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Weak Maximum Principle for Cooperative Systems: the Degenerate Elliptic Case

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Dedicated to Umberto Mosco on the occasion of his 80th birthday.

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We consider cooperative systems of nonlinear degenerate elliptic equations. Conditions for the validity of the weak Maximum Principle are obtained through a reduction to a single scalar equation. A suitable index related to the principal eigenvalues of the Dirichlet problems for the operators involved in the system is introduced. The positivity of this index enforces the validity of the weak Maximum Principle.

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1. Introduction and main results

We consider systems of elliptic partial differential inequalities of the form

$$F[u] + C(x)u \geq 0 \tag{1}$$

Here $u = (u_1 \dots u_N)$ is a vector-valued function $u : \mathbb{R}^n \to \mathbb{R}^N$, which is intended either as a row or as a column on the occasion. Furthermore, $C(x) = (c_{ij}(x))$ is a $N \times N$ matrix-valued function and $F = (F_1, ..., F_N)$ are second order operators acting on u, possibly in some weak sense, of the form

$$F_i[u] = F_i(x, u_i, Du_i, D^2 u_i), \ \ i = 1 \dots N. \tag{2}$$

We will often refer to a solution of (1) as a subsolution of the system $F[u]+C(x)u = 0$.

The vector differential inequality (1) is meant to hold componentwise :

$$F_i[u] + \sum_{j=1}^{N} c_{ij}(x)u_j \geq 0, \ \ i = 1 \dots N. \tag{3}$$

A number of papers, see for example [1], [10], [29] for linear F_i and [13] for fully nonlinear operators, considered such systems in order to establish the validity of the

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weak Maximum Principle, namely the sign propagation property:

$$u_i \leq 0 \text{ on } \partial\Omega \text{ for all } i = 1 \dots N \text{ implies } u_i \leq 0 \text{ in } \Omega \text{ for all } i = 1 \dots N, \qquad (4)$$

where Ω is a bounded domain of $\mathbb{R}^n$. The boundary condition $u_i \leq 0$ is understood as

$$\limsup_{x \to \partial\Omega} u_i \leq 0 \qquad (5)$$

The results obtained in those papers require uniform ellipticity of the operators F_i, that is there are positive constants λ and Λ such that

$$\lambda \mathrm{Tr}(Y - X) \leq F_i(x, t, \xi, Y) - F_i(x, t, \xi, X) \leq \Lambda \mathrm{Tr}(Y - X) \qquad (6)$$

whenever $X \leq Y$, in the sense that $Y - X$ is a positive semidefinite matrix, and that $C(x) = (c_{ij}(x))_{i,j=1\dots N}$ is a cooperative matrix. By this we mean that the following conditions hold (see in particular [13]).

$$c_{ij}(x) \geq 0 \;\; \forall\, i \neq j, \quad \sum_{j=1}^{N} c_{ij}(x) \leq 0, \quad i = 1 \dots N\,, \qquad (7)$$

In this paper we present some new results about the sign propagation property for such systems in the more general context of operators F_i satisfying the weaker degenerate ellipticity condition

$$F_i(x, t, \xi, X) \leq F_i(x, t, \xi, Y) \;\; \forall\, (x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n. \qquad (8)$$

whenever $X \leq Y$. Basic examples of such operators are linear operators of the form

$$F_i(x, u_i, Du_i, D^2u_i) = \mathrm{Tr}\big(A^i(x) D^2 u_i\big) + b^i(x) \cdot Du_i + c^i(x) u_i, \qquad (9)$$

where $A^i(x)$ are symmetric positive semidefinite matrices, $b^i(x)$ vectors in $\mathbb{R}^n$ and $c^i(x) \in \mathbb{R}$ for all $x \in \Omega$. More general examples are provided by the Bellman type operators

$$F_i(x, u_i, Du_i, D^2u_i) = \sup_{\gamma \in \Gamma} \big[\mathrm{Tr}\big(A^i_\gamma(x) D^2 u_i\big) + b^i_\gamma(x) \cdot Du_i + c^i_\gamma(x) u_i\big] \qquad (10)$$

where γ is a parameter running in an arbitrary set Γ. Other examples comprise the partial trace operators

$$P_{(\alpha)}[u_i] = \sum_{h=1}^{k} \lambda_{\alpha_h}(D^2 u_i) \qquad (11)$$

where $\alpha = \{\alpha_1 \dots \alpha_k\}$ is a combination of k among the first n positive integers and λ_h is the h-th greatest eigenvalue of the Hessian matrix $D^2 u_i$. Such a class of Hessian operators contains as subclass the extremal partial trace operators

$$P_k^-[u_i] = \sum_{h=1}^{k} \lambda_h(D^2 u_i), \quad P_k^+[u_i] = \sum_{h=n-k+1}^{n} \lambda_h(D^2 u_i) \qquad (12)$$

which received a growing attention in the last decades. Such operators have been introduced in connection with geometric problems of partial mean curvature and more recently interpreted as models of stochastic differential games, see for instance [11], [12], and [23].

