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Homogenization of the Elasticity Problem for a Material with Fractures

Riuji Sato, Bogdan Vernescu
Dept. of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, U.S.A.
risato@wpi.edu, vernescu@wpi.edu

Dedicated to Umberto Mosco on the occasion of his 80th birthday.

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We consider the stationary linear elasticity problem for a solid with fractures, and review, in the framework of Legendre-Fenchel duality, three equivalent formulations for the problem in terms of displacement, stress, and strain. For periodically distributed fractures, we prove a homogenization result using Mosco convergence in the L^2 topology.

Keywords: Homogenization, Mosco convergence, Gamma convergence, elasticity.

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1. Introduction

We consider a linear elasticity problem for a homogeneous solid with periodically distributed fractures. This problem was considered by Sanchez-Palencia in [10]. He proved that the problem, which we describe later, has a unique solution. Assuming that the solution has an asymptotic expansion, by formally taking limits, he obtained in the limit the homogenized problem *(without fractures)*. Properties of the homogenized stress are given in [10].

A few authors have considered proving using different methods this homogenization result. Attouch and Murat in [2] have considered a more general form of the problem but in the scalar case. Their stresses are assumed to be subgradients of a convex energy functional with other suitable properties, e.g., ellipticity, boundedness, etc. They proved that the energy functionals Γ-converge to the energy functional for the homogenized problem, in the strong $L^2(\Omega)$ topology. A major hurdle in this problem is to find a suitable space that will contain $H^1(\Omega_\epsilon)$ for all $\epsilon > 0$. $L^2(\Omega)$ does this but to accomplish Γ-convergence in this topology, one needs to prove that *(under certain conditions)* limit points of $\{u_\epsilon\}_\epsilon$, where $u_\epsilon \in H^1(\Omega_\epsilon)$, must be in $H^1(\Omega)$. To do this, the authors in [2] constructed a suitable restriction-extension operator that allowed them to prove this under the condition that $\sup_\epsilon \|u_\epsilon\|_{H^1(\Omega_\epsilon)} < +\infty$. The construction of the restriction-extension operator required that the fracture in the unit cell, Y, has a neighborhood with smooth boundary that is compactly contained in the cell. A challenge there is justifying the limits of certain functionals.

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In particular, taking limits of functions of the form,

$$\int_{\Omega_\epsilon} j(Z + \nabla w_Z(\frac{x}{\epsilon}))\, dx,$$

where w_Z is a periodic solution to a unit cell problem, given a constant tensor Z, as one cannot simply use weak convergence in $L^2(\Omega)$ to the average in the unit cell since the integrand is not necessarily in $L^2(\Omega)$, (*w_Z is only in $H^1(Y \setminus \Gamma)$*). To prove this assertion we use a partitioning of the domain into cells compactly contained in Ω and those touching the boundary. We adapt standard arguments to proving the *limsup inequality* from [1].

Taking a different approach, Pastukhova in [9], argued that $u_\epsilon \to u$ in $L^2(\Omega)$ and that the energies also converge, assuming that the limit point of the sequence must be in $H^1(\Omega)$. We show that this is true using properties of the restriction-extension operator found in [2]. We have also justified some limits using the partitioning previously discussed. Our proof of the liminf inequality is based on her arguments in [9].

In this paper we formulate the stationary linear elasticity problem for a solid with fractures, using a weighted trace space as in [5] and [6]. We discuss the dual formulations in terms of stess and strain form in the framework of Legendre-Fenchel duality developed in [11]. We prove the homogenization result using Γ-convergence in the strong $L^2(\Omega)$ topology *(for the vector-valued case)*. We prove that the Mosco-convergence in the $L^2(\Omega)$ topology also holds.

2. Problem formulation

Let Ω be an open bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The fracture is assumed to be a smooth surface which may or may not be connected, and is denoted by Γ. We define $\Omega_\Gamma := \Omega \setminus \Gamma$.

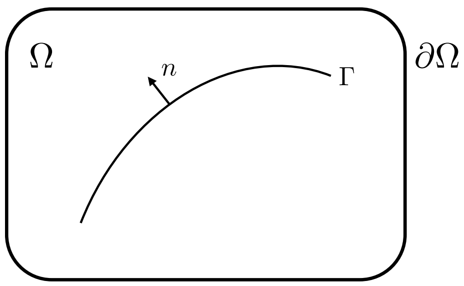

Figure 2.1: Elastic solid with fracture

The classical formulation of the problem in terms of displacements is:

find u such that:

$$\operatorname{div} \sigma + f = 0 \quad \text{in } \Omega_\Gamma \tag{1}$$

$$\sigma = A\nabla_S(u) \quad \text{in } \Omega_\Gamma \tag{2}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{3}$$

$$[u \cdot n] \geq 0 \quad \text{on } \Gamma \tag{4}$$

$$\sigma n|_1 = \sigma_{nn} n;\ \sigma n|_2 = -\sigma_{nn} n;\ \sigma_{nn} \leq 0 \quad \text{on } \Gamma \tag{5}$$

$$\text{if } [u \cdot n] > 0 \,\text{on}\, \Gamma, \text{then } \sigma_{nn} = 0. \tag{6}$$

Here, n refers to the unit normal on Γ, n is the outward unit normal on the boundary of Ω_Γ, $[\phi] = \phi|_1 - \phi|_2$ refers to the jump of the field ϕ across the fracture Γ, where the subscripts 1 and 2 denote the faces of F, $\sigma_{nn} = \sigma n \cdot n$. $A = [a_{ijkl}]$ is the elasticity tensor, assumed to have symmetry and positivity properties, i.e.,

$$AB : B > 0, \quad B \in \mathbb{R}^{3\times 3}, \tag{7}$$

$$a_{ijkl} = a_{ijlk} = a_{jikl} = a_{jilk}, \tag{8}$$

and f represents the body forces acting on the body. We denote by $\nabla_S(\cdot)$ the linearized strain tensor.

