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Uniqueness and Estimates for a Parabolic Equation with L^1 Data

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Dedicated to Umberto Mosco on the occasion of his 80th birthday.

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We investigate on the uniqueness property of the solutions to a parabolic problem with the data and the coefficient of zero order term only summable functions. Despite all this lack of regularity and without using any notion of entropy or renormalized solutions or the property to be a solution obtained by approximations we prove an uniqueness result. Then, we study the behavior in time of the unique solution. Finally, we estimate the distance between the unique solution and the solutions of other parabolic or stationary problems showing cases when the same asymptotic behavior in time appears.

Keywords: Uniqueness, asymptotic behavior, regularity of solutions, linear parabolic equations.

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1. Introduction

Let us consider the following parabolic problem

$$\begin{cases} u_t - \operatorname{div}(M(x,t)\nabla u) + a(x,t)u = f(x,t) & \text{in } \Omega_T = \Omega \times (0,T), \\ u(x,t) = 0 & \text{on } \Gamma_T = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ $(N > 2)$, T is a positive constant and M is a bounded and uniformly elliptic matrix, i.e. there exist $\alpha > 0$ and $\beta > 0$ such that

$$M(x,t)\xi \cdot \xi \geq \alpha|\xi|^2, \quad |M(x,t)| \leq \beta, \tag{2}$$

for almost every (x,t) in Ω_T and for every ξ in $\mathbb{R}^N$. On the data $u_0(x)$, $a(x,t)$ and $f(x,t)$ we assume

$$u_0 \in L^1(\Omega), \tag{3}$$

$$a(x,t) \geq 0, \qquad a(x,t) \in L^1(\Omega_T), \qquad f(x,t) \in L^1(\Omega_T), \tag{4}$$

and that there exists a non negative constant Q such that

$$|f(x,t)| \leq Q\, a(x,t) \quad \text{a.e. } (x,t) \in \Omega_T. \tag{5}$$

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We recall that there exists a weak solution of the previous problem (see [9]) where by a weak solution of (1) we mean a function u belonging to $L^1(0,T;W_0^{1,1}(\Omega))$ satisfying $a(x,t)u \in L^1(\Omega_T)$ and such that

$$\iint_{\Omega_T} \{-u\,\varphi_t + M(x,t)\nabla u\cdot\nabla\varphi + a(x,t)u\,\varphi\} = \iint_{\Omega_T} f(x,t)\,\varphi + \int_\Omega u_0\,\varphi(0)\,, \quad (6)$$

for every $\varphi \in W^{1,\infty}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$ satisfying $\varphi(T)=0$.

Indeed, although the data f, a and u_0 are only summable functions, the previous structure assumptions (2)-(5) guarantee also the existence of a weak solution u of problem (1) that becomes "immediately bounded", i.e. such that

$$u \in L^\infty(\Omega\times(\varepsilon,T)) \quad \text{for every } \varepsilon > 0\,,$$

and verifying the following estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le C_1 \max\left\{\frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}}, Q\right\} \qquad \text{a.e.} \quad t\in(0,T)\,, \quad (7)$$

where C_1 depends only on the constant α in (2) and on N (see Theorem 1.1 in [9]).

Notice that for $Q=0$ estimate (7) becomes the well known decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le C_1\,\frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}} \qquad \text{a.e.} \quad t\in(0,T)\,, \quad (8)$$

satisfied not only by the solution of the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega_T,\\ u(x,t)=0 & \text{on } \Gamma_T,\\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases}$$

but also by the solutions of all the following linear modifications of the heat equation

$$\begin{cases} u_t - \operatorname{div}(M(x,t)\nabla u) = 0 & \text{in } \Omega_T,\\ u(x,t)=0 & \text{on } \Gamma_T,\\ u(x,0)=u_0(x) & \text{in } \Omega, \end{cases} \quad (9)$$

with M satisfying (2) (see [19] if $M(x,t)=M(x)$, [6] if $u_0\in L^2(\Omega)$ and [12] in the remaining cases).

Note that the previous L^∞-result is false if $f\in L^1(\Omega_T)$ doesn't satisfy assumption (5).

Hence the linear equation (1) has a very strong regularizing effect since, although all the data u_0, $a(x,t)$ and $f(x,t)$ are assumed to be only summable functions, there exists a solution u that becomes immediately bounded and satisfies (7).

We point out that if u_0 is bounded then the previous solution u is bounded up to $t=0$, i.e. u is in $L^\infty(\Omega_T)$. Moreover, u belongs also $L^2(0,T;H_0^1(\Omega))$ and satisfies the following L^∞-estimate

$$\|u\|_{L^\infty(\Omega_T)} \le \max\{\|u_0\|_{L^\infty(\Omega)}, Q\}\,,$$

(see Remark 2.1 and Remark 3.1 in [9]).

This solution u of problem (1) in $L^\infty(\Omega_T)\cap L^2(0,T;H_0^1(\Omega))$ is also the unique solution of (1) belonging to $L^\infty(\Omega_T)\cap L^2(0,T;H_0^1(\Omega))$ and hence if u_0 is in $L^\infty(\Omega)$ problem (1) admits one and only one solution in $L^\infty(\Omega_T)\cap L^2(0,T;H_0^1(\Omega))$. The proof of this uniqueness result is very easy. As a matter of fact, if by contradiction, u_1 and u_2 are both solutions of (1) in $L^\infty(\Omega_T)\cap L^2(0,T;H_0^1(\Omega))$, choosing as a test function u_1-u_2 first in the equation satisfied by u_1 and then in the equation satisfied by u_2, subtracting the results and using assumption (2) we get

$$\frac{1}{2}\int_\Omega |u_1-u_2|^2(t)+\alpha\int_0^t\!\!\int_\Omega |\nabla(u_1-u_2)|^2+\int_0^t\!\!\int_\Omega a(x,t)(u_1-u_2)^2\le 0$$

which implies the assertion.

It is well known that also in the stationary case a strong regularization appears since the following problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla w)+a(x)w=f(x) & \text{in } \Omega,\\ \qquad\qquad w(x)=0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

has a solution $w\in L^\infty(\Omega)\cap H_0^1(\Omega)$ if $a(x)$ is only a nonnegative summable function and f a summable function satisfying (see [1]),

$$|f(x)|\le Q\,a(x) \quad \text{a.e. } x\in\Omega\,, \tag{11}$$

See also [2], [3] and [4]. Moreover, such a bounded solution w is also unique.

We point out that, differently from the stationary case which always admits a bounded solution, in the evolution case there exists a bounded solution of the problem only if the initial datum u_0 belongs to $L^\infty(\Omega)$.

Therefore, to conclude that problem (1) admits an unique bounded solution we need the further assumption that u_0 is a bounded function in Ω.

But what happens if u_0 is not bounded and for example is only a summable function? It is still possible to prove an uniqueness result?

As recalled above, also in this case a regularization phenomenon appears since there exists at least a solution u in $L^\infty(\Omega\times(\varepsilon,T))$ (for every $\varepsilon>0$). Unfortunately this regularity of u is not enough to proceed as above and to conclude the uniqueness of solutions of (1) in $L^\infty(\Omega\times(\varepsilon,T))$.

Thus, first purpose of this paper is to investigate the uniqueness of solutions in this general case when f, a and u_0 are only summable functions.

It is well-known that if $a(x,t)\equiv 0$ and $f\in L^1(\Omega_T)$ ($f\not\equiv 0$) then problem (1) has not a unique solution (see for example [17]) and that in order to have uniqueness of solutions further requirement on the solutions are needed like, for example, that u is an entropy solution or a renormalized solution or a solution constructed by approximation.

Anyway, despite the lack of regularity of the data (f, a and u_0) we prove here that to guarantee the uniqueness of solutions of (1) we can avoid to introduce the further requirements on the solutions recalled above and although these solutions generally are not bounded in all the set Ω_T there exists a unique "suitably regular" solution of (1) (see Theorem 2.1 below). We observe that also different kinds of evolution problems like, for example, the degenerate p-laplacian, the porous media equation or a class of nonlinear Leray-Lions problems exhibit unique "regular" solutions also in presence of irregular data like, for example, only summable initial data (see [5], [11], [14], [15] and [18], and the references therein).