Note that these operators are uniformly elliptic only in the case $k = n$, when they reduce to the Laplace operator Δ, while they are degenerate, non uniformly elliptic for $k < n$.

For maximum principles and other basic topics in the scalar case we refer to [2], [15], [23], [25], [31]. Similar features are shared by the directional elliptic operators

$$D_{(\alpha)}[u_i] = \sum_{h=1}^{k} D_{\alpha_h \alpha_h} u_i + b(x) \cdot Du_i \tag{13}$$

where $D_{\alpha_h \alpha_h} = \partial^2 / \partial x_{a_h}^2$. These operators model some stationary directional convection diffusion processes, see for instance [33]. For maximum principles in narrow domains, possibly unbounded, we refer to [15],[20],[21].

To be more specific on our framework and notations, let n and N be positive integers and denote by $\mathcal{S}^n$ the set of all $n \times n$ real symmetric matrices endowed with the partial order $X \le Y$ if and only if $Y - X$ positive semidefinite. We assume a separate dependence of the components of $F = (F_1 \dots F_N)$ on the respective set of variables $(t_i, \xi_i, X_i) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ for $i = 1 \dots N$, that is $F_i : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ where Ω is a bounded domain of $\mathbb{R}^n$.

We will also assume on $F = (F_1 \dots F_N)$ the minimal amount of ellipticity:

(A1) F_i is degenerate elliptic for all $i = 1 \dots N$, that is (8) holds.

The above condition allows first order operators $F_i(x, t, \xi, X) = F_i(x, t, \xi)$, that include those ones of transport or Hamilton-Jacobi type.

We also make technical assumptions

(A2) $F_i[0] = F_i(x, 0, 0, 0) = 0$ for all $x \in \Omega$

(A3) $F_i = F(x, t, \xi, X)$ continuous in $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N$

Concerning $C(x) = (c_{ij}(x)) \in \mathcal{M}^N$, the space of the $N \times N$ real matrices, we assume:

(C1) $C \in C(\Omega; \mathcal{M}^N)$, that is C is a continuous mapping from Ω to $\mathcal{M}^N$

(C2) C is cooperative, that is (7) holds.

Observe that (C2) implies $c_{ii}(x) \le -\sum_{j \ne i} c_{ij}(x) \le 0$ for $i = 1 \dots N$ in agreement with what is well-known in relation with the weak Maximum Principle in the diagonal case $c_{ij} \equiv 0$ for $i \ne j$. See [7] for further properties of such matrices.

In the present paper we revisit the argument developed in [13] for uniformly elliptic cooperative systems in order to adapt it to the general case of degenerate elliptic ones.

Actually, we consider the the extremal scalar nonlinear operator

$$F^*(x, t, \xi, X) = F_1(x, t, \xi, X) \vee \dots \vee F_N(x, t, \xi, X) \equiv \max_{i=1 \dots N} F_i(x, t, \xi, X) \tag{14}$$

Our first result in this setting is as follows:

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that F is a vector mapping with components F_i satisfying conditions* (A1)–(A3) *for $i = 1 \dots N$ and C a matrix valued function satisfying conditions* (C1)–(C2).

Assume that the sign propagation property

$$w \leq 0 \text{ on } \partial\Omega \;\Rightarrow\; w \leq 0 \text{ in } \Omega$$

holds for all viscosity subsolutions $w \in C(\Omega;\mathbb{R})$ *of the scalar equation* $F^*[w] = 0$. *Then the same property* (4) *holds for all viscosity subsolutions* $u \in C(\Omega;\mathbb{R}^N)$ *of the vectorial equation* $F[u] + C(x)u = 0$ *in* Ω.

This theorem generalises the result of [13] for uniformly elliptic F_i since it allows to obtain the sign propagation property (4) in cases of degenerate, non-uniformly elliptic F_i.

Example 1.2. Suppose that the F_i's are elliptic operators of type $P_{(\alpha)}$ or $D_{(\alpha)}$, see (11) and (13). Then

$$F^*(x,\xi,X) := \max_{i=1\ldots N} F_i(x,\xi,X) \leq n\lambda_n(X) + b|\xi|, \tag{15}$$

where b is a common upper bound for all $\|b^h\|$. Let d be the diameter of Ω. From the results of [24] it follows that the weak Maximum Principle holds for the viscosity subsolutions $w \in USC(\overline{\Omega})$ of the equation $n\lambda_n(D^2w) + b|Dw| = 0$, provided that $bd < 1/n$. Under this assumption, by Theorem 1.1, the propagation sign property holds for the viscosity subsolutions of a cooperative system $F[u] + C(x)u = 0$. □

In the scalar case $N = 1$, for operators G which are positively homogeneous of degree 1 the validity of the weak Maximum Principle is guaranteed by and in fact characterised, by the positivity of the number [1]

$$\lambda_1(G,\Omega) = \sup\{\lambda \in \mathbb{R} : \exists\, \psi \in LSC(\Omega),\ \psi > 0 : G[\psi] + \lambda\psi \leq 0 \text{ in } \Omega\}\,, \tag{16}$$

see for example, in the case of linear uniformly elliptic operators, results in [27] for classical solutions and [6] in a more general non smooth setting. In this framework, $\lambda_1(G,\Omega)$ turns out to be, thanks to compactness properties guaranteed by uniform ellipticity, the principal eigenvalue of the homogeneous Dirichlet problem in Ω. In [8] similar results were generalised to a class of fully nonlinear uniformly elliptic positively homogeneous operators.

