

On BV Homeomorphisms

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We obtain the rectifiability of the graph of a bounded variation homeomorphism f in the plane and relations between gradients of f and its inverse. Further, we show an example of a bounded variation homeomorphism f in the plane which satisfies the (N) and (N^{-1}) properties and strict positivity of Jacobian of both itself and its inverse, but neither f nor f^{-1} is Sobolev.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f: \Omega \rightarrow \mathbb{R}^n$ be a Sobolev or BV homeomorphism. Then it is natural to ask under what condition the inverse is also Sobolev or BV . For planar Sobolev mapping it has been solved by Hencl and Koskela [12]. Then Hencl, Koskela and Onninen [15] proved that the inverse of a planar BV homeomorphism is also BV . For first results in the spatial case see Hencl, Koskela and Malý [14], Onninen [22] and Hencl, Koskela and Onninen [15]. The BV -regularity of the inverse of $W^{1,n-1}$ mappings has been obtained by Csörnyei, Hencl and Malý [4] and even a more precise result for $n = 3$ by Hencl, Kauranen and Luisto [10].

The above mentioned results when the inverse of a Sobolev mapping is also Sobolev were accompanied by the formula for the the gradient of the inverse; in fact it is

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the same as in the smooth case. The corresponding result for BV mappings is more difficult even in the planar case. For $f = (f_1, f_2) \in BV(\Omega, \mathbb{R}^2)$, the identity

$$|D_1 f^{-1}|(f(U)) = |Df_1|(U), \quad U \subset \Omega \text{ open}$$

has been shown by D'Onofrio and Schiattarella [6]. We prove the full characterization of Df^{-1} :

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and $f: \Omega \rightarrow \mathbb{R}^2$ be a bounded sense preserving BV -homeomorphism. Let $U \subset \Omega$ be a Borel set. Then*

- (a) $Df^{-1}(f(U)) = \text{adj } Df(U)$,
- (b) $\text{Det } Df(U) = |f(U)|$.

Here
$$\text{adj } Df = \begin{pmatrix} D_2 f_2 & -D_2 f_1 \\ -D_1 f_2 & D_1 f_1 \end{pmatrix}.$$

Note that this is a measure. The part (a) has been proved first by Quittnerová in [23]. She also obtained the corresponding result in the spatial case for $W^{1,n-1}$ -homeomorphisms. Unfortunately, her thesis has been never published.

Our proof is entirely different and shorter. While our project was in preparation, Hencl, Kauranen and Malý [11] proved a similar formula to (a) in dimension 3 under the assumption that both f and f^{-1} are BV . They use a distributional adjugate.

The expression $\text{Det } Df$ appearing in (b) is the distributional Jacobian introduced by Ball [3], see Definition 2.14. Also the part (b) follows from the degree formula proved in [10] and [11]. We present our original proof which is simpler as we consider only the homeomorphic case.

Whereas the main novelty of Theorem 1.1 is in new proofs, our main result on rectifiability is entirely new. The rectifiability of the graph has been known only for scalar BV functions (Federer [7, 4.5.9], see also Giaquinta, Modica and Souček [9, Section 4.1.5]). For vector valued function it is available only due to the assumption that our function is a homeomorphism. The spatial case is open.

The graph of a Sobolev mapping f on an n -dimensional domain is rectifiable if the graph mapping satisfies Luzin's (N) -condition, it means

$$E \subset \Omega, |E| = 0 \implies \mathcal{H}^n(f(E)) = 0. \quad (1)$$

This is commonly the case when f itself satisfies the (N) -condition, as the proof of (N) for f usually gives (N) for the graph mapping. For further discussion and typical results on the (N) -condition see [21], [19], [17], [20]. In our situation of a planar homeomorphism, previously known results give rectifiability of the graph if $f \in W^{1,2}(\Omega)$ as follows from the result by Reshetnyak [25].

To describe our rectifiability result, we need to introduce some notation. The symbol Γ stands for the mapping $x \mapsto (x, f(x))$, so that $\Gamma(\Omega)$ is the full graph of f . We consider a measure μ supported on $\Gamma(\Omega)$ with values in $\Lambda^2(\mathbb{R}^4)$; we write in coordinates

$$\mu = \mu_{12} \mathbf{e}_1 \wedge \mathbf{e}_2 + \mu^{12} \mathbf{e}^1 \wedge \mathbf{e}^2 + \sum_{i,j=1}^2 \mu_i^j \mathbf{e}_i \wedge \mathbf{e}^j,$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ is the canonical basis of the domain space and $(\mathbf{e}^1, \mathbf{e}^2)$ is the canonical basis of the target space.

This means that given a smooth differential form with compact support in $\Omega \times f(\Omega)$,

$$\omega = \omega_{12} dx_1 dx_2 + \omega^{12} dy_1 dy_2 + \sum_{i,j=1}^2 \omega_i^j dx_i dy_j,$$

we have
$$\langle \mu, \omega \rangle = \int_{\Gamma(\Omega)} \omega_{12} d\mu_{12} + \omega^{12} d\mu^{12} + \sum_{i,j=1}^2 \omega_i^j d\mu_i^j.$$

The measure μ is defined as push forward of measures on Ω through the graph mapping. Namely, if $E \subset \Omega$ is a Borel set then

$$\begin{aligned} \mu_{12}(\Gamma(E)) &= |E|, & \mu_1^j(\Gamma(E)) &= D_2 f_j(E), \\ \mu_2^j(\Gamma(E)) &= -D_1 f_j(E), & \mu^{12}(\Gamma(E)) &= |f(E)| = \text{Det } Df(E). \end{aligned} \quad (2)$$

If $M \subset \mathbb{R}^4$ is a 2-rectifiable set (see Definition 2.5), then it admits an orientation, see Definition 2.10. The structure of an orientation on M can be alternatively expressed by an integer multiplicity 2-rectifiable current of multiplicity 1 on M , which is a measure on M with values in $\Lambda_2(\mathbb{R}^4)$. For the definition of integer multiplicity rectifiable current see Definition 2.12. In our situation we want the orientation to coincide with the topological orientation; this is expressed by the property that the current is boundaryless, see Definition 2.13.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and $f: \Omega \rightarrow \mathbb{R}^2$ be a sense preserving bounded BV-homeomorphism. Then the graph $\Gamma(\Omega)$ of f is a 2-rectifiable set and μ is a boundaryless rectifiable current of multiplicity 1 on $\Gamma(\Omega)$.*

Our second main result is a surprising example.

Let $\Omega \subset \mathbb{R}^2$ be an open set. For a Sobolev homeomorphism $f \in W^{1,1}(\Omega; \mathbb{R}^2)$, Luzin's (N) -condition is a useful property, as it implies the *area formula* for Borel sets $E \subset \Omega$,

$$\int_E J_f(x) dx = |f(E)| \quad (3)$$

and the change of variables can be used as for smooth transformations. Moreover, the (N) -condition implies that the pointwise Jacobian determinant $J_f(x) = \det Df(x)$ coincides with the distributional Jacobian, $\text{Det } Df(x)$, see [5].

The formula
$$\det Df(x) = \text{Det } Df(x) \quad (4)$$

should be interpreted that the distributional Jacobian is an absolutely continuous measure and the pointwise Jacobian acts as its density. For the definition of the distributional Jacobian see Definition 2.14.

The positivity of J_f ,
$$J_f(x) > 0 \quad \text{for a.e. } x \in \Omega, \quad (5)$$

is also very useful because it guarantees that also f^{-1} is Sobolev map (we say that f is bi-Sobolev, see [16]) and the usual formula for the gradient of the inverse holds. Notice that the condition $J_f > 0$ a.e. is equivalent to the (N) -condition for f^{-1} ([13]).

Hence, if a Sobolev homeomorphism f and its inverse f^{-1} satisfy the (N) -condition, then $J_f > 0$ a.e., $J_{f^{-1}} > 0$ a.e., f is bi-Sobolev, and (3), (4) hold true for f and f^{-1} .