Another purpose of this paper is to investigate the behavior in time of such a unique solution. To do this, we first show that there exists a unique "suitably regular" global solution of (1) (see Definition 2.2 below) which is actually the extension to Ω_∞ of the unique "suitably regular" solution of (1). Then we show that, it is possible to estimate the distance between the unique global "suitably regular" solution u of (1) and the unique global "suitably regular" solution v of

$$\begin{cases} v_t - \operatorname{div}(M(x,t)\nabla v) + a(x,t)v = f(x,t) & \text{in } \Omega_\infty = \Omega \times (0,+\infty), \\ v(x,t) = 0 & \text{on } \Gamma_\infty = \partial\Omega \times (0,+\infty), \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

(i.e. v solves the same problem of u but with a different initial datum v_0) showing that if $v_0 \in L^1(\Omega)$ this leads to

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \le C_2 \frac{\|u_0 - v_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}} e^{\sigma t}} \qquad \text{a.e. } t > 0\,,$$

(see Theorem 2.4 below). Hence, for large value of t the solutions u and v "forget their initial data" u_0 and v_0 and have the same behavior.

Indeed, we show that it is also possible to estimate the difference between the unique global "suitably regular" solution u of (1) and the the unique global "suitably regular" solution v of a different evolution problem of the type

$$\begin{cases} v_t - \operatorname{div}(M_0(x,t)\nabla v) + a_0(x,t)v = f_0(x,t) & \text{in } \Omega_\infty, \\ v(x,t) = 0 & \text{on } \Gamma_\infty, \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases}$$

where M_0, a_0 and f_0 satisfy assumptions at all similar to (2)–(5). In this last case, we prove that if the problems are "not too far away" (see Theorem 2.5 below) this leads to

$$\lim_{t\to+\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0$$

and hence, once again u and v have the same behavior for large value of t. Moreover, we prove estimates on the difference $u-v$ that show a sort of continuous dependence of these solutions from the data in their structure conditions (see again Theorem 2.5).

Finally, we additionally estimate the distance between u and the unique solution $w \in L^\infty(\Omega) \cap H_0^1(\Omega)$ of the stationary problem

$$\begin{cases} -\operatorname{div}(M_1(x)\nabla w) + a_1(x)w = f_1(x) & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

where we assume that the summable functions a_1 and f_1 satisfy an assumption analogous to (11), i.e. there exists $Q_1 > 0$ such that

$$|f_1(x)| \le Q_1 a_1(x) \quad \text{a.e. } x \in \Omega\,.$$

Also in this case, if the problems considered are "not too far" (see Theorem 2.11 below) it results

$$\lim_{t\to+\infty} \|u(t) - w\|_{L^2(\Omega)} = 0$$

and hence, for large value of t the global solution u of the evolution problem (1) behaves as the solution w of the above stationary problem.

The paper is organized as follows. In the next section we give in detail all our results. In Section 3 we recall some preliminary results proved in [12] and in [10] that will be essential in the proofs of the results presented above. Finally, in Section 4 we prove all our results.

2. Statement of the results

Theorem 2.1. *Assume* (2)–(5). *Then problem* (1) *has one and only one weak solution u belonging to V_T where*

$$V_T \equiv L^\infty_{loc}((0,T];L^\infty(\Omega)) \cap L^2_{loc}((0,T];H^1_0(\Omega)) \cap C([0,T];L^1(\Omega))\,. \tag{12}$$

Moreover, u belongs also to $L^q(0,T;W^{1,q}_0(\Omega)) \cap C_{loc}((0,T];L^2(\Omega))$ for every q with $1 < q < 2 - \frac{1}{N+1}$, and satisfies the following estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le C_1 \max\left\{ \frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}}, Q\right\} \qquad \text{a.e.} \quad t \in (0,T)\,, \tag{13}$$

where C_1 depends only on the coerciveness coefficient α in (2) *and on N.*

We observe that the previous result improves the existence result proved in [9] showing that a higher regularity of the solutions appear. Moreover, Theorem 2.1 establishes that there aren't other "such regular" solutions.

We show now that it is also possible to prove the existence and uniqueness of a suitably regular global solution of (1). To this aim, we give the following definition.

Definition 2.2. We say that u is a global solution of (1), or equivalently, that u is a solution of

$$\begin{cases} u_t - \operatorname{div}(M(x,t)\nabla u) + a(x,t)u = f(x,t) & \text{in } \Omega_\infty, \\ u(x,t) = 0 & \text{on } \Gamma_\infty, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{14}$$

if u is a weak solution of (1) for every $T > 0$ arbitrarily chosen.

Theorem 2.3. *Assume that* (2)–(5) *hold true for every fixed $T > 0$. Then problem* (1) *has one and only one global solution u belonging to V where*

$$V \equiv L^\infty_{loc}((0,+\infty);L^\infty(\Omega)) \cap L^2_{loc}((0,+\infty);H^1_0(\Omega)) \cap C_{loc}([0,+\infty);L^1(\Omega))\,.$$

Moreover, u belongs to $L^\infty(\Omega \times (t_0,+\infty))$ for every $t_0 > 0$ and estimate (13) *holds for almost every $t > 0$. Besides, u is in $L^q(0,T;W^{1,q}_0(\Omega)) \cap C_{loc}((0,+\infty);L^2(\Omega))$ for every $1 < q < 2 - \frac{1}{N+1}$ and for every $T > 0$.*

Therefore, problem (1) *has one and only one global solution belonging to* Y *where*

$$Y = V \cap Z \cap C_{loc}((0, +\infty); L^2(\Omega)), \tag{15}$$

$$Z = L^q_{loc}([0, +\infty); W_0^{1,q}(\Omega)) \cap \{\varphi \in L^\infty(\Omega \times (t_0, +\infty)) \text{ for every } t_0 > 0\}.$$

Finally, if $f \in L^1(\Omega_\infty)$ u *belongs also to* $V(t_0)$ *for every* $t_0 > 0$*, where*

$$V(t_0) \equiv L^2(t_0, +\infty; H_0^1(\Omega)) \cap C([0, +\infty); L^1(\Omega)) \cap L^q(0, +\infty; W_0^{1,q}(\Omega)), \tag{16}$$

and hence u *is the unique global solution of* (1) *that for every* $t_0 > 0$ *belongs to*

$$L^\infty(\Omega \times (t_0, +\infty)) \cap V(t_0) \cap C_{loc}((0, +\infty); L^2(\Omega)).$$

We state now some results concerning the behavior in time of the unique global solution u of (1) described in the previous theorem. The first and following result shows that it is possible to estimate the distance between global solutions satisfying different initial data. In details, we have.

Theorem 2.4. *Assume that* (2)–(5) *hold true for every fixed* $T > 0$*. Let* u *be the unique global solution of* (1) *belonging to* V *and* v *the unique global solution in* V *of the following problem*

$$\begin{cases} v_t - \operatorname{div}(M(x,t)\nabla v) + a(x,t)v = f(x,t) & \textit{in } \Omega_\infty, \\ v(x,t) = 0 & \textit{on } \Gamma_\infty, \\ v(x,0) = v_0(x) & \textit{in } \Omega, \end{cases} \tag{17}$$

where $v_0 \in L^1(\Omega)$*. Then the following estimate holds true*

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \le C_2 \frac{\|u_0 - v_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}} e^{\sigma t}} \qquad \text{a.e. } t > 0, \tag{18}$$

where C_2 *depends only on* α *and* N*, and* $\sigma = c(\alpha)|\Omega|^{-\frac{2}{N}}$*. Hence, it results*

$$\lim_{t \to +\infty} \|u(t) - v(t)\|_{L^\infty(\Omega)} = 0. \tag{19}$$

We point out that (19) says that all the solutions of (1), independently on the value of their initial datum u_0, have the same asymptotic behavior as t becomes large.

