nu}{\nu-\varepsilon_0})$,

$$\begin{aligned}&|\{x\in B(z,\sigma R)\,:\,u(x)\geqslant 1\}|^{(\frac{1}{q}-\frac{1}{q^2})}\leq\\ &\leq cq^{\frac{1}{q}}q^{\frac{p-1}{p}}|B|^{(\frac{1}{q}-\frac{1}{q^2})}R^{\frac{\nu}{p}}\left(\frac{1}{|B(z,R)|}\int_{B(z,R)}d\mathcal{L}^{(p)}[u]\right)^{1/p},\end{aligned} \tag{32}$$

where c is a constant depending only on c_E, c_T, c_D, ν, p, τ, c_P, ε_0 and σ.

Proof. We can assume $\int_{B(z,R)}d\mathcal{L}^{(p)}[u]<\infty$. Set

$$G=\{x\in B(z,\sigma R)\,:\,u(x)\geq 1\},$$

where σ satisfies $0<c_T\tau\sigma<1$. We consider a point $x\in G$, (according to Section 2) we denote $B=B(z,R)$, $\hat{B}=B(z,R/\tau)$, and $B_j=B(x,r_j)$ where $r_j=\theta R2^{1-j}$.

For every element $u\in\mathcal{C}$ there exists a radius r_L such that for every ball $\breve{B}\subseteq B(x,r_L)$ we have $|u(x)-u_{\breve{B}}|\leq 1/4$ and, as the point x belongs to G, we have

$$|u(x)-u_{\hat{B}}|=u(x)-u_{\hat{B}}\geq\frac{1}{2}$$

hence $$|u_{\hat{B}}-u_{\breve{B}}|\geq|u_{\hat{B}}-u(x)|-|u(x)-u_{\breve{B}}|\geq 1/4\geq|u(x)-u_{\breve{B}}|$$

and then $$|u(x)-u_{\hat{B}}|\leq 2|u_{\hat{B}}-u_{\breve{B}}|.$$

Now we choose the parameters r and R^* of our chain. We set $r=r_K=\theta R2^{1-K}$ where the integer K is such that $\theta R2^{1-K}\leq r_L$ (see Section 2) and $R^*=\frac{R\theta}{8\tau c_T}$ where

θ is defined in (14). Let $B_k^* = B(x_k^*, r_k^*)$, $k = 0, \cdots, h$ where $r_k^* = \frac{\varepsilon 8R^* c_T}{2^j}$ and $x_k^* \in A_j$ for some $j \in \{1, 2, .., M^*\}$, $\varepsilon = \frac{1}{5c_T\tau}$; we deduce that $\tau B_k^* \subseteq B_j$. Moreover $B_{k+1}^* \bigcup B_k^* \subseteq B_j$, $k = 0, \cdots, h-1$ and there exist at most N^* balls B_k^* of the chain having centers in the annulus A_j.

We consider the chain associated with x, r_K, R^*, hence

$$\frac{1}{2} \leq |u(x) - u_{\hat{B}}| \leq 2|\sum_{k=0}^{h-1}(u_{B_{k+1}^*} - u_{B_k^*}) + u_{B_0^*} - u_{\hat{B}}| \leq$$

$$\leq 2\left(\sum_{k=0}^{h-1}\left(|u_{B_{k+1}^*} - u_{\tilde{B}_k}| + |u_{B_k^*} - u_{\tilde{B}_k}|\right) + |u_{B_0^*} - u_{\hat{B}}|\right).$$

