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Anisotropy Versus Inhomogeneity in the Calculus of Variations

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Dedicated to Umberto Mosco on the occasion of his 80th birthday.

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For an energy integral of the calculus of variations we can read anisotropies and inhomogeneities looking at the analytic expression of the integrand. The non-homogeneous case give rise to homogenization, also well known under the names Gamma-convergence, G-convergence, H-convergence; we give some details, as well as we describe some connections between these notions and the Mosco-convergence. We also describe the connection of the Gamma-convergence with the convergence of eigenvalues and eigenfunctions. The anisotropic case for an energy integral of the calculus of variations appears when the integrand has different behaviors in different directions of the space; we consider anisotropic energy integrals as well as integrals with more general growth.

Keywords: Gamma-convergence, Mosco-convergence, convergence of eigenvalues and eigenfunctions, elliptic equations, calculus of variations, regularity of solutions, p,q-growth conditions, general growth conditions.

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1. Introduction

In the classical 3-dimensional Euclidian space *anisotropy* is related to the *direction*, while *inhomogeneity* is related to the position. For an *energy integral* of the calculus of variations of the type

$$F(u) = \int_{\Omega} f(x, Du(x))\, dx \tag{1}$$

we can read *anisotropies* and *inhomogeneities* looking at the analytic expression of the integrand $f(x, Du(x))$. Here $u = u(x)$ is a real function defined for x in a bounded open set $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, $Du = Du(x)$ is its gradient in Ω, and $f : \Omega \times \mathbb{R}^n \to \mathbb{R}$ is a Carathéodory function, i.e., $f = f(x, \xi)$ is *convex* with respect to the gradient variable $\xi \in \mathbb{R}^n$ and measurable with respect to $x \in \Omega$.

More precisely, the x-dependence of the integrand f corresponds to an energy which is not homogeneous with respect to the body Ω. This is the *inhomogeneous*, or the *non-homogeneous case*. In this context an interesting and relevant aspect is to consider the inhomogeneity spread in Ω in a random way, like for two or more materials mixed together. A mathematical relevant approach is to consider a *periodic* distribution of the two (or more) materials with a small n-dimensional period, say depending on a positive (small) parameter ε, or equivalently on an integer number

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$k \to +\infty$, with $\varepsilon = \frac{1}{k}$; any corresponding minimizer u_ε of the energy integral of the type in (1) with integrand $f = f_\varepsilon(x,\xi) = f\left(\frac{x}{\varepsilon},\xi\right)$ describes the microscopic behavior of the physical system. Then a natural approach is to go from *microscopic* to *macroscopic*. The mathematical process corresponds to consider a minimizer u_ε of the energy integral as in (1), related to the integrand $f_\varepsilon(x,\xi)$, and let ε go to zero, describing the *weak* limit $u_\varepsilon \rightharpoonup u$, where u is a minimizer of an *homogeneous* energy of the form $\int_\Omega f_0(Du(x))\,dx$. This is the method of *homogenization*, also well known under the names Γ*-convergence*, G*-convergence*, H*-convergence*. In the next section we give some details, as well as we describe some connections between these notions and the *Mosco-convergence*.

The *anisotropic* case for an *energy integral* of the calculus of variations as in (1) appears when the integrand $f(x,\xi)$ has different behaviors in different directions of $\mathbb{R}^n$, for instance of the type

$$f(x,\xi) = \sum_{i=1}^{n} a_i(x)\,|\xi_i|^{p_i}, \qquad x \in \Omega,\ \xi = (\xi_i)_{i=1,2,\dots,n} \in \mathbb{R}^n, \tag{2}$$

for some positive coefficients $(a_i(x))_{i=1,2,\dots,n}$ and exponents $p_1, p_2, \dots, p_n$ greater than 1. If the p_i are not all equal each other, they describe the *anisotropy* of the problem, governed by an energy integral with different behavior in different directions of the space. We consider in more details this case in Sections 3.1 and 3.2.

2. The non-homogeneous case and the Γ-convergence

2.1. In the origin it was named G-convergence

We consider the *energy integral* $F(u)$ as in (1). For instance, the *energy* $F(u)$ may have the expression

$$F(u) = \int_\Omega \sum_{i=1}^{n} a_i(x)\left(\frac{\partial u}{\partial x_i}\right)^2 dx \tag{3}$$

for some *bounded measurable* coefficients $a_1(x), a_2(x), \dots, a_n(x)$ greater than some positive constant. The x-dependence corresponds to a not homogeneous body Ω. This is a *non-homogeneous case***.** If we fix the potential at the boundary $\partial\Omega$, say $u(x) = u_0(x)$ for every $x \in \partial\Omega$, then an *equilibrium potential* u satisfies the Dirichlet problem

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i(x)\frac{\partial u}{\partial x_i}\right) = 0, & x \in \Omega\,, \\ u(x) = u_0(x)\,, & x \in \partial\Omega\,. \end{cases} \tag{4}$$