The constitutive relation for the elastic body is given by (2), (3) says that on the outer boundary, the displacement is fixed, (4) implies that the body cannot penetrate itself on the crack, (5) shows that there is no friction on the crack and there is compression on it. Finally, (6) says that if the crack is open, there are no stresses on Γ.

To define traces on the fracture, we follow the development in [6]. We assume that Γ can be extended into a smooth closed surface Σ that divides Ω into two disjoint sets and that Γ does not intersect itself. Observe that for a function $u \in H^1(\Omega \setminus \Gamma)$, we have that the jump of u, denoted by $[u]$, is zero in $\Sigma \setminus \Gamma$.

We introduce the following trace space:

$$H^{\frac{1}{2}}_{00}(\Gamma) := \{v \in H^{\frac{1}{2}}(\Gamma) \mid d^{-\frac{1}{2}} v \in L^2(\Gamma)\},$$

where $d \in C^{1,1}(\overline{\Sigma}_c)$, $d > 0$, $d = 0$ on $\partial\Gamma$, and $\lim_{x\to x_0} \frac{d(x)}{\operatorname{dist}(x,\partial\Gamma)} = \alpha \neq 0$ for every $x_0 \in \partial\Gamma$. $\operatorname{dist}(x, \partial\Gamma)$ refers to the distance from $x \in \Gamma$ to $\partial\Gamma$. This space is a Hilbert space with the norm

$$\|v\|^2_{00,\Gamma} = \|v\|^2_{\frac{1}{2},\Gamma} + \left\| d^{-\frac{1}{2}} v \right\|^2_{0,\Gamma}.$$

One can prove [6] that $u \in H^{\frac{1}{2}}_{00}(\Gamma)$ if and only if the extension function

$$\bar{u} := \begin{cases} u & \text{on } \Gamma, \\ 0 & \text{on } \Sigma \setminus \Gamma, \end{cases}$$

belongs to $H^{\frac{1}{2}}(\Sigma)$. This characterization will be used here to describe the jump $[u] \in H^{\frac{1}{2}}_{00}(\Gamma)$ on Γ for u in $H^1(\Omega_\Gamma)$. We define

$$\mathbb{H}(\operatorname{div}, \Omega_\Gamma) := \left\{\sigma \in L^2(\Omega_\Gamma) | \ \operatorname{div} \sigma \in L^2(\Omega)\right\}.$$

and denote the duality pairing on $H^{\frac{1}{2}}(\Sigma)$ and its dual by $\langle \cdot, \cdot \rangle_{\frac{1}{2},\Sigma}$. Similarly, the duality pairing between $H^{\frac{1}{2}}_{00}(\Gamma)$ and its dual is denoted by $\langle \cdot, \cdot \rangle_{00,\Gamma}$.

With the notations above, one can prove [6] the following trace theorem and Green's theorem to describe functions and functionals on the fracture.

Lemma 2.1. *Let the boundary Γ belong to the class $C^{0,1}$, and let a function u belong to $H^1(\Omega_\Gamma)$. Then there exists a linear continuous operator which uniquely defines at $\partial(\Omega_\Gamma)$ the values*

$$u|_{\partial\Omega} \in H^{\frac{1}{2}}(\partial\Omega), \ \ u|_1, u|_2 \in H^{\frac{1}{2}}(\Gamma), \ \ [u] \in H^{\frac{1}{2}}_{00}(\Gamma).$$

Conversely, there exists a linear continuous operator such that for any given

$$\psi \in H^{\frac{1}{2}}(\partial\Omega), \ \ \varphi|_1, \varphi|_2 \in H^{\frac{1}{2}}(\Gamma), \ \ [\varphi] \in H^{\frac{1}{2}}_{00}(\Gamma),$$

a function $u \in H^1(\Omega_\Gamma)$ can be found such that

$$u = \psi \text{ on } \partial\Omega, \ \ u|_i = \varphi|_i \text{ on } \Gamma, \ \ i = 1, 2.$$

Lemma 2.2. *Let the boundary $\partial(\Omega_\Gamma)$ belong to the class $C^{1,1}$, let σ belong to $\mathbb{H}(div, \Omega_\Gamma)$. Then there is a linear continuous operator $\mathbb{H}(div, \Omega) \to \left(H^{\frac{1}{2}}_{00}(\Gamma)\right)^*$ which uniquely defines on the crack Γ the values*

$$\sigma_n \in \left(H^{\frac{1}{2}}_{00}(\Gamma)\right)^*, \ \ \sigma_\tau \in \left(H^{\frac{1}{2}}_{00}(\Gamma)\right)^*, \ \ \sigma_\tau \cdot n = 0,$$

and for all $v \in V$, the generalized Green formula holds:

$$\int_\Omega \sigma : \nabla_S(v)\, dx = -\int_\Omega div\ \sigma \cdot v\, dx - \langle \sigma_n, [v_n] \rangle_{00,\Gamma} - \langle \sigma_\tau, [v_\tau] \rangle_{00,\Gamma} \tag{9}$$

We can now define the following spaces:

$$V := \{v \in H^1(\Omega_\Gamma) \mid v = 0 \text{ in } H^{\frac{1}{2}}(\partial\Omega)\},$$

$$K := \{v \in V \mid [v_n] \geq 0 \text{ on } H^{\frac{1}{2}}_{00}(\Gamma)).$$

The problem (1)–(6) describing a linear elastic solid with a fracture is then shown to be equivalent to the following variational formulation:
Find $u \in K$ such that

$$\int_{\Omega_\Gamma} A\nabla_S(u) : \nabla_S(v-u)\, dx \geq \int_{\Omega_\Gamma} f \cdot (v-u)\, dx \ \ \forall v \in K. \tag{10}$$

Sanchez-Palencia in [10] showed that a unique solution to problem (10) exists. It is standard to show that the above variational inequality is equivalent to the following minimization problem:

Problem 2.3. (Displacement formulation) *Find $\bar{v} \in K$ such that*

$$j(\bar{v}) = \inf_{v \in K_\Gamma} j(v),$$

where $j(v) := \frac{1}{2}\int_{\Omega_\Gamma} A\nabla_S(v) : \nabla_S(v) - \int_{\Omega_\Gamma} f \cdot v.$

3. Dual formulations

In addition to the displacement formulation, it is known that one can reformulate the problem describing the behavior of a linear elastic body in terms of stress, or in terms of strain. In [3], using the Legendre-Fenchel duality theory, it is shown that these formulations are dual problems of each other.