In this spirit, a numerical criterion for the validity of the weak Maximum Principle in a scalar degenerate fully nonlinear elliptic setting has been established in [5]. In order to deal with degeneracies a new index was introduced by setting

$$\mu_1(G,\Omega) = \sup\left\{\lambda \in \mathbb{R} : \begin{array}{c} \exists\,\Omega' \Supset \Omega \text{ and } \psi \in C(\Omega'),\ \psi > 0 \\ \text{such that } G[\psi] + \lambda\psi \leq 0 \text{ in } \Omega' \end{array}\right\} \tag{17}$$

This rather implicit definition of the index $\mu_1(G,\Omega)$, requiring G to be defined on larger set, is motivated, in particular, by possible degeneracies occurring on $\partial\Omega$.

[1]It is worth to note for historical reasons that definition (16) is clearly reminiscent of the well-known Collatz-Wielandt representation formula for the Perron eigenvalue of a matrix with positive entries, see [4]. The first author became acquainted with this notion, thanks are due to [26], while working at the papers [17], [18] on degenerate complementarity systems arising in some ergodic control problems for Markov chains.

Observe however that using the monotonicity of λ_1 with respect to set inclusion, it is possible to show that

$$\mu_1(G,\Omega) = \sup_{\Omega' \supset \overline{\Omega}} \lambda_1(G,\Omega'). \tag{18}$$

The main result in [5] is that the weak Maximum Principle, where the boundary condition is intended in the viscosity sense, holds for functions, $u \in USC(\overline{\Omega})$, such that $G[u] \geq 0$ in the viscosity sense if and only if $\mu_1(G,\Omega) > 0$, provided G satisfies the assumptions which guarantee comparison between viscosity sub and supersolutions satisfying boundary conditions in the viscosity sense, see [22].

One more comment is in order: the numerical index $\mu_1(G,\Omega)$ fails in general to be a genuine principal eigenvalue for the homogeneous Dirichlet problem for G in Ω. Indeed, such an eigenvalue may not exist in the degenerate cases, even linear ones, due to the lack of a priori estimates which enforce the compactness properties needed to apply Krein-Rutman type theorems to prove the existence of an associated principal eigenfunction. See however [28] for some specific results in the linear case and also the recent paper [9] for a class of fully nonlinear operators like (12).

In order to transpose Theorem 1.1 in terms of this notion of, so to say, generalised principal eigenvalues, we will suppose indeed that conditions (A1)–(A3) hold in a slightly larger open set such that $\Omega' \Supset \Omega$. In addition, the comparison principle between viscosity sub and supersolution will be used in the proof of Theorem 1.3 below. In view of this, the further conditions (A4)–(A6) will be required, see Section 3.

Relying on Theorem 1.1 we use the approach of [5] to obtain the following result:

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that C a matrix valued function satisfying conditions* (C1)–(C2) *and F is a vector mapping with components F_i satisfying conditions* (A1)–(A6) *for $i = 1 \ldots N$ on an open set Ω' such that $\Omega' \Supset \Omega$. If $\mu_1(F^*,\Omega) > 0$ then the sign propagation property* (4) *holds for any $u \in C(\Omega;\mathbb{R}^N)$ such that $F[u] + C(x)u \geq 0$ in Ω in the viscosity sense.*

It is worth to remark that since $F_i \leq F^*$, definition (17) implies

$$\mu_1(F^*,\Omega) \leq \mu_1(F_i,\Omega) \quad \forall\, i = 1 \ldots N. \tag{19}$$

Consider now the operators F_i and the sets

$$E_\lambda(F_i,\Omega) = \{\psi \in C(\Omega') : \psi > 0 \text{ and} : F_i[\psi] + \lambda\psi \leq 0 \text{ in some } \Omega' \Supset \Omega\}, \tag{20}$$

so that

$$\mu_1(F_i,\Omega) = \sup\{\lambda \in \mathbb{R} : E_\lambda(F_i,\Omega) \neq \emptyset\}\,. \tag{21}$$

Next, we introduce a new index associated to the vector operator $F = (F_1 \ldots F_N)$ by setting

$$\overline{\mu}_1(F,\Omega) = \sup\{\lambda \in \mathbb{R} : E_\lambda(F_1,\Omega) \cap \cdots \cap E_\lambda(F_N,\Omega) \neq \emptyset\}, \tag{22}$$

It is easy to check that

$$\mu_1(F^*,\Omega) \leq \overline{\mu}_1(F,\Omega). \tag{23}$$

We will prove in Lemma 3.1 in Section 3 that also the reverse inequality is true when $\overline{\mu}_1(F,\Omega) > 0$. As a consequence of this, Theorem 1.3 can be equivalently restated by replacing $\mu_1(F^*,\Omega)$ with $\overline{\mu}_1(F,\Omega)$.