We show now that it is also possible to estimate the distance between the global solutions of (1) and the solutions of problems which are similar to (1). In detail, let us consider the following problem

$$\begin{cases} v_t - \operatorname{div}(M_0(x,t)\nabla v) + a_0(x,t)v = f_0(x,t) & \text{in } \Omega_\infty, \\ v(x,t) = 0 & \text{on } \Gamma_\infty, \\ v(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{20}$$

where M_0, a_0, f_0 and v_0 satisfy assumptions that are at all similar to those satisfied by, respectively, M, a, f and u_0 in (1), i.e.

$$M_0(x,t)\xi \cdot \xi \ge \alpha_0 |\xi|^2, \quad |M_0(x,t)| \le \beta_0, \tag{21}$$

for almost every (x,t) in Ω_T and for every ξ in $\mathbb{R}^N$, with α_0 and β_0 positive constants,

$$v_0 \in L^1(\Omega)\,, \tag{22}$$

$$a_0(x,t) \geq 0\,, \qquad a_0(x,t) \in L^1(\Omega_T)\,, \qquad f_0(x,t) \in L^1(\Omega_T)\,, \tag{23}$$

and there exists a non negative constant Q_0 such that

$$|f_0(x,t)| \leq Q_0\, a_0(x,t)\,. \tag{24}$$

We have the following result in which we use *Poincare's inequality:*

$$c_P \int_\Omega w^2 \leq \int_\Omega |\nabla w|^2 \quad \text{for every } w \in H_0^1(\Omega), \tag{25}$$

where c_P is a positive constant depending only on Ω.

Theorem 2.5. *Assume that* (2)–(5) *and* (21)–(24) *hold true for every* $T > 0$. *Let* u *be the unique global solution of* (1) *belonging to* V *and let* v *be the unique global solution of* (20) *belonging* V. *Hence, by Theorem 2.3,* u *and* v *belong also to* Y. *Then the following estimate holds true for every* $t \geq t_0 > 0$

$$\int_\Omega |u(t) - v(t)|^2 \leq e^{-\alpha c_P (t-t_0)} \int_\Omega |u(t_0) - v(t_0)|^2 + \int_{t_0}^t g(s)ds\,, \tag{26}$$

where c_P *is the Poincare's constant and*

$$g(s) = \int_\Omega \frac{1}{\alpha} |M(x,s) - M_0(x,s)|^2 |\nabla v(x,s)|^2\, dx + \\ + 2 \int_\Omega [C_3 |a(x,s) - a_0(x,s)| + C_4 |f(x,s) - f_0(x,s)|]\, dx, \tag{27}$$

where C_3 *and* C_4 *are positive constants depending only on* N, t_0, Q, $\|u_0\|_{L^1(\Omega)}$ *and* $\|v_0\|_{L^1(\Omega)}$ *(see formula* (87)*).*

Moreover, if g *belongs to* $L^1((t_0, +\infty))$ *for a suitable* $t_0 > 0$, *then also the following estimate holds true for every* $t \geq 2t_0$

$$\int_\Omega |u(t) - v(t)|^2 \leq \Lambda e^{-\frac{\alpha c_P}{2} t} + \int_{\frac{t}{2}}^t g(s)ds\,, \tag{28}$$

where we have set $$\Lambda = \int_\Omega |u(t_0) - v(t_0)|^2 + \int_{t_0}^{+\infty} g(s)ds\,. \tag{29}$$

In particular, if g *belongs to* $L^1((t_0, +\infty))$ *(for a suitable* $t_0 > 0$*) it results*

$$\lim_{t \to +\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0\,. \tag{30}$$

Notice that the previous result implies that if the parabolic problems (1) and (20) are "not too far" (i.e. if g is in $L^1((t_0, +\infty))$ for a suitable $t_0 > 0$) then the relative global solutions u and v have the same behavior in time. We show some particular cases of "not too far problems".

Corollary 2.6. *Let* (2)–(5) *and* (22)–(24) *hold true for every* $T > 0$. *Assume that* $M(x,t) \equiv M_0(x,t)$ *in* Ω_∞. *Finally, let* u *be the unique global solution of* (1) *belonging to* V *and let* v *be the unique global solution of* (20) *belonging* V *or, that is the same, let* u *be the unique global solution of* (1) *belonging to* Y *and let* v *be the unique global solution of* (20) *belonging* Y. *If it results*

$$[|a(x,t) - a_0(x,t)| + |f(x,t) - f_0(x,t)|] \in L^1(\Omega \times (t_0, +\infty))$$

for a suitable $t_0 > 0$, *then estimate* (28) *holds true and therefore we have*

$$\lim_{t \to +\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0 \,. \tag{31}$$

In particular, if $a(x,t) = a_0(x,t)$ *and if*

$$|f(x,t) - f_0(x,t)| \in L^1(\Omega \times (t_0, +\infty))$$

then estimate (28) *becomes*

$$\int_\Omega |u(t) - v(t)|^2 \leq \Lambda e^{-\frac{\alpha c_P}{2} t} + 2C_3 \int_{\frac{t}{2}}^t \int_\Omega |f(x,t) - f_0(x,t)| \,,$$

and (31) *holds true.*

Corollary 2.7. *Assume that* (2)–(5) *and* (21)–(24) *hold true for every* $T > 0$. *Let* u *be the unique global solution of* (1) *belonging to* V *and let* v *be the unique global solution of* (20) *belonging* V. *Then* u *and* v *belong to* Y. *Moreover, if* f *and* f_0 *belong to* $L^1(\Omega \times (t_0, +\infty))$ *and if it results*

$$|a(x,t) - a_0(x,t)| \in L^1(\Omega \times (t_0, +\infty)) \tag{32}$$

for a suitable $t_0 > 0$, *then* (28) *and* (30) *hold true. In particular, if* f *belongs to* $L^1(\Omega \times (t_0, +\infty))$ *and* u *is the unique global solution of* (1) *belonging to* V *it results*

$$\lim_{t \to +\infty} \int_\Omega |u(t)|^2 = 0 \,. \tag{33}$$

Indeed, it is also possible to estimate the distance between the global solution $u \in V$ of our problem (1) and the solutions of problems where the lower order term $a(x,t)$ doesn't appear. In details, let us consider the following parabolic problem:

$$\begin{cases} v_t - \operatorname{div}(M_0(x,t)\nabla v) = f_0(x,t) & \text{in } \Omega_\infty, \\ v(x,t) = 0 & \text{on } \Gamma_\infty, \\ v(x,0) = v_0(x) & \text{in } \Omega \,, \end{cases} \tag{34}$$

i.e. problem (20) with $a_0(x,t) \equiv 0$ in Ω_∞. We recall that if the data f and u_0 are only summable functions and in absence of the lower order term a_0 satisfying (23) and (24) it is no more true that there exist solutions of (34) that become immediately bounded. Anyway, as proved in [14], if f_0 is sufficiently regular and although u_0 is only in $L^1(\Omega)$, there exists only one global "regular" solution of (34). We prove now that such a "regular" solution of (34), under suitable "proximity conditions" on the structure of these problems and on their data, has the same asymptotic behavior of the unique global solution $u \in V$ of (1). More precisely, we have the following result

Theorem 2.8. *Let* (2)–(5) *and* (21)–(22) *be true for every* $T > 0$. *Assume that* $f_0 \in L^m_{loc}([0, +\infty); L^s(\Omega))$ *with* m *and* s *satisfying*

$$\frac{1}{m} + \frac{N}{2s} < 1 \qquad m, s \in [1, +\infty]\,, \tag{35}$$

where (here and throughout the paper) we adopt the convention that $\frac{1}{+\infty} = 0$. *Let* u *be the unique global solution in* V *of* (1) *and let* v *the unique global solution of* (34) *belonging to* $W \cap L^\infty(\Omega \times (t_0, +\infty))$ *for every* $t_0 > 0$ *where*

$$W \equiv C_{loc}([0, +\infty); L^1(\Omega)) \cap C_{loc}((0, \infty); L^2(\Omega)) \cap L^2_{loc}((0, \infty); H^1_0(\Omega))\,. \tag{36}$$

Then u *belongs to* Y *and estimate* (26) *holds true for every* $t \geq t_0 > 0$ *with now* $g(s)$ *defined by*

$$g(s) = \int_\Omega \frac{1}{\alpha}|M(x,s) - M_0(x,s)|^2 |\nabla v(x,s)|^2\, dx + \tag{37}$$

$$+\, 2\int_\Omega \left[c(u,v)|a(x,s)| + c_0(u,v)|f(x,s) - f_0(x,s)|\right]\, dx$$

with $\qquad c(u,v) = \|v\|_{L^\infty(\Omega\times(t_0,+\infty))}\, c_0(u,v)\,,$

and $\qquad c_0(u,v) = \|u\|_{L^\infty(\Omega\times(t_0,+\infty))} + \|v\|_{L^\infty(\Omega\times(t_0,+\infty))}\,.$

Moreover, if g *belongs to* $L^1((t_0, +\infty))$ *for a suitable* $t_0 > 0$, *then also estimate* (28) *holds true for every* $t \geq 2t_0$ *and hence it results*

$$\lim_{t\to+\infty} \|u(t) - v(t)\|_{L^2(\Omega)} = 0\,.$$

Remark 2.9. We observe that in the above definition of g it is possible eliminate the dependence on u and v in the functions $c(u,v)$ and $c_0(u,v)$ simply by using the L^∞-estimate (13) on u proved in Theorem 2.3 and the estimates on v proved in [14].

