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Duality Arguments for Linear Elasticity Problems with Incompatible Deformation Fields

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Dedicated to Umberto Mosco on the occasion of his 80th birthday.

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We prove existence and uniqueness for solutions to equilibrium problems for free-standing, traction-free, non homogeneous crystals in the presence of plastic slips. Moreover we prove that this class of problems is closed under G-convergence of the operators. In particular the homogenization procedure, valid for elliptic systems in linear elasticity, depicts the macroscopic features of a composite material in the presence of plastic deformation.

Keywords: Boundary value problems for elliptic systems, elastic problems, duality solutions, homogenization.

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1. Introduction

In this paper we consider linear boundary value problems for systems of the form

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)\beta) = 0, & \text{in } \Omega, \\ \operatorname{curl}\beta = \mu, & \text{in } \Omega, \\ \mathbb{C}(x)\beta \cdot n = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded, smooth, open subset of $\mathbb{R}^3$, $\mathbb{C}(x)$ is a symmetric tensor-valued function in Ω with VMO coefficients, satisfying the standard hypotheses of elasticity theory (see Definition 2.1), and μ is a matrix-valued bounded Radon measure in Ω.

In linear elasto-plastic models $\mathbb{C}(x)$ is the elastic tensor for a non homogeneous elastic body and the gradient of the displacement field is decomposed in plastic and elastic strain. The field $\beta \in L^1(\Omega; \mathbb{R}^{3\times 3})$ in (1) represents the elastic strain which, in the presence of a non trivial plastic deformation (possibly due to non homogenous plastic slips), may be non compatible, i.e., it may not be curl free. The incompatibility is the effective result of a distribution of crystals defects (the dislocations) and the measure μ represents the dislocation density. Dislocations are topological defects in the crystalline structure that, at a mesoscopic level, can be identified with loops along which the strain has a singularity. In particular the strain field is curl free outside the loops and has a non trivial circulation around the lines. Therefore these singularities are nicely described by measures supported on 1-rectifiable closed curves

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with matrix valued multiplicities that depend on the underlined crystalline structure and the orientation of the line, i.e., $\operatorname{curl}\beta = b \otimes \tau\mathcal{H}^1|\gamma$ (see e.g. [5], Section 2.2., and the references therein, for a detailed description of the kinematics of plastic deformations). The topological nature of these defects is transparent in the fact that the line γ is a collection of closed curves with constant multiplicity b (the Burgers vector) and it is translated in the constraint $\operatorname{div}\mu = 0$. The collective effect of dislocations produces an effective strain field β whose curl is given by an arbitrary Radon measure as in (1).

We then prove that for every μ such that $\operatorname{div}\mu = 0$, there exists a unique distributional solution $\beta \in L^{3/2}(\Omega, \mathbb{R}^{3\times3})$ to the system (1).

Following [5] the result is obtained by decoupling the problem and finding the solution in the form $\beta = \beta^\mu + Du$, where $\operatorname{curl}\beta^\mu = \mu$ and u is the solution of the elliptic system

$$\begin{cases} -\operatorname{div}(\mathbb{C}Du) = \operatorname{div}(\mathbb{C}\beta^\mu), & \text{in } \Omega, \\ \mathbb{C}Du \cdot n = -\mathbb{C}\beta^\mu \cdot n, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The existence of $\beta^\mu \in L^{3/2}(\Omega, \mathbb{R}^{3\times3})$ is guaranteed by the celebrated result by Bourgain and Brezis [4], while the existence and uniqueness (up to rigid infinitesimal rotations, i.e., antisymmetric matrices) of a distributional solution $u \in W^{1,3/2}(\Omega; \mathbb{R}^3)$ to the non variational problem (2) is obtained by adapting the method of duality solutions for elliptic equations with measure forcing terms (see [6], [15], and, e.g., [2], [3], [9], [12], [13]).

In the second part of the paper we deal with the asymptotic behavior, as $h \to 0$, of the solution β_h of the problems

$$\begin{cases} -\operatorname{div}(\mathbb{C}_h(x)\beta_h) = 0, & \text{in } \Omega, \\ \operatorname{curl}\beta_h = \mu, & \text{in } \Omega, \\ \mathbb{C}_h(x)\beta_h \cdot n = 0, & \text{on } \partial\Omega. \end{cases} \tag{3}$$

Assuming that the linear elliptic operators associated to the coefficients $\mathbb{C}_h$ G-converge to the operator associated to $\mathbb{C}_0$, under suitable uniform conditions for $\mathbb{C}_h$, we prove the weak convergence in $L^{3/2}(\Omega, \mathbb{R}^{3\times3})$ of β_h to the unique (up to rigid infinitesimal rotations) solution β_0 to the problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}_0(x)\beta_0) = 0, & \text{in } \Omega, \\ \operatorname{curl}\beta_0 = \mu, & \text{in } \Omega, \\ \mathbb{C}_0(x)\beta_0 \cdot n = 0, & \text{on } \partial\Omega. \end{cases} \tag{4}$$

This result is also obtained by decoupling the boundary value problems and by investigating the asymptotic behaviour of the duality solutions of elliptic systems with non regular forcing terms under the assumption of G-convergence of the differential operators. We conclude by discussing the special case of the homogenization.