We use the properties of the chain condition (in particular, $\tilde{B}_k \subseteq B_{k+1}^* \cap B_k^*$, $B_{k+1}^* \bigcup B_k^* \subseteq M\tilde{B}_k \subseteq B(z, R_0)$, $k = 0, \cdots h-1$, $B_0^* \subseteq \hat{B} \setminus B(x, R^*)$) and assumptions (3) (its consequence (5)) and (4) to obtain

$$\frac{1}{2} \leq |u(x) - u_{\hat{B}}| \leq 2\left(2M^\nu c_D \sum_{k=0}^{h} \frac{1}{|B_k^*|}\int_{B_k^*}|u - u_{B_k^*}| + c_E(\frac{2NR}{\tau R^*})^\nu \frac{1}{|\hat{B}|}\int_{\hat{B}}|u - u_{\hat{B}}|\right) \leq$$

$$\leq 2c_P\Big\{2M^\nu c_D \sum_{k=1}^{h}(r_k^*)^{\frac{\nu}{p}}\Big(\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\Big)^{1/p} +$$

$$+ 2M^\nu c_D (r_0^*)^{\frac{\nu}{p}}\Big(\frac{1}{|\tau B_0^*|}\int_{\tau B_0^*} d\mathcal{L}^{(p)}[u]\Big)^{1/p} + c_E(\frac{2NR}{\tau R^*})^\nu R^{\frac{\nu}{p}}\Big(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\Big)^{1/p}\Big\} \leq$$

$$\leq 2c_P\left\{2M^\nu c_D \sum_{k=1}^{h}(r_k^*)^{\frac{\nu}{p}}\left(\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\right)^{1/p} +\right.$$

$$\left. + \left(2M^\nu c_D (c_E)^{\frac{1}{p}}\tau^{\frac{-\nu}{p}} + c_E(\frac{2NR}{\tau R^*})^\nu\right) R^{\frac{\nu}{p}}\left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p}\right\}$$

(we recall that $\frac{R}{R^*} = \frac{8c_T\tau}{\theta}$).

We evaluate the first term in the previous estimate as in the proof of Proposition 4.2, more precisely for $q > p$

$$\sum_{k=1}^{h}(r_k^*)^{\frac{\nu}{p}}\left(\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\right)^{1/p} =$$

$$= \sum_{k=1}^{h}(r_k^*)^{\frac{\nu}{p} - \frac{\nu}{q^2}}\left(\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p} - \frac{1}{q}}\left((r_k^*)^{q\frac{\nu}{q^2}}\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{q}} =$$

$$= \sum_{k=1}^{h}|\tau B_k^*|^{\frac{1}{q} - \frac{1}{p}}(r_k^*)^{\frac{\nu}{p} - \frac{\nu}{q^2}}\left(\int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p} - \frac{1}{q}}\left((r_k^*)^{\frac{\nu}{q}}\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{q}}.$$

By Hölder's inequality we obtain

$$\sum_{k=1}^{h}(r_k^*)^{\frac{\nu}{p}}\left(\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*}d\mathcal{L}^{(p)}[u]\right)^{1/p}\le$$

$$\le\left(\sum_{k=1}^{h}(|\tau B_k^*|^{\frac{1}{q}-\frac{1}{p}}(r_k^*)^{\frac{\nu}{p}-\frac{\nu}{q^2}})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\left(\sum_{k=1}^{h}\int_{\tau B_k^*}d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p}-\frac{1}{q}}\cdot$$

$$\cdot\left(\sum_{k=1}^{h}(r_k^*)^{\frac{\nu}{q}}\frac{1}{|\tau B_k^*|}\int_{\tau B_k^*}d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{q}}.$$

The first term can be estimated by summing over the balls B_k^* corresponding to an annulus $A_j = A(\frac{4c_TR^*}{2^j}, \frac{4c_TR^*}{2^{j-1}})$. By the construction of the chain there are at most N^* balls of radii equal to $\varepsilon\frac{4c_TR^*}{2^j}$ with N^* independent of x and j. Let be $I_{k,j}$ the set of indices k corresponding to A_j. Then