Similarly, for the linear elastic solid with fractures one can use the Legendre-Fenchel duality theory to obtain stress and strain formulations. We extended the results in [3], in a forthcoming paper [11], to the case of a linear elastic solid with fractures. We first describe the analog of the stress formulation.

To do this, we define the set of admissible stresses, denoted by $\mathbb{S}$, as

$$\mathbb{S} = \left\{ \sigma \in \mathbb{H}_S(\mathrm{div}, \Omega_\Gamma) \;\middle|\; \begin{array}{l} \mathrm{div}\ \sigma + f = 0 \text{ in } L^2(\Omega_\Gamma),\ \sigma_n \leq 0 \text{ on } \left(H_{00}^{\frac{1}{2}}(\Gamma)\right)^* \\ \sigma_\tau = 0 \text{ on } \left(H_{00}^{\frac{1}{2}}(\Gamma)\right)^*,\ [\sigma_n] = 0 \text{ on } \left(H^{\frac{1}{2}}(\Gamma)\right)^* \end{array} \right\},$$

where $\mathbb{H}_S(\mathrm{div}, \Omega_\Gamma)$ is the set of symmetric tensors in $\mathbb{H}(\mathrm{div}, \Omega_\Gamma)$. We now have the *stress* formulation of the elasticity problem with fractures:

Problem 3.1. (Stress formulation) *Find $\bar{\sigma} \in \mathbb{S}$ such that*

$$g(\bar{\sigma}) = \inf_{\sigma \in \mathbb{S}} g(\sigma), \quad \textit{where} \;\; g(\sigma) := \frac{1}{2} \int_{\Omega_\Gamma} B\sigma : \sigma.$$

To describe the strain problem, we define

$$\mathbb{M} := \{\sigma \in \mathbb{H}_S(\mathrm{div}, \Omega_\Gamma) \mid \langle \nabla_S^*(\sigma), v \rangle_{V^*,V} \geq 0 \ \forall v \in K\},$$

and let $\mathbb{M}^+$ be its polar cone in V and $\nabla_S^*(\cdot)$ is the adjoint of $\nabla_S(\cdot)$. We now have the analog of the strain formulation for the elasticity problem with fractures.

Problem 3.2. (Strain formulation) *Find $\bar{u} \in \mathbb{M}^+$ such that*

$$\tilde{j}(\bar{\mu}) = \inf_{\mu \in \mathbb{M}^+} \tilde{j}(\mu),$$

where $\tilde{j}(\mu) := \dfrac{1}{2} \displaystyle\int_{\Omega_\Gamma} A\mu : \mu - \int_{\Omega_\Gamma} f \cdot \mathcal{L}\mu$ *and $\mathcal{L}$ is a vector in K such that $\nabla_S(\mathcal{L}) = \mu$.*

We prove in [11] that the image of K_Γ under the strain operator is $\mathbb{M}^+$. This shows that $\mathbb{M}^+$ is indeed the set of admissible strains for the elasticity problem with fractures.

Finally, in [11], we show that, in the framework of Fenchel duality, these minimization problems are, up to a change in sign, dual to each other. Indeed, we have

Theorem 3.3. $\quad -\inf_{v \in K} j(v) = \inf_{\sigma \in \mathbb{S}} g(\sigma) = -\inf_{\mu \in \mathbb{M}^+} \tilde{j}(\mu).$

We also prove the following relationship among the minimizers of these problems.

Theorem 3.4. *Let $\bar{\sigma} \in \mathbb{S}$, $\bar{v} \in K$, and $\bar{\mu}$ such that*

$$g(\bar{\sigma}) = \inf_{\sigma \in \mathbb{S}} g(\sigma), \quad j(\bar{v}) = \inf_{v \in K} j(v), \quad \tilde{j}(\bar{\mu}) = \inf_{\mu \in \mathbb{M}^+} \tilde{j}(\mu).$$

Then

$$\bar{\sigma} = A\nabla_S(\bar{v}) = A\bar{\mu}. \tag{11}$$

We note that since in this case, the solution space for the minimization problems is a convex set, rather than a linear one as that in [3], extending those results is not straightforward. More details can be found in [11].

4. Periodic problem

In this section we describe the periodic problem. Let $Y := (0,1)^3$ be the unit cell in $\mathbb{R}^3$. Let $\Gamma \subset Y$ be a smooth surface such that there exists an open neighborhood $\eta \subset\subset Y$ with smooth boundary containing Γ.

We denote by $\mathbb{T}_\epsilon := \{z \in \mathbb{Z}^3 | \epsilon(Y+z) \subset\subset \Omega\}$. Let $\Gamma_\epsilon := \cup_{z\in\mathbb{T}_\epsilon} \epsilon(\Gamma + z)$ denote the periodically distributed fractures in Ω, and $\Omega_\epsilon := \Omega \setminus \Gamma_\epsilon$. The trace theorem can be extended naturally to this case, hence we can define similar function spaces:

$$V_\epsilon = \{v \in H^1(\Omega_\epsilon) \mid v = 0 \,\text{in}\, H^{\frac{1}{2}}(\partial\Omega)\},$$
$$K_\epsilon = \{v \in V_\epsilon \mid [v_n] \geq 0 \,\text{in}\, H^{\frac{1}{2}}_{00}(\Gamma_\epsilon)\}.$$