Finally we remark that the duality arguments and therefore the regularity properties assumed for the elastic tensor $\mathbb{C}$ are needed in order to deal with the cases in which the curl of the strain β is assigned to be singular. This is the case when the plastic strain is concentrated on low dimensional sets in $\mathbb{R}^3$. In particular in the presence of single dislocations, when the measure μ is concentrated on 1-rectifiable lines, the field β is not in L^2. On the other hand if $\mu \in H^{-1}(\Omega, \mathbb{R}^{3\times 3})$ the duality argument is not necessary, the problem is variational and it can be studied minimizing the corresponding elastic energy and the asymptotics can be obtained via Γ-convergence.

2. Notations and basic hypotheses on the operators

In what follows $\Omega \subseteq \mathbb{R}^3$ will be a open, bounded, simply connected set with C^1 boundary, and $\mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3})$ will denote the set of all bounded matrix-valued Radon measures on Ω. The subspaces of $\mathbb{R}^{3\times 3}$ of all symmetric matrices and of all skew-symmetric matrices will be denoted by $\mathcal{S}$ and $\mathcal{A}$, respectively.

If $\mathbb{C} = (a_{ij}^{hk})$, $i, j, h, k \in \{1, 2, 3\}$, is a fourth-order tensor, and $\xi = (\xi_{ij})$, $\eta = (\eta_{ij})$ $i, j \in \{1, 2, 3\}$ are square matrices of order 3, we set

$$\mathbb{C}\xi = \left(\sum_{h,k=1}^{3} a_{ij}^{hk} \xi_{hk} \right)_{i,j\in\{1,2,3\}} \qquad \mathbb{C}\xi \cdot \eta = \sum_{i,j,h,k=1}^{3} a_{ij}^{hk} \xi_{hk} \eta_{ij} \qquad \|\xi\| = \left(\sum_{i,j=1}^{3} |\xi_{ij}|^2 \right)^{\frac{1}{2}}$$

Definition 2.1. Given $c_0, c_1 > 0$, we denote by $\mathcal{E}(c_0, c_1, \Omega)$ the set of all tensor-valued functions $\mathbb{C}(x) = (a_{ij}^{hk}(x))$, $x \in \Omega$, such that the following hold:

(1) $a_{ij}^{hk} \in L^\infty(\Omega)$ for all $i, j, h, k \in \{1, 2, 3\}$;

(2) $a_{ij}^{hk} = a_{ji}^{hk} = a_{hk}^{ij}$ for all $i, j, h, k \in \{1, 2, 3\}$;

(3) $c_0 \|\xi + \xi^T\|^2 \le \mathbb{C}(x)\xi \cdot \xi \le c_1 \|\xi + \xi^T\|^2$ for a.e. $x \in \Omega$, and for every $\xi \in \mathbb{R}^{3\times 3}$.

Remark 2.2. The symmetry assumption in Definition 2.1 implies that $\mathbb{C}\xi = 0$ for every $\xi \in \mathcal{A}$. On the other hand, if $\mathbb{C} \in \mathcal{E}(c_0, c_1, \Omega)$, $\mathbb{C}(x)\xi \cdot \xi$ is a positive definite continuous quadratic form on $\mathcal{S}$. □

For a matrix valued distribution $V : \Omega \to \mathbb{R}^{3\times 3}$, $\operatorname{div} V$ and $\operatorname{curl} V$ denote the row-wise distributional divergence and curl of V respectively. In particular, given a tensor-valued function $\mathbb{C} : \Omega \to \mathbb{R}^{(3\times 3)^2}$ and a matrix-valued function $\beta : \Omega \to \mathbb{R}^{3\times 3}$, the vector $\operatorname{div}(A(x)\beta(x)) : \Omega \to \mathbb{R}^3$ has components

$$(\operatorname{div}(\mathbb{C}\beta))_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \left(\sum_{h,k=1}^{3} a_{ij}^{hk} \beta_{hk} \right), \qquad i \in \{1, 2, 3\}.$$

3. Preliminary results on elliptic systems

In this section we recall the basic existence and regularity results concerning boundary value problems for elliptic systems of PDEs of the form

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)Dv) = \operatorname{div} G, & \text{in } \Omega, \\ \mathbb{C}Dv \cdot n = -G \cdot n, & \text{on } \partial\Omega. \end{cases} \tag{5}$$