$$\left(\sum_{k=1}^{h}(|\tau B_k^*|^{\frac{1}{q}-\frac{1}{p}}(r_k^*)^{\frac{\nu}{p}-\frac{\nu}{q^2}})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}=\left(\sum_{j=1}^{M^*}\sum_{I_{k,j}}(|\tau B_k^*|^{\frac{1}{q}-\frac{1}{p}}(r_k^*)^{\frac{\nu}{p}-\frac{\nu}{q^2}})^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\le$$

$$\le(c_E)^{1/p-1/q}(N^*)^{\frac{p-1}{p}}\left(\sum_{j=1}^{M^*}\left(|B|^{\frac{1}{q}-\frac{1}{p}}(\frac{\tau\varepsilon 4c_TR^*}{2^jR})^{\nu(\frac{1}{q}-\frac{1}{p})}(\frac{\varepsilon 4c_TR^*}{2^j})^{\frac{\nu}{p}-\frac{\nu}{q^2}}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\le$$

$$\le(N^*)^{\frac{p-1}{p}}(c_E)^{1/p-1/q}(\varepsilon 4c_TR^*)^{\frac{\nu}{q}-\frac{\nu}{q^2}}(\tau)^{-\frac{\nu}{p}+\frac{\nu}{q}}|B|^{\frac{1}{q}-\frac{1}{p}}R^{\frac{\nu}{p}-\frac{\nu}{q}}\left(\frac{1}{2^{(\frac{\nu}{q}-\frac{\nu}{q^2})\frac{p}{p-1}}-1}\right)^{\frac{p-1}{p}}\le$$

$$\le(4c_T)^{\frac{\nu}{q}-\frac{\nu}{q^2}}(N^*)^{\frac{p-1}{p}}(c_E)^{1/p-1/q}|B|^{\frac{1}{q}-\frac{1}{p}}R^{\frac{\nu}{p}-\frac{\nu}{q^2}}\left(\frac{1}{2^{(\frac{\nu}{q}-\frac{\nu}{q^2})\frac{p}{p-1}}-1}\right)^{\frac{p-1}{p}}=$$

$$=(c(q))^{\frac{p-1}{p}}|B|^{\frac{1}{q}-\frac{1}{p}}R^{\frac{\nu}{p}-\frac{\nu}{q^2}}\le cq^{\frac{p-1}{p}}|B|^{\frac{1}{q}-\frac{1}{p}}R^{\frac{\nu}{p}-\frac{\nu}{q^2}},$$

with the constant c depending only on $c_E,\ \nu,\ p,\ N^*$. Here we have used the fact that $\varepsilon\le 1,\ \ \tau\ge 1,\ \ R\ge R^*$ and

$$\lim_{q\to\infty}\frac{c(q)}{q}=N^*c_E^{\frac{1}{p-1}}\ \frac{p-1}{\nu p\ln 2}.$$

The second term can be estimated by using property (4) of the chain condition. We note that the inclusion $\tau B_k^*\subseteq B_j$ holds for every $k=1,\cdot,\cdot,h$ for some $j\in\{1,....M^*\}$ due to the choice of the external radius R^* and we have $B_j\subseteq B$.

Hence we obtain $$\left(\sum_{k=1}^{h}\int_{\tau B_k^*}d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p}-\frac{1}{q}}\le\left(L\int_B d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p}-\frac{1}{q}}.$$

The third term can be estimated by using the previous subdivision according annuli A_j, taking into account that in the chain there are at most N^* balls of radii equal to $\varepsilon\frac{4c_TR^*}{2^j}$ with N^* independent of j. We recall that according to definition (15)

$$I_\beta^B\mathcal{L}^{(p)}[u](x)=\int_B I_\beta^B(x,y)d\mathcal{L}^{(p)}[u](y).$$