On the contrary, for a planar BV -homeomorphism f , whose inverse is automatically in BV_{loc} (see [15], [4]), the assumption that f and f^{-1} verify the (N) -condition is not sufficient to gain $f^{-1} \in W_{\text{loc}}^{1,1}$.

The following example shows that for a planar BV -homeomorphism, positivity of Jacobian together with the (N) -condition do not guarantee absolute continuity of the gradient.

Theorem 1.3. *There exists a BV -homeomorphism $f: [-1, 1]^2 \rightarrow [-1, 1]^2$ such that both f and f^{-1} satisfy the (N) -condition, $J_f > 0$ a.e., $J_{f^{-1}} > 0$ a.e., but neither f nor f^{-1} are Sobolev.*

There is a close link between the positive result and the example. Indeed, the most difficult part of the proof of rectifiability consists in the analysis of the part of the graph which is seen as vertical for both f and f^{-1} . For illustration of this behavior we can use the part of the graph of f from the example over the Cantor set S (see (28)).

2. Preliminaries

We denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$ by $|E|$ and the k -dimensional Hausdorff measure of E by $\mathcal{H}^k(E)$.

Let $\Omega \subset \mathbb{R}^n$ be an open set. If $\mathbf{m}$ is a finite signed Radon measure on Ω and u is an $\mathbf{m}$ -integrable function, we define

$$\langle \mathbf{m}, u \rangle = \int_{\Omega} u \, d\mathbf{m};$$

this is an extension of the duality between $\mathbf{m} \in \mathcal{C}_0(\Omega)^*$ and $u \in \mathcal{C}_0(\Omega)$.

Let Ω be a domain in $\mathbb{R}^n$. A function $u \in L^1(\Omega)$ is of bounded variation, $u \in BV(\Omega)$, if the distributional partial derivatives of u are measures with finite total variation in Ω : there exist Radon signed measures $D_i u$ in Ω such that for $i = 1, \dots, n$, $|D_i u|(\Omega) < \infty$ and

$$\langle D_i u, \varphi \rangle = - \int_{\Omega} u D_i \varphi \, dx, \quad \varphi \in \mathcal{C}_c^1(\Omega).$$

The gradient of u is then the vector-valued measure $Du = (D_1 u, \dots, D_n u)$, $|Du|$ stands for its total variation. We have

$$|Du|(\Omega) = \sup \left\{ \int_{\Omega} u \, \text{div} \varphi \, dx : \varphi \in \mathcal{C}_0^1(\Omega, \mathbb{R}^n), \|\varphi\|_{\infty} \leq 1 \right\} < \infty.$$

We say that u is a Sobolev function if its distributional gradient can be represented as a locally integrable function, this means, $u \in W_{\text{loc}}^{1,1}(\Omega)$. The Sobolev space $W^{1,1}(\Omega)$ is contained in $BV(\Omega)$ as $L^1(\Omega)$ can be regarded as a closed subspace of $\mathcal{C}_0(\Omega)^*$.

We say that $f \in L^1(\Omega; \mathbb{R}^m)$ belongs to $BV(\Omega; \mathbb{R}^m)$ if each component of f is a function of bounded variation. The total variation of Df is then computed as

$|Df_1| + \cdots + |Df_n|$. Finally we say that $f \in BV_{\text{loc}}(\Omega; \mathbb{R}^m)$ if $f \in BV(U; \mathbb{R}^m)$ for every open $U \subset\subset \Omega$. The space $BV(\Omega, \mathbb{R}^m)$ is endowed with the norm

$$\|f\|_{BV} := \int_{\Omega} |f(x)| dx + |Df|(\Omega).$$

Now, we recall some useful tools.

Proposition 2.1. (Morse-Sard theorem, [18, Lemma 13.15]) *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $\eta \in C^\infty(\mathbb{R}^n)$ and $E = \{x \in \mathbb{R}^n : D\eta(x) = 0\}$, then $|\eta(E)| = 0$. In particular, $\{\eta = t\} = \{x \in \mathbb{R}^n : \eta(x) = t\}$ is a C^∞ hypersurface in $\mathbb{R}^n$ for a.e. $t \in \mathbb{R}$.*

Proposition 2.2. (Luzin approximation in BV, [2, Thm. 5.34]) *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f \in BV_{\text{loc}}(\Omega, \mathbb{R}^m)$. Then there are Lipschitz functions $f_j : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that*

$$\left| \bigcap_j \{x \in \Omega : f_j(x) \neq f(x)\} \right| = 0.$$

Proposition 2.3. (Co-area formula, [7, 4.5.9]) *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $u \in BV(\Omega)$ be a continuous function and η be a Borel function on Ω , $\eta \geq 0$. Then*

$$\langle |Du|, \eta \rangle = \int_{-\infty}^{\infty} \left(\int_{\{u=t\}} \eta d\mathcal{H}^{n-1} \right) dt. \quad (6)$$

Given a smooth map f from $\Omega \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and $U \subset\subset \Omega$ we can define the topological degree as

$$\deg(f, U, y_0) = \sum_{\{x \in U : f(x) = y_0\}} \text{sgn}(J_f(x))$$

if $J_f(x) \neq 0$ for each $x \in f^{-1}(y_0)$. This definition can be extended to an arbitrary continuous mapping and each point of $\mathbb{R}^n \setminus f(\partial U)$, see e.g. [8] for the definition and properties of the degree.

A continuous mapping $f : \Omega \rightarrow \mathbb{R}^n$ is called *sense-preserving* if

$$\deg(f, U, y_0) > 0$$

for all domains $U \subset\subset \Omega$ and all $y_0 \in f(U) \setminus f(\partial U)$. Similarly we call f *sense-reversing* if $\deg(f, U, y_0) < 0$ for all U and y_0 . Let us recall that each homeomorphism on a domain is either sense-preserving or sense-reversing, see [24, II.2.4., Thm. 3].

Proposition 2.4. (Degree formula, [8]) *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \rightarrow \mathbb{R}^n$ be a C^1 -mapping. Let $U \subset\subset \Omega$ be an open set. Then*

$$\int_U \eta(f(x)) J_f(x) dx = \int_{\mathbb{R}^n} \eta(y) \deg(f, U, y) dy$$

for each continuous function η with compact support in $\mathbb{R}^n \setminus f(\partial U)$.

Definition 2.5. We say that a set $M \subset \mathbb{R}^n$ is *countably k -rectifiable* if M can be written as $N \cup \bigcup_{j=1}^{\infty} \psi_j(E_j)$, where $E_j \subset \mathbb{R}^k$ are measurable, $\psi_j : E_j \rightarrow \mathbb{R}^n$ are Lipschitz, and $\mathcal{H}^k(N) = 0$. We say that M is *k -rectifiable* if it is countably k -rectifiable and $\mathcal{H}^k(M) < \infty$.

Remark 2.6. We can reduce $(\psi_j)_j$ to a single locally Lipschitz mapping $\psi : E \rightarrow \mathbb{R}^n$, where $E \subset \mathbb{R}^n$ is measurable. Indeed, it is enough to split the sets E_j into small pieces and reorganize them by translations to be mutually distant. Then we have

$$M = \psi(E) \cup N$$

where $\mathcal{H}^k(N) = 0$ and we call ψ to be a *parametrization* of M . The set E can be very wild and scattered even if M is topologically nice.

Definition 2.7. We denote the space of k -vectors on $\mathbb{R}^n$ by $\Lambda_k(\mathbb{R}^n)$ and the space of k -covectors on $\mathbb{R}^n$ by $\Lambda^k(\mathbb{R}^n)$. A differential k -form on an open set $W \subset \mathbb{R}^n$ is a mapping $\omega : W \rightarrow \Lambda^k(\mathbb{R}^n)$. For more information we refer to [7].