For $G \in L^2(\Omega, \mathbb{R}^{3\times 3})$ we deal with variational solutions in $W^{1,2}(\Omega, \mathbb{R}^3)$. More precisely, a function v is a weak solution of (5) if $v \in W^{1,2}(\Omega, \mathbb{R}^3)$, and

$$\int_\Omega \mathbb{C}Dv \cdot D\varphi \, dx = -\int_\Omega G \cdot D\varphi \, dx, \qquad \forall \varphi \in W^{1,2}(\Omega; \mathbb{R}^3).$$

Choosing as test function the rigid movement $\varphi(x) = \xi x + b$, with $b \in \mathbb{R}^n$ and $\xi \in \mathcal{A}$, we obtain

$$\int_\Omega \mathbb{C} Dv \cdot \xi \, dx = -\int_\Omega G \cdot \xi \, dx.$$

If we assume that the coefficients $\mathbb{C}$ belong to the class $\mathcal{E}(c_0, c_1, \Omega)$, we have that

$$\int_\Omega \mathbb{C} Dv \cdot \xi \, dx = \int_\Omega \mathbb{C}\xi \cdot Dv \, dx = 0$$

so that the existence of a weak solution v to (5) implies that

$$\int_\Omega G \, dx \in \mathcal{S}. \tag{6}$$

The compatibility condition (6) on the forcing term G must be required and the solutions of (5) will be defined up to additive rigid transformations belonging to

$$\mathcal{R} = \{\varphi(x) = \xi x + b, \ b \in \mathbb{R}^n, \ \xi \in \mathcal{A}\}.$$

In what follows $\mathbb{X}_p$, $p > 1$, will denote the set of admissible forcing terms in L^p

$$\mathbb{X}_p = \{G \in L^p(\Omega, \mathbb{R}^{3\times 3}) \text{ such that (6) holds true}\},$$

and, with a little abuse of notation, $\mathcal{R}^\perp$ will denote the following set

$$\mathcal{R}^\perp := \{u \in W^{1,1}(\Omega, \mathbb{R}^3) : \int_\Omega u \, dx = 0, \ \int_\Omega Du \, dx \in \mathcal{S}\}. \tag{7}$$

The existence result below for (5) is based on the second Korn inequality (see, [11], Theorem 2.5)

$$\int_\Omega |u|^2 \, dx + \int_\Omega |Du|^2 \, dx \le C \int_\Omega |Du + (Du)^T|^2 \, dx, \qquad \forall u \in W^{1,2}(\Omega, \mathbb{R}^3) \cap \mathcal{R}^\perp, \tag{8}$$

and Lax-Milgram Theorem (see e.g.[13], Theorem 1.4.4).

Theorem 3.1. *Suppose that Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, and let $\mathbb{C} \in \mathcal{E}(c_0, c_1, \Omega)$. Then for every $G \in \mathbb{X}_2$ there exists a weak solution $v \in H^1(\Omega, \mathbb{R}^3)$ to problem* (5)*, unique up to rigid displacements in $\mathcal{R}$, i.e., unique in $\mathcal{R}^\perp$.*

In what follows we will need a $W^{1,p}$ estimate for the weak solution to (5) with forcing term in L^p, $p > 2$. The higher summability of the solution, valid for operators with constant coefficients, fails to be true for general elliptic systems (see, e.g., [1]).

Hence, from now on, we assume in addition that the coefficients belong to VMO, that is, setting

$$\omega_\Omega(\mathbb{C}, r) := \sup_{B_\rho \subseteq \Omega, \ \rho \le r} \frac{1}{|B_\rho|} \int_{B_\rho} \left| \mathbb{C}(s) - \frac{1}{|B_\rho|} \int_{B_\rho} \mathbb{C}(t) \, dt \right| ds$$

we assume that

$$\lim_{r \to 0^+} \omega_\Omega(\mathbb{C}, r) = 0. \tag{9}$$

Theorem 3.2. *Assume that* Ω *is a bounded* C^1 *domain in* $\mathbb{R}^3$, $p \geq 2$, *and that* $\mathbb{C} \in \mathcal{E}(c_0, c_1, \Omega)$ *is such that* (9) *holds. Then for every* $G \in \mathbb{X}_p$ *there exists a unique weak solution* v *to* (5) *in* $W^{1,p}(\Omega; \mathbb{R}^3) \cap \mathcal{R}^\perp$ *which satisfies*

$$\int_\Omega \mathbb{C} Dv \cdot D\varphi \, dx = -\int_\Omega G \cdot D\varphi \, dx, \qquad \forall \varphi \in W^{1,p'}(\Omega; \mathbb{R}^3).$$

Moreover, there exists a constant $C > 0$, *depending only on* p, c_0, c_1 Ω, *and* $\omega_\Omega(\mathbb{C}, r)$, *such that*

$$\|v\|_{W^{1,p}(\Omega,\mathbb{R}^3)} \leq C \|G\|_{L^P(\mathbb{R}^{3\times 3})}. \tag{10}$$

Proof. See [13], Theorem 5.6.4. □

Remark 3.3. In what follows, we will consider as the unique weak solution to (5) the one orthogonal to the set of rigid transformations $\mathcal{R}$. □

4. Existence

This section is devoted to the proof of the following result.