A similar, but more general energy in this *inhomogeneous* context is

$$F(u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx\,, \tag{5}$$

where $A(x)$ is the square $n \times n$ matrix

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & a_{22}(x) & \cdots & a_{2n}(x) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(x) & a_{n2}(x) & \cdots & a_{nn}(x) \end{pmatrix},$$

while in the previous case (3) the vector $(a_1(x), \dots, a_n(x))$ deserves to be represented as the diagonal $n \times n$ matrix

$$\begin{pmatrix} a_1(x) & 0 & \cdots & 0 \\ 0 & a_2(x) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_n(x) \end{pmatrix}.$$

From this relevant example the *G-convergence, Gamma-convergence, homogenization* theories took their origin, in the '70, with the contribution of De Giorgi, Marino, Spagnolo et al. The first references on this subject is the article [44] published by De Giorgi-Spagnolo in 1973, a paper which followed the first pioneering attempts by Spagnolo [83] and Marino-Spagnolo [75] in 1968-69, all of them being related to second order linear elliptic (and parabolic) operators of the form $\sum_{ij=1}^{n} D_i(a_{ij}(x)D_j)$, whose *energy functional* is expressed in (5). The celebrated Marino-Spagnolo [75] result *in loose form* states that solutions of Dirichlet problems associated to the general energy integral in (5) can be approximated, in the strong L^2-topology, as well as in the weak topology of $W^{1,2}(\Omega)$, to solutions of Dirichlet problems related to simpler energy integrals as in (3).

This is nowadays a well known fact, which gave origin to the G-convergence and to the Γ-convergence theories, and we will not go further here. A reader can see the well known books by Dal Maso [35], Braides [21], and also Attouch [1], Attouch-Buttazzo-Michaille [2] and Buttazzo [22]. Here we observe that the described phenomenon related to matrices and to *linear* Dirichlet problems was discovered, at the very beginning of the Γ-convergence theory, also when lower order terms are considered; see for instance C. Sbordone [74], [80], [81], [82], and Luc Tartar [84],[85],[86], who gave also several other relevant contributions to the *G-convergence, H-convergence, homogenization*. For the linear case with lower order terms see also [63]. The first attempts of Γ-*convergence* are due to Ennio De Giorgi [43] in 1975 in collaboration with Tullio Franzoni. In the years 1973–1976 the first nonlinear attempts for the Γ-*convergence* in the nonlinear context of *convex energy functionals* can be found in [8],[60],[61],[62], in connection with the *Mosco convergence* too and some aspects are described in the next sections.

2.2. Connections between Γ-convergence and Mosco convergence

Relations between Γ-convergence and the Mosco convergence are well known, although few details seem to be less known and maybe can be pointed out. We refer to the approaches started in the 1970s, in the period when the Γ-convergence was initiated at Pisa, mainly by the fundamental work of Ennio De Giorgi, as described in the previous section. We emphasize here some of the Mosco's results where his contribution is more related to the Γ-convergence; in particular we refer to the Mosco's papers [76],[77]; see Dal Maso-Mosco [36],[37] too.

We recall the *sequential* definition of Γ-convergence. Details can be found in the reference books by Dal Maso [35] and Braides [21]. See also Attouch [1], Attouch-Buttazzo-Michaille [2] and Buttazzo [22].

Definition 2.1. (Γ-convergence) Let (X, τ) be a topological space and $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions defined in X with values in $\mathbb{R} \cup \{+\infty\}$. We say that f_k Γ-*converges* to $f\colon X \to \mathbb{R} \cup \{+\infty\}$ in the (sequential) topology τ if the following two conditions hold:

(i) (equi-lower semicontinuity) for every $x \in X$ and every sequence $(x_k)_{k\in\mathbb{N}}$ converging to x in the topology τ
$$\liminf_{k\to+\infty} f_k(x_k) \geq f(x) \ ;$$

(ii) (optimality) for every $x \in X$ there exist a sequence $(x_k)_{k\in\mathbb{N}}$ converging to x in the topology τ such that the equality holds; i.e.,
$$\lim_{k\to+\infty} f_k(x_k) = f(x) \ .$$

Definition 2.2. (Mosco convergence) Let X be a real Banach space and $(f_k)_{k\in\mathbb{N}}$ be a sequence of functions defined in X with values in $\mathbb{R} \cup \{+\infty\}$. We say that f_k M-*converges* (Mosco-converges) to $f : X \to \mathbb{R} \cup \{+\infty\}$ in X if f_k Γ-converges to f in the strong topology as well as in the weak topology of X.

Of course many properties satisfied by sequences Γ-converging either in the weak or in the strong topology of a real Banach space are satisfied also by Mosco-converging sequences. Clearly the reverse not always is true, for instance the compactness properties of weakly Γ-converging sequences, which are typical and which characterize the Γ-convergence. However the Mosco-convergence, useful for instance in the convergence of solutions to variational inequalities which was one of Mosco's main motivations (see [76]), is very elegant because it allows to treat variational problems in a symmetric general way, for instance in Mosco's original description of the continuity of the *Young-Fenchel transform* (see the definition below in (6) and details in Mosco' paper [77]).