Definition 2.8. If σ is a Radon measure on $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $r > 0$, we denote by $\sigma_{x,r}$ the measure which acts on any function $\varphi \in \mathcal{C}_c(\mathbb{R}^n)$ as

$$\langle \sigma_{x,r}, \varphi \rangle = \int_{\mathbb{R}^n} \varphi\left(x + \frac{y-x}{r}\right) d\sigma(y). \quad (7)$$

Proposition 2.9. ([26, Thm. 11.8]) *Let $M \subset \mathbb{R}^n$ be a set and σ be a Radon measure on $\mathbb{R}^n$ such that $\sigma(\mathbb{R}^n \setminus M) = 0$. Suppose that for each $x \in M$ there exists a k -dimensional linear subspace $\mathbf{V} \subset \mathbb{R}^n$ such that*

$$\lim_{r \rightarrow 0^+} \int_{\mathbb{R}^n} \varphi d\sigma_{x,r} = \int_{\mathbf{V}} \varphi d\mathcal{H}^k$$

for any $\varphi \in \mathcal{C}_c(\mathbb{R}^n)$. Then M is countably k -rectifiable and $\sigma = \mathcal{H}^k \llcorner M$.

Definition 2.10. Let $R \subset \mathbb{R}^n$ be a countably k -rectifiable set and $x \in R$. A linear space $\mathbf{V} \subset \mathbb{R}^n$ is called an *approximate tangent space* to R at x if there exists $M \subset R$ and a measure σ such that the situation from Proposition 2.9. occurs. By [26, Thm. 11.6], an approximate tangent plane exists at $\mathcal{H}^k$ -a.e. $x \in R$; by [26, Remark 11.5] it is uniquely determined up to a $\mathcal{H}^k$ -null set.

We can consider an *orientation* on M , by this we mean that the approximate tangent space is oriented at $\mathcal{H}^k$ -a.e. $x \in M$, so that the k -vector field ξ on M , where $\xi(x)$ is obtained as the exterior product of an ortonormal basis of the approximate tangent space to M at x , is $\mathcal{H}^k$ -measurable. We write the structure of orientation on M as (M, ξ) and call (M, ξ) an *oriented k -rectifiable set*. A locally Lipschitz mapping $\psi : E \rightarrow \mathbb{R}^n$, where $E \subset \mathbb{R}^k$, is measurable, is called a *positive parametrization* of (M, ξ) if ψ is a parametrization of M and $(D_1\psi_1(t), \dots, D_k\psi(t))$ is a positive basis of the approximate tangent space to M at $\psi(t)$ for a.e. $t \in E$. Here, if E is not a neighborhood of t , we use derivatives of a Lipschitz extension of ψ , they are independent of the choice of the extension a.e. in E .

Remark 2.11. Similarly to the argument of Remark 2.6, it can be shown that a positive parametrization of an oriented k -rectifiable set always exists.

Definition 2.12. Let ν be a measure with values in $\Lambda_k(\mathbb{R}^n)$. We say that ν is an *integer multiplicity k -rectifiable current* if it acts on differential k -forms as

$$\langle \nu, \omega \rangle = \int_M \theta(x) \langle \omega(x), \xi(x) \rangle d\mathcal{H}^k(x),$$

where (M, ξ) is an oriented countably k -rectifiable set and $\theta: M \rightarrow \mathbb{N}$ is a $\mathcal{H}^k$ -measurable function (multiplicity). We write $\nu = \nu(M, \xi, \theta)$. We call ν a *simple k -rectifiable current*, and simplify to $\nu(M, \xi)$, if $\theta \equiv 1$ (this is our situation throughout the paper). $\square$

If we have a structure of a simple k -rectifiable current on M where M is a connected topological manifold, we may want to express that the orientation is compatible with the topological orientation of M . The right interpretation of this is the property of being boundaryless.

Definition 2.13. Let $W \subset \mathbb{R}^n$ be an open set and $\nu = \nu(M, \xi)$ is an integer multiplicity k -rectifiable current with $M \subset W$. We say that ν is *boundaryless in W* if $\langle \nu, d\omega \rangle = 0$ for each smooth differentiable $(k-1)$ -form ω with compact support in W .

Definition 2.14. Let $\Omega \subset \mathbb{R}^2$ be an open set and $f: \Omega \rightarrow \mathbb{R}^2$ be a continuous BV mapping. Then we define the *distributional Jacobian* $\text{Det } Df$ as

$$\langle Df, \varphi \rangle = \lim_{j \rightarrow \infty} \int_{\Omega} \varphi(x) J_{f_j}(x) dx,$$

where f_j are standard convolution approximations of f . The limit has a good sense as integration by parts leads to a valid duality between measures and continuous functions, see Remark 2.15.

If $u \in BV(\Omega)$ is continuous and $\varphi: \Omega \rightarrow \mathbb{R}$ is smooth, we need neither approximation nor distributional differentiation to give sense to the determinant $\det(D\varphi, Du)$. Therefore we denote it by lowercase “det” although it is a measure. Namely,

$$\langle \det(D\varphi, Du), \eta \rangle = \langle D_2 u, \eta D_1 \varphi \rangle - \langle D_1 u, \eta D_2 \varphi \rangle, \quad \eta \in \mathcal{C}_c(\Omega).$$

Remark 2.15. If $\Phi: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a $\mathcal{C}^1$ -mapping satisfying $\text{div } \Phi \equiv 1$ and $\varphi \in \mathcal{C}_c^\infty(\Omega)$, then

$$\langle \text{Det } Df, \varphi \rangle = \langle \det(D\varphi, Df_1), \Phi_2 \circ f \rangle - \langle \det(D\varphi, Df_2), \Phi_1 \circ f \rangle, \quad \varphi \in \mathcal{C}_c^\infty(\Omega).$$

In particular, $\langle \text{Det } Df, \varphi \rangle = -\langle \det(D\varphi, Df_2), f_1 \rangle$, $\varphi \in \mathcal{C}_c^\infty(\Omega)$, which is frequently used as a definition.

3. Gradient of the inverse

In this section we prove the part (a) of Theorem 1.1. We proceed similarly to [14]. Recall that $f: \Omega \rightarrow \mathbb{R}^2$ is a sense preserving BV homeomorphism. Without loss of generality, we can restrict our attention to one coordinate, namely we want to prove that

$$D_2(f^{-1})_2(f(U)) = D_1 f_1(U) \tag{8}$$

for each open set $U \subset \mathbb{R}^2$. To this end, we use an auxiliary mapping

$$g(x) = (f_1(x), x_2).$$

Lemma 3.1. *Let $U \subset\subset \Omega$ be an open set. Then*

$$\int_{\mathbb{R}^2} \eta(z) \deg(g, U, z) dz = \langle D_1 f_1, \eta \circ g \rangle$$

for each $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ with $\text{spt } \eta \cap g(\partial U) = \emptyset$ and

$$\int_{\mathbb{R}^2 \setminus g(\partial U)} |\deg(g, U, z)| dz \leq |D_1 f_1|(U).$$

Proof. Assume first that g is smooth. Using the degree formula (Proposition 2.4) we compute

$$\int_{\mathbb{R}^2} \eta(z) \deg(g, U, z) dz = \int_U \eta(g(x)) J_g(x) dx = \int_U \eta(g(x)) D_1 f_1(x) dx.$$