Theorem 4.1. *Suppose that*

(1) $\mathbb{C} \in \mathcal{E}(c_0, c_1, \Omega)$ *satisfying* (9);

(2) $\mu \in \mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3})$ *with* $\operatorname{div} \mu = 0$.

Then there is a distributional solution $\beta \in L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$ *to the system*

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)\beta) = 0, & \textit{in } \Omega, \\ \operatorname{curl} \beta = \mu, & \textit{in } \Omega, \\ \mathbb{C}\beta \cdot n = 0, & \textit{on } \partial\Omega. \end{cases} \tag{11}$$

The solution is unique (up to an additive constant antisymmetric matrix), and there exists a constant $c > 0$, *depending only on* Ω, $\omega_\Omega(\mathbb{C}, r)$, *and* c_0, c_1, *such that*

$$\|\beta - \bar{\beta}^a\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})} \leq c|\mu|(\Omega), \tag{12}$$

where $\bar{\beta}^a$ *denotes the average of the antisymmetric part of* β, *i.e.,*

$$\bar{\beta}^a = \frac{1}{2|\Omega|} \int_\Omega (\beta - \beta^T) dx.$$

The proof is based on a suitable decomposition $\beta = \beta^\mu + Du$, with β^μ such that $\operatorname{curl} \beta^\mu = \mu$, and u weak solution of an elliptic problem. The uniqueness then follows by the linearity of the problem.

Concerning the purely incompatible part of β, we use the following well-known result by Bourgain and Brezis

Theorem 4.2. *For every* $\mu \in \mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3})$ *with* $\operatorname{div} \mu = 0$ *there exists a field* $\beta^\mu \in L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$ *such that*

(1) $\operatorname{curl} \beta^\mu = \mu$,

(2) $\|\beta^\mu\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})} \leq c|\mu|(\Omega)$.

Proof. See [4], [5]. □

Concerning the potential part Du of β, we adapt to the problems of linear elasticity the method of duality solutions for elliptic equations with measure forcing terms (see [15], [5], [13]), in order to obtain a selected distributional solution u to the non variational elliptic problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)Du) = \operatorname{div}(\mathbb{C}\beta^{\mu}), & \text{in } \Omega, \\ \mathbb{C}Du \cdot n = -\mathbb{C}\beta^{\mu} \cdot n, & \text{on } \partial\Omega, \end{cases}$$

with forcing term given by a field belonging to $L^{\frac{3}{2}}(\Omega; R^3)$.

The starting point for the formulation by duality of elliptic problems is the following. Let F and $G \in \mathbb{X}_2$, and let w, v be the weak solutions to (5) with datum F and G, respectively. Choosing w as test function in the equation solved by v and conversely, thanks to the symmetry of the tensor $\mathbb{C}$ we obtain

$$\int_\Omega G \cdot Dw\, dx = \int_\Omega F \cdot Dv\, dx. \tag{13}$$

If, in addition, $G \in L^3(\Omega, \mathbb{R}^{3\times 3})$, then, by Theorem 3.2, $v \in W^{1,3}(\Omega, \mathbb{R}^3)$, so that (13) is well defined when the forcing term F belongs to $L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$ and the corresponding "solution" w belongs to $W^{1,3/2}(\Omega, \mathbb{R}^3)$. This fact inspires the following definition of weak solution for (5) when the forcing term is in $L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$.

Definition 4.3. Let $\mathbb{C} \in \mathcal{E}(c_0, c_1, \Omega)$, and $F \in \mathbb{X}_{3/2}$. A function u is a *duality solution* to the elliptic problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)Du) = \operatorname{div} F, & \text{in } \Omega, \\ \mathbb{C}Dv \cdot n = -F \cdot n, & \text{on } \partial\Omega, \end{cases}$$

if $u \in W^{1,3/2}(\Omega, \mathbb{R}^3)$, and $\displaystyle\int_\Omega G \cdot Du\, dx = \int_\Omega F \cdot Dv\, dx$ for every $G \in \mathbb{X}_3$, where v is the weak solution to (5).