By the choice of the external radius R^* we have $\tau B_k^* \subseteq B_j \subseteq B_1$

$$\left(\sum_{k=1}^{h} (r_k^*)^{\frac{\nu}{q}} \frac{1}{|\tau B_k^*|} \int_{\tau B_k^*} d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{q}} \leq$$

$$\leq \left(c_E\, N^* \tau^{-\nu} (\frac{2\theta R}{\varepsilon 4 c_T R^*})^{\nu(1-\frac{1}{q})} \sum_{j=1}^{M^*} (r_j)^{\frac{\nu}{q}} \frac{1}{|B_j|} \int_{B_j} d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{q}} \leq c \left(I^B_{\frac{\nu}{q}} \mathcal{L}[u](x)\right)^{\frac{1}{q}}$$

with the constant c depending only on c_E, c_T, ν, σ, p, N^*, τ and θ. Then, we have

$$1 \leq c R^{\frac{\nu}{p}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{1/p} +$$
$$+ c q^{\frac{p-1}{p}} R^{\frac{\nu}{p} - \frac{\nu}{q^2}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p} - \frac{1}{q}} \left(I^B_{\frac{\nu}{q}} \mathcal{L}^{(p)}[u](x)\right)^{\frac{1}{q}}. \tag{33}$$

If the first term on the right hand side of (33) exceeds the second one, that is,

$$1 \leq c R^{\frac{\nu}{p}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{1/p}$$

then we obtain immediately

$$|G|^{(\frac{1}{q} - \frac{1}{q^2})} \leq |B|^{(\frac{1}{q} - \frac{1}{q^2})} c R^{\frac{\nu}{p}} \left(\frac{1}{|B(z,R)|} \int_{B(z,R)} d\mathcal{L}^{(p)}[u]\right)^{1/p}. \tag{34}$$

If the second term on the right hand side of (33) exceeds the first one, that is,

$$1 \leq c q^{\frac{p-1}{p}} R^{\frac{\nu}{p} - \frac{\nu}{q^2}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p} - \frac{1}{q}} \left(I^B_{\frac{\nu}{q}} \mathcal{L}^{(p)}[u](x)\right)^{\frac{1}{q}}, \tag{35}$$

then by Hölder's inequality, we obtain

$$|G| \leq c q^{\frac{p-1}{p}} R^{\frac{\nu}{p} - \frac{\nu}{q^2}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{\frac{1}{p} - \frac{1}{q}} \int_G \left(I^B_{\frac{\nu}{q}} \mathcal{L}^{(p)}[u](x)\right)^{\frac{1}{q}} dm(x) \leq$$
$$\leq c q^{\frac{p-1}{p}} R^{\frac{\nu}{p} - \frac{\nu}{q^2}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{1/p - 1/q} \left(\int_G I^B_{\frac{\nu}{q}} \mathcal{L}^{(p)}[u](x) dm(x)\right)^{\frac{1}{q}} |G|^{1 - \frac{1}{q}}.$$

We use Fubini's theorem and Theorem 3.1 for $\beta = \frac{\nu}{q}$ with $q > \max(p, \frac{\nu}{\nu - \varepsilon_0})$ and we obtain

$$|G| \leq c q^{\frac{p-1}{p}} R^{\frac{\nu}{p} - \frac{\nu}{q^2}} \left(\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]\right)^{1/p - 1/q} \left(\int_B \left(\int_G I^B_{\frac{\nu}{q}}(x,y) dm(x)\right) d\mathcal{L}^{(p)}[u](y)\right)^{\frac{1}{q}} |G|^{1 - \frac{1}{q}} \leq$$
$$\leq c (\frac{q}{\nu})^{\frac{1}{q}} q^{\frac{p-1}{p}} R^{\frac{\nu}{p} - \frac{\nu}{q^2}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{1/p - 1/q} \left(\int_B d\mathcal{L}^{(p)}[u](y)\right)^{\frac{1}{q}} |G|^{1 - \frac{1}{q}} R^{\frac{\nu}{q^2}} \left(\frac{|G|}{|B|}\right)^{\frac{1}{q^2}} \leq$$
$$\leq c q^{\frac{1}{q}} q^{\frac{p-1}{p}} R^{\frac{\nu}{p}} \left(\frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u]\right)^{1/p} |G|^{1 - \frac{1}{q}} \left(\frac{|G|}{|B|}\right)^{1/q^2} |B|^{\frac{1}{q}},$$