In the general case we approximate g by convolution and pass to the limit. Now, we use the fact that

$$\int_{\mathbb{R}^2 \setminus g(\partial U)} |\deg(g, U, z)| dz = \sup \left\{ \int_{\mathbb{R}^2} \eta(z) \deg(g, U, z) dz : \eta \in \mathcal{C}_c^\infty(\mathbb{R}^2 \setminus g(\partial U)), |\eta| \leq 1 \right\},$$

which completes the proof. $\square$

Lemma 3.2. *Let $U \subset\subset \Omega$ be an open set and $|g(\partial U)| = 0$. Then*

$$\int_{\mathbb{R}^2} \deg(g, U, z) dz = D_1 f_1(U).$$

Proof. Let $\eta_j \in \mathcal{C}_c^\infty(\mathbb{R}^2)$ be smooth functions satisfying $\text{spt } \eta_j \cap g(\partial U) = \emptyset$ and $\eta_j \nearrow 1$ on $g(U) \setminus g(\partial U)$. By Lemma 3.1,

$$\int_{\mathbb{R}^2} \deg(g, U, z) dz = \lim_{j \rightarrow \infty} \int_{\mathbb{R}^2} \eta_j(z) \deg(g, U, z) dz = \lim_{j \rightarrow \infty} \langle D_1 f_1, \eta_j \circ g \rangle = D_1 f_1(U).$$

The passage to the limit is justified as $\deg(g, U, \cdot)$ is integrable by Lemma 3.1 and $D_1 f_1$ is a finite measure. $\square$

Proof of Theorem 1.1(a). Let $U \subset\subset \Omega$ be an open set. We find a monotone exhaustion $U = \bigcup_j U_j$ where, for each $j = 1, 2, \dots$, $U_j \subset\subset U$ is an open set with a smooth boundary and g has a BV trace on ∂U_j . Then $|g(\partial U_j)| = 0$. By Lemma 3.2 we have

$$\text{adj}_{22} Df(U_j) = D_1 f_1(U_j) = \int_{\mathbb{R}^2} \deg(g, U_j, z) dz.$$

Now, we apply Lemma 3.2 to the inverse, but instead of $g(x) = (f_1(x), x_2)$ we consider $h(y) = (y_1, (f^{-1})_2(y))$. We obtain

$$D_2(f^{-1})_2(f(U_j)) = \int_{\mathbb{R}^2} \deg(h, f(U_j), z) dz.$$

However, by the formula on composition of degree, we have

$$\deg(g, U_j, z) = \deg(h, f(U_j), z), \quad z \in \mathbb{R}^2 \setminus g(\partial U_j),$$

as $g = h \circ f$ and f is a sense preserving homeomorphism. Therefore

$$D_2(f^{-1})_2(f(U_j)) = \text{adj}_{22} Df(U_j), \quad j = 1, 2, \dots,$$

and passing to the limit we obtain

$$D_2(f^{-1})_2(f(U)) = \text{adj}_{22} Df(U).$$

By a routine measure-theoretic approximation we obtain the formula for each Borel set. $\square$

4. Distributional Jacobian

Let $\Omega \subset \mathbb{R}^2$ be a bounded open set and $f = (f_1, f_2): \Omega \rightarrow \mathbb{R}^2$ be a bounded sense-preserving BV -homeomorphism.

Let $G \subset\subset \Omega$ be an open set with a smooth boundary. Then each connected component T of the boundary of G is a smooth Jordan curve which can be parametrized as $\gamma: [0, 1] \rightarrow \mathbb{R}^2$, where γ is smooth, regular, and oriented so that $\det(\mathbf{n}^G(\gamma(t)), \gamma'(t)) > 0$, where $\mathbf{n}^G$ is the outward normal to G . Given a continuous function u and a BV function v on T , we consider the Stieltjes integral $\int_T u dv$ (and thus also the Stieltjes integral over ∂G) which is defined via the parametrization γ but does not depend on it.

Lemma 4.1. *Let $u, v: \Omega \rightarrow \mathbb{R}$ be continuous BV functions and $\eta \in \mathcal{C}_c^\infty(\Omega)$ be nonnegative. Then*

$$\langle \det(Dv, D\eta), u \rangle = \int_0^\infty \left(\int_{\{\eta=t\}} u dv \right) dt \quad (9)$$

$$\text{and} \quad \langle \det(Du, D\eta), v \rangle = \int_0^\infty \left(\int_{\{\eta=t\}} v du \right) dt. \quad (10)$$

Proof. By the Morse-Sard theorem (Proposition 2.1), there is a set $Z \subset \mathbb{R}$ of measure zero such that $\{\eta > t\}$ has a smooth boundary for all $t \in (0, +\infty) \setminus Z$. Let $t > 0$ be such that $\{\eta = t\}$ has a smooth boundary. Then we have

$$D\eta(x) = -\mathbf{n}^{\{\eta>t\}}(x) |D\eta(x)|, \quad x \in \{\eta = t\}.$$

To prove (9) and (10), let us first assume that v is smooth. Applying the co-area formula (6) we obtain

$$\begin{aligned} \langle \det(Dv, D\eta), u \rangle &= \int_\Omega u \det(Dv, D\eta) dx = \\ &= - \int_0^\infty \left(\int_{\{\eta=t\}} u \det(Dv, \mathbf{n}^{\{\eta>t\}}) d\mathcal{H}^1 \right) dt = \int_0^\infty \left(\int_{\{\eta=t\}} u dv \right) dt. \end{aligned}$$

This proves (9) in the case of smooth v . If u is also smooth, we observe that

$$\langle \det(Dv, D\eta), u \rangle = -\langle \det(Du, D\eta), v \rangle \quad (11)$$

and

$$\int_{\{\eta=t\}} u dv = - \int_{\{\eta=t\}} v du \quad (12)$$

holds for $t > 0$, $t \notin Z$. A simple approximation argument shows that (11) holds in general for u, v which are both continuous and BV on Ω and (12) holds for $t > 0$ such that $t \notin Z$ and u and v are continuous BV on $\{\eta = t\}$. Then, assuming still that v is smooth, using (11) and (12) we obtain

$$\begin{aligned} \langle \det(Du, D\eta), v \rangle &= \langle \det(Dv, D\eta), u \rangle \\ &= \int_0^\infty \left(\int_{\{\eta=t\}} u dv \right) dt = \int_0^\infty \left(\int_{\{\eta=t\}} v du \right) dt. \end{aligned}$$

This proves (10) in the case of smooth v . Now, in the general case we use convolution approximation of v by smooth functions and obtain (10) as well. For the passages to limits on the right hand side we use the dominated convergence theorem and the fact that the integral over t of the variation of v on $\{\eta = t\}$ is finite. Now, interchanging the roles of u and v we obtain (9) in the general case. $\square$

Theorem 4.2. *Let $f: \Omega \rightarrow \mathbb{R}^2$ be a bounded sense-preserving BV homeomorphism. If $E \subset \Omega$ is a Borel set, then $\text{Det } Df(E) = |f(E)|$.*

Proof. We compare the distribution $\text{Det } Df$ and the measure $\rho: E \mapsto |f(E)|$. Choose a nonnegative test function $\eta \in C_c^\infty(\Omega)$. By Lemma 6, Green's theorem and integration by means of the distribution function we have

$$\begin{aligned} \langle \text{Det } Df, \eta \rangle &= -\langle \det(D\eta, Df_2), f_1 \rangle = \int_0^\infty \left(\int_{\{\eta=t\}} f_1 df_2 \right) dt \\ &= \int_0^\infty \left(\int_{\{\eta \circ f^{-1}=t\}} y_1 dy_2 \right) dt = \int_0^\infty \left| \{\eta \circ f^{-1} > t\} \right| dt = \int_{f(\Omega)} \eta(f^{-1}(y)) dy = \int_\Omega \eta d\rho. \end{aligned}$$

Therefore the measures $\text{Det } Df$ and ρ coincide. $\square$

5. Fundamental estimate

Theorem 5.1. *Let $f: \Omega \rightarrow \mathbb{R}^2$ be a sense-preserving BV homeomorphism and $(x, y) \in \Gamma(\Omega)$. Then there exists $r_0 > 0$ such that for each $r \in (0, r_0)$ we have*

$$r^2 \leq C|\mu|(B(x, r) \times B(y, r)).$$

Proof. Choose $x_0 \in \Omega$ and denote $y_0 = f(x_0)$. Set

$$r_0 = \min\{\text{dist}(x_0, \Omega^c), \text{dist}(y_0, f(\Omega)^c)\}$$

and choose $r \in (0, r_0)$. If $f(B(x_0, \frac{1}{2}r)) \subset B(y_0, r)$, then

$$\mu_{12}(B(x_0, r) \times B(y_0, r)) \geq |B(x_0, \frac{1}{2}r)|.$$

If $f(B(x_0, r)) \supset B(y_0, \frac{1}{2}r)$, then $\mu^{12}(B(x_0, r) \times B(y_0, r)) \geq |B(y_0, \frac{1}{2}r)|$.