Remark 4.4. Given $u \in W^{1,3/2}(\Omega, \mathbb{R}^3) \cap \mathcal{R}^\perp$, and $H \in L^3(\Omega, \mathbb{R}^3)$, and setting

$$G = H - \overline{H}^a \qquad \text{with} \qquad \overline{H}^a = \frac{1}{|\Omega|} \int_\Omega \frac{H - H^T}{2},$$

we have that $G \in \mathbb{X}_3$ and

$$\int_\Omega H Du\, dx = \int_\Omega G Du\, dx + \overline{H}^a \int_\Omega Du\, dx = \int_\Omega G Du\, dx$$

where in the last equality we have used the fact that $\overline{H}^a \in \mathcal{A}$, and $\int_\Omega Du\, dx \in \mathcal{S}$. □

The next result shows that the duality solution exists, is unique (up to rigid transformations in $\mathcal{R}$), and it is the unique solution in the sense of distributions which can be obtained as limit of variational solutions of the same problem.

Theorem 4.5. *Let $\mathbb{C} \in \mathcal{E}(c_0, c_1, \Omega)$ satisfying* (9)*, and $F \in \mathbb{X}_{3/2}$. Then there exists a unique function $u \in W^{1,3/2}(\Omega; \mathbb{R}^3) \cap \mathcal{R}^\perp$, such that the following holds:*

(1) *u is a duality solution to*

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)Du) = \operatorname{div} F, & \text{in } \Omega, \\ \mathbb{C}Du \cdot n = -F \cdot n, & \text{on } \partial\Omega; \end{cases} \tag{14}$$

(2) *u is a solution obtained by approximation: for every sequence $(F_k) \subseteq \mathbb{X}_3$ converging to F in $L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$, the sequence v_k of solutions in $W^{1,2}(\Omega, \mathbb{R}^3) \cap \mathcal{R}^\perp$, to the problems*

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)Dv_k) = \operatorname{div} F_k, & \text{in } \Omega, \\ \mathbb{C}Dv_k \cdot n = -F_k \cdot n, & \text{on } \partial\Omega, \end{cases} \tag{15}$$

converges to u in the weak topology of $W^{1,3/2}(\Omega; \mathbb{R}^3)$.

(3) *u is a distributional solution to* (14)*, and there exists a constant $c > 0$, depending only on c_0, c_1, Ω and $\omega_\Omega(\mathbb{C}, r)$, such that*

$$\|u\|_{W^{1,3/2}(\Omega;\mathbb{R}^3)} \leq c\|F\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})}. \tag{16}$$

Proof. For any given $G \in \mathbb{X}_3$, let $v \in W^{1,3}(\Omega; \mathbb{R}^3)$ be the solution to (5) with right hand side $\operatorname{div} G$. Let v_k be the solutions to (15).

Since both v and v_k are variational solutions, by the symmetry assumption on $\mathbb{C}$, we obtain that the equality

$$\int_\Omega F_k \cdot Dv\,dx = \int_\Omega \mathbb{C}Dv_k \cdot Dv\,dx = \int_\Omega \mathbb{C}Dv \cdot Dv_k\,dx = \int_\Omega G \cdot Dv_k\,dx \tag{17}$$

holds for every $k \in \mathbb{N}$, and hence we get the estimate

$$\left| \int_\Omega G \cdot Dv_k\,dx \right| \leq \|F_k\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})} \|v\|_{W^{1,3}(\Omega;R^3)} \leq M\|G\|_{L^3(\Omega;R^3)} \tag{18}$$

for every $G \in \mathbb{X}_3$. Using the fact that the sequence F_k is equibounded in $L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$, the regularity estimate (10) for v, and Remark 4.4, we conclude that

$$\|Dv_k\|_{L^{3/2}(\Omega;R^3)} = \sup_{G \in L^3(\Omega;R^3)} \frac{1}{\|G\|_{L^3(\Omega;R^3)}} \left| \int_\Omega G \cdot Dv_k\,dx \right| \leq M,$$

so that there exists a subsequence (still denoted by v_k) converging to a function u in the weak topology of $W^{1,3/2}(\Omega; \mathbb{R}^3)$.

Since u is the weak limit of distributional solutions, it follows that it is also a distributional solution, while the fact that u is a duality solution follows passing to the limit in (17) as $k \to \infty$. Moreover, since $W^{1,3/2}(\Omega; \mathbb{R}^3) \cap \mathcal{R}^\perp$ is a weakly closed subspace of $W^{1,3/2}(\Omega; \mathbb{R}^3)$, we also obtain that $u \in \mathcal{R}^\perp$.

The estimate (16) follows from the very definition of duality solution. Specifically as above we have

$$\left| \int_\Omega G \cdot Du\,dx \right| \leq \|F\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})} \|v\|_{W^{1,3}(\Omega;R^3)} \leq c\|F\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})} \|G\|_{L^3(\Omega;R^3)}$$

for every $G \in \mathbb{X}_3$, and hence, by Remark 4.4, for every $G \in L^3(\Omega; R^3)$, which gives us equation (16).