$$\text{that is,}\quad |G|^{(\frac{1}{q}-\frac{1}{q^2})} \leq c q^{\frac{1}{q}} q^{\frac{p-1}{p}} |B|^{(\frac{1}{q}-\frac{1}{q^2})} R^{\frac{\nu}{p}} \left(\frac{1}{|B(z,R)|} \int_{B(z,R)} d\mathcal{L}^{(p)}[u] \right)^{1/p} \tag{36}$$

with the constant c depending only on c_E, c_T, c_D, ν, σ, τ, p, N, M, L and c_P. This concludes the proof. □

Now we prove *sharp Trudinger type inequalities* for measure-valued Lagrangeans. More precisely, we establish the following theorem.

Theorem 5.3. *Suppose that* (L1), (L2), (L3), *and* (L4) *hold with* $\alpha p = \nu$, $p > 1$, $\nu > 1$, $\tau \geq 1$ *and* $0 < c_T \sigma \tau < 1$. *Assume that* $B(z,R)$ *is connected and* $w \in A_\infty(m)$ (*see* (9)). *Then*

$$\inf_{C\in\mathbb{R}} \left(\int_0^\infty \frac{t^{p-1}}{(\ln(\frac{ew(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u-C|>t\})}))^{p-1}}\, dt \right) \leq c R^\nu \frac{1}{|B|} \int_B d\mathcal{L}^{(p)}[u] \tag{37}$$

for every $u \in \mathcal{C}$, *with the constant* c *depending only on* c_E, c_T, c_D, ν, p, c_P, σ, τ , s *and the constant* C_{A_∞} *of the weight* w.

Remark 5.4. We note that inequality (37) can be compared to embedding (2). In fact we have

$$\int_0^{|\Omega|} (u^*(t))^p \Big(\ln(\frac{|\Omega|}{t}) \Big)^{-p} \frac{dt}{t} = \frac{p}{p-1} \int_0^\infty \Big(\ln(\frac{|\Omega|}{|\{x \in \Omega : |u(x)| > y\}|}) \Big)^{1-p} y^{p-1} dy,$$

where $u^*(t)$ denotes the non-increasing rearrangement of u (see, for example, [18] and Theorem 2.1 in [6]). □

Proof. Since $w \in A_\infty(m)$ (see (9)) we deduce from inequalities (32) and (3) that for every $v \in \mathcal{C}$ such that $\frac{1}{|B(z,R/\tau)|} \int_{B(z,R/\tau)} v\, dm \leq \frac{1}{2}$

$$\begin{aligned} &(w(B(z,\sigma R) \cap \{v \geq 1\})^{\frac{p(q-1)}{q^2 s}} \leq \\ &\leq c^p \left(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu} \right)^{\frac{p(q-1)}{q^2}} q^{\frac{p}{q}+\frac{p}{p'}} R^\nu w(B(z,\sigma R))^{\frac{p(q-1)}{q^2 s}} \frac{1}{|B(z,R)|} \int_{B(z,R)} d\mathcal{L}^{(p)}[v], \end{aligned} \tag{38}$$

where $q > \max\{p, \frac{\nu}{\nu-\varepsilon_0}\}$, we denote $p' = \frac{p}{p-1}$ and the constant c depends only on c_E, c_T, c_D, ν, p, c_P, σ, ε_0 and τ. As in the proof of Proposition 4.3 let u_M be a constant (depending both on the ball B and the function u) such that

$$|B(z,R/\tau) \cap \{u \geq u_M\}| \geq \frac{1}{2} |B(z,R/\tau)| \text{ and } |B(z,R/\tau) \cap \{u \leq u_M\}| \geq \frac{1}{2} |B(z,R/\tau)|.$$