In the remaining case, for each $t \in [\frac{1}{2}r, r]$, $K_t := f^{-1}(\partial B(y_0, t))$ is a continuum which intersects both $\partial B(x_0, \frac{1}{2}r)$ and $\partial B(x_0, r)$ and thus its diameter is at least $\frac{1}{2}r$. Using the co-area formula (Proposition 2.3), we obtain

$$\begin{aligned} |\mu|(B(x_0, r) \times B(y_0, r)) &\geq |Df|(B(x_0, r)) \geq |D|f - y_0||((B(x_0, r))) \\ &\geq \int_{r/2}^r \mathcal{H}^1(\{|f - y_0| = t\}) \geq \frac{1}{4}r^2. \end{aligned} \quad \square$$

Theorem 5.2. *Let $f: \Omega \rightarrow \mathbb{R}^2$ be a sense-preserving BV homeomorphism and $E \subset \Omega$ be a Borel set.*

Then $\mathcal{H}^2(\Gamma(E)) \leq C|\mu|(\Gamma(E))$. In particular, $\mathcal{H}^2(\Gamma(\Omega)) < \infty$.

Proof. It follows from Theorem 5.1 by a Vitali type covering argument. $\square$

6. Rectifiability

Theorem 6.1. *Let $f: \Omega \rightarrow \mathbb{R}^2$ be a sense-preserving BV homeomorphism and Γ be its graph. Let $\tau(x, f(x))$ be the Radon-Nikodym derivative of $D\Gamma$ with respect to $|D\Gamma|$ at x and $\tau_1(x, f(x))$, $\tau_2(x, f(x))$ be the columns of this matrix. Then $\Gamma(\Omega)$ is 2-rectifiable and (τ_1, τ_2) is a basis of the approximate tangent plane to $\Gamma(\Omega)$ $\mathcal{H}^2$ -a.e.*

Proof. We use the blow-up idea of De Giorgi's proof of rectifiability of the reduced boundary of a set of finite perimeter, see e.g. [2, Thm. 3.59], however, our situation is more complicated. Let μ be as in (2) and $\sigma = |\mu|$. Let $\kappa = \frac{d\mu}{d\sigma}$. Let M_0 be the set of all Lebesgue points of κ with respect to σ . By the Lebesgue-Besicovitch differentiation theorem, $\sigma(\Gamma \setminus M_0) = 0$. We split M_0 as follows:

$$\begin{aligned} M_{12} &= \{x \in M_0 : \kappa_{12}(x) > 0\}, \\ M^{12} &= \{x \in M_0 : \kappa^{12}(x) > 0\}, \\ M^* &= \{x \in M_0 : \kappa_{12}(x) = \kappa^{12}(x) = 0\}. \end{aligned}$$

We want to prove 2-rectifiability of these sets.

By the Luzin approximation (Proposition 2.2), there is a set $E \subset \Omega$ such that $|E| = 0$ and $\Gamma(\Omega \setminus E)$ can be covered by countably many Lipschitz graphs. Since $\mu_{12}(\Gamma(E)) = 0$, we have $\kappa_{12} = 0$ σ -a.e. in $\Gamma(E)$. Therefore σ -almost all of M_{12} is 2-rectifiable. By Theorem 5.2, the exceptional set has $\mathcal{H}^2$ -measure zero and thus M_{12} is 2-rectifiable.

We have $D_1\Gamma(x) = (1, 0, D_1f_1(x), D_1f_2(x))$ and $D_2\Gamma(x) = (0, 1, D_2f_1(x), D_2f_2(x))$ a.e. (here the derivatives have the meaning of pointwise representatives of the absolutely continuous part), so that κ is a positive multiple of $\tau_1 \wedge \tau_2$ $\mathcal{H}^2$ -a.e. on M_{12} .

Passing to the inverse using Theorem 1.1 we see that also M^{12} is 2-rectifiable and (τ_1, τ_2) is a basis of the approximate tangent plane to $\Gamma(\Omega)$ $\mathcal{H}^2$ -a.e. on M^{12} .

We proceed to the set M^* . Due to its definition, σ is the push-forward of $|D\Gamma|$ but also of $|Df|$ through Γ on M^* , as μ_{12} and μ^{12} vanish on M^* and the remaining

coordinates are obtained from the coordinates of $D_j f_i$ by reordering and adjusting signs, see (2).

It follows that $\tau_1 = (0, 0, -\kappa_2^1, -\kappa_2^2)$ and $\tau_2 = (0, 0, \kappa_1^1, \kappa_1^2)$ holds σ -a.e. on M^* . We want to verify the assumptions of Proposition 2.9. To this end, observe that for σ -almost all $(x, y) \in M^*$ we have

$$\lim_{r \rightarrow 0^+} \frac{\sigma(B((x, y), r) \setminus M^*)}{\sigma(B((x, y), r))} = 0. \quad (13)$$

This is a consequence of the Lebesgue-Besicovitch differentiation theorem. Therefore it does not matter that σ is not carried by M^* . By Alberti's rank-one theorem [1], the Radon-Nikodym derivative of $D_s f$ with respect to $|Df|$ is rank-one. It follows that the Radon-Nikodym derivative of $(\mu_i^j)_{i,j=1}^2$ with respect to σ is also rank-one. Consequently, $(\kappa_i^j)_{i,j=1}^2$ is rank-one σ -a.e. in M^* . Pick $(x_0, y_0) \in M^*$ with this rank-one property and such that $|\kappa(x_0, y_0)| = 1$ and (13) holds. For simplicity let us assume that $x_0 = y_0 = 0$. By a rotation both in domain and range we may assume that

$$\kappa_1^2(0, 0) = \kappa_2^2(0, 0) = \kappa_2^1(0, 0) = 0, \quad \kappa_1^1(0, 0) = 1. \quad (14)$$

We want to prove that $\mathbf{V} = \{(x, y) \in \mathbb{R}^2 \times \mathbb{R}^2 : x_2 = y_2 = 0\}$ satisfies the assumptions of Proposition 2.9. For $r > 0$ small enough write

$$f^r(x) = \frac{f(rx)}{r}.$$

Consider functions $\varphi_1, \varphi_2 \in \mathcal{C}_c^\infty(\mathbb{R})$ and denote their indefinite integrals by Φ_1 and Φ_2 , respectively, normalized by $\Phi_i(0) = 0$. Let $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^2)$. Let B be a ball containing the support of $(x, y) \mapsto \eta(x)\varphi_1(y_1)\varphi_2(y_2)$.

A change of variables yields

$$\int_{\Gamma(\Omega)} \varphi_1\left(\frac{\bar{y}_1}{r}\right) \varphi_2\left(\frac{\bar{y}_2}{r}\right) \eta\left(\frac{\bar{x}}{r}\right) d\mu_1^1(\bar{x}, \bar{y}) = -r^2 \langle D_1 f_1^r, (\varphi_1 \circ f_1^r) (\varphi_2 \circ f_2^r) \eta \rangle, \quad (15)$$

similarly for other coordinates of μ .

We want to prove that $r^{-2}\sigma(B(0, r) \times [-r, r]^2)$ remains bounded as $r \rightarrow 0^+$.