It remains to show that the duality solution is unique in $W^{1,3/2}(\Omega;\mathbb{R}^3)\cap\mathcal{R}^\perp$. Suppose that both u and w belong to $\mathcal{R}^\perp$ and satisfy (13). Then we have

$$\int_\Omega G[Du - Dv]\, dx = 0, \qquad \forall G \in \mathbb{X}_3,$$

and, by Remark 4.4,

$$\int_\Omega G[Du - Dv]\, dx = 0, \qquad \forall G \in L^3(\Omega; R^3).$$

This implies that $u = v$ in Ω. In particular, we recover that the whole sequence v_k weakly converges to the solution u of (13).

Finally, if $(\widetilde{F}_k) \subseteq L^3(\Omega;\mathbb{R}^{3\times 3})$ is another sequence converging to F in $L^{3/2}(\Omega,\mathbb{R}^{3\times 3})$, the previous arguments show that the sequence $(\widetilde{v}_k)$ of solutions of the problems with forcing term $\widetilde{F}_k$ converge to the duality solution u of (14), which turns out to be the unique distributional solution of problem (14) that can be obtained by approximation. □

As a consequence of Propositions 4.2 and 4.5, we obtain the existence result in elasticity stated in Theorem 4.1

Proof of Theorem 4.1. Let $\beta^\mu \in L^{3/2}(\Omega,\mathbb{R}^{3\times 3})$ be as in Theorem 4.2. Since $F = \mathbb{C}\beta^\mu$ belongs to $L^{3/2}(\Omega,\mathbb{R}^{3\times 3})$ and satisfies the compatibility condition (6), by Theorem 4.5 there exists u duality solution to the problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}(x)Du) = \operatorname{div}(\mathbb{C}\beta^\mu), & \text{in } \Omega,\\ \mathbb{C}Du\cdot n = -\mathbb{C}\beta^\mu\cdot n, & \text{on } \partial\Omega. \end{cases}$$

The field $\beta = \beta^\mu + Du$ provides a distributional solution to (11). The uniqueness (up to constant antisymmetric matrices) then follows from the linearity of the problem.

Finally, by Theorem 4.2(ii), (16), and the boundedness of the coefficients $\mathbb{C}$, we get

$$\|\beta\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times3})} \le \|\beta^\mu\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times3})} + \|Du\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times3})} \le C|\mu|(\Omega).$$

In particular, $\beta \in L^1(\Omega,\mathbb{R}^{3\times 3})$, and

$$|\overline{\beta}^a| = \left|\frac{1}{2|\Omega|}\int_\Omega (\beta - \beta^T)\, dx\right| \le C\|\beta\|_{L^1(\Omega,\mathbb{R}^{3\times3})} \le C|\mu|(\Omega),$$

and estimate (12) follows. □

Remark 4.6. The unique solution β to problem (11) admits infinitely many representations of the form $\beta = \beta^\mu + Du$, depending on the choice of β^μ. □

5. G-convergence for problems in elasticity

We are now interested on the behaviour of the solutions to (11) corresponding to varying operators. In the framework of elliptic systems, the convergence of solutions is encoded in the notion of G-convergence (see, e.g. [7], [8], [10], [14]).

Definition 5.1. A sequence of tensor-valued functions $(\mathbb{C}_h) \in \mathcal{E}(c_0, c_1, \Omega)$ is said to be *G-convergent* to $\mathbb{C}_0 \in \mathcal{E}(c_0, c_1, \Omega)$ if for any $f \in W^{-1,2}(\Omega, \mathbb{R}^3)$ the solution $v_h \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ to the Dirichlet problems

$$\begin{cases} -\operatorname{div}(\mathbb{C}_h(x) D v_h) = f, & \text{in } \Omega, \\ v_h = 0, & \text{on } \partial\Omega \end{cases}$$

converge in weak topology of $W_0^{1,2}(\Omega, \mathbb{R}^3)$, as $h \to 0$, to the solution v_0 of the problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}_0(x) D v_0) = f, & \text{in } \Omega, \\ v_0 = 0, & \text{on } \partial\Omega. \end{cases}$$

The main properties of G-convergence are the following (see, e.g., [8], Section 12.2).

Theorem 5.2.

(i) *The G-limit $\mathbb{C}_0$ is uniquely defined.*

(ii) *Let $(\mathbb{C}_h)$ be G-converging to $\mathbb{C}_0$ and let $w_h \in W^{1,2}(\Omega, \mathbb{R}^3)$ be such that we have $-\operatorname{div}(\mathbb{C}_h D w_h) = g$ in $W^{-1,2}(\Omega, \mathbb{R}^3)$. If w_h converge to w weakly in $W^{1,2}(\Omega, \mathbb{R}^3)$, then $\mathbb{C}_h D w_h$ converge to $\mathbb{C}_0 D w$ weakly in L^2.*

(iii) *The class $\mathcal{E}(c_0, c_1, \Omega)$ is compact with respect to G-convergence;*

(iv) *The G-limit $\mathbb{C}_0$ satisfies the two-sided estimate of Voigt-Reiss:*

$$\left(\lim_{h\to 0} (\mathbb{C}_h)^{-1}\right)^{-1} \leq \mathbb{C}_0 \leq \lim_{h \to 0} \mathbb{C}_h,$$

where the limits are understood in the sense of weak convergence in L^2.