Without loss of generality we assume that $u_M = 0$. Let Ψ be a C^1 function on $\mathbb{R}$ such that $\Psi = 0$ on $(-\infty, \frac{1}{2}]$, $\Psi = 1$ on $[1,\infty)$, $0 < \Psi < 1$ on $(\frac{1}{2}, 1)$, $0 < \Psi' \leq 4$ on $(\frac{1}{2}, 1)$. For the given function u and a positive number t we set

$$v_t^+(x) = \Psi\left(\frac{u(x)}{t}\right) \text{ if } u \geq 0, \text{ and } v_t^-(x) = \Psi\left(\frac{-u(x)}{t}\right) \text{ if } u < 0.$$

Since $v_t^+ \le 1$, we have

$$\frac{1}{|B(z,R/\tau)|}\int_{B(z,R/\tau)} v_t^+\,dt \le \frac{|B(z,R/\tau)\cap\{v_t>0\}|}{|B(z,R/\tau)|}\,dt \le$$
$$\le \frac{|B(z,R/\tau)\cap\{u>0\}|}{|B(z,R/\tau)|} \le \frac{1}{2}.$$

Applying (38) to v_t^+ we obtain

$$(w(B(z,\sigma R)\cap\{u>t\}))^{\frac{q-1}{q^2 s}} \le (w(B(z,\sigma R)\cap\{v_t^+\ge 1\}))^{\frac{q-1}{q^2 s}} \le$$
$$\le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{\frac{q-1}{q^2}} q^{\frac{1}{q}} q^{\frac{p-1}{p}} w(B(z,\sigma R))^{\frac{q-1}{q^2 s}} R^{\frac{\nu}{p}}\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big)^{1/p}.$$

In a similar way, applying (38) to v_t^-, since

$$\frac{1}{|B(z,R/\tau)|}\int_{B(z,R/\tau)} v_t^-\,dt \le \frac{1}{2}$$

we obtain

$$(w(B(z,\sigma R)\cap\{-u>t\}))^{\frac{q-1}{q^2 s}} \le (w(B(z,\sigma R)\cap\{v_t^-\ge 1\})|^{\frac{q-1}{q^2 s}} \le$$
$$\le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{\frac{q-1}{q^2}} q^{\frac{1}{q}} q^{\frac{p-1}{p}} w(B(z,\sigma R))^{\frac{q-1}{q^2 s}} R^{\frac{\nu}{p}}\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-]\Big)^{1/p}.$$

Then, we obtain that

$$(w(B(z,\sigma R)\cap\{|u|>t\}))^{\frac{q-1}{q^2 s}} \le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{\frac{q-1}{q^2}} q^{\frac{1}{q}} q^{\frac{p-1}{p}} w(B(z,\sigma R))^{\frac{q-1}{q^2 s}}\; R^{\frac{\nu}{p}}\cdot$$
$$\cdot\bigg(\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big)^{1/p} + \Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-]\Big)^{1/p}\bigg)$$

that is
$$\Big(\frac{w(B(z,\sigma R)\cap\{|u|>t\})}{w(B(z,\sigma R))}\Big)^{\frac{q-1}{q^2 s}} \le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{\frac{q-1}{q^2}} q^{\frac{1}{q}} q^{\frac{p-1}{p}} R^{\frac{\nu}{p}}\cdot$$
$$\cdot\bigg(\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big)^{1/p} + \Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-]\Big)^{1/p}\bigg). \tag{39}$$