Using similar methods as those in the case for a single fracture, one can prove that a unique solution exists to the problem: Find $u_\epsilon \in K_\epsilon$ such that

$$\int_{\Omega_\epsilon} A\nabla_S(u_\epsilon) : \nabla_S(v-u)\, dx \geq \int_{\Omega_\epsilon} f\cdot(v-u_\epsilon)\, dx \;\; \forall v \in K_\epsilon. \tag{12}$$

This can be written equivalently as the following minimization problem:

$$\min_{v\in K_\epsilon} \left\{ \frac{1}{2}\int_{\Omega_\epsilon} A\nabla(v) : \nabla(v) - \int_{\Omega_\epsilon} f\cdot v \right\}. \tag{13}$$

We are now interested in the limit as $\epsilon \to 0$. Sanchez-Palencia [10] used an asympotic expansion for u_ϵ of the form

$$u_\epsilon(x) = u_0(x, \frac{x}{\epsilon}) + \epsilon u_1(x, \frac{x}{\epsilon}) + \epsilon^2 u_2(x, \frac{x}{\epsilon}) + ...$$

and calculated formally the limit problem. The condition that $u_\epsilon \in K_\epsilon$ implies that $u_0(x,y) = u_0(x)$ and that $[u_1 \cdot n] \geq 0$ on Γ. Moreover, formally letting $\epsilon \to 0$, he obtains that $u_0 \in H^1_0(\Omega)$ is a solution to the homogenized problem:

$$\text{div}\; \bar{\sigma}^\circ(\nabla u_0) + f = 0, \;\text{in}\; \Omega, \quad \text{where}\;\; \bar{\sigma}^\circ(\nabla u_0) := \int_Y A(\nabla_x u_0 + \nabla_y u_1)\, dy,$$

and $u_1 \in H^1_0(Y\setminus\Gamma)$ solves the unit cell problem:

$$\int_{Y\setminus\Gamma} A(\nabla_x u_0 + \nabla_y u_1)\nabla_y(w-u_1)\, dy \geq 0, \tag{14}$$

for all $w \in H^1_0(Y\setminus\Gamma)$ such that $[w\cdot n] \geq 0$ on Γ.

The homogenized problem is then equivalent to the following minimization problem:

$$\min_{v\in H^1_0(\Omega)} \left\{ \frac{1}{2}\int_\Omega \bar{\sigma}^\circ(\nabla_S(v)) : \nabla_S(v) \right\} \tag{15}$$

Our goal now is to obtain a homogenization result by proving Mosco convergence [7] of the energy functionals found in (13) and (15).

5. Auxiliary lemmas

We list some lemmas which will be used to prove our main result.

Lemma 5.1. [2] *For any sequence* $\{u_\epsilon\}_{\epsilon>0}$ *satisfying* $\sup_{\epsilon>0} \|u_\epsilon\|_{H^1(\Omega_\epsilon)} < \infty$*, there exists a bounded sequence* $\{Q_\epsilon(u_\epsilon)\}_{\epsilon>0}$ *in* $H^1(\Omega)$ *such that*

$$\lim_{\epsilon\to 0} \|u_\epsilon - Q_\epsilon(u_\epsilon)\|_{L^2(\Omega)} = 0.$$

Remark 5.2. The assumption that the Γ has a neighborhood η with smooth boundary that is compactly contained in Y allowed the authors in [2] to construct Q_ϵ. Their approach is to construct a restriction-extension operator that first restricts a function defined on $Y \setminus \Gamma$ on $Y \setminus \eta$ and extend it to the whole of Y. Doing the appropriate scaling and translations, one obtains the operator $Q_\epsilon \colon H^1(\Omega_\epsilon) \to H^1(\Omega)$. $\square$

Following [9], we define $C^\infty_{per}(Y_\Gamma)$ to be the set of smooth 1-periodic functions defined on $\mathbb{R}^N \setminus \Gamma_\epsilon$. We set $L^2_{per}(Y_\Gamma)$ to be the closure of this set in $L^2(Y)$. We then define $H^1_{per}(Y_\Gamma)$ to be the closure of the set of functions in $C^\infty_{per}(Y_\Gamma)$ that has support outside a neighborhood of Γ with respect to the $H^1(Y)$ norm. We then have the following lemmas from [9]:

Lemma 5.3. *Suppose that* $a(y) \in L^2_{per}(Y \setminus \Gamma)$ *and* $\bar{a} = \int_{Y\setminus\Gamma} a\, dy = 0$. *Then there exists* $w \in H^1_{per}(Y, \Gamma)$ *such that div* $w(y) = a(y)$ *and* $\|w\|_{H^1(Y)} \leq C(\Gamma)\|a\|_{L^2(Y)}$.

Lemma 5.4. *If* $u_E(y)$ *is a solution of the variational inequality* (14) *for a given tensor* E *with constant components, then the tensor function* Z *such that* $Z = A(E + \nabla_S(u_E))$ *has the following properties:*

$$div\, Z = 0 \quad in\, Y \setminus \Gamma,$$

and the orthogonal decomposition of the vector $Z_n = Zn = Z_{nn}n + Z_\tau$ *satisfies the following conditions on* Γ, *where* $\bar{\sigma}^\circ(E) = \bar{Z} = \int_Y Z\, dy$:

$$Z_n|_1 = Z_{nn}|_1 n, \quad Z_n|_2 = -Z_{nn}|_1 n, \quad Z_{nn}|_1 \leq 0;$$

$$\int_{Y\setminus\Gamma} A(E + \nabla_S(u_E)) : (E + \nabla_S(u_E))\, dy = \bar{\sigma}^\circ(E)E = \bar{Z}E.$$

6. Homogenization result

We extended the proof of the *limsup inequality* found in [2] to the vector-valued case, specialized to our particular case of linear elasticity. We provided necessary justifications for some of the convergences of the energy functionals. The proof of the *liminf inequality* is adapted from [9]. We adjusted her proof to our case and used Lemma 5.1 in some of the arguments. More details can be found below.