Let X be a *reflexive* real Banach space and let $f : X \to \mathbb{R} \cup \{+\infty\}$, not identically $+\infty$. The *conjugate*, or the Young-Fenchel *transform* $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$, is the function defined in the dual Banach space X^* of X by
$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : \ x \in X\} \,. \tag{6}$$

The following Theorem 2.3 is a modification of a similar result given by Mosco [76],[77] in the version proposed in [60],[61]; see in particular Lemma 1 in [61]. We make use of the following *coercivity assumption*: there exist constants $m > 0$ and $p > 1$ such that
$$f(x) \geq m\,\|x\|_X^p \ , \quad \forall\, x \in X. \tag{7}$$

In the original terminology by Mosco Theorem 2.3 below is one of the main steps for the so-called continuity of the Young-Fenchel transform with respect to the Mosco-convergence. Despite of this fact, we give below essentially the original results by Umberto Mosco about the convergence of convex sets and of convex functions, which he studied in [76],[77], and we emphasize that neither in the statement of Theorem 2.3, nor in its proof, there is the assumption or the use of the convexity of the functions f_k, $f : X \to \mathbb{R}\cup\{+\infty\}$. As shown here, *this result* not only *is independent of the convexity* of the functions f_k, but it also gives the convergence of the minimum values when the sequence f_k Γ-converges.

In fact, if each function $f_k : X \to \mathbb{R} \cup \{+\infty\}$ is also weakly lower semicontinuous on X and it is not identically equal to $+\infty$, then the coercivity assumption (7) ensures the existence of the minimum value

$$\min \{ f_k (x) - \langle x^*, x \rangle : \ x \in X \} \tag{8}$$

for any linear perturbation $x^* \in X^*$ and the minimum value holds $-f_k^* (x^*)$ by the same definition of the *Young-Fenchel transform* f^* in (6). The limit condition (10) is in the statement of Theorem 2.3 below; it states that, as $k \to +\infty$, the minimum value in (8) converges to the corresponding minimum value for f. Under the coercivity condition (7) we can also obtain the weak convergence (up to a subsequence) of the minimizers. For instance we refer to the well known context in pdes when $X = W^{1,2} (\Omega)$ and the notation $f_k \colon X \to \mathbb{R} \cup \{+\infty\}$ is modified into $F_k : W^{1,2} (\Omega) \to \mathbb{R} \cup \{+\infty\}$ and it is given, for example, by the *Dirichlet integral*

$$F_k (u) = \int_\Omega \sum_{i,j=1}^{n} a_{ij}^k (x)\, u_{x_i} u_{x_j} \, dx \, , \qquad u \in W^{1,2} (\Omega) \, , \tag{9}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $n \geq 1$, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and $Du = (u_{x_i})_{i=1.2,\dots,n}$ its gradient. For every $k \in \mathbb{N}$ the $n \times n$ symmetric matrix $\left(a_{ij}^k\right)$ is positive definite and bounded, in the sense that

$$m \, |\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}^k (x) \, \xi_i \xi_j \leq M \, |\xi|^2$$

for some positive constants m, M, for all $\xi \in \mathbb{R}^n$, a.e. $x \in \Omega$ and for every $k \in \mathbb{N}$. With the notation of this section, x^* is an element of the dual space of $W_0^{1,2} (\Omega)$ of functions $u \in W^{1,2} (\Omega)$ with zero boundary value on $\partial\Omega$; for instance x^* is a generic function $h \in L^2 (\Omega)$. The minimization problem corresponding to (8) is

$$\min \left\{ \int_\Omega \left[\sum_{i,j=1}^{n} a_{ij}^k (x) \, u_{x_i} u_{x_j} - h (x) \, u (x) \right] dx : \ u \in W_0^{1,2} (\Omega) \right\}.$$

As well known, in this context the Γ-convergence of the sequence of Dirichlet integrals $F_k : W_0^{1,2} (\Omega) \to \mathbb{R}$ in (9) implies the convergence of the minimum values $F_k (u_k)$ and also the convergence in the weak topology of $W_0^{1,2} (\Omega)$ and of the minimizers u_k. In fact in this context the Γ-convergence for every $h \in L^2 (\Omega)$ of the minimum values and of the minimizers are equivalent facts.

The following Theorem 2.3 is the main result of this section. As already said, it is a modification of a similar result given by Mosco [76],[77] in the version proposed in [60],[61]; see in particular Lemma 1 in [61]. As already noticed, although originally stated under convexity assumptions, Theorem 2.3 holds independently of the convexity of the functions f_k, $f : X \to \mathbb{R} \cup \{+\infty\}$.