Remark 2.10. We point out that the results of Theorem 2.8 remain true (together with Remark 2.9) also if v is the regular solution of a suitable nonlinear problem. More in detail, let v be the unique global solution belonging to $W \cap L^\infty(\Omega\times(t_0,+\infty))$ (for every $t_0 > 0$) of the following nonlinear problem

$$\begin{cases} v_t - \mathrm{div}(a(x,t,\nabla v)) = f_0(x,t) & \text{in } \Omega_\infty, \\ v(x,t) = 0 & \text{on } \Gamma_\infty, \\ v(x,0) = v_0(x) & \text{in } \Omega\,, \end{cases} \tag{38}$$

where the function $a(x,t,\xi) : \Omega \times (0,+\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ is a Caratheodory function, satisfying for a.e. $(x,t) \in \Omega_\infty$ and for every ξ and $\eta \in \mathbb{R}^N$ the structure conditions

$$\alpha|\xi|^2 \leq a(x,t,\xi)\xi, \quad \alpha > 0, \tag{39}$$

$$[a(x,t,\xi) - a(x,t,\eta)][\xi - \eta] \geq \gamma|\xi - \eta|^2, \qquad \gamma > 0\,, \tag{40}$$

$$|a(x,t,\xi)| \leq \beta[|\xi| + h(x,t)], \quad \beta > 0, \tag{41}$$

with $h \in L^{2m_1}_{loc}([0,+\infty); L^{2s_1}(\Omega))$, $m_1\,,\ s_1 \in (1, +\infty]$, $\frac{1}{m_1} + \frac{N}{2s_1} < 1$.

Recall that a Carathéodory function is continuous with respect to ξ for almost every $(x,t) \in \Omega_\infty$, and measurable with respect to (x,t) for every $\xi \in \mathbb{R}^N$.
Then all the results in Theorem 2.8 remain true simply replacing (34) with (38) and $g(s)$ with the following function

$$g(s) = \int_\Omega \frac{1}{\alpha}|M(x,s)\nabla v(x,s) - a(x,s,\nabla v(x,s))|^2\,dx + \\ + \int_\Omega [c(u,v)|a(x,s)| + c_0(u,v)|f(x,s) - f_0(x,s)|]\,dx\,.$$

In particular, if g belongs to $L^1((t_0,+\infty))$ for a suitable $t_0>0$, then it results

$$\lim_{t\to+\infty} \|u(t)-v(t)\|_{L^2(\Omega)} = 0\,.$$

See Remark 4.1 for further details. □

Finally, it is also possible to estimate the distance between the global solution u of (1) and the solution of a "suitable near" stationary problem. In details, let us consider the following elliptic problem

$$\begin{cases} -\mathrm{div}(M_1(x)\nabla w) + a_1(x)w = f_1(x) & \text{in } \Omega, \\ w(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{42}$$

We assume the following structure conditions

$$M_1(x)\xi\cdot\xi \geq \alpha_1|\xi|^2\,, \quad |M_1(x)| \leq \beta_1\,, \tag{43}$$

for almost every x in Ω and for every ξ in $\mathbb{R}^N$, with α_1 and β_1 positive costants

$$a_1(x) \geq 0\,, \qquad a_1(x) \in L^1(\Omega)\,, \qquad f_1(x) \in L^1(\Omega)\,, \tag{44}$$

and that there exists a non negative constant Q_1 such that

$$|f_1(x)| \leq Q_1\, a_1(x) \quad \text{ a.e. } x \in \Omega\,. \tag{45}$$

We have the following result

Theorem 2.11. *Assume that* (2)–(5) *hold true for every $T>0$. Assume also that* (43)–(45) *are satisfied. Let u be the unique global solution of* (1) *belonging to V and let w be the unique solution of* (42) *belonging to $L^\infty(\Omega)\cap H_0^1(\Omega)$. Then u belongs to Y and the following estimate holds true for every $t \geq t_0 > 0$*

$$\int_\Omega |u(t)-w|^2 \leq e^{-\alpha c_P(t-t_0)}\int_\Omega |u(t_0)-w|^2 + \int_{t_0}^t g_1(s)ds\,, \tag{46}$$

where c_P is the Poincare's constant defined in (25) *and* $g_1(s) =$

$$= \int_\Omega \left[\frac{1}{\alpha}|M(x,s)-M_1(x)|^2|\nabla w|^2 + 2C_5|a(x,s)-a_1(x)| + 2C_6|f(x,s)-f_1(x)|\right]dx \tag{47}$$

where C_5 and C_6 are positive constants depending only on α, N, t_0, $\|u_0\|_{L^1(\Omega)}$, Q and Q_1 (*see formula* (91)).

Moreover, if g_1 belongs to $L^1((t_0,+\infty))$ for a suitable $t_0>0$, then also the following estimate holds true for every $t\geq 2t_0$

$$\int_\Omega |u(t)-w|^2 \leq \Lambda_1 e^{-\frac{\alpha c_P}{2}t} + \int_{\frac{t}{2}}^{t} g_1(s)ds\,, \tag{48}$$

where we have set $$\Lambda_1 = \int_\Omega |u(t_0)-w|^2 + \int_{t_0}^{+\infty} g_1(s)ds\,. \tag{49}$$

Therefore, if g_1 belongs to $L^1((t_0,+\infty))$ it results

$$\lim_{t\to+\infty} \|u(t)-w\|_{L^2(\Omega)} = 0\,. \tag{50}$$

Remark 2.12. We observe that the particular autonomous case

$$M(x,t)=M_1(x)\,,\quad a(x,t)=a_1(x)\,,\quad f(x,t)=f_1(x)\,,$$

was already investigated in [9] and that in such a particular case it is also possible to estimate the $L^\infty(\Omega)$-norm of the difference $u(t)-w$.

3. Preliminary results

To be self-contained we recall here two results that, as said above, will be an essential tool in the proof of the stated results. The first one (see Theorem 3.1 below) is a particular case of Theorem 2.2 in [12] (applied here with $r=2$, $r_0=1$, $b=2$ and $q=2^*$). The second one (see Proposition 3.2 below) can be found in [10].

Theorem 3.1. *Let us assume that*

$$v\in C((0,T);L^2(\Omega))\cap L^2(0,T;L^{2^*}(\Omega))\cap C([0,T);L^1(\Omega)) \tag{51}$$

where Ω is an open bounded set of $\mathbb{R}^N$ with $N\geq 3$ and $0<T\leq+\infty$. Suppose that v satisfies for every $k>0$ and $0<t_1<t_2<T$ the following inequalities

$$\int_\Omega |G_k(v)|^2(t_2)dx - \int_\Omega |G_k(v)|^2(t_1)dx + c_1\int_{t_1}^{t_2} \|G_k(v)(\tau)\|^2_{L^{2^*}(\Omega)}d\tau \leq 0 \tag{52}$$

$$\|G_k(v)(t)\|_{L^1(\Omega)} \leq c_2\|G_k(v)(t_0)\|_{L^1(\Omega)} \quad \text{for every } \ 0\leq t_0<t<T, \tag{53}$$

where c_1 and c_2 are positive constants independent of k.

Finally, let us define $$v_0\equiv v(x,0)\in L^1(\Omega). \tag{54}$$

Then the following exponential decay estimate occurs

$$\|v(t)\|_{L^\infty(\Omega)} \leq C_2\frac{\|v_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}e^{\sigma t}} \quad \text{for every } \ t\in(0,T),$$

where v_0 is as in (54)*, C_2* (*see formula* (4.17) *in* [12]) *is a positive constant depending only on N, c_1 and c_2 while σ is given by the following formula*

$$\sigma=\frac{c_1\kappa}{4|\Omega|^{\frac{2}{N}}},\quad \kappa \textit{ arbitrarily fixed in } \Big(0,\frac{1}{2}\Big),$$

where $|\Omega|$ denotes the measure of Ω.