Theorem 6.1. *Define for* $v \in L^2(\Omega)$,

$$J_\epsilon(v) := \frac{1}{2}\int_{\Omega_\epsilon} A\nabla_S(v) : \nabla_S(v) - \int_{\Omega_\epsilon} f \cdot v + \chi_{K_\epsilon}(v)$$

$$J_{hom}(v) := \frac{1}{2}\int_{\Omega} \bar{\sigma}^\circ(\nabla_S(v))\nabla_S(v) - \int_{\Omega} f \cdot v + \chi_{H^1_0(\Omega)}(v).$$

Then $J_{hom} = \Gamma - \lim J_\epsilon$, *in the strong* $L^2(\Omega)$ *topology.*

Proof. Observe that the map $v \mapsto \int_{\Omega_\epsilon} f \cdot v = \int_\Omega f \cdot v$ is continuous on $L^2(\Omega)$. Hence if we show that $F_{hom} = \Gamma - \lim F_\epsilon$ in the strong $L^2(\Omega)$ topology, where

$$F_\epsilon(v) := \frac{1}{2}\int_{\Omega_\epsilon} A\nabla_S(v) : \nabla_S(v) + \chi_{K_\epsilon}(v), \quad F_{hom}(v) := \frac{1}{2}\int_{\Omega} \bar{\sigma}^\circ(\nabla_S(v))\nabla_S(v) + \chi_{H^1_0(\Omega)}(v),$$

then the assertion of the theorem follows.

We first prove the *limsup inequality*, i.e., for every $v \in L^2(\Omega)$, there is a sequence $\{v_\epsilon\}_{\epsilon>0}$ in $L^2(\Omega)$ such that

$$\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq F_{hom}(v), \tag{16}$$

As $F_{hom}(v) = +\infty$ for $v \notin H_0^1(\Omega)$, it suffices to prove the inequality for $v \in H_0^1(\Omega)$. We proceed in several steps.

Step 1. Suppose v is affine, i.e., for some $Z \in \mathbb{R}^{N\times N}$ and $\alpha \in \mathbb{R}^N$,

$$v(x) = Zx + \alpha.$$

For $Z \in \mathbb{R}^{N\times N}$, let $w_Z \in H_0^1(Y \setminus \Gamma)$ be the unique solution of

$$\int_{Y\setminus\Gamma} A(Z + \nabla w_Z) : \nabla(w - w_Z)\, dy \geq 0,$$

for all $w \in H_0^1(Y \setminus \Gamma)$ such that $[w \cdot n] \geq 0$ on Γ.

We define $v_\epsilon(x) := v(x) + \epsilon w_Z(\frac{x}{\epsilon})$. As $[w_Z \cdot n] \geq 0$ on Γ and $v \in H_0^1(\Omega)$, we have that $v_\epsilon \in K_\epsilon$. Also, observe that $w_Z(\frac{x}{\epsilon}) \rightharpoonup \int_{Y\setminus\Gamma} w_Z(y)\, dy$ weakly in $L^2(\Omega)$ and hence is bounded in $L^2(\Omega)$. Thus, $v_\epsilon \to v$ in $L^2(\Omega)$.

Observe that we can find a finite number of translates of ϵY with disjoint interiors, $\{Y_i\}_{i=1}^{N(\epsilon)}$ and $\{Y_i'\}_{i=1}^{N'(\epsilon)}$ (see Figure 6.1) such that

$$\Omega \subset \left(\cup_{i=1}^{N(\epsilon)} Y_i\right) \cup \left(\cup_{i=1}^{N'(\epsilon)} Y_i'\right),\ Y_i \subset\subset \Omega\ i = 1, ..., N(\epsilon),\ Y_i' \cap \partial\Omega \neq \emptyset\ i = 1, ..., N'(\epsilon).$$

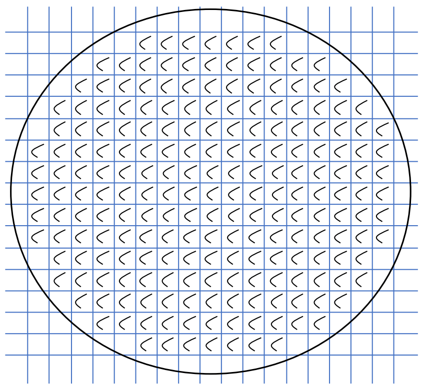

Figure 6.1:

Denoting by $Y_{i\epsilon}$ and $Y_{i\epsilon}$ the ϵY translates without the fractures, we now calculate,

$$\begin{aligned} F_\epsilon(v_\epsilon) &= \frac{1}{2}\int_{\Omega_\epsilon} A\nabla_S(v_\epsilon) : \nabla_S(v_\epsilon) = \frac{1}{2}\int_{\Omega_\epsilon} A\nabla(v_\epsilon) : \nabla(v_\epsilon) \\ &= \frac{1}{2}\int_{\Omega_\epsilon} A\left(Z + \nabla w_Z\left(\frac{x}{\epsilon}\right)\right) : \left(Z + \nabla w_Z\left(\frac{x}{\epsilon}\right)\right) \\ &\leq \frac{1}{2}\sum_i \left(\int_{Y_{i\epsilon}} A(Z + \nabla w_Z(\frac{x}{\epsilon})) : (Z + \nabla w_Z(\frac{x}{\epsilon}))\right) \end{aligned}$$

$$+\frac{1}{2}\sum_i\left(\int_{Y'_{i_\epsilon}} A(Z+\nabla w_Z(\frac{x}{\epsilon})):(Z+\nabla w_Z(\frac{x}{\epsilon}))\right)$$
$$\leq \frac{1}{2}\left(N(\epsilon)\epsilon^N + N'(\epsilon)\epsilon^N\right)\int_{Y\setminus\Gamma} A(Z+\nabla w_Z(\frac{x}{\epsilon})):(Z+\nabla w_Z(\frac{x}{\epsilon})),$$

where we have used a change of variables to obtain the last inequality. Similar to arguments found in (Theorem 2.6, [4]), we have that

$$N(\epsilon)\epsilon^N \to \frac{|\Omega|}{|Y|} = |\Omega| \quad\text{and}\quad N'(\epsilon)\epsilon^{N-1} \leq N\frac{|\Omega|}{|Y|} = N|\Omega|.$$