Theorem 1.3'. *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that C a matrix valued function satisfying conditions* (C1)–(C2) *and F is a vector mapping with components F_i satisfying conditions* (A1)–(A6) *for $i = 1 \dots N$ on an open set Ω' such that $\Omega' \Supset \Omega$. If $\overline{\mu}_1(F, \Omega) > 0$ then the sign propagation property* (4) *holds for any $u \in C(\Omega; \mathbb{R}^N)$ such that $F[u] + C(x)u \geq 0$ in Ω in the viscosity sense.*

This allows to improve the conclusion of Example 1.2, at least when the first-order coefficients $b_h(x)$ of $D_{(\alpha)}$ in (13) identically vanish for $h \notin \{\alpha_1 \dots \alpha_k\}$ that is when the convection occurs only in the directions of the diffusion. In this case, being $\overline{\mu}_1(F, \Omega) > 0$, the sign propagation property holds for the viscosity subsolutions of the cooperative system (1) without any restriction on the relative sizes of the diameter d of Ω and the first order coefficients b^h, differently from the conclusion of Example 1.2 which was obtained through a coarser argument. See Example 3.4 in Section 3.

The paper is organized as follows: in Section 2 we introduce the basic notions of ellipticity, viscosity solutions and cooperative systems used to obtain a few preliminary results and the proof of our first main result, Theorem 1. In Section 3 we present the proofs of the other main results of this paper, as described above. The section contains also some computations in order to get positive lower bounds for the index μ_1, whose positivity yields the weak Maximum Principle.

2. Cooperativity and its consequences in the viscosity framework

We consider again the vectorial differential inequality

$$F[u] + C(x)u \geq 0 \tag{24}$$

for the vector valued function $u = (u_1 \dots u_N)$ or, component-wise,

$$F_i(x, u_i, Du_i, D^2u_i) + \sum_{j=1}^{N} c_{ij}(x)u_j \geq 0, \ i = 1 \dots N. \tag{25}$$

The components of $u(x)$ are coupled only through the matrix $C(x)$ so that the system is cooperative in the sense of [13], that is, for all $j = 1 \dots N$:

$$u_i \leq 0, \ i = 1 \dots N, \ u_j = 0 \quad \Rightarrow \quad (Cu)_j := \sum_{h=1}^{N} c_{jh}u_h \leq 0\,. \tag{26}$$

This will be the key condition allowing to reduce the weak Maximum Principle for the system to the case of a suitable scalar equation.

The differential inequality (24) can be understood in the classical way when $u \in C^2(\Omega; \mathbb{R}^N)$ or in the almost everywhere sense for Sobolev functions in $W^{2,p}(\Omega)$. Adopting the viscosity approach, (25) can be interpreted in a weak sense even for $u \in USC(\Omega; \mathbb{R}^N)$, the space of vector functions $u = (u_1 \dots u_N)$ which are upper semicontinuous in Ω.

Let us recall for convenience this notion in the scalar case. Let $G = G(x, t, \xi, X)$: $\Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ be a scalar operator, $u \in USC(\Omega; \mathbb{R})$ and $g \in C(\Omega; \mathbb{R})$.

The function u is a viscosity solution of $G[u] \geq g$ (or, u is a subsolution of $G[u] = g$) in Ω if the following holds:

for all $x_0 \in \Omega$ and for all $\varphi \in C^2(\Omega)$ such that $u \le \varphi$ in a neighbourhood of x_0 and $u(x_0) = \varphi(x_0)$ then

$$G(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \ge g(x_0). \tag{27}$$

Symmetrically, $u \in LSC(\Omega; \mathbb{R})$ is a viscosity solution of $G[u] \le g$ (or, u is a supersolution of $G[u] = g$) if

$$G(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \le g(x_0). \tag{28}$$

for all $x_0 \in \Omega$ and for all $\varphi \in C^2(\Omega)$ such that $u \ge \varphi$ in a neighbourhood of x_0 and $u(x_0) = \varphi(x_0)$.

A viscosity solution of $G[u] = g$ is a continuous function which is both a subsolution and a supersolution.