Let
$$q = \gamma\frac{p\ln\big(\frac{e\,w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}\big)}{p-1} \tag{40}$$

with $\gamma = \max(\frac{1}{s},\, p-1,\, \frac{\nu(p-1)}{p(\nu-1)})$ then (in particular) $\frac{\gamma}{q} \ge \frac{q-1}{q^2 s}$ and from (39) we derive

$$\Big(\frac{w(B(z,\sigma R)\cap\{|u|>t\})}{e\,w(B(z,\sigma R))}\Big)^{\frac{\gamma}{q}} \le e^{\frac{-\gamma}{q}}\Big(\frac{w(B(z,\sigma R)\cap\{|u|>t\})}{w(B(z,\sigma R)}\Big)^{\frac{q-1}{q^2 s}} \le$$
$$\le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{\frac{q-1}{q^2}} e^{\frac{-\gamma}{q}} q^{\frac{1}{q}} q^{\frac{p-1}{p}} R^{\frac{\nu}{p}}\cdot$$
$$\cdot\bigg(\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big)^{1/p} + \Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-]\Big)^{1/p}\bigg).$$

From (40) we derive
$$e^{\frac{p-1}{p}} = \Big(\frac{e\, w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}\Big)^{\frac{\gamma}{q}} \tag{41}$$

and hence (taking into account (41))

$$1 \le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{\frac{q-1}{q^2}} e^{\frac{p-1}{p}}\, e^{\frac{-\gamma}{q}}\, q^{\frac{1}{q}} q^{\frac{p-1}{p}} R^{\frac{\nu}{p}} \cdot$$
$$\cdot\Bigg(\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big)^{1/p} + \Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-]\Big)^{1/p}\Bigg). \tag{42}$$

Again from (40) and (42) we derive

$$\frac{1}{(\ln(\frac{e\, w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}))^{\frac{p-1}{p}}} \le c\Big(\frac{c_D\, C_{A_\infty}^{1/s}}{\sigma^\nu}\Big)^{\frac{q-1}{q^2}} \Big(\frac{p\gamma}{p-1}\Big)^{\frac{(p-1)}{p}} e^{\frac{p-1}{p}}\, e^{\frac{-\gamma}{q}}\, q^{\frac{1}{q}} R^{\frac{\nu}{p}} \cdot$$
$$\cdot\Bigg(\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big)^{1/p} + \Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-]\Big)^{1/p}\Bigg),$$

and taking into account that $q^{\frac{1}{q}} \le e^{\frac{1}{e}}$ and $e^{\frac{-\gamma}{q}} \le 1$ also

$$\frac{1}{(\ln(\frac{e\, w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}))^{\frac{p-1}{p}}} \le \tag{43}$$
$$\le c\, R^{\frac{\nu}{p}}\Bigg(\Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^+]\Big)^{1/p} + \Big(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[v_t^-]\Big)^{1/p}\Bigg),$$

where the constant c depends only on c_E, c_T, c_D, ν, c_P, σ, p, τ, s and C_{A_∞}. By property $(L4)$, we have that

$$\mathcal{L}[v_t^+] \le 4^p t^{-p}\mathcal{L}[u]\chi_{\{t/2<u<t\}} \tag{44}$$

and
$$\mathcal{L}[v_t^-] \le 4^p t^{-p}\mathcal{L}[u]\chi_{\{t/2<-u<t\}}. \tag{45}$$

We obtain by using (44) and (45)

$$\frac{1}{(\ln(\frac{e\, w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}))^{p-1}} \le \tag{46}$$
$$\le cR^\nu t^{-p}\frac{1}{|B|}\Big(\int_{B\cap\{t/2<u<t\}} d\mathcal{L}^{(p)}[u] + \int_{B\cap\{-t<u<-t/2\}} d\mathcal{L}^{(p)}[u]\Big).$$

We define

$$F^+(t) = \int\limits_{B\cap\{0<u<t\}} d\mathcal{L}^{(p)}[u] \ \text{ for } t>0, \quad \text{and} \quad F^-(t) = \int\limits_{B\cap\{-t<u<0\}} d\mathcal{L}^{(p)}[u] \text{ for } t>0.$$