Thus, it follows that

$$\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq \frac{1}{2}|\Omega|\int_{Y\setminus\Gamma} A(Z+\nabla w_Z(\frac{x}{\epsilon})):(Z+\nabla w_Z(\frac{x}{\epsilon}))$$
$$= \frac{1}{2}\int_\Omega\int_{Y\setminus\Gamma} A(Z+\nabla w_Z(\frac{x}{\epsilon})):(Z+\nabla w_Z(\frac{x}{\epsilon})) = F_{hom}(v).$$

Step 2. Suppose v is a continuous piecewise affine function, i.e.,

$$v(x) = \sum_{i=1}^{l}(Z_i x + \alpha_i)\,\mathbb{1}_{\Omega_\epsilon^i}(x),$$

where Ω_ϵ^i forms a partition of Ω_ϵ.

We set $v_\epsilon^i := v(x) + \epsilon w_{Z_i}(\frac{x}{\epsilon})$ in Ω_ϵ^i. As $\nabla w_{Z_i}(\frac{\cdot}{\epsilon})$ is not necessarily equal to $\nabla w_{Z_j}(\frac{\cdot}{\epsilon})$ on the interface between Ω_ϵ^i and Ω_ϵ^j, we introduce smooth cut-off functions to obtain an appropriate sequence in $H^1(\Omega_\epsilon)$. We do this in the case $l = 2$. The general case follows similarly. Let $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$. For $\delta > 0$ small enough, we define $\Sigma_\delta := \{x \in \Omega | d(x,\Sigma) < \delta\}$. Let $\varphi_\delta \in C_0^\infty(\Omega)$ such that $0 \leq \varphi_\delta \leq 1$ and

$$\varphi_\delta = \begin{cases} 1 & \text{in } \Sigma_\delta, \\ 0 & \text{in } \Omega\setminus\Sigma_{2\delta}. \end{cases}$$

Set $v_\epsilon^\delta := (1-\varphi_\delta)v_\epsilon^i + \varphi_\delta v$. Observe that $v_\epsilon^\delta = v$ in Σ_δ and $[v_\epsilon^\delta\cdot n] = (1-\varphi_\delta)[w_{Z_i}\cdot n] \geq 0$. Hence, $v_\epsilon^\delta \in K_\epsilon$. Note that,

$$\nabla v_\epsilon^\delta = v_\epsilon^i \otimes \nabla(1-\varphi_\delta) + (1-\varphi_\delta)\nabla v_\epsilon^i + v\otimes\nabla\varphi_\delta + \varphi_\delta\nabla v.$$

Using the convexity of the map $Z \to AZ : Z$ for $Z \in \mathbb{R}^{N\times N}$, we have for $0 < t < 1$,

$$F_\epsilon(tv_\epsilon^\delta) = \frac{1}{2}\sum_i\int_{\Omega_\epsilon^i} A\left(t(1-\varphi_\delta)\nabla v_\epsilon^i + t\varphi_\delta\nabla v + (1-t)\frac{t}{1-t}(v-v_\epsilon^i)\otimes\nabla\varphi_\delta\right):$$
$$\left(t(1-\varphi_\delta)\nabla v_\epsilon^i + t\varphi_\delta\nabla v + (1-t)\frac{t}{1-t}(v-v_\epsilon^i)\otimes\nabla\varphi_\delta\right)dx,$$
$$\leq \frac{1}{2}\sum_i\int_{\Omega_\epsilon^i} t(1-\varphi_\delta)A\nabla v_\epsilon^i:\nabla v_\epsilon^i + t\varphi_\delta A\nabla v:\nabla v$$
$$+ (1-t)A\left(\frac{t}{1-t}(v-v_\epsilon^i)\otimes\nabla\varphi_\delta\right):\left(\frac{t}{1-t}(v-v_\epsilon^i)\otimes\nabla\varphi_\delta\right)dx,$$

$$\leq \frac{1}{2}\sum_i \int_{\Omega_\epsilon^i} A\nabla v_\epsilon^i : \nabla v_\epsilon^i + \int_{\Omega_\epsilon^i} A\nabla v : \nabla v$$
$$+ (1-t)\int_{\Omega_\epsilon^i} A\left(\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right) : \left(\frac{t}{1-t}(v - v_\epsilon^i)\otimes\nabla\varphi_\delta\right).$$

As $v_\epsilon^i \to v$ in $L^2(\Omega)$, we get

$$\limsup_{\epsilon\to 0} F_\epsilon(tv^{\delta_\epsilon})$$
$$\leq \frac{1}{2}\sum_i \int_\Omega \int_{Y\setminus\Gamma} A\left(Z_i + \nabla w_{Z_i}(y)\right) : \left(Z_i + \nabla w_{Z_i}(y)\right)\, dy + \frac{1}{2}\int_{\Sigma_{2\delta}} A\nabla v : \nabla v,$$

where we used *Step 1* to get the first term of the right-hand side of the inequality. As $v \in H^1(\Omega)$, we have that

$$\int_{\Sigma_{2\delta}} A\nabla v : \nabla v \to 0, \quad as\ \delta \to 0.$$

Thus,

$$\limsup_{\substack{\delta\to 0\\ t\to 1}} \limsup_{\epsilon\to 0} F_\epsilon(tv_\epsilon^\delta) \leq \frac{1}{2}\sum_i \int_\Omega \int_{Y\setminus\Gamma} A\left(Z_i + \nabla w_{Z_i}(y)\right) : \left(Z_i + \nabla w_{Z_i}(y)\right)\, dy$$
$$= F_{hom}(v).$$

By a standard diagonalization argument, we find a sequence $\delta(\epsilon) \to 0$ and $t(\epsilon) \to 1$ as $\epsilon \to 0$ such that

$$\limsup_{\epsilon\to 0} F(t(\epsilon)v_\epsilon^{\delta(\epsilon)}) \leq \limsup_{\substack{\delta\to 0\\ t\to 1}} \limsup_{\epsilon\to 0} F_\epsilon(tv_\epsilon^\delta).$$

Hence, taking $v_\epsilon := t(\epsilon)v_\epsilon^{\delta(\epsilon)}$, we obtain $\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq F_{hom}(v)$.