Assume for a while that φ_1, φ_2 and η have the shape of cut-off functions. For simplicity, assume that $\varphi_1 \in \mathcal{C}_c^\infty(\mathbb{R})$ is even, $0 \leq \varphi_1 \leq 1$, $\varphi_1 = 1$ on $[-1, 1]$, $\varphi_1 = 0$ outside $(-2, 2)$, and $\varphi_2 = \varphi_1$, whereas $\eta(x) = \varphi_1(|x|)$ on $\mathbb{R}^2$.

For the estimate, it is enough to consider r such that

$$r^{-2}\sigma(B(0, r) \times [-r, r]^2) \geq (2r)^{-2}\sigma(B(0, 2r) \times [-2r, 2r]^2).$$

Given $\varepsilon > 0$, for r small enough we have

$$r^{-2}\sigma(B(0, r) \times [-r, r]^2) \leq 2r^{-2}\mu_1^1(B(0, r) \times [-r, r]^2)$$

and, by (14), $|\mu_1^2|(B(0, 2r) \times [-2r, 2r]^2) \leq \varepsilon\sigma(B(0, 2r) \times [-2r, 2r]^2)$.

We estimate

$$\begin{aligned}
r^{-2}\sigma(B(0, r) \times [-r, r]^2) &\leq 2r^{-2}\mu_1^1(B(0, r) \times [-r, r]^2) \\
&\leq -2\langle D_1 f_1^r, (\varphi_1 \circ f_1^r)(\varphi_2 \circ f_2^r) \eta \rangle \\
&= -2\langle D_1(\Phi_1 \circ f_1^r), (\varphi_2 \circ f_2^r) \eta \rangle \\
&= 2\langle D_1(\varphi_2 \circ f_2^r), (\Phi_1 \circ f_1^r) \eta \rangle + 2 \int_B (\Phi_1 \circ f_1^r)(\varphi_2 \circ f_2^r) D_1 \eta \, dx \\
&\leq C(r^{-2}|\mu_1^2|(B(0, 2r) \times [-2r, 2r]^2) + 1) \\
&\leq C(\varepsilon r^{-2}\sigma(B(0, 2r) \times [-2r, 2r]^2) + 1) \\
&\leq 4C\varepsilon r^{-2}\sigma(B(0, r) \times [-r, r]^2) + C.
\end{aligned}$$

Taking ε such that $8C\varepsilon < 1$ we conclude that

$$r^{-2}\sigma(B(0, r) \times [-r, r]^2) \leq C. \quad (16)$$

Consider $r_j \searrow 0$ and write $f^{(j)} = f^{r_j}$. In view of (16), (15) and (14) we observe that

$$\langle |Df_2^{(j)}|, (\varphi_1 \circ f_1^{(j)})(\varphi_2 \circ f_2^{(j)}) \eta \rangle \rightarrow 0,$$

$$\langle |D_2 f_1^{(j)}|, (\varphi_1 \circ f_1^{(j)})(\varphi_2 \circ f_2^{(j)}) \eta \rangle \rightarrow 0.$$

and

$$\sup_j \langle |Df_1^{(j)}|, (\varphi_1 \circ f_1^{(j)})(\varphi_2 \circ f_2^{(j)}) \eta \rangle < \infty.$$

Hence $|D(\Phi_1 \circ f_1^{(j)})| \rightarrow 0$ and $\sup_j |D(\Phi_1 \circ f_1^{(j)})| < \infty$. Passing if necessary to a subsequence, the sequence $g_j := -\Phi_1 \circ f_1^{(j)}$ converges weakly* in $BV(B(0, 1))$ and strongly in $L^1(B(0, 1))$ (in view of compact embedding) to a limit function g . Since $\|D_2 g_j\| \rightarrow 0$, the function g depends only on the x_1 -variable. Let $R = \Phi_1(2)$. Since the measure of $\{x: |g_j(x)| < R\}$ is estimated by

$$Cr_j^{-2}\mu_{12}(B(0, 2r_j) \times [-2r_j, 2r_j]^2) \rightarrow 0,$$

passing to the limit we have $|g| = R$ a.e. In view of Theorem 5.1, $(\|Dg_j\|)_j$ is bounded away from 0 and by (14) we see that the negative parts of $D_1 g_j$ tend to 0. It follows that there is $c \in \mathbb{R}$ such that $g(x) = -R$ for $x_1 < c$ and $g(x) = R$ for $x_1 > 0$. Assuming that $c > 0$, we would get that $g \equiv -R$ when replacing r_j with $\frac{1}{2}cr_j$. This would contradict the estimate of Theorem 5.1. Similarly we would exclude $c < 0$. Hence $c = 0$ and

$$g = \begin{cases} R, & x_1 > 0, \\ -R, & x_1 < 0. \end{cases}$$

Now, let φ_1, φ_2 and η be general (this means, smooth with compact support, but not necessarily cut-off functions). We may assume that the support of φ_i is contained in $[-1, 1]$, $i = 1, 2$, and the support of η is contained in $B(0, 1)$.

Let, again, $g_j = -\Phi_1 \circ f_1^{(j)}$. Then, arguing as above, $g_j \rightarrow g$ weakly* in $BV(B(0, 1))$ and strongly in $L^1(B(0, 1))$, where

$$g = \begin{cases} \Phi_1(1), & x_1 > 0, \\ \Phi_1(-1), & x_1 < 0. \end{cases}$$

Then

$$\begin{aligned} -\langle D_1 f_1^{(j)}, (\varphi_1 \circ f_1^{(j)})(\varphi_2 \circ f_2^{(j)}) \eta \rangle &\rightarrow (\Phi_1(1) - \Phi_1(-1))\varphi_2(0) \int_{\mathbb{R}} \eta(x_1, 0) dx_1 \\ &= \int_{\mathbb{R}^2} \varphi_1(y_1)\varphi_2(0) \eta(x_1, 0) dx_1 dy_1. \end{aligned}$$

Under the notation as in (7) we observe that

$$\int_{B(0,1) \times [-1,1]^2} \Psi(x, y) d(\mu_1^1)_{0,r_j}(x, y) \rightarrow \int_{x_2=y_2=0} \Psi(x, y) d\mathcal{H}^2(x, y),$$

if $\Psi(x, y)$ is of the form $\varphi_1(y_1)\varphi_2(y_2)\eta(x)$ with φ_1, φ_2 and η smooth with compact support. However, since linear combinations of such functions are dense in $\mathcal{C}_0(\mathbb{R}^4)$ and the sequence $((\mu_1^1)_{0,r_j})_j$ is locally bounded, it follows that it converges weakly* to $\mathcal{H}^2|_{\{x_2=y_2=0\}}$ on any bounded ball in $\mathbb{R}^4$. We have proved that from any sequence $(r_j)_j$ we can extract a subsequence $(r_{j_k})_k$ such that

$$(\mu_1^1)_{0,r_{j_k}} \rightarrow \mathcal{H}^2|_{\{x_2=y_2=0\}} \quad \text{weakly*}$$

on any bounded ball in $\mathbb{R}^4$.

This is enough to conclude that the weak* convergence occurs as $r \rightarrow 0^+$, since the weak* topology is metrizable on bounded subsets of $\mathcal{C}_0(\mathbb{R}^4)^*$. Finally, as σ behaves asymptotically as μ_1^1 at the origin, we conclude that

$$\int_{B(0,1) \times [-1,1]^2} \Psi(x, y) d\sigma_{0,r_j}(x, y) \rightarrow \int_{x_2=y_2=0} \Psi(x, y) d\mathcal{H}^2(x, y)$$

for any test function $\Psi \in \mathcal{C}_c(\mathbb{R}^4)$.