Theorem 2.3. *Let f_k, $f \colon X \to \mathbb{R} \cup \{+\infty, \}$, not identically $+\infty$, and let the coercivity assumption* (7) *hold uniformly in k. If f_k Γ-converges to f in the weak topology of X then f_k^* Γ-converges to f^* in the strong topology of X^*. Moreover*

$$\lim_{k \to +\infty} f_k^* (x^*) = f^* (x^*) \, . \tag{10}$$

Proof. First we follow Lemma 1b in [61]. In order to obtain the equi-lower semicontinuity of f_k^*, for every $x \in X$ with $f(x)$ finite by (ii) in the Definition 2.1 we can consider $(x_k)_{k\in\mathbb{N}}$ converging to x in the weak topology of X such that $\lim_{k\to+\infty} f_k(x_k) = f(x)$. If x_k^* converges to x^* in the strong topology of X^* then

$$f_k^*(x_k^*) = \sup_{x\in X}\{\langle x_k^*, x\rangle - f(x)\} \geq \langle x_k^*, x_k\rangle - f(x_k). \tag{11}$$

As $k \to +\infty$ we obtain

$$\liminf_{k\to+\infty} f_k^*(x_k^*) \geq \langle x^*, x\rangle - f(x). \tag{12}$$

Since (12) trivially holds if $f(x) = +\infty$, then it is satisfied for all $x \in X$ and

$$\liminf_{k\to+\infty} f_k^*(x_k^*) \geq \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\} = f^*(x^*). \tag{13}$$

With the aim to prove the optimality condition in (ii), we observe that the limit condition (10) imply (ii); therefore we go directly to prove (10). At this stage we use the assumption (7), while in the previous computation to get (13) we did not used it. We start from the definition $f_k^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f_k(x)\}$; if $f_k^*(x^*)$ is finite there exists $x_k \in X$ such that

$$\langle x^*, x_k\rangle - f_k(x_k) > f_k^*(x^*) - \frac{1}{k}.$$

We assume that f is finite somewhere in X, say $f(y_0) \in \mathbb{R}$ and let y_k be a sequence in X converging in the weak topology of X such that $\lim_{k\to+\infty} f_k(y_k) = f(y_0)$; then, for every $x^* \in X^*$

$$\inf_{x\in X}\{f_k(x) - \langle x^*, x\rangle\} \leq [f_k(x) - \langle x^*, x\rangle]_{x=y_k} = f_k(y_k) - \langle x^*, y_k\rangle. \tag{14}$$

Let us first consider the case $\limsup_{k\to+\infty} f_k^*(x^*) < +\infty$. Then $f_k^*(x^*)$ is finite for every large value of k and there exists a sequence $x_k \in X$ such that

$$\inf_{x\in X}\{f_k(x) - \langle x^*, x\rangle\} + \frac{1}{k} > f_k(x_k) - \langle x^*, x_k\rangle. \tag{15}$$

From (14),(15) we get $f_k(y_k) - \langle x^*, y_k\rangle \geq f_k(x_k) - \langle x^*, x_k\rangle - \dfrac{1}{k}$.

By the coercivity condition (7) we obtain

$$\begin{aligned} f_k(y_k) + \|x^*\|_{X^*}\|y_k\|_X &\geq f_k(y_k) - \langle x^*, y_k\rangle \geq f_k(x_k) - \langle x^*, x_k\rangle - \frac{1}{k} \\ &\geq m\|x_k\|_X^p - \|x^*\|_{X^*}\|x_k\|_X - \frac{1}{k} \geq \|x_k\|_X\left(m\|x_k\|_X^{p-1} - \|x^*\|_{X^*}\right) - 1. \end{aligned} \tag{16}$$

Since $\lim_{k\to+\infty} f_k(y_k) = f(y_0) \in \mathbb{R}$ and since y_k converges in the weak topology of X , the left hand side of (16) is bounded. Therefore also the right hand side remains bounded and, being $p > 1$, $\|x_k\|_X$ is bounded too. Up to a subsequence, which we continue to denote with the same symbol, as $k \to +\infty$ the sequence x_k weakly converges to some $x_0 \in X$.

We rewrite (15) by using the definition of the Young-Fenchel transform (6), namely $f_k^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f_k(x)\}$, in the form

$$-f_k^*(x^*) + \frac{1}{k} > f_k(x_k) - \langle x^*, x_k\rangle$$

and we go to the $\liminf$ as $k \to +\infty$. We obtain

$$-\limsup_{k\to+\infty} f_k^*(x^*) \geq \liminf_{k\to+\infty}[f_k(x_k) - \langle x^*, x_k\rangle]$$

that is, since the equi-lower semicontinuity (i) in the Definition 2.1,

$$\limsup_{k\to+\infty} f_k^*(x^*) \le \langle x^*, x_0\rangle - \liminf_{k\to+\infty} f_k(x_k) \le \langle x^*, x_0\rangle - f(x_0) \ .$$

Since $f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\} \ge \langle x^*, x_0\rangle - f(x_0)$ we finally get

$$\limsup_{k\to+\infty} f_k^*(x^*) \le \langle x^*, x_0\rangle - f(x_0) \le f^*(x^*) \ . \tag{17}$$

The limit in (17) together with the limit condition in (13) gives the conclusion (10).