Step 3. We argue using density that (16) holds for $v \in H_0^1(\Omega)$. Indeed, let $v \in H_0^1(\Omega)$. Then there exists a sequence of continuous piecewise affine functions, $\{v_k\}_{k=1}^\infty$, that converges to v in $H^1(\Omega)$. From *Step 2*, for each k there is a sequence $\{v_{k,\epsilon}\}_{\epsilon>0}$ such that $v_{k,\epsilon} \to v_k$ in $L^2(\Omega)$ as $\epsilon \to 0$ and $\limsup_{\epsilon\to 0} F_\epsilon(v_{k,\epsilon}) \leq F_{hom}(v_k)$. Arguing similarly as in (Theorem 1.20, [1]), it can be shown that F_{hom} is convex and finitely valued on $H_0^1(\Omega)$ and hence is continuous. Thus, we obtain

$$F_{hom}(v) = \lim_{k\to\infty} F_{hom}(v_k) \geq \limsup_{k\to\infty} \limsup_{\epsilon\to 0} F_\epsilon(v_{k,\epsilon}) \geq \limsup_{\epsilon\to 0} F_\epsilon(v_{k(\epsilon),\epsilon}),$$

where the last inequality follows from a diagonalization argument. Taking $v_\epsilon := v_{k(\epsilon),\epsilon}$, we get the desired result.

We now prove the *liminf inequality*, i.e., if $v \in L^2(\Omega)$ and $\{v_\epsilon\}_{\epsilon>0}$ is a sequence in $L^2(\Omega)$ such that $v_\epsilon \to v$ in $L^2(\Omega)$, then

$$F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon). \tag{17}$$

If $\liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon) = +\infty$, we are done. Suppose now that $\liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq M$ for some $M > 0$. We first need to show that $F_{hom}(v) < \infty$, i.e., $v \in H_0^1(\Omega)$. By the assumption, there is a subsequence such that $F_{\epsilon'}(v_{\epsilon'}) \leq M$ for all ϵ'. Necessarily, $v_{\epsilon'}$ belongs to $K_{\epsilon'}$ for each ϵ'. By coercivity of $F_{\epsilon'}$, we obtain

$$\sup_{\epsilon'} \|v_{\epsilon'}\|_{H^1(\Omega_\epsilon)} < \infty.$$

Using Lemma 5.1, we obtain a bounded sequence $\{Q_{\epsilon'}(v_{\epsilon'})\}_{\epsilon'}$ in $H^1(\Omega)$ such that

$$\|v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'})\|_{L^2(\Omega)} \to 0, \quad as\ \epsilon' \to 0.$$

As $H^1(\Omega) \subset\subset L^2(\Omega)$, up to a subsequence, for some $u \in H^1(\Omega)$, $Q_{\epsilon'}(v_{\epsilon'}) \to u$ in $L^2(\Omega)$. Then,

$$\|v - u\|_{L^2(\Omega)} \leq \|v - v_{\epsilon'}\|_{L^2(\Omega)} + \|v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'})\|_{L^2(\Omega)} + \|Q_{\epsilon'}(v_{\epsilon'}) - u\|_{L^2(\Omega)} \to 0,$$

Hence, $v \in H^1(\Omega)$. As v_ϵ has trace zero on $\partial\Omega$ for every ϵ, we have that $v \in H_0^1(\Omega)$. This guarantees that $F_{hom}(v) < \infty$. We now show that $F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon)$. Indeed, let E be an arbitrary symmetric tensor in $\mathbb{R}^{N\times N}$. Let Z be as in Lemma 5.4. We define $Z_\epsilon(x) := Z(\frac{x}{\epsilon})$. By Young's inequality, we obtain

$$\frac{1}{2} A\nabla v_\epsilon : \nabla_\epsilon \geq Z_\epsilon : \nabla v_\epsilon - \frac{1}{2} BZ_\epsilon : Z_\epsilon.$$

Let $\omega \subset\subset \Omega$ be a subdomain and $\varphi \in C_0^\infty(\omega)$, $0 \leq \varphi \leq 1$. Set $\omega_\epsilon := \omega \setminus \Gamma_\epsilon$. Then,

$$\frac{1}{2}\int_{\omega_\epsilon} \varphi A\nabla v_\epsilon : \nabla_\epsilon \geq \int_{\omega_\epsilon} \varphi Z_\epsilon : \nabla v_\epsilon - \frac{1}{2}\int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon. \tag{18}$$

Now, using Lemma 5.4 and Lemma 2.2 *(Green's formula)*, we obtain

$$\begin{aligned}
&\int_{\omega_\epsilon} \varphi Z_\epsilon : \nabla v_\epsilon = \int_{\omega_\epsilon} Z_\epsilon : (\nabla(\varphi v_\epsilon) - v_\epsilon \otimes \nabla\varphi) \\
&= -\int_{\omega_\epsilon} \operatorname{div} Z_\epsilon \cdot (\varphi v_\epsilon) - \langle (Z_\epsilon)_n, [\varphi v_\epsilon]_n\rangle_{\frac{1}{2},\Gamma_\epsilon} - \langle (Z_\epsilon)_\tau, [\varphi v_\epsilon]_\tau\rangle_{\frac{1}{2},\Gamma_\epsilon} - \int_{\omega_\epsilon} Z_\epsilon : (v_\epsilon \otimes \nabla\varphi) \\
&\geq -\int_{\omega_\epsilon} \nabla\varphi Z_\epsilon v_\epsilon \, dx = -\int_{\omega_\epsilon} \nabla\varphi \left(Z_\epsilon - \bar{Z}\right) v_\epsilon - \bar{Z}\int_{\omega_\epsilon} \nabla\varphi v_\epsilon.
\end{aligned}$$