For a pair of real numbers s and t we set $s \vee t = \max(s,t)$ and $s \wedge t = \min(s,t)$. We will use in the sequel the following properties of viscosity solutions:

$$\begin{aligned} G[u] \ge f \,,\ G[v] \ge h \ &\Rightarrow G[u \vee v] \ge f \wedge h; \\ G[u] \le f \,,\ G[v] \le h \ &\Rightarrow G[u \wedge v] \le f \vee h. \end{aligned} \tag{29}$$

In view of the above definition, $u = (u_1 \dots u_N) \in C(\Omega; \mathbb{R}^N)$ is a viscosity subsolution of the differential system $F[u] + C(x)u \ge 0$ if the function $u_i \in C(\Omega; \mathbb{R})$ is a viscosity subsolution of the i-th equation:

$$F_i(x, u_i, Du_i, D^2u_i) + c_{ii}(x)u_i \ge -\sum_{j \ne i} c_{ij}(x)u_j(x). \tag{30}$$

for each $i = 1 \dots N$, Using this observation, we obtain differential inequalities for the $u_i^+ = u_i \vee 0$:

Lemma 2.1. *Assume that F_i satisfies conditions $(A1) \div (A3)$, and $C(x)$ satisfies conditions $(C1) - (C2)$. If $u = (u_1 \dots u_N)$ is a viscosity solution of the differential system* (30)*, then*

$$F_i(x, u_i^+, Du_i^+, D^2u_i^+) \ge -\sum_{j=1}^{N} c_{ij}(x)u_j^+(x) \tag{31}$$

Proof. Since by (7), the off-diagonal coefficients $c_{ij}(x)$ are nonnegative ($i \ne j$), from (30) we deduce

$$F_i(x, u_i, Du_i, D^2u_i) + c_{ii}(x)u_i \ge -\sum_{j \ne i} c_{ij}(x)u_j^+(x), \tag{32}$$

while, by (A2) $\quad F_i(x, 0, D0, D^20) + c_{ii}(x)0 = F_i(x, 0, 0, 0) = 0. \qquad (33)$

Using the observation before this lemma, we have therefore

$$F_i(x, u_i^+, Du_i^+, D^2u_i^+) + c_{ii}(x)u_i^+ \ge -\sum_{j \ne i} c_{ij}(x)u_j^+(x), \tag{34}$$

and we are done. □

Next, we will reduce to a single scalar equation which resumes the information of the system. To do this we construct, starting from the F_i's, the scalar operator

$$F^* = F_1 \vee \cdots \vee F_N. \tag{35}$$

and observe that, in view of Lemma 2.1:

$$F^*[u_i^+](x) \geq -\sum_{j=1}^{N} c_{ij}(x) u_j^+(x), \quad i = 1 \ldots N, \tag{36}$$

in the viscosity sense. Consider now the continuous scalar function

$$u^* = u_1^+ \vee \cdots \vee u_N^+ \tag{37}$$

Lemma 2.2. *Assume that F_i and $C(x)$ satisfy the conditions of Lemma* 2.1. *Let $u = (u_1 \ldots u_N)$ be a viscosity subsolution of system* (1). *Then,*

$$F^*[u^*] = F^*(x, u^*, Du^*, D^2 u^*) \geq 0 \tag{38}$$

in the viscosity sense in Ω.

Proof. We argue by contradiction, supposing on the contrary to (38) that there exists some point $x_0 \in \Omega$ and a smooth function φ touching u^* from above at x_0, but

$$F^*[\phi](x_0) = F^*(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) < 0. \tag{39}$$

Let $i \in \{1 \ldots N\}$ be such that $u^*(x_0) = u_i^+(x_0)$. Then, from (39) and cooperativity (7) we deduce

$$F^*[\phi](x_0) + u_i^+(x_0) \sum_{j=1}^{N} c_{ij}(x_0) < 0$$

Now, (7) implies that at point x_0

$$u_i^+ \sum_{j=1}^{N} c_{ij} = c_{ii} u_i^+ + \sum_{j \neq i} c_{ij} u_i^+ \geq c_{ii} u_i^+ + \sum_{j \neq i} c_{ij} u_j^+ = \sum_{j=1}^{N} c_{ij} u_j^+$$

Hence,

$$F^*[\phi](x_0) < -\sum_{j=1}^{N} c_{ij}(x_0) u_j^+(x_0) \tag{40}$$

Since φ touches u_i^+ from above at x_0, then (36) implies by viscosity

$$F^*[\phi](x_0) \geq -\sum_{j=1}^{N} c_{ij}(x_0) u_j^+(x_0)\ , \tag{41}$$

which yields a contradiction with (40) and (38) is proved. $\square$

We conclude this section with the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose $u = (u_1 \dots u_N)$ satisfies $F_i[u] + \sum_{j=1}^{N} c_{ij}(x) u_j \geq 0$, $i = 1 \dots N$, in the viscosity sense and the boundary condition (5). By Lemma 2.2 we know that, for $F^* = F_1 \vee \cdots \vee F_N$ and $u^* = u_1^+ \vee \cdots \vee u_N^+$, the scalar differential inequality $F^*[u^*] \geq 0$ holds in Ω and, of course, the boundary condition $u^* \leq 0$ on $\partial\Omega$ is satisfied as well. Using the assumption with $w = u^*$ we obtain $u^* \leq 0$. A fortiori, $u_i \leq 0$ in Ω for all $i = 1 \dots N$, as we needed to prove. $\square$