We have shown that the condition of Proposition 2.9 is verified at σ -almost all points $(x, y) \in M^*$. By Theorem 5.2, it holds at $\mathcal{H}^2$ -a.e. $(x, y) \in M^*$ and thus M^* is 2-rectifiable. Getting all together we conclude that $\Gamma(\Omega)$ is 2-rectifiable. $\square$

Theorem 6.2. *Let $f: \Omega \rightarrow \mathbb{R}^2$ be a bounded sense-preserving BV homeomorphism and $\Gamma(\Omega)$ be its graph. Then $(\Gamma(\Omega), \mu)$ is a 2-dimensional 1-multiplicity rectifiable current.*

Proof. Consider the orientation making the basis (τ_1, τ_2) positive on M , where τ is as in Theorem 6.1. Given a positive parametrization

$$\psi = (\psi_1, \psi_2, \psi^1, \psi^2): E \rightarrow \Gamma(\Omega),$$

where $E \subset \mathbb{R}^2$ is measurable, we need to show that

$$\int_E \det(D\psi_1, D\psi_2) dt = \mu_{12}(\psi(E)), \quad (17)$$

$$\int_E \det(D\psi^1, D\psi^2) dt = \mu^{12}(\psi(E)), \quad (18)$$

$$\int_E \det(D\psi_j, D\psi^i) dt = \mu_i^j(\psi(E)), \quad i, j = 1, 2. \quad (19)$$

However, (17) is just a change of variables, (18) follows from Theorem 4.2. Concerning (19), we use the fact that the formula holds for scalar BV-functions, see [7, 4.5.9], [9, Section 4.1.5]. Let, for example, $i = j = 1$. Set $\tilde{\psi} = (\psi_1, \psi_2, \psi^1)$. Then $\tilde{\psi}$ is a Lipschitz mapping to the graph of the scalar BV-function f_1 and thus

$$\int_E \det(D\psi_1, D\psi^1) dt = -D_2 f_1((\psi_1, \psi_2)(E)) = \mu_1^1(\psi(E)). \quad \square$$

Theorem 6.3. *The current from Theorem 6.2 is boundaryless.*

Proof. Consider a differential form $\omega dx_1 = \eta(x)\varphi(y) dx_2$ where $\eta \in \mathcal{C}_c^\infty(\Omega)$ and $\varphi \in \mathcal{C}_c^\infty(f(\Omega))$. Then by the chain rule [2, Thm. 3.96] we have

$$\begin{aligned} \int_{\Gamma(\Omega)} d(\omega dx_2) &= \int_{\Gamma(\Omega)} \eta(x) D_1 \varphi(y) dy_1 dx_2 + \eta(x) D_2 \varphi(y) dy_2 dx_2 \\ &\quad + \varphi(y) D_1 \eta(x) dx_1 dx_2 \\ &= \langle D_1 f_1, \eta(D_1 \varphi) \circ f \rangle + \langle D_1 f_2, \eta(D_2 \varphi) \circ f \rangle + \int_{\Omega} D_1 \eta(x) \varphi(f(x)) dx. \end{aligned}$$

The expression on the right hand side vanishes if f is smooth (and not necessarily invertible), it is the standard integration by parts. In the general case we can use a routine approximation argument, as we do not need invertibility at this point. Note that by mollification we obtain uniform convergence at the part $(D_i \varphi) \circ f$ and weak* convergence of the gradients. As a next step, we observe that linear combinations of functions ω of type $\omega(x, y) = \eta(x)\varphi(y)$ are dense in $\mathcal{C}_c^1(\Omega \times f(\Omega))$, so that we obtain

$$\int_{\Gamma(\Omega)} d(\omega dx_2) = 0$$

for the differential forms of type $\omega(x, y) dx_2$ with compact support in $\Gamma(\Omega)$. Similarly we handle differential forms $\omega(x, y) dx_1$. For differential forms of type $\omega(x, y) dy_1$ and $\omega(x, y) dy_2$ we pass to the inverse mapping, as we already know the results of Theorem 1.1. $\square$

Proof of Theorem 1.2. The assertions are contained in Theorems 6.1, 6.2 and 6.3. $\square$

7. Example

In this section we construct a BV-homeomorphism $f: [-1, 1]^2 \rightarrow [-1, 1]^2$ such that both f and f^{-1} satisfy the (N)-condition, $J_f > 0$ a.e., $J_{f^{-1}} > 0$ a.e., but neither f nor f^{-1} are Sobolev.

7.1. Auxiliary mappings

Set
$$a_k = \frac{1 + 2^{1-k}}{2}, \quad b_k = 2^{1-k}.$$

Then
$$1 = a_1 > a_2 > \dots, \quad \lim_k a_k = \frac{1}{2},$$

$$1 = b_1 > b_2 > \dots, \quad \lim_k b_k = 0,$$

$$b_k = 2b_{k+1} \quad \text{and} \quad b_k - b_{k+1} = 2(a_k - a_{k+1}), \quad k = 1, 2, \dots \quad (20)$$

We are going to construct a mapping g_k of

$$P_k := [-2^{-k}a_k, 2^{-k}a_k] \times [-2^{-k}b_k, 2^{-k}b_k]$$

onto itself. Define

$$Q_k = [-2^{-k}a_{k+1}, 2^{-k}a_{k+1}] \times [-2^{-k}b_{k+1}, 2^{-k}b_{k+1}],$$

$$\xi_k(x) = \frac{a_k - 2^k|x_1|}{a_k - a_{k+1}}, \quad \eta_k(x) = \frac{b_k - 2^k|x_2|}{b_k - b_{k+1}},$$

$$A_k = \{x \in P_k \setminus Q_k : \xi_k(x) \geq \eta_k(x)\}, \quad B_k = \{x \in P_k \setminus Q_k : \xi_k(x) \leq \eta_k(x)\},$$

$$\mathbf{T}_k^t = \begin{pmatrix} 0, & \frac{a_k + t(a_{k+1} - a_k)}{b_k + t(b_{k+1} - b_k)} \\ \frac{b_k + t(b_{k+1} - b_k)}{a_k + t(a_{k+1} - a_k)}, & 0 \end{pmatrix}.$$

We set
$$g_k(x) = (g_k^1(x), g_k^2(x)) = \begin{cases} \mathbf{T}_k^1 x, & x \in Q_k, \\ \mathbf{T}_k^{\min\{\xi_k(x), \eta_k(x)\}} x, & x \in P_k \setminus Q_k. \end{cases}$$

In particular, $g_k(x) = \mathbf{T}_k^0 x$ on ∂P_k . The function g_k is bi-Lipschitz, indeed, $g_k^{-1} = g_k$. Let us estimate the gradient of g_k . If $x \in A_k$, we have

$$t := \min\{\xi_k(x), \eta_k(x)\} = \eta_k(x),$$

$$2^k|x_2| = b_k + t(b_{k+1} - b_k), \quad 2^k|x_1| \leq a_k + t(a_{k+1} - a_k)$$

and

$$g_k^1(x) = \frac{a_k + t(a_{k+1} - a_k)}{b_k + t(b_{k+1} - b_k)} x_2 = \pm 2^{-k} (a_k + t(a_{k+1} - a_k)),$$

$$g_k^2(x) = \frac{b_k + t(b_{k+1} - b_k)}{a_k + t(a_{k+1} - a_k)} x_1 = \pm 2^k \frac{x_1 x_2}{a_k + t(a_{k+1} - a_k)}.$$