Here the function $G_k(s)$ $(k \geq 0)$ is defined by

$$G_k(s) = (|s| - k)_+ \mathrm{sign}(s) \text{ for every } s \in \mathbb{R}\,. \tag{55}$$

We point out that a more general version of Theorem 3.1 (including the previous theorem) is proved in section 3 of [16] while further extensions to decay estimates for unbounded solutions can be found in [13].

Proposition 3.2. *Assume $T \in (t_0, +\infty]$ and let $\phi(t)$ a continuous and non negative function defined in $[t_0, T)$ verifying*

$$\phi(t_2) - \phi(t_1) + M \int_{t_1}^{t_2} \phi(t)dt \leq \int_{t_1}^{t_2} g(t)dt$$

for every $t_0 \leq t_1 \leq t_2 < T$ where M is a positive constant and g is a non negative function in $L^1_{loc}([t_0, T))$. Then for every $t_0 \leq t < T$ we get

$$\phi(t) \leq \phi(t_0)e^{-M(t-t_0)} + \int_{t_0}^{t} g(s)ds, . \tag{56}$$

Moreover, if $T = +\infty$ and g belongs to $L^1((t_0, +\infty))$ there exists $t_1 > t_0$ $(t_1 = 2t_0)$ such that

$$\phi(t) \leq \Lambda e^{-\frac{M}{2}t} + \int_{\frac{t}{2}}^{t} g(s)ds \qquad \forall t \geq t_1\,, \tag{57}$$

where $\Lambda = \phi(t_0) + \displaystyle\int_{t_0}^{+\infty} g(s)ds$. *In particular, we get that* $\lim_{t \to +\infty} \phi(t) = 0$.

Throughout the paper we denote with c and c(s,v) (respectively) positive constants and functions depending only on the variable in brackets, that can change from one line to the other. Moreover, the use of some test function in the proofs can be made rigorous by means of Steklov averaging process (see [7] and [8]).

4. Proofs of the results

4.1. Proof of Theorem 2.1

The proof is divided in two steps: first we prove the existence of a solution having the stated regularity and then we prove the uniqueness of these regular solutions.

Step 1: Existence. We recall that the weak solution u of (1) constructed in [9] belongs to $L^\infty_{loc}((0,T]; L^\infty(\Omega)) \cap L^2_{loc}((0,T]; H^1_0(\Omega)) \cap L^q(0,T; W^{1,q}_0(\Omega))$ for every $1 < q < 2 - \frac{1}{N+1}$ and satisfies (13). Hence, it is sufficient to prove that such a solution u belongs also to $C([0,T]; L^1(\Omega)) \cap C_{loc}((0,T]; L^2(\Omega))$. To this purpose, we recall that u is obtained as the a.e. limit in Ω_T of the solutions

$$u_n \in C([0,T]; L^2(\Omega)) \cap C(\overline{\Omega_T}) \cap L^2(0,T; H^1_0(\Omega)) \ (n \text{ positive integer})$$

of the following approximating problems

$$\begin{cases} (u_n)_t - \mathrm{div}(M(x,t)\nabla u_n) + a_n(x,t)u_n = f_n & \text{in } \Omega_T, \\ u_n(x,t) = 0 & \text{on } \Gamma_T, \\ u_n(x,0) = u_{0,n}(x) & \text{in } \Omega. \end{cases} \tag{58}$$

where a_n and f_n are bounded functions defined as follows

$$f_n(x,t) = \frac{f(x,t)}{1+\frac{1}{n}|f(x,t)|}, \qquad a_n(x,t) = \frac{a(x,t)}{1+\frac{Q}{n}|a(x,t)|}, \tag{59}$$

and $u_{0,n}(x) \in C^1(\bar{\Omega})$ satisfies

$$||u_{0,n}||_{L^1(\Omega)} \leq ||u_0||_{L^1(\Omega)} \quad u_{0,n} \to u_0 \text{ in } \; L^1(\Omega),$$

(see Theorems 4.1, 4.2 and 7.1 in chapter III of [8]).

Indeed, in [9] it is proved that also the following convergences hold true

$$a_n(x,t)u_n \;\to a(x,t)u \quad \text{in} \;\; L^1(\Omega_T), \tag{60}$$

$$\nabla u_n \;\to \nabla u \quad \text{in} \;\; (L^q(\Omega_T))^N \;\text{ for every }\; 1 < q < 2 - \frac{N}{N+1}. \tag{61}$$

To prove that u belongs to $C([0,T];L^1(\Omega))$ we show that u_n is a Cauchy sequence in $C([0,T];L^1(\Omega))$. To this aim, as in [17], we take as test function $\varphi = T_1(u_n - u_m)$ in the problems solved by u_n and u_m, where $T_k(s)$ $(k>0)$ is the usual truncation function at level k defined by

$$T_k(s) = \min\{|s|, k\}\mathrm{sign}(s). \tag{62}$$

Subtracting the equations obtained in this way we get for every $0 < t \leq T$

$$\int_\Omega [\Phi_1(u_n - u_m)](t) - \int_\Omega \Phi_1(u_{0,n} - u_{0,m}) + \int_0^t\!\!\int_\Omega M(x,t)\nabla(u_n - u_m)\nabla T_1(u_n - u_m) +$$
$$+ \int_0^t\!\!\int_\Omega [a_n(x,t)u_n - a_m(x,t)u_m]T_1(u_n - u_m) \leq \int_0^t\!\!\int_\Omega (f_n - f_m)T_1(u_n - u_m), \tag{63}$$

where
$$\Phi_k(s) = \int_0^s T_k(\sigma)d\sigma.$$

We estimate now the integrals in (63). We have

$$\int_\Omega \Phi_1(u_{0,n} - u_{0,m}) \leq \int_\Omega |u_{0,n} - u_{0,m}| \tag{64}$$

since $\Phi_1(s) \leq |s|$. Moreover, the following estimate holds true

$$\frac{1}{2}\int_{|u_n - u_m|\leq 1} |u_n - u_m|^2(t) + \frac{1}{2}\int_{|u_n-u_m|>1} |u_n - u_m|(t) \leq \int_\Omega [\Phi_1(u_n - u_m)](t), \tag{65}$$

since we have $\Phi_1(s) \geq \frac{1}{2}\left[s^2\chi_{\{|s|\leq 1\}} + |s|\chi_{\{|s|>1\}}\right]$.

By (63)–(65) and using assumption (2) we deduce

$$\frac{1}{2}\int_{|u_n - u_m|\leq 1} |u_n - u_m|^2(t) + \frac{1}{2}\int_{|u_n-u_m|>1} |u_n - u_m|(t)$$
$$\leq \int_\Omega |u_{0,n} - u_{0,m}| + \int_0^t\!\!\int_\Omega |a_n(x,t)u_n - a_m(x,t)u_m| + \int_0^t\!\!\int_\Omega |f_n - f_m|. \tag{66}$$

Observe that we have

$$\int_{\Omega} |u_n - u_m|(t) \leq \left(\int_{|u_n-u_m|\leq 1} |u_n - u_m|^2(t) \right)^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} + \int_{|u_n-u_m|>1} |u_n - u_m|(t)\,. \tag{67}$$

By (66)–(67) it follows that u_n is a Cauchy sequence in $C([0,T];L^1(\Omega))$ since f_n and $a_n(x,t)u_n$ are convergent in $L^1(\Omega_T)$ and $u_{0,n}$ is convergent in $L^1(\Omega)$.

To conclude this step we recall that in [9] the proof that u satisfies (13) is done by showing that it results

$$\|u_n(t)\|_{L^\infty(\Omega)} \leq C_1 \max\left\{ \frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}}, Q \right\} \qquad \text{for every} \quad t \in (0,T)\,.$$

Hence the sequence u_n is bounded in $\Omega \times [t_0, T]$, for every arbitrarily fixed $t_0 > 0$. Thanks to the previous results it follows that u_n is also a Cauchy sequence in $C([t_0,T];L^2(\Omega))$ (for every arbitrarily fixed $t_0 > 0$) and therefore u belongs to $C_{loc}((0,T];L^2(\Omega))$. In conclusion, u has all the regularity properties stated in Theorem 2.1.