From (46) we obtain

$$\frac{1}{(\ln(\frac{e\, w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}))^{p-1}} \le cR^\nu t^{-p}\frac{1}{|B|}((F^+(t)-F^+(t/2))+(F^-(t)-F^-(t/2))) \tag{47}$$

and then
$$\int_0^\infty \frac{t^{p-1}}{(\ln(\frac{e\,w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}))^{p-1}}\,dt \le$$
$$\le cR^\nu \frac{1}{|B|}\int_0^\infty ((F^+(t)-F^+(t/2))+(F^-(t)-F^-(t/2)))\frac{dt}{t}.$$

Since
$$\int_0^\infty (F^+(t)-F^+(t/2))\frac{dt}{t} = \lim_{T\to\infty}\left(\int_{2/T}^{2T} F^+(t)\frac{dt}{t} - \int_{2/T}^{2T} F^+(t/2)\frac{dt}{t}\right) =$$
$$= \lim_{T\to\infty}\left(\int_{2/T}^{2T} F^+(t)\frac{dt}{t} - \int_{1/T}^{T} F^+(t)\frac{dt}{t}\right) = \lim_{T\to\infty}\left(\int_{T}^{2T} F^+(t)\frac{dt}{t} - \int_{1/T}^{2/T} F^+(t)\frac{dt}{t}\right)$$
$$\le \ln 2\, F^+(\infty)$$

and, in a similar way, $\displaystyle\int_0^\infty (F^-(t)-F^-(t/2))\frac{dt}{t} \le \ln 2F^-(\infty)$ we obtain
$$\int_0^\infty \frac{t^{p-1}}{(\ln(\frac{e\,w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u|>t\})}))^{p-1}}\,dt \ \le cR^\nu\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u].$$

Hence
$$\int_0^\infty \frac{t^{p-1}}{(\ln(\frac{e\,w(B(z,\sigma R))}{w(B(z,\sigma R)\cap\{|u-u_M|>t\})}))^{p-1}}\,dt \le cR^\nu\frac{1}{|B|}\int_B d\mathcal{L}^{(p)}[u]$$

for every $u\in\mathcal{C}$, with the constant c depending only on c_E, c_T, c_D, ν, p, c_P, σ, τ, s and C_{A_∞} of the weight w. This completes the proof of Theorem 5.3. □

It is well know that estimate (37) in Theorem 5.3 implies the following Trudinger type inequality: we refer e.g. to the paper [2]. Actually estimate (37) is *stronger* than estimate (48) in fact the integral in (48) may be finite in cases when the integral in (37) diverges, see e.g. Section 3 in [11].

Theorem 5.5. *Suppose that* (L1), (L2), (L3), (L4) *hold with* $\nu>1$, $\alpha p=\nu$, $p>1$, $\nu>1$, *and* $\tau\ge 1$, $0<c_T\,\sigma\,\tau<1$. *Let* $w\in A_\infty(m)$ *and* $B(z,R)$ *be connected. Then for every* $u\in\mathcal{C}$,
$$\inf_{C\in\mathbb{R}}\frac{1}{|B(z,\sigma R)|}\int\limits_{B(z,\sigma R)}\exp\left(\frac{|u-C|}{c_3R^{\frac{\nu}{p}}\left(\frac{1}{|B(z,R)|}\int_{B(z,R)} d\mathcal{L}^{(p)}[u]\right)^{1/p}}\right)^{\frac{p}{p-1}} w\,dm \le c_4, \quad (48)$$

where the constants c_3 *and* c_4 *depend only on* c_E, c_T, c_D, c_P, σ, τ, ν, p, s *and on the constant* C_{A_∞} *of the weight* w.

Remark 5.6. Theorem 5.5 can be *directly* proved by repeating the steps of the proof of Theorem 4.1 and by replacing Proposition 4.2 by Theorem 5.2. □

Acknowledgements. The authors express their gratitude to Professor Umberto Mosco for being an inexhaustible source of ideas and perspectives.

The authors wish to thank the anonymous referees for many suggestions, which were useful to improve the overall presentation.

The authors are members of GNAMPA (INdAM) and are partially supported by Grant Ateneo "Sapienza" 2019.

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