It follows

$$|Dg_k^1(x)| \leq 2^{-k}(a_k - a_{k+1})|D\eta_k(x)| \leq \frac{a_k - a_{k+1}}{b_k - b_{k+1}} = \frac{1}{2},$$

$$|Dg_k^2(x)| \leq 2 + 2^k \frac{|x_1 x_2|(a_k - a_{k+1})}{(a_k + t(a_{k+1} - a_k))^2} |D\eta_k(x)|$$

$$\leq 2 + \frac{b_k(a_k - a_{k+1})}{a_{k+1}(b_k - b_{k+1})} \leq 3,$$

so that $|Dg_k(x)| \leq 4, \quad x \in A_k. \quad (21)$

Since $|A_k| \leq 2^{1-2k} a_k (b_k - b_{k+1}) \leq 2^{2-3k},$

we obtain
$$\int_{A_k} |Dg_k| dx \leq 2^{4-3k}. \quad (22)$$

If $x \in B_k$, we have $t := \min\{\xi_k(x), \eta_k(x)\} = \xi_k(x),$

$$2^k |x_2| \leq b_k + t(b_{k+1} - b_k), \quad 2^k |x_1| = a_k + t(a_{k+1} - a_k)$$

and

$$g_k^1(x) = \frac{a_k + t(a_{k+1} - a_k)}{b_k + t(b_{k+1} - b_k)} x_2 = \pm 2^k \frac{x_1 x_2}{b_k + t(b_{k+1} - b_k)},$$

$$g_k^2(x) = \frac{b_k + t(b_{k+1} - b_k)}{a_k + t(a_{k+1} - a_k)} x_1 = \pm 2^{-k} (b_k + t(b_{k+1} - b_k)).$$

It follows

$$|Dg_k^1(x)| \leq 1 + \frac{a_k}{b_{k+1}} + 2^k \frac{|x_1 x_2| (b_k - b_{k+1})}{(b_k + t(b_{k+1} - b_k))^2} |D\xi_k(x)|$$

$$\leq 1 + 2^k + \frac{a_k (b_k - b_{k+1})}{b_{k+1} (a_k - a_{k+1})} \leq 2^{k+2},$$

$$|Dg_k^2(x)| \leq 2^{-k} (b_k - b_{k+1}) |D\eta_k(x)| \leq \frac{b_k - b_{k+1}}{a_k - a_{k+1}} \leq 2,$$

so that $|Dg_k(x)| \leq 2^{k+3}, \quad x \in B_k. \quad (23)$

Since $|B_k| \leq 2^{1-2k} b_k (a_k - a_{k+1}) = 2^{1-4k},$

we obtain
$$\int_{B_k} |Dg_k| dx \leq 2^{4-3k}. \quad (24)$$

Of course $|Dg_k(x)| = \frac{a_{k+1}}{b_{k+1}}, \quad x \in Q_k. \quad (25)$

Since $|Q_k| = 4^{1-k} a_{k+1} b_{k+1}$, we have

$$\int_{Q_k} |Dg_k| dx = 4^{1-k} a_{k+1}^2 \leq 4^{1-k}. \quad (26)$$

7.2. The construction of the approximating sequence

Consider multiindices $\alpha = (\alpha_1, \dots, \alpha_k) \in \{-1, 1\}^k$ and $\beta = (\beta_1, \dots, \beta_k) \in \{-1, 1\}^k$, $k = 1, 2, \dots$. Then we define

$$u_\alpha = \sum_{j=1}^k 2^{-j} \alpha_j a_j, \quad v_\alpha = \sum_{j=1}^k 2^{-j} \alpha_j b_j, \quad \alpha \in \{-1, 1\}^k,$$

$$u_\beta = \sum_{j=1}^k 2^{-j} \beta_j a_j, \quad v_\beta = \sum_{j=1}^k 2^{-j} \beta_j b_j, \quad \beta \in \{-1, 1\}^k.$$

$$\begin{aligned}
\text{Set } X_\alpha &= [u_\alpha - 2^{-k}a_k, u_\alpha + 2^{-k}a_k], \\
X'_\alpha &= [u_\alpha - 2^{-k}a_{k+1}, u_\alpha + 2^{-k}a_{k+1}], & \alpha \in \{-1, 1\}^k, \\
Y_\beta &= [v_\beta - 2^{-k}b_k, v_\beta + 2^{-k}b_k], \\
Y'_\beta &= [v_\beta - 2^{-k}b_{k+1}, v_\beta + 2^{-k}b_{k+1}], & \beta \in \{-1, 1\}^k.
\end{aligned}$$

Now, we are ready to construct our approximating mappings by gluing the copies of g_k . Namely, set

$$P_{\alpha,\beta} = X_\alpha \times Y_\beta, \quad Q_{\alpha,\beta} = X'_\alpha \times Y'_\beta, \quad \alpha, \beta \in \{-1, 1\}^k,$$

which are translated copies of P_k and Q_k , respectively, and

$$f_k(x) = (u_\beta, v_\alpha) + g_k(x - (u_\alpha, v_\beta)), \quad x \in P_{\alpha,\beta}, \quad \alpha, \beta \in \{-1, 1\}^k. \quad (27)$$

This defines f_k on $S_k := \bigcup_{\alpha,\beta \in \{-1,1\}^k} P_{\alpha,\beta}$.

We have $S_k = Q_0 = [-1, 1]^2$ if $k = 1$, otherwise we let $f_k = f_{k-1}$ on $Q_0 \setminus S_k$.

7.3. Sobolev estimates

Since the functions f_k on $P_{\alpha,\beta}$ are only translates of g_k , we use (22), (24) and (26) to estimate for $j = 1, \dots, k$,

$$\int_{S_j \setminus S_{j+1}} |Df_k| dx = \sum_{\alpha,\beta \in \{-1,1\}^j} \int_{P_{\alpha,\beta} \setminus Q_{\alpha,\beta}} |Df_j| dx = 4^k \int_{P_j \setminus Q_j} |Dg_j| dx \leq 2^{4-j},$$

$$\text{and} \quad \int_{S_{k+1}} |Df_k| dx \leq \sum_{\alpha,\beta \in \{-1,1\}^k} \int_{Q_{\alpha,\beta}} |Df_k| dx \leq 4^k 4^{1-k} = 4.$$

$$\text{All together} \quad \int_{Q_0} |Df_k| dx \leq 20.$$

7.4. Passing to the limit

We see that the sequence $(f_k)_k$ is bounded in $W^{1,1}(Q_0)$. At the same time, the sequence converges in $\mathcal{C}(Q_0)$ to a continuous mapping f . It is obvious from the construction that $f_k^{-1} = f_k$ for each k , so that $f^{-1} = f$, f is a BV homeomorphism. We have $|S_k| = 4^k |P_k| = 4^k 4^{1-k} a_k b_k \leq 2^{3-k}$. The intersection

$$S = \bigcap_k S_k \quad (28)$$

is a Cantor set of measure zero. Now, the function f is locally Lipschitz on $Q_0 \setminus S$ and maps S to itself, so that f satisfies the Luzin (N)-condition. Also it is obvious that $J_f > 0$ a.e. in $Q_0 \setminus S$, so simply $J_f > 0$ a.e. Now, define

$$X = \bigcap_{k=1}^{\infty} \bigcup_{\alpha \in \{-1,1\}^k} X_\alpha, \quad \text{and} \quad Y = \bigcap_{k=1}^{\infty} \bigcup_{\beta \in \{-1,1\}^k} Y_\beta.$$

$$\begin{aligned} \text{Then } S = X \times Y, \quad |X| &= \lim_{k \rightarrow \infty} \sum_{\alpha \in \{-1,1\}^k} |X_\alpha| = 2 \lim_{k \rightarrow \infty} a_k = 1, \\ |Y| &= \lim_{k \rightarrow \infty} \sum_{\beta \in \{-1,1\}^k} |Y_\beta| = 2 \lim_{k \rightarrow \infty} b_k = 0. \end{aligned}$$

Consider a vertical segment $\{z_1\} \times [-1, 1]$, where $z_1 \in X$. Then the function $x_2 \mapsto f(z_1, x_2)$ maps Y onto X , so that it fails to satisfy the Luzin (N)-condition, in particular, it fails to be absolutely continuous. It follows that f is not a Sobolev mapping, in other words, the singular part of Df is nontrivial.

Proof of Theorem 1.3. The existence of a function with required properties follows from our construction. $\square$

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