In the case $\limsup_{k\to+\infty} f_k^*(x^*) = +\infty$ then up to a subsequence, which we continue to denote with the same symbol, $\lim_{k\to+\infty} f_k^*(x^*) = +\infty$. For every $M \in \mathbb{R}$ (but we limit ourselves to $M \ge 0$) there exists $k_M \in \mathbb{N}$ such that $f_k^*(x^*) > M$ for all $k > k_M$. Then, being $f_k^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f_k(x)\} > M$, for every $k > k_M$ there exists $x_k \in X$ such that

$$\langle x^*, x_k\rangle - f_k(x_k) > M\,. \tag{18}$$

By the coercivity assumption (7)

$$-M > f_k(x_k) - \langle x^*, x_k\rangle \ge \|x_k\|_X \left(m\,\|x_k\|_X^{p-1} - \|x^*\|_{X^*}\right)$$

which, when $M \ge 0$ implies $\|x_k\|_X^{p-1} \le \frac{1}{m}\|x^*\|_{X^*}$. Up to a subsequence, which we continue to denote with the same symbol, as $k \to +\infty$ the sequence x_k weakly converges to some $x_0 \in X$. Passing to the limit as $k \to +\infty$ in (18), i.e. in the inequality $M < \langle x^*, x_k\rangle - f_k(x_k)$, by the equi-lower semicontinuity (i) in the Definition 2.1 we get

$$\begin{aligned} M &\le \limsup_{k\to+\infty}\left[\langle x^*, x_k\rangle - f_k(x_k)\right] = \langle x^*, x_0\rangle - \liminf_{k\to+\infty} f_k(x_k) \\ &\le \langle x^*, x_0\rangle - f(x_0) \le f^*(x^*) \ . \end{aligned}$$

The arbitrarily of M gives $f^*(x^*) = +\infty$ which implies

$$\limsup_{k\to+\infty} f_k^*(x^*) = +\infty = f^*(x^*)\,. \tag{19}$$

Again together with the other limit condition in (13), the limit in (19) gives the conclusion (10). □

2.3. Γ-convergence and the convergence of eigenvalues and eigenfunctions

We point out a property of the Γ-convergence related to the convergence of *eigenvalues* and *eigenfunctions*. Following Boccardo-Marcellini [8], for every $k \in \mathbb{N}$ we consider the energy integral $F_k : W_0^{1,2}(\Omega) \to \mathbb{R}$

$$F_k(u) = \int_\Omega \sum_{i,j=1}^n a_{ij}^k(x)\, u_{x_i} u_{x_j}\, dx\,, \quad u \in W_0^{1,2}(\Omega)\ , \tag{20}$$

where, for every $k \in \mathbb{N}$, $\left(a_{ij}^k(x)\right)_{i,j=1,2,\dots,n}$ is an $n \times n$ symmetric square matrix of measurable functions in Ω, satisfying the ellipticity and boundedness conditions

$$m\,|\xi|^2 \le \sum_{i,j=1}^n a_{ij}^k(x)\,\xi_i\xi_j \le M\,|\xi|^2\,, \tag{21}$$

for some positive constants m, M independent of $k \in \mathbb{N}$, a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^n$. For every $k \in \mathbb{N}$ we denote by $\left(\lambda_h^k\right)_{h\in\mathbb{N}}$ the increasing sequence of *eigenvalues* of the elliptic operator $A^k : W_0^{1,2}(\Omega) \to H^{-1}(\Omega)$ (as usual, we denote by $H^{-1}(\Omega)$ the topological dual space of $W_0^{1,2}(\Omega)$) given by

$$A^k = -\text{div}\left(\sum_{j=1}^{n} a_{ij}^k(x)\frac{\partial}{\partial x_j}\right) = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}^k(x)\frac{\partial}{\partial x_j}\right). \tag{22}$$

We count the eigenvalues with their multiplicity; i.e., if for instance one eigenvalue has multiplicity 2 then it is repeated twice in the sequence $\left(\lambda_h^k\right)_{h\in\mathbb{N}}$. We also denote by $\left(u_h^k\right)_{h\in\mathbb{N}}$ the sequence in $W_0^{1,2}(\Omega)$ of the corresponding normalized *eigenfunctions, i.e.* $\left\|u_h^k\right\|_{L^2(\Omega)} = 1$ for all $h, k \in \mathbb{N}$. Similarly we denote by $(\lambda_h)_{h\in\mathbb{N}}$ and $(u_h)_{h\in\mathbb{N}}$ the sequences of eigenvalues and eigenfunctions of the operator

$$A = -\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right), \tag{23}$$

whose energy integrals are

$$F(u) = \int_{\Omega}\sum_{i,j=1}^{n} a_{ij}(x)\,u_{x_i}u_{x_j}\,dx\,, \quad u \in W_0^{1,2}(\Omega)\,, \tag{24}$$

related to the $n \times n$ symmetric square matrix $(a_{ij}(x))_{i,j=1,2,\ldots,n}$ of measurable functions in Ω satisfying the same ellipticity and boundedness conditions as in (21).

It is well known that F_k in (20) Γ-converges to F in (24) if and only if the sequence of matrices $\left(a_{ij}^k(x)\right)_{i,j=1,2,\ldots,n}$, or equivalently the sequence of operators A^k in (22), G-converge respectively to the matrix $(a_{ij}(x))_{i,j=1,2,\ldots,n}$ or to the operator A in (23) in the sense of Spagnolo [83] and De Giorgi-Spagnolo [44]. Less known is the following characterization of the Γ-convergence in terms of *eigenvalues* and *eigenfunctions*. The following result has been obtained by Boccardo-Marcellini (Theorem 4.1 in [8]).