3. Weak Maximum Principle and generalised principal eigenvalues

As indicated in the introduction and on the basis of the previous section it seems natural, in the perspective of a possible numerical criterion for the validity of the weak Maximum Principle, to associate to the vector mapping F the number $\mu_1(F^*, \Omega)$ defined, in agreement with [5], by

$$\mu_1(F^*, \Omega) = \sup\{\lambda \in \mathbb{R} : \exists\, \Omega' \Supset \Omega \text{ and } \psi \in C(\Omega'), \psi > 0 : F^*[\psi] + \lambda\psi \leq 0 \text{ in } \Omega'\}$$

where F^* is the scalar mapping $F^* = F_1 \vee \cdots \vee F_N$, see (17) .
In order to establish the relationship between strict positivity of the generalised principal eigenvalue $\mu_1(F^*, \Omega)$ and the validity of the weak Maximum Principle we need to introduce, as announced in Section 1, further conditions on $F = (F_1 \dots F_N)$ which are essential in order to guarantee in full generality the validity of the comparison principle between viscosity sub and supersolutions, see [22].

(A4) F_i positively homogeneous of degree 1 with respect to (t, ξ, X),

(A5) $t \to F_i(x, t, \xi, X)$ continuous, uniformly with respect to (x, ξ, X),

(A6) for all $R > 0$ there exists $\omega \in C(\mathbb{R}_+)$ such that $\omega(s) \to 0$ as $s \to 0^+$ and

$$F_i(x, t, \alpha(x-y), X) - F_i(y, t, \alpha(x-y), Y) \leq \omega(\alpha|x-y|^2 + |x-y|) \tag{42}$$

for all $X, Y \in \mathcal{S}^n$ such that, for some $\alpha > 0$,

$$-3\alpha \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3\alpha \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}. \tag{43}$$

Observe that (A6) implies in particular degenerate ellipticity. We will suppose that conditions $(A_1) \div (A6)$ are satisfied for $(t, \xi, X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$ and for $x \in \Omega'$, where Ω' is such that $\Omega \Subset \Omega'$. Observe also that if each F_i satisfies $(A_1) \div (A6)$ then the same is true for F^*.

Below follows the proof of Theorem 1.3.

Proof of Theorem 1.3. If $u = (u_1 \dots u_N)$ is a subsolution of (1) with $u_i \leq 0$ on $\partial\Omega$, then by Lemma 2.2 the function $u^* = u_1^+ \vee \cdots \vee u_N^+$ satisfies

$$u^* = 0 \text{ on } \partial\Omega \;,\; F^*[u^*] \geq 0 \text{ in } \Omega$$

Now, the assumption $\mu_1(F^*, \Omega) > 0$ guarantees through Theorem 1.3 in [5] that sign propagation holds for u^* so that $u_i \leq 0$ in Ω. $\square$

Recall that we have introduced in (22) the index

$$\overline{\mu}_1(F,\Omega) = \sup\{\lambda \in \mathbb{R} : E_\lambda(F_1,\Omega) \cap \cdots \cap E_\lambda(F_n,\Omega) \neq \emptyset\},$$

where $E_\lambda(G,\Omega)$ is defined by (20) as

$$E_\lambda(F_i,\Omega) = \{\psi \in C(\Omega') : \psi > 0 \text{ and } F_i[\psi] + \lambda\psi \leq 0 \text{ in some } \Omega' \Supset \Omega\}.$$

We have already observed, see (23), that $\overline{\mu}_1(F,\Omega)$ is larger than $\mu_1(F^*,\Omega)$. The following lemma states that the positivity of $\overline{\mu}_1(F,\Omega)$ is equivalent to the positivity of $\mu_1(F^*,\Omega)$.

Lemma 3.1. *Assume that conditions* $(A1)\div(A6)$ *are satisfied in an open set* $\Omega' \Supset \Omega$. *Then* $\overline{\mu}_1(F,\Omega) > 0$ *implies* $\overline{\mu}_1(F,\Omega) = \mu_1(F^*,\Omega)$.

Proof. Since $\overline{\mu}_1(F,\Omega) \geq \mu_1(F^*,\Omega)$, see (23), we are left with showing that

$$\overline{\mu}_1(F,\Omega) \leq \mu_1(F^*,\Omega). \tag{44}$$

Assume therefore $\overline{\mu}_1(F,\Omega) > 0$, and take $0 < \lambda < \overline{\mu}_1(F,\Omega)$. By definition (22) we have $\cap_{i=N} E_\lambda(F_i,\Omega) \neq \emptyset$. Let $\psi \in \cap_{i=1}^N E_\lambda(F_i,\Omega)$, then $\psi \in C(\Omega')$ for some $\Omega' \Supset \Omega$, $\psi > 0$ in Ω' and

$$F_i[\psi] + \lambda\psi \leq 0 \quad \text{in } \Omega' \quad \text{for all } i = 1 \ldots N. \tag{45}$$

Consequently $\psi \in E_\lambda(F^*,\Omega)$, so that $\lambda \leq \mu_1(F^*,\Omega)$. Passing to the supremum over $\lambda < \overline{\mu}_1(F,\Omega)$, we get (44) and the two indexes have the same value. □

Let us conclude this section by describing a few significant model examples involving degenerate elliptic operators where the condition $\mu_1(F^*,\Omega) > 0$ is easily checked through a positive lower bound of $\mu_1(F^*,\Omega) = \overline{\mu}_1(F,\Omega)$, so enforcing through Theorem 2 the validity of the weak Maximum Principle for the cooperative system $F[u] + C(x)u \geq 0$.