Step 2: Uniqueness. The proof proceed by contradiction. Hence, let u and v weak solutions of (1) belonging to V. Let $\delta > 1$ and take as test function in the equation satisfied by u and by v the function

$$\varphi = \left\{ 1 - \frac{1}{[1+|u-v|]^\delta} \right\} \mathrm{sign}(u-v).$$

Subtracting the equations obtained in this way and using assumption (2) we get for every $0 < t_0 < t \leq T$

$$\begin{aligned} &\int_\Omega |u-v|(t) + \delta\alpha \int_{t_0}^t\!\!\int_\Omega \frac{|\nabla(u-v)|^2}{[1+|u-v|]^{\delta+1}} - \frac{1}{\delta-1}\int_\Omega \left\{ 1 - \frac{1}{[1+|u-v|(t)]^{\delta-1}} \right\} \\ &\qquad + \int_{t_0}^t\!\!\int_\Omega a(x,t)|u-v| \left\{ 1 - \frac{1}{[1+|u-v|]^\delta} \right\} \\ &\leq \int_\Omega |u-v|(t_0) - \frac{1}{\delta-1}\int_\Omega \left\{ 1 - \frac{1}{[1+|u-v|(t_0)]^{\delta-1}} \right\}. \end{aligned} \tag{68}$$

Dropping the positive terms we deduce

$$\int_\Omega |u-v|(t) \leq \int_\Omega |u-v|(t_0) + \frac{1}{\delta-1}\int_\Omega \left\{ 1 - \frac{1}{[1+|u-v|(t)]^{\delta-1}} \right\}$$

which implies
$$\int_\Omega |u-v|(t) \leq \int_\Omega |u-v|(t_0) + \frac{|\Omega|}{\delta-1}\,.$$

Letting $\delta \to +\infty$, we obtain that for every $0 < t_0 < t \leq T$

$$||(u-v)(t)||_{L^1(\Omega)} \leq ||(u-v)(t_0)||_{L^1(\Omega)}\,. \tag{69}$$

Now the assertion follows letting $t_0 \to 0$ thanks to the regularity $C([0,T];L^1(\Omega))$ of both the solutions u and v. □

4.2. Proof of Theorem 2.3

For every $(x,t) \in \Omega_\infty$ we define $u(x,t)$ as the value of the unique weak solution of (1) belonging to V_T where $T > t$ is arbitrarily fixed. We observe explicitly that the definition is well posed since, if $v \in V_{T_0}$ is a weak solution of our problem in Ω_{T_0}, with $T_0 > t$, by the uniqueness result of Theorem 2.1 it follows that $u(x,t) = v(x,t)$ in $\Omega \times (0, \min\{T, T_0\})$. Notice that (thanks again to Theorem 2.1) u satisfies estimate (13) for almost every $t > 0$ and therefore belongs to $L^\infty(\Omega \times (t_0, +\infty))$ for every $t_0 > 0$.

Moreover, u is in $L^q(0,T; W_0^{1,q}(\Omega)) \cap C_{loc}((0,T]; L^2(\Omega))$ for every $1 < q < 2 - \frac{1}{N+1}$ and for every $T > 0$ and it is the unique global solution belonging to V_T for every $T > 0$, i.e. u is the unique global solution in V. Hence, to conclude the proof it remains to prove that if f is in $L^1(\Omega_\infty)$ then u belongs also to $V(t_0)$ for every $t_0 > 0$. We observe that it is possible to proceed in the proof by using the construction of u done in [9] but we prefer to give a direct and self contained proof. To this aim, choosing u as a test function in (1), using assumption (2) and dropping the positive terms we deduce that for every $0 < t_0 < t < +\infty$

$$\frac{1}{2}\int_\Omega u^2(t) - \frac{1}{2}\int_\Omega u^2(t_0) + \alpha \int_{t_0}^t\int_\Omega |\nabla u|^2 \le \int_{t_0}^t\int_\Omega |f||u|\,.$$

Hence, using the L^∞-estimate (13) we deduce

$$\begin{aligned} &||u||^2_{L^\infty(t_0,t;L^2(\Omega))} + ||\nabla u||^2_{L^2(\Omega\times(t_0,t))} \\ &\le \min\left\{\frac{1}{2}, \alpha\right\}^{-1} A(t_0)\left[||f||_{L^1(\Omega_t)} + \frac{|\Omega|}{2} A(t_0)\right], \end{aligned} \tag{70}$$

where
$$A(t) \equiv C_1 \max\left\{\frac{\|u_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}}}, Q\right\} \tag{71}$$

with $C_1 = C(\alpha, N)$ the same constant in (13). By (70) it follows that if f is in $L^1(\Omega_\infty)$ then u belongs to $L^2(t_0, +\infty; H_0^1(\Omega))$ for every $t_0 > 0$. Thus, it remains to prove the regularity of $C([0,+\infty); L^1(\Omega)) \cap L^q(0,+\infty; W_0^{1,q}(\Omega))$. Since by construction $u \in C_{loc}([0,+\infty); L^1(\Omega))$, the regularity of $C([0,+\infty); L^1(\Omega))$ follows by proving that u belongs to $L^\infty(0,+\infty; L^1(\Omega))$. Notice that it is sufficient to show the following estimate

$$||u||_{L^\infty(0,T;L^1(\Omega))} \le ||u_0||_{L^1(\Omega)} + ||f||_{L^1(\Omega_T)}\,, \tag{72}$$

since, by the arbitrariness of T, if $f \in L^1(\Omega_\infty)$ it implies

$$||u||_{L^\infty(0,\infty;L^1(\Omega))} \le ||u_0||_{L^1(\Omega)} + ||f||_{L^1(\Omega_\infty)}\,. \tag{73}$$

To prove (72) let $\delta > 0$ and take

$$\varphi = \left(1 - \frac{1}{(1+|u|)^\delta}\right)\mathrm{sign}(u)$$

as a test function in (1).

Using assumption (2) (and also the fact that $|\varphi| \le 1$) we deduce for every interval $0 < t < T < +\infty$ arbitrarily fixed:

$$\int_\Omega |u|(t) + \delta\alpha \int_0^t\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{\delta+1}} - \frac{1}{\delta-1}\int_\Omega \left\{1 - \frac{1}{(1+|u|(t))^{\delta-1}}\right\} + \int_0^t\int_\Omega a(x,t)|u|\left(1-\frac{1}{(1+|u|)^\delta}\right) \le \int_\Omega |u_0| - \frac{1}{\delta-1}\int_\Omega\left\{1-\frac{1}{(1+|u_0|)^{\delta-1}}\right\} + ||f||_{L^1(\Omega_T)}\,.$$

Hence, by the previous estimate (dropping the positive term coming from the lower order term) we deduce

$$\begin{aligned}&\int_\Omega |u|(t) + \delta\alpha \int_0^t\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{\delta+1}} - \frac{1}{\delta-1}\int_\Omega \left\{1 - \frac{1}{(1+|u|(t))^{\delta-1}}\right\} \\ &\le \int_\Omega |u_0| - \frac{1}{\delta-1}\int_\Omega\left\{1-\frac{1}{(1+|u_0|)^{\delta-1}}\right\} + ||f||_{L^1(\Omega_T)}\,.\end{aligned} \tag{74}$$

Now, choosing $\delta > 1$, from (74) we obtain for every $0 < t < T < +\infty$

$$\int_\Omega |u|(t) \le \int_\Omega |u_0| + \frac{|\Omega|}{\delta-1} + ||f||_{L^1(\Omega_T)}\,, \tag{75}$$

which, letting $\delta \to +\infty$, implies (72).