Theorem 2.4. ([8]) *The sequence of Dirichlet energy integrals F_k in* (20) Γ*-converges to F in* (24) *in the weak topology of $W_0^{1,2}(\Omega)$, or equivalently in the strong topology of $L^2(\Omega)$, if and only if the following conditions hold:*

(i) *the sequence of the corresponding eigenvalues converge; i.e.,*

$$\lim_{k\to+\infty}\lambda_h^k = \lambda_h \quad \forall\, h \in \mathbb{N};$$

(ii) *for every $h \in \mathbb{N}$, if λ_h has multiplicity $l \geq 1$ (thus $\lambda_{h-1} < \lambda_h = \lambda_{h+l-1} < \lambda_{h+l}$), the linear space of dimension l generated by the eigenfunctions $\left\{u_h^k, \ldots, u_{h+l-1}^k\right\}$ as $k \to +\infty$ converges in the weak topology of $W_0^{1,2}(\Omega)$, or equivalently in the strong topology of $L^2(\Omega)$, to the linear space of dimension l corresponding to the eigenvalue λ_h.*

The meaning of convergence of a linear space of l-dimension is in the sense of *Kuratowski's convergence of sets* (see [59], § 25, VI):

Definition 2.5. (Kuratowski [59], §25, VI) A sequence $\left(S^k\right)_{k\in\mathbb{N}}$ of sets of a topological space X *converges to a set* $S \subset X$ as $k \to +\infty$ if the following two conditions hold:

(i) for all $x \in S$ there exists a sequence $\left(x^k\right)_{k\in\mathbb{N}}$ converging as $k \to +\infty$ to x and such that $x^k \in S^k$ for all $k \in \mathbb{N}$;

(ii) for all subsequences $\left(S^{k_r}\right)_{r\in\mathbb{N}}$ of $\left(S^k\right)_{k\in\mathbb{N}}$, if $x_r \in S^{k_r}$ for all $r \in \mathbb{N}$ and if $x_r \to x$ as $r \to +\infty$ then $x \in S$.

The proof of Theorem 2.4 is in Section 4 of [8].

3. The anisotropic case

We consider *anisotropic energy integrals* of the type

$$F(u) = \int_\Omega \left\{ \left| \frac{\partial u}{\partial x_1} \right|^{p_1} + \left| \frac{\partial u}{\partial x_2} \right|^{p_2} + \ldots + \left| \frac{\partial u}{\partial x_n} \right|^{p_n} \right\} dx\,, \tag{25}$$

or, more generally

$$F(u) = \int_\Omega \left\{ a_1(x) \left| \frac{\partial u}{\partial x_1} \right|^{p_1} + a_2(x) \left| \frac{\partial u}{\partial x_2} \right|^{p_2} + \ldots + a_n(x) \left| \frac{\partial u}{\partial x_n} \right|^{p_n} \right\} dx\,; \tag{26}$$

here the x-dependence describes the *lack of homogeneity*, while different exponents $p_1, p_2, \ldots, p_n$ describe the *anisotropy*. More generally, for $i = 1, 2, \ldots, n$ the exponent $p_i = p_i(x)$ could depend on $x \in \Omega$ as in the following Section 3.1, while in Section 3.2 we go back to constant exponents. Finally in Section 3.3 we consider more general growth conditions.

3.1. Anisotropic energy integrals

With the expression of *anisotropic integrals* we mean, for instance, integrals of the calculus of variations of the form

$$F(u) = \int_\Omega \sum_{i=1}^n a_i(x) \left| u_{x_i}(x) \right|^{p_i(x)} dx\,, \tag{27}$$

for some bounded measurable functions $a_i(x)$ with $m \le a_i(x) \le M$, a.e. $x \in \Omega$, $i = 1, 2, \ldots, n$, for some positive constants m, M and bounded measurable exponents $p_i(x)$, $i = 1, 2, \ldots, n$, such that $1 < p \le p_i(x) \le q$ for some constant p, q and for all $i = 1, 2, \ldots, n$. The Euler's first variation of the integral F in (27) takes the form of the *anisotropic* elliptic partial differential equation

$$\sum_{i=1}^n \left(a_i(x)\, p_i(x) \left| u_{x_i} \right|^{p_i(x)-2} u_{x_i} \right)_{x_i} = 0\,. \tag{28}$$

We know that this type of anisotropic pdes may have *not smooth, even unbounded, minimizers*; see for instance [65],[66],[67]. This happens also in the case of *constant* exponents p_i, $i = 1, \ldots, n$, if they are spread out; i.e., if the ratio $\max\{p_i\}/\min\{p_i\}$ is not close enough to 1 in dependence on n; for example when $n > 3$ and

$$p_1 = \ldots = p_{n-1} = 2, \qquad p_n = q > \frac{2(n-1)}{n-3}\,.$$

Following an original idea in 1990 of Boccardo-Marcellini-Sbordone [9] (see Boccardo-Gallouet-Marcellini [7] too), Cupini-Marcellini-Mascolo [28]–[34] considered exponents greater than or equal to 1: q and p_i, $i = 1, \ldots, n$, such that

$$p_i \leq p_i\,(x) \leq q, \quad \text{a.e. } x \in B_r, \quad 1 \leq i \leq n,$$

where B_r is a ball of radius $r > 0$ contained in Ω. Then $\overline{p}$ is the *harmonic average* of the $\{p_i\}$ and $\overline{p}^*$ is the *Sobolev conjugate exponent* of $\overline{p}$; i.e.

$$\frac{1}{\overline{p}} := \frac{1}{n} \sum_{i=1}^{n} \frac{1}{p_i}\,, \qquad \overline{p}^* = \frac{n\overline{p}}{n - \overline{p}} \tag{29}$$

if $\overline{p} < n$, while $\overline{p}^*$ is any fixed real number greater than $\overline{p}$ if $\overline{p} \geq n$.