Example 3.2. Consider linear operators as in (9) with positive semidefinite matrices A^i with, just for simplicity, constant entries and null lower order terms. Let Ω be contained in B_R, the ball of radius R centered at the origin.

The function $\psi(x) = \dfrac{R^2}{2} - \dfrac{|x|^2}{2}$ is strictly positive in B_R and $D^2\psi = -I$. Assume that $\mathrm{Tr}(A^i) > 0$ for each $i = 1 \ldots N$. Then

$$\mathrm{Tr}(A^i D^2\psi) + \lambda\psi = -\mathrm{Tr}(A^i) + \frac{\lambda}{2}\left(R^2 - |x|^2\right) \leq 0 \text{ in } B_R,$$

provided that $\lambda \leq 2\mathrm{Tr}(A^i)/R^2$ for all $i = 1 \ldots N$. Therefore

$$\overline{\mu}_1(F,\Omega) \geq \frac{2}{R^2}\,\mathrm{Tr}(A^1) \wedge \cdots \wedge \mathrm{Tr}(A^n).$$

Since the A^i are positive semidefinite matrices, the extra assumption $\mathrm{Tr}(A^i) > 0$ amounts to the requirement that each linear operator is strictly elliptic at least in one coordinate direction, as for instance the directional elliptic operators $D_{(\alpha)}$ considered in the introduction. See [20], [21] for the validity of the scalar weak Maximum Principle in this setting. □

Example 3.3. Consider $F_i = P^-_{k_i}$, $i = 1 \dots N$, as in (12). Take as before $B_R \Supset \Omega$ and $\psi(x) = \frac{R^2}{2} - \frac{|x|^2}{2} > 0$ in B_R. Then,

$$P^-_{k_i}(D^2\psi) + \lambda\psi = -k_i + \frac{\lambda}{2}(R^2 - |x|^2) \le 0,$$

provided that $\lambda \le 2k_i/R^2$ for all $i = 1 \dots N$. Therefore

$$\overline{\mu}_1(F, \Omega) \ge \frac{2}{R^2}\, k_1 \wedge \cdots \wedge k_N \ge \frac{2}{R^2}. \qquad \square$$

Example 3.4. Let us turn back to Example 1.2 of the introduction, where F_i is either a partial trace operator $P_{(\alpha)}$ or a directional diffusion operator $D_{(\alpha)}$. In the latter case we suppose that the drift term contains at most the partial derivatives with respect to x_{α_h} such that α_h is one of the integers of $\alpha = (\alpha_1 \dots \alpha_k)$, that is the convection occurs only in the direction of the diffusion. Supposing that $\Omega \Subset]0, d[^n$, we define $\psi(x) = \sum_{j=1}^n (e^{\beta d} - e^{\beta x_j})$ with $\beta > 0$ to be chosen, so that $\psi > 0$ in the cube $]0, d[^n$. Then

$$P_{(\alpha)}[\psi] \le -\beta^2, \quad D_{(\alpha)}[\psi] \le -\beta^2 \sum_{h=1}^k e^{\beta x_{\alpha_h}} + \beta b \sum_{h=1}^k e^{\beta x_{\alpha_h}},$$

where $b \ge \max \|b_{(\alpha)}\|_{L^\infty(\Omega)}$. Taking $\beta > b$, we get then

$$\begin{aligned} P_{(\alpha)}[\psi] + \lambda\psi &\le -\beta^2 + \lambda\, n(e^{\beta d} - 1) \le 0 \\ D_{(\alpha)}[\psi] + \lambda\psi &\le -\beta(\beta - b) + \lambda\, n(e^{\beta d} - 1) \le 0, \end{aligned}$$

provided that $\lambda \le [\beta(\beta - b)]/[n(e^{\beta d} - 1)]$. Therefore

$$\overline{\mu}_1(F, \Omega) \ge \frac{\beta(\beta - b)}{n(e^{\beta d} - 1)}. \qquad \square$$

Example 3.5. Assume that F_i are completely degenerate first order operators of the form $F_i[u_i] = b^i \cdot Du_i + c^i u_i$. Suppose there exists $\psi > 0$ such that $\nabla\psi \ne 0$ in $\Omega' \Supset \Omega$.
If $b^i \cdot \frac{\nabla\psi}{|\nabla\psi|} \le -\beta^i$ in Ω' for all $i = 1 \dots N$, with $\beta^i > 0$ and $c^i \le 0$, then

$$F_i[\psi] + \lambda\psi \le -\beta^i|\nabla\psi| + c^i\psi + \lambda\psi \le -\beta^i|\nabla\psi| + \lambda\psi \le 0 \ \text{ in } \Omega',$$

provided that $\lambda \le \kappa := \inf_{\Omega'} |\nabla\psi|/\psi > 0$. Therefore

$$\overline{\mu}_1(F, \Omega) \ge \kappa \left(\beta^1 \wedge \cdots \wedge \beta^N\right) > 0\,. \qquad \square$$