To conclude the proof it remains to prove the global $(L^q(\Omega_\infty))^N$ regularity of the gradient of u. To this aim we notice that by (72), it follows that for every $\delta \in (0,1)$ and $0 < t \le T < +\infty$

$$\begin{aligned}\frac{1}{\delta-1}\int_\Omega \left\{1-\frac{1}{(1+|u|(t))^{\delta-1}}\right\} &= \frac{1}{1-\delta}\int_\Omega \left\{(1+|u|(t))^{1-\delta}-1\right\} \\ &\le \frac{1}{1-\delta}\int_\Omega |u|(t) \le \frac{1}{1-\delta}\left(\int_\Omega |u_0| + ||f||_{L^1(\Omega_T)}\right)\,.\end{aligned}$$

Hence, by the previous estimate and (74) it follows that for every $\delta \in (0,1)$ and $0 < t \le T < +\infty$

$$\delta\alpha\int_0^t\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{\delta+1}} \le \int_\Omega |u_0| + ||f||_{L^1(\Omega_T)} + \frac{1}{1-\delta}\left(\int_\Omega |u_0| + ||f||_{L^1(\Omega_T)}\right)\,,$$

which implies $\quad \delta\alpha\displaystyle\int_0^T\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{\delta+1}} \le C_2 \quad \text{for every} \quad \delta \in (0,1)\,, \qquad (76)$

where $C_2 = \left(1+\frac{1}{1-\delta}\right)\left[\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega_T)}\right]$, and consequently (thanks to the arbitrarieness of T) we have

$$\delta\alpha\int_0^\infty\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^{\delta+1}} \le C_3 \quad \text{for every} \quad \delta \in (0,1)\,, \tag{77}$$

where $C_3 = \left(1+\frac{1}{1-\delta}\right)\left[\|u_0\|_{L^1(\Omega)} + \|f\|_{L^1(\Omega_\infty)}\right]$. Now, thanks to (73) and (77) it is possible to proceed as in Lemma 3.2 in [11] to deduce the assertion. □

4.3. Proof of Theorem 2.4

Let u and v be the global solutions of, respectively, (1) and (17), belonging to V. Thanks to the regularity of u and v we can take as test function in the equation satisfied by u and by v the function $G_k(u-v)$, where $G_k(s)$ $(k \geq 0)$ is as in (55). Subtracting the equations obtained in this way and using assumption (2) we get for every $0 < t_1 < t_2 < +\infty$

$$\frac{1}{2}\int_\Omega |G_k(u-v)|^2(t_2) - \frac{1}{2}\int_\Omega |G_k(u-v)|^2(t_1) + \tag{78}$$

$$+\alpha \int_{t_1}^{t_2}\int_\Omega |\nabla G_k(u-v)|^2 + \int_{t_1}^{t_2}\int_\Omega a(x)(u-v)G_k(u-v) \leq 0$$

Using the Sobolev inequality and recalling that the last integral in (78) is non negative we deduce that for every positive k and $0 < t_1 < t_2 \leq T$ it results

$$\frac{1}{2}\int_\Omega |G_k(u-v)|^2(t_2) - \frac{1}{2}\int_\Omega |G_k(u-v)|^2(t_1) + \tag{79}$$

$$+\alpha c_S \int_{t_1}^{t_2}\int_\Omega \|G_k(u-v)\|^2_{L^{2^*}(\Omega)} \leq 0$$

Moreover, using as a test function

$$\varphi = \left\{1 - \frac{1}{(1+|G_k(u-v)|)^\delta}\right\} \operatorname{sign}(u-v) \quad (\delta > 0)$$

in (1) and in (17), subtracting the results obtained in this way we deduce

$$\begin{aligned}
&\int_\Omega |G_k(u-v)|(t) + \delta\alpha \int_{t_0}^{t}\int_\Omega \frac{|\nabla G_k(u-v)|^2}{(1+|G_k(u-v)|)^{\delta+1}} - \\
&\quad - \frac{1}{\delta-1}\int_\Omega \left\{1 - \frac{1}{(1+|G_k(u-v)|(t))^{\delta-1}}\right\} + \\
&\quad + \int_{t_0}^{t}\int_\Omega a(x,t)|u-v| \left\{1 - \frac{1}{(1+|G_k(u-v)|)^\delta}\right\} \\
&\leq \int_\Omega |G_k(u-v)|(t_0) - \frac{1}{\delta-1}\int_\Omega \left\{1 - \frac{1}{(1+|G_k(u-v)|(t_0))^{\delta-1}}\right\}.
\end{aligned}$$

Now, reasoning as in the proof of inequality (69) we get for every $0 \leq t_0 < t < +\infty$ and $k > 0$

$$\|G_k(u-v)(t)\|_{L^1(\Omega)} \leq \|G_k(u-v)(t_0)\|_{L^1(\Omega)}\,. \tag{80}$$

Hence the assumptions of Theorem 3.1 are satisfied and we deduce that the following estimate holds true

$$\|u(t) - v(t)\|_{L^\infty(\Omega)} \leq C_2 \frac{\|u_0 - v_0\|_{L^1(\Omega)}}{t^{\frac{N}{2}} e^{\sigma t}} \qquad \text{for every} \quad t > 0\,,$$

where C_2 is a positive constant depending only on α and N and $\sigma = c(\alpha)|\Omega|^{-\frac{2}{N}}$. $\square$

4.4. Proof of Theorem 2.5

Let $t_0 > 0$ arbitrarily fixed and take as test function in (1) and in (20) the function $u - v$. Subtracting the results obtained in this way we get for every interval $0 < t_0 \le t_1 < t_2 < +\infty$:

$$\frac{1}{2}\int_\Omega |u(t_2) - v(t_2)|^2 - \frac{1}{2}\int_\Omega |u(t_1) - v(t_1)|^2 + \tag{81}$$
$$+ \int_{t_1}^{t_2}\int_\Omega [M(x,t)\nabla u \nabla(u-v) - M_0(x,t)\nabla v \nabla(u-v)] +$$
$$+ \int_{t_1}^{t_2}\int_\Omega [a(x,t)u(u-v) - a_0(x,t)v(u-v)] = \int_{t_1}^{t_2}\int_\Omega [f(x,t) - f_0(x,t)](u-v)\,.$$

We estimate the terms in the previous equation. It results, using assumption (2) and the Poincare's inequality (25)

$$\int_{t_1}^{t_2}\int_\Omega [M(x,t)\nabla u \nabla(u-v) - M_0(x,t)\nabla v \nabla(u-v)] \tag{82}$$
$$= \int_{t_1}^{t_2}\int_\Omega [M(x,t)\nabla(u-v)\nabla(u-v) + \int_{t_1}^{t_2}\int_\Omega [M(x,t) - M_0(x,t)]\nabla v \nabla(u-v)]$$
$$\ge \frac{\alpha}{2}\int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 + \frac{\alpha c_P}{2}\int_{t_1}^{t_2}\int_\Omega |u-v|^2 +$$
$$+ \int_{t_1}^{t_2}\int_\Omega [M(x,t) - M_0(x,t)]\nabla v \nabla(u-v)]\,.$$

We estimate the last term in (82). Using Young's inequality we deduce

$$\left|\int_{t_1}^{t_2}\int_\Omega [M(x,t) - M_0(x,t)]\nabla v \nabla(u-v)]\right| \le \tag{83}$$
$$\le \frac{\alpha}{2}\int_{t_1}^{t_2}\int_\Omega |\nabla(u-v)|^2 + \frac{1}{2\alpha}\int_{t_1}^{t_2}\int_\Omega |M(x,t) - M_0(x,t)|^2 |\nabla v|^2\,.$$

Hence, by the last estimate we deduce that

$$\int_{t_1}^{t_2}\int_\Omega [M(x,t)\nabla u \nabla(u-v) - M_0(x,t)\nabla v \nabla(u-v)] \ge \tag{84}$$
$$\ge \frac{\alpha c_P}{2}\int_{t_1}^{t_2}\int_\Omega |u-v|^2 - \frac{1}{2\alpha}\int_{t_1}^{t_2}\int_\Omega |M(x,t) - M_0(x,t)|^2 |\nabla v|^2\,.$$

Moreover, it results

$$\int_{t_1}^{t_2}\int_\Omega [a(x,t)u(u-v) - a_0(x,t)v(u-v)] = \int_{t_1}^{t_2}\int_\Omega a(x,t)(u-v)^2 + \tag{85}$$
$$+ \int_{t_1}^{t_2}\int_\Omega [a(x,t) - a_0(x,t)]v(u-v)] \ge -c(u,v)\int_{t_1}^{t_2}\int_\Omega |a(x,t) - a_0(x,t)|\,,$$

where we have set $\qquad c(u,v) = c_0(u,v)\|v\|_{L^\infty(\Omega\times(t_0,+\infty))}$

and
$$c_0(u,v) = \|u\|_{L^\infty(\Omega\times(t_0,+\infty))} + \|v\|_{L^\infty(\Omega\times(t_0,+\infty))}\,. \tag{86}$$