By Theorem 5.2(iii) every sequence $\mathbb{C}_h$ admits a G-converging subsequence. In what follows we only consider G-converging sequences $\mathbb{C}_h \in \mathcal{E}(c_0, c_1, \Omega)$ with coefficients satisfying the following uniform VMO estimate: there exists a decreasing function $\omega\colon [0,1] \to \mathbb{R}$ such that $\lim_{r\to 0} \omega(r) = 0$, and for every $r \in [0,1]$

$$\sup_{B_\rho \subseteq \Omega,\ \rho \leq r} \frac{1}{|B_\rho|} \int_{B_\rho} \left| \mathbb{C}_h(s) - \frac{1}{|B_\rho|} \int_{B_\rho} \mathbb{C}_h(t)\, dt \right| ds \leq \omega(r), \qquad \forall h. \tag{19}$$

We show that the G-convergence implies also the convergence of the solutions to elasticity problems.

Theorem 5.3. *Assume that the sequence* $(\mathbb{C}_h) \in \mathcal{E}(c_0, c_1, \Omega)$ *satisfies the uniform VMO condition* (19). *If* $(\mathbb{C}_h)$ *G-converges to* $\mathbb{C}_0$*, then for every* $\mu \in \mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3})$ *the sequence* β_h *of solutions to*

$$\begin{cases} -\operatorname{div}(\mathbb{C}_h(x)\beta_h) = 0, & \text{in } \Omega, \\ \operatorname{curl}\beta_h = \mu, & \text{in } \Omega, \\ \mathbb{C}_h\beta_h \cdot n = 0, & \text{on } \partial\Omega \end{cases} \tag{20}$$

converges weakly in $L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$ *to the solution* β_0 *to*

$$\begin{cases} -\operatorname{div}(\mathbb{C}_0(x)\beta_0) = 0, & \text{in } \Omega, \\ \operatorname{curl}\beta_0 = \mu, & \text{in } \Omega, \\ \mathbb{C}_0\beta_0 \cdot n = 0, & \text{on } \partial\Omega. \end{cases} \tag{21}$$

Proof. Given β^μ as in Proposition 4.2, let u_h be the duality solution to the problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}_h(x)Du_h) = \operatorname{div}(\mathbb{C}_h\beta^\mu), & \text{in } \Omega, \\ \mathbb{C}_hDu_h \cdot n = -\mathbb{C}_h\beta^\mu \cdot n, & \text{on } \partial\Omega, \end{cases}.$$

Then $u_h \in W^{1,3/2}(\Omega, \mathbb{R}^{3\times 3}) \cap \mathcal{R}^\perp$, there exists $c > 0$, independent of h, such that

$$\|u_h\|_{W^{1,3/2}(\Omega;\mathbb{R}^3)} \leq c\|\beta^\mu\|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})}, \tag{22}$$

and

$$\int_\Omega GDu_h\,dx = \int_\Omega \mathbb{C}_h\beta^\mu(x)Dv_h\,dx \qquad \forall\, G \in \mathbb{X}_3, \tag{23}$$

where $v_h \in W^{1,3}(\Omega, \mathbb{R}^3) \cap \mathcal{R}^\perp$ is the solution to

$$\begin{cases} -\operatorname{div}(\mathbb{C}_h(x)Dv_h) = \operatorname{div} G, & \text{in } \Omega, \\ \mathbb{C}_hDv_h \cdot n = -G \cdot n, & \text{on } \partial\Omega. \end{cases}.$$

On the other hand, given $G \in \mathbb{X}_3$, by (10) there exists $C > 0$, independent of h, such that

$$\|v_h\|_{W^{1,3}(\Omega;\mathbb{R}^3)} \leq C\|G\|_{L^3(\Omega;\mathbb{R}^{3\times 3})},$$

so that v_h converges, up to a subsequence, to a function v_0 in the weak topology of $W^{1,3}(\Omega; \mathbb{R}^3)$. Then, by Theorem 5.2(ii) the sequence $(\mathbb{C}_hDv_h)$ converges to $\mathbb{C}_0Dv_0$, and v_0 is the solution to the boundary value problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}_0(x)Dv_0) = \operatorname{div} G, & \text{in } \Omega, \\ \mathbb{C}_0Dv_0 \cdot n = -G \cdot n, & \text{on } \partial\Omega. \end{cases} \tag{24}$$

As a matter of fact, the whole sequence v_h converges to v_0, due to the uniqueness of the solution of the limit problem (24).