Theorem 3.1. *Let u be a local minimizer of the anisotropic integrals of the calculus of variations, as described above, and let*

$$q \leq \overline{p}^* = \frac{n\overline{p}}{n - \overline{p}}\,. \tag{30}$$

If $q = \overline{p}^*$ we also assume that $u \in L^{\overline{p}^*}_{\mathrm{loc}}(\Omega)$. Then u is locally bounded in Ω. Moreover, if $q < \overline{p}^*$, then the following estimate holds

$$\|u - u_r\|_{L^\infty(B_{r/(2\sqrt{n})}(x_0))} \leq c \left\{ 1 + \int_{B_r(x_0)} \sum_{i=1}^{n} |u_{x_i}(x)|^{p_i(x)}\, dx \right\}^{\frac{1+\theta}{p}}, \tag{31}$$

for some constant $c > 0$, where

$$u_r = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u\, dx, \quad p = \min_{1 \leq i \leq n} \{p_i\} \quad \text{and} \quad \theta = \frac{\overline{p}^*(q - p)}{p(\overline{p}^* - q)}.$$

Observe that if $p_1 = \ldots = p_{n-1} = 2$ and $p_n = q \geq 2$, then the assumption $q \leq \overline{p}^*$ is equivalent to

$$q \leq \frac{2\,(n-1)}{n-3}. \tag{32}$$

This inequality is *exactly the opposite* to condition (29) in the counterexample to regularity. Thus, the L^∞_{loc}-regularity result in Theorem 3.1 is sharp.

3.2. Anisotropic energy integrals with constant exponents

We consider a local minimizer of energy integrals with constant exponents p_i

$$F\,(u) = \int_\Omega \sum_{i=1}^{n} a_i\,(x)\, |u_{x_i}(x)|^{p_i}\, dx\,, \tag{33}$$

or equivalently a weak solution to the elliptic equation

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(p_i a_i\,(x) \left| \frac{\partial u}{\partial x_n} \right|^{p_i - 2} \frac{\partial u}{\partial x_i} \right) = 0\,. \tag{34}$$

A nonzero right hand side can be considered too. We have here bounded *measurable* coefficients (not necessarily *continuous* coefficients), satisfying $m \leq a_i\,(x) \leq M$ a.e. in Ω for some positive constants m, M and for all $i = 1, 2, \ldots, n$. By Theorem 3.1, minimizers and weak solutions are *locally bounded* in Ω if $q \leq \overline{p}^*$.

It is of interest at this stage the problem to establish the Hölder continuity of the local minimizers of the integral in (33) and of the weak solutions to the pde (34), in the spirit of the De Giorgi's theory, valid under natural growth conditions $q = p$.

Some attempts have been recently made to obtain the local Hölder continuity of weak solutions, however the problem can be considered still partially open, at least in its full generality. The known references about local Hölder continuity in the anisotropic case $p = p_1 \leq p_2 \leq \ldots \leq p_{n-1} \leq p_n = q$, when at least one the the above inequalities is a *strict inequality*, are Liskevich-Skrypnik [58]; Düzgün-Marcellini-Vespri [48],[49]; DiBenedetto-Gianazza-Vespri [45]; Liao-Skrypnik-Vespri [57].

We make a comparison among the four results which state that if u is a *bounded* weak solution in Ω to the elliptic anisotropic equation (34) then u is locally Hölder continuous in Ω, under a further assumption which is different in the quoted papers. Since Theorem 3.1 above, the assumption of boundedness of the weak solution u implicitly replaces the bound (30) $q \leq \overline{p}^*$ in dependence of the dimension n. This is the scheme of the different assumptions made in the four cases: Liskevich-Skrypnik [58] assume $p = 2 = p_1 < p_2 = \ldots = p_{n-1} = p_n = q$; Düzgün-Marcellini-Vespri [48],[49] consider $p = p_1 < p_2 = \ldots = p_{n-1} = p_n = q$; DiBenedetto-Gianazza-Vespri [45] assume that there exists $\varepsilon_0 > 0$ (depending only on the data) such that $q - p < \varepsilon_0$, where $p = \min\{p_i \ : i = 1, 2, \ldots, n\}$ and $q = \max\{p_i \ : i = 1, 2, \ldots, n\}$; finally recently Liao-Skrypnik-Vespri [57] obtain the Hölder continuity of the weak solution to the anisotropic prototype equation

$$u_{x_n x_n} + \sum_{i=1}^{n-1} \left(a_i(x) \left| u_{x_i} \right|^{p-2} u_{x_i} \right) = 0, \quad x \in \Omega\,, \tag{35}$$

when one of the exponents is equal to 2 and all the others are equal each other, say $p_i = p$ for all $i = 1, 2, \ldots, n-1$, and $1 < p < 2$.