Using Lemma 5.3, we obtain $W \in H^1_{per}(Y, \Gamma)$ such that $\|W\|_{H^1(Y)} \leq C(\Gamma)$ and

$$\operatorname{div}_y W(y) = Z(y) - \bar{Z}.$$

Thus,
$$Z_\epsilon(x) - \bar{Z} = \operatorname{div}_y W(y)|_{y=\frac{x}{\epsilon}} = \operatorname{div}_x W(\frac{x}{\epsilon}).$$

Since,
$$\begin{aligned}
\int_{\omega_\epsilon} \left(\operatorname{div}_x W\left(\frac{x}{\epsilon}\right)\right)^2 &\leq C\int_{\omega_\epsilon} \left|\nabla W(\frac{x}{\epsilon})\right|^2 = C\epsilon^N \int_Y |\nabla W(y)|^2 \, dy \\
&\leq C\int_Y |\nabla W(y)|^2 \, dy,
\end{aligned}$$

it follows that $\int_{\omega_\epsilon} \nabla\varphi \left(Z_\epsilon - \bar{Z}\right) v_\epsilon = \int_{\omega_\epsilon} \nabla\varphi \left(\epsilon \mathrm{div}_x W(\frac{x}{\epsilon})\right) v_\epsilon \to 0.$

As $v_\epsilon \to v$ in $L^2(\Omega)$,

$$\bar{Z} \int_{\omega_\epsilon} \nabla\varphi v_\epsilon = \bar{Z} \int_\omega \nabla\varphi v_\epsilon \to \bar{Z} \int_\omega \nabla\varphi v = -\bar{Z} \int_\omega \varphi\nabla v. \tag{19}$$

We thus obtain,
$$\liminf_{\epsilon\to 0} \int_{\omega_\epsilon} \varphi Z_\epsilon : \nabla v_\epsilon \geq \bar{Z} \int_\omega \varphi\nabla v. \tag{20}$$

Using the definition of Z and the technique of partitioning the cells into those inside the domain and those intersecting the boundary, we get

$$\begin{aligned}
\int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon &= \int_{\omega_\epsilon} A\left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) : \left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) \\
&\leq \sum_i \int_{Y_{i\epsilon}} A\left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) : \left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) \\
&\qquad + \sum_i \int_{Y'_{i\epsilon}} A\left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) : \left(E + \nabla u_E\left(\frac{x}{\epsilon}\right)\right) \\
&\leq \epsilon^N \left(N(\epsilon) + N'(\epsilon)\right) \int_{Y\setminus\Gamma} A\left(E + \nabla u_E(y)\right) : \left(E + \nabla u_E(y)\right) dy,
\end{aligned}$$

where $N(\epsilon)$ and $N'(\epsilon)$ counts the cells inside ω and those intersecting $\partial\omega$, respectively. Thus, we have

$$\begin{aligned}
\limsup_{\epsilon\to 0} \int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon &\leq |\omega| \int_{Y\setminus\Gamma} A\left(E + \nabla u_E(y)\right) : \left(E + \nabla u_E(y)\right) dy \\
&= \int_\omega \int_{Y\setminus\Gamma} A\left(E + \nabla u_E(y)\right) : \left(E + \nabla u_E(y)\right) dy\, dx = \int_\omega \bar{Z} E\, dx.
\end{aligned}$$

Combining this with (18) and (20), we obtain

$$\liminf_{\epsilon\to 0} \frac{1}{2} \int_{\omega_\epsilon} \varphi A\nabla v_\epsilon : \nabla v_\epsilon \geq \int_\omega \varphi\bar{Z}\nabla v - \liminf_{\epsilon\to 0} \frac{1}{2} \int_{\omega_\epsilon} BZ_\epsilon : Z_\epsilon \geq \int_\omega \varphi\bar{Z}\nabla v - \frac{1}{2}\int_\omega \bar{Z}E.$$

As φ is arbitrary, it holds that for any subdomain ω,

$$\liminf_{\epsilon\to 0} \frac{1}{2} \int_{\omega_\epsilon} A\nabla v_\epsilon : \nabla v_\epsilon \geq \int_\omega \bar{Z}\nabla v - \frac{1}{2}\int_\omega \bar{Z}E = \int_\omega \bar{\sigma}^\circ(E)\nabla v - \frac{1}{2}\int_\omega \bar{\sigma}^\circ(E)E.$$

If g is a continuous piecewise affine function, we can extend the above estimate to

$$\liminf_{\epsilon\to 0} \frac{1}{2} \int_{\omega_\epsilon} A\nabla v_\epsilon : \nabla v_\epsilon \geq \int_\omega \bar{\sigma}^\circ(\nabla_S(g))\nabla v - \frac{1}{2}\int_\omega \bar{\sigma}^\circ(\nabla_S(g))\nabla_S(g). \tag{21}$$

Since $v \in H_0^1(\Omega)$, we can choose a sequence of continuous piecewise affine functions $\{g^\delta\}_{\delta>0}$ such that $g^\delta \to v$ in $H^1(\Omega)$. The map $E \mapsto \bar{\sigma}^\circ(E)$ is Lipschitz continuous on $\mathbb{R}^{N\times N}$ (Theorem 7.2, [10]).

Thus, we have $\bar{\sigma}^\circ(\nabla_S(g^\delta)) \to \bar{\sigma}^\circ(\nabla_S(v))$ in $L^2(\Omega)$, which gives together with (