Example 3.6. Consider the uniformly elliptic Pucci maximal operators

$$\mathcal{P}^+_{\lambda_i,\Lambda_i}(X) := \Lambda_i \mathrm{Tr}(X^+) - \lambda_i \mathrm{Tr}(X^-)$$

where $0 < \lambda_i \le \Lambda_i$ and suppose that $\Omega \subset]0, d[^n$ for some $d > 0$. Here we used the standard orthogonal decomposition $X = X^+ - X^-$ where X^+, X^- are positive semidefinite matrices such that $X^+X^- = 0$.

If each F_i occurring in system (1) satisfies the following control from above

$$F_i(t,x,\xi,X) \le \mathcal{P}^+_{\lambda_i,\Lambda_i}(X) + b(x)|\xi| + c(x)t^+$$

(let us observe explicitly that these one-sided assumptions do not require F_i to be uniformly elliptic), then

$$F^*(t,x,\xi,X) := \max_{i=1\dots N} F_i(t,x,\xi,X) \le \mathcal{P}^+_{\lambda_F,\Lambda_F}(X) + b|\xi| + ct^+ \text{ in } \Omega \Subset]0,d[^n$$

where $\lambda_F = \lambda_1 \wedge \cdots \wedge \lambda_n$, $\Lambda_F = \Lambda_1 \vee \cdots \vee \Lambda_n$, $|b^i(x)| \le b$ and $c^+(x) \le c$ in $]0,d[^n$.

Hence, $\mu_1(F^*,\Omega) \ge \mu_1(\mathcal{P}_{\lambda_F,\Lambda_F},\Omega)$. Let $\beta > 0$ a parameter to be chosen later and consider $\psi(x) = e^{\beta d} - e^{\beta x_1}$. The smooth function ψ is strictly positive in $]0,d[^n$ and

$$\begin{aligned}\mathcal{P}^+_{\lambda_F,\Lambda_F}(D^2\psi) + b|D\psi| + c\psi + \lambda\psi &= -\lambda_F\beta^2 e^{\beta x_1} + b\beta e^{\beta x_1} + (c+\lambda)(e^{\beta d} - e^{\beta x_1}) \\ &\le \beta e^{\beta x_1}\left(-\lambda_F\beta + b + (c+\lambda)\,\frac{e^{\beta d}-1}{\beta}\right) \qquad (46)\end{aligned}$$

Let us consider first the case $c(x) \le 0$ and so $c = 0$.
In this event the right-hand side can be made non positive by choosing $\beta > \frac{b}{\lambda_F}$ and, for $\lambda > 0$, λ such that $0 < \lambda < \frac{\beta}{e^{\beta d}-1}(\beta\lambda_F - b)$. With these choices, ψ is a positive supersolution of $P^+_{\lambda_F,\Lambda_F}[\psi] + b|D\psi| + \lambda\psi = 0$ and, a fortiori, of $F^*[\psi] + \lambda\psi = 0$, so that $\mu_1(F^*,\Omega) > 0$.
Observe that we obtain indeed a more precise lower bound, namely

$$\mu_1(F^*,\Omega) \ge \frac{\beta}{e^{\beta d}-1}(\lambda_F\beta - b).$$

Let us look now to the case $c > 0$. The right-hand side of (46) can still be made non positive for some $\lambda > 0$ provided β satisfies the more restrictive condition

$$c\,\frac{e^{\beta d}-1}{\beta} < \lambda_F\beta - b$$

which could not be satisfied whatever positive β is taken.
Indeed, for a given $c > 0$, such a β can be found only if d is sufficiently small.
Conversely, for a fixed d, the above condition is satisfied, for all $\beta > \frac{b}{\lambda_F}$, provided c is suitably small.
Loosely speaking, the weak Maximum Principle holds if either the domain is narrow or the zero order coefficient is positive, but small, see [3], [6], [15], [19], [20], [27], [30]. Using this smallness condition on the relative sizes of c and d, we obtain the following positive lower bound:

$$\mu_1(F^*,\Omega) \ge \frac{\beta}{e^{\beta d}-1}(\beta\lambda_F - b) - c > 0$$

and the weak Maximum Principle holds by virtue of Theorem 1.3.
Note that, for $c = 0$ the last inequality reduces to the previous lower bound with $c = 0$. No restriction is needed in this case on the size of d.
In the special case of uniformly elliptic operators F_i the same conclusions can be obtained via the results in [8] which guarantee the existence of a pair eigenvalue-eigenfunction $\psi^* > 0, \mu_1(F^*,\Omega) > 0$ such that $F^*[\psi^*] + \mu_1(F^*,\Omega)\psi^* = 0$ in Ω and $\psi^* = 0$ on $\partial\Omega$. $\square$

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