3.3. Some examples with more general growth

The mathematical literature about regularity for solutions to partial differential equations and systems of elliptic and parabolic type under p, q-growth conditions and general growth conditions is very large, and it is mostly dedicated to energy integrals, for instance, modelized according to the following model examples, for some $p > 1$ and $q > p$,

$$\int_\Omega |Du(x)|^{p(x)}\, dx\,, \tag{36}$$

for some continuous exponent $p(x)$, and

$$\int_\Omega |Du(x)|^p \log(1 + |Du(x)|)\, dx\,, \tag{37}$$

$$\int_\Omega \{a(x)\,|Du(x)|^p + b(x)\,|Du(x)|^q\}\, dx\,, \tag{38}$$

the last one being with some nonnegative continuous coefficients $a(x)$, $b(x)$, possibly equal to 0 in Ω but not both equal to 0 at the same $x \in \Omega$. All the above examples are special cases of the so-called *p, q-growth conditions*; on the contrary in (39) we have a further case with exponential growth

$$\int_\Omega e^{|Du(x)|^2}\, dx\,, \tag{39}$$

which also cannot be studied with the standard thecniques of regularity, but which needs some special methods, such as, for instance, that ones considered in [69],[73], and more recently in [6],[47].

Without being exhaustive, in the general context of L^∞-*gradient estimates* under p, q-*growth conditions* (started in 1989 in [65],[66],[67]) we refer to the recent researches about the so-called *double phase energy-integrals*, explored from 2015 in a series of papers by Mingione, in collaboration with P. Baroni and M. Colombo [3],[4],[5],[25],[26], and by Eleuteri-Marcellini-Mascolo [50]–[53]. About *Orlicz-type functionals*, as in [24], see Rădulescu-Zhang [79] and De Filippis et al. [39],[40],[41].

General growth conditions even for the one-dimensional case $n = 1$ have been studied in [19], [54]. For related recent references we quote [70],[71],[72], Bousquet-Brasco [20], Cupini-Giannetti-Giova-Passarelli [27], Carozza-Giannetti-Leonetti-Passarelli [23], Cupini-Marcellini-Mascolo [28]–[34], Diening-Harjulehto-Hasto-Ruzicka [46], Harjulehto-Hästö-Toivanen [55], Hästö-Ok [56] and others in the references below.

Finally, specific for partial differential equations and systems of *parabolic type*, under p, q-growth conditions and general growth conditions, we refer to an approach started in 2013 in a joint research project by Bögelein-Duzaar-Marcellini; more precisely, see the references [10]-[18], some of these articles also in collaboration with B. Dacorogna, C. Scheven and S. Signoriello.

A model regularity result under p, q-growth conditions is stated in the following Theorem 3.2 about the gradient boundedness of local minimizers, i.e. about the local Lipschitz regularity of local minimizers. For more details and proofs we refer to [66],[68],[72]. We consider here an energy integral of the type $\int_\Omega f(Du\,(x))\,dx$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function of class $C^2\,(\mathbb{R}^n)$ satisfying the p, q-growth conditions

$$m\,|\xi|^{p-2}\,|\lambda|^2 \le \sum_{i,j=1}^{n} \frac{\partial^2 f\,(\xi)}{\partial \xi_i \partial \xi_j} \lambda_i \lambda_j \le M\left(1 + |\xi|^2\right)^{\frac{q-2}{2}} |\lambda|^2\,, \tag{40}$$

for some exponents $q \ge p \ge 2$, some positive constants m, M and for every $\lambda, \xi \in \mathbb{R}^n$.

Under these growth condition the integrand is coercive, in the sense that we have $f\,(\xi) \ge c_1\,|\xi|^p - c_2$ for all $\xi \in \mathbb{R}^n$ and for some constants $c_1 > 0$ and $c_2 \in \mathbb{R}$. We denote by B_R a generic ball of radius R compactly contained in Ω and by B_ϱ a ball of radius $\varrho < R$ concentric with B_R.

Theorem 3.2. ([66],[68],[72]) *Let u be a local minimizer in $W^{1,p}\,(\Omega)$ of the energy integral $\int_\Omega f(Du\,(x))\,dx$ with $f : \mathbb{R}^n \to \mathbb{R}$ satisfying the p, q-growth conditions in* (40) *and*

$$\frac{q}{p} < 1 + \frac{2}{n}\,. \tag{41}$$

Then u is of class $W^{1,\infty}_{\mathrm{loc}}\,(\Omega)$ and there exists constants c, δ depending only on p, q, n such that, for all ϱ, R with $0 < \varrho < R \le \varrho + 1$,

$$\|Du\,(x)\|_{L^\infty(B_\varrho;\mathbb{R}^n)} \le c\left(\frac{1}{(R-\varrho)^\delta} \int_{B_R} \{1 + f\,(Du)\}\,dx\right)^{\frac{2}{(n+2)p-nq}}. \tag{42}$$

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