Finally, by (22), also the sequence u_h converges, up to a subsequence, to a function u_0 in the weak topology of $W^{1,3/2}(\Omega; \mathbb{R}^3)$, and a passage to the limit in (23) shows that u_0 is the duality solution to the problem

$$\begin{cases} -\operatorname{div}(\mathbb{C}_0Du_0) = \operatorname{div}(\mathbb{C}_0\beta^\mu), & \text{in } \Omega, \\ \mathbb{C}_0u_0 \cdot n = -\mathbb{C}_0\beta^\mu \cdot n, & \text{on } \partial\Omega. \end{cases}$$

The convergence of the solutions β_h of the problems (20) to the solution β_0 of the limit problem (21) now follows from the fact that $\beta_h = \beta^\mu + Du_h$. $\square$

As an example we finally observe that in the case of periodic rapidly oscillating coefficients the effective behaviour of the corresponding incompatible fields is described by the homogenized effective tensor characterized by the homogenization procedure of the elliptic systems in elasticity.

Specifically, if Y denotes the reference cell $Y = [0,1]^3$, let $\mathbb{C} = \mathbb{C}(y)$ be a Y-periodic tensor valued function in $\mathcal{E}(c_0, c_1, Y)$ satisfying (9), and let us consider the asymptotic behavior as $\varepsilon \to 0$ of the solutions to the linear problems with rapidly oscillating periodic coefficients $\mathbb{C}_\varepsilon(x) = \mathbb{C}(\frac{x}{\varepsilon})$

$$\begin{cases} -\operatorname{div}(\mathbb{C}_\varepsilon(x)\beta_\varepsilon) = 0, & \text{in } \Omega, \\ \operatorname{curl}\beta_\varepsilon = \mu, & \text{in } \Omega, \\ \mathbb{C}_\varepsilon\beta_\varepsilon \ \cdot n = 0, & \text{on } \partial\Omega \end{cases} \tag{25}$$

where $\mu \in \mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3})$. It is well known (see, e.g. [13]) that the sequence $(\mathbb{C}_\varepsilon)$ is G-convergent to an effective operator with constant coefficients $\widehat{\mathbb{C}}$, and that $\widehat{\mathbb{C}} \in \mathcal{E}(c_0, c_1, \Omega)$.

Moreover, under the assumption that $\mathbb{C}$ is *VMO*, by Theorem 4.3.1 in [13], for every $G \in \mathbb{X}_3$ the variational solution $v_\varepsilon \in W^{1,3}(\Omega, \mathbb{R}^3)$ to

$$\begin{cases} -\operatorname{div}(\mathbb{C}_\varepsilon(x)Dv_\varepsilon) = \operatorname{div} G, & \text{in } \Omega, \\ \mathbb{C}_\varepsilon Dv_\varepsilon \ \cdot n = -G\cdot n, & \text{on } \partial\Omega \end{cases} \tag{26}$$

satisfies the estimate

$$\|v_\varepsilon\|_{W^{1,3}(\Omega,\mathbb{R}^3)} \le C\|G\|_{L^3(\mathbb{R}^{3\times 3})}$$

with a constant $C > 0$ independent of ε. Hence, following the lines of the proof of Theorem 4.5, we obtain that the duality solutions to the problems

$$\begin{cases} -\operatorname{div}(\mathbb{C}_\varepsilon(x)Du_e) = \operatorname{div} F, & \text{in } \Omega, \\ \mathbb{C}_\varepsilon Du_e \cdot n = -F\cdot n, & \text{on } \partial\Omega, \end{cases}.$$

satisfy the estimate

$$\|u_e\|_{W^{1,3/2}(\Omega;\mathbb{R}^3)} \le c\|F|_{L^{3/2}(\Omega,\mathbb{R}^{3\times 3})}$$

with a constant $C > 0$ independent of ε.

In conclusion, following the lines of Theorem 5.3 we obtain that, if $\mathbb{C} = \mathbb{C}(y)$ is a Y-periodic tensor valued function in $\mathcal{E}(c_0, c_1, Y)$ satisfying (9), then for every $\mu \in \mathcal{M}_b(\Omega; \mathbb{R}^{3\times 3})$, the solutions β_ε to (25) converge weakly in $L^{3/2}(\Omega, \mathbb{R}^{3\times 3})$ to the solution β to the problem

$$\begin{cases} -\operatorname{div}(\widehat{\mathbb{C}}\beta) = 0, & \text{in } \Omega, \\ \operatorname{curl}\beta = \mu, & \text{in } \Omega, \\ \widehat{\mathbb{C}}\beta\cdot n = 0, & \text{on } \partial\Omega. \end{cases}$$

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