Notice that since u and v are the unique global solutions of respectively, (1) and (20), which belong to V they satisfy estimate (13) for almost every $t>0$. Hence, it results

$$\begin{aligned} c_0(u,v) &\le C_4 \equiv C_1\left[\max\left\{\frac{\|u_0\|_{L^1(\Omega)}}{t_0^{\frac{N}{2}}}, Q\right\} + \max\left\{\frac{\|v_0\|_{L^1(\Omega)}}{t_0^{\frac{N}{2}}}, Q\right\}\right], \\ c(u,v) &\le C_3 \equiv C_4 C_1 \max\left\{\frac{\|v_0\|_{L^1(\Omega)}}{t_0^{\frac{N}{2}}}, Q\right\}, \end{aligned} \tag{87}$$

where $C_1 = C_1(\alpha, N)$ is the same constant in (13). Finally, we have

$$\int_{t_1}^{t_2}\int_\Omega [f(x,t)-f_0(x,t)](u-v) \le C_4\int_{t_1}^{t_2}\int_\Omega |f(x,t)-f_0(x,t)|\,.$$

By the previous estimates we deduce

$$\int_\Omega |u(t_2)-v(t_2)|^2 - \int_\Omega |u(t_1)-v(t_1)|^2 + \alpha c_P\int_{t_1}^{t_2}\int_\Omega |u-v|^2 \le \int_{t_1}^{t_2} g(s)\,ds \tag{88}$$

where
$$\begin{aligned} g(s) = &\int_\Omega \frac{1}{\alpha}|M(x,s)-M_0(x,s)|^2|\nabla v(x,s)|^2\,dx + \\ &+\int_\Omega [2C_3|a(x,s)-a_0(x,s)| + 2C_4|f(x,s)-f_0(x,s)|]\,dx. \end{aligned} \tag{89}$$

Notice that by the structure assumptions it follows that $g(s)$ belongs to $L^1_{loc}([t_0,+\infty))$. Thus we can apply Proposition 3.2 with $\phi(t) = \int_\Omega |u(t)-v(t)|^2$ obtaining estimate (26). Finally, if g belongs to $L^1((t_0,+\infty))$, again by Proposition 3.2 we deduce that also (28) holds true. □

4.5. Proof of Corollary 2.6

The assertions are immediate consequences of Theorem 2.5 applied with $M_0(x,t)\equiv M(x,t)$. □

4.6. Proof of Corollary 2.7

Also in this case (28) and (30) follow once we show that all the assumptions of Theorem 2.5 are satisfied. We start observing that by the assumption that both the functions f and f_0 belong to $L^1([t_0,+\infty))$ it results

$$G_0(s) = \int_\Omega |f(x,s)-f_0(x,s)| \in L^1([t_0,+\infty))\,.$$

Thus, since (32) holds true, to prove that the function $g(s)$ defined in (27) belongs to $L^1([t_0,+\infty))$ it is sufficient to prove that

$$G_1(s) \equiv \int_\Omega |M(x,s)-M_0(x,s)|^2|\nabla v(x,s)|^2 \in L^1([t_0,+\infty))\,.$$

To this aim we recall that by assumptions (2) and (21) it follows that

$$|M(x,s) - M_0(x,s)|^2 \in L^\infty(\Omega_\infty)\,.$$

Moreover, since we are assuming $f_0 \in L^1([t_0,+\infty))$, by Theorem 2.3 we deduce that

$$|\nabla v(x,t)|^2 \in L^1(\Omega \times [t_0,+\infty))\,.$$

Hence, it follows that $G_1(s)$ is in $L^1([t_0,+\infty))$ and applying Theorem 2.5 we deduce that (28) and (30) hold true.

To conclude the proof it remains to prove that (33) holds true. To this aim it is sufficient to choose in (20) $a_0(x,t) = a(x,t)$, $f_0 \equiv 0$ and $v_0 \equiv 0$. As a matter of fact, with such a choice assumption (32) is satisfied and the unique solution of (20) is the zero function and hence (33) follows by (30). □

4.7. Proof of Theorem 2.8

We start observing that by Theorem 2.3 the unique global solution u in V of (1) belongs also to Y. Moreover, thanks to the assumptions done and to Theorems 2.5 and 2.6 in [14], problem (34) admits a unique global solution belonging to $W \cap V$. Hence, we can proceed exactly as in the proof of Theorem 2.5 simply replacing $a_0(x,t)$ with the zero function and we deduce the following estimate

$$\int_\Omega |u(t_2) - v(t_2)|^2 - \int_\Omega |u(t_1) - v(t_1)|^2 + \alpha c_P \int_{t_1}^{t_2}\int_\Omega |u-v|^2 \le \int_{t_1}^{t_2} g(s)\,ds$$

where
$$g(s) = \int_\Omega \frac{1}{\alpha}|M(x,s) - M_0(x,s)|^2 |\nabla v(x,s)|^2\,dx + \\ + 2\int_\Omega [c(u,v)|a(x,s)| + c_0(u,v)|f(x,s) - f_0(x,s)|]\,dx$$

with $c(u,v)$ and $c_0(u,v)$ as in (86).

Notice that by the structure assumptions it follows that $g(s)$ belongs to $L^1_{loc}([t_0,+\infty))$. Thus, as before, choosing $\phi(t) = \int_\Omega |u(t) - v(t)|^2$ and applying Proposition 3.2 we obtain estimate (26). Finally, if $g \in L^1((t_0,+\infty))$, again by Proposition 3.2 we conclude that also (28) holds true. □

Remark 4.1. We point out that thanks to the results in [14], if (39)–(41) are satisfied and if $v_0 \in L^1(\Omega)$ and $f \in L^m_{loc}([0,+\infty); L^s(\Omega))$ with m and s satisfying (35), then there exists a unique global weak solution v of (38) belonging to the set $W \cap L^\infty(\Omega \times [t_0,+\infty))$, for every $t_0 > 0$. Hence it is possible to proceed exactly as in the proof of the previous Theorem showing that the results of Theorem 2.8 remain true (with v now unique regular solution of (38)) simply replacing g in all the statements with the following function

$$g(s) = \int_\Omega \frac{1}{\alpha}|M(x,s)\nabla v(x,s) - a(x,s,\nabla v(x,s))|^2\,dx + \\ + 2\int_\Omega [c(u,v)|a(x,s)| + c_0(u,v)|f(x,s) - f_0(x,s)|]\,dx\,.$$

4.8. Proof of Theorem 2.11

We observe that by Theorem 2.3 the unique global solution u in V of (1) belongs also to Y. Moreover, the unique solution $w \in H_0^1(\Omega) \cap L^\infty(\Omega)$ of (42) is also the unique global solution in V of the following evolution problem

$$\begin{cases} w_t - \operatorname{div}(M_1(x)\nabla w) + a_1(x)v = f_1(x) & \text{in } \Omega_\infty, \\ w(x,t) = 0 & \text{on } \Gamma_\infty, \\ w(x,0) = w(x) & \text{in } \Omega, \end{cases} \tag{90}$$

where $w(x,t) \equiv w(x)$. Thus, being w the unique global solution of (20) with $M_0(x,t) = M_1(x)$, $a_0(x,t) = a_1(x)$, $f_0(x,t) = f_1(x)$ and $v_0(x) = w(x)$, all the assumptions of Theorem 2.5 are satisfied with $v = w$. Now applying Theorem 2.5 the assertions follow with g as in (27) and hence with the constants $C_5 = C_3$ and $C_6 = C_4$ possibly depending also on $\|v_0\|_{L^1(\Omega)} = \|w\|_{L^1(\Omega)}$. To avoid this last dependence in C_5 and C_6 (and to obtain a sharper estimate) it is sufficient to observe that being now $v = w$ it results

$$\|w\|_{L^\infty(\Omega)} \le Q_1$$

(see [3]) and hence it is possible to replace the constants C_3 and C_4 in (87) with, respectively, C_5 and C_6 defined as follows

$$c_0(u,v) \le C_6 \equiv C_1 \max\left\{ \frac{\|u_0\|_{L^1(\Omega)}}{t_0^{\frac{N}{2}}}, Q \right\} + Q_1 \,,$$

and
$$c(u,v) \le C_5 \equiv Q_1 C_6 \,, \tag{91}$$

where we recall that $c(u,v)$ and $c_0(u,v)$ are as in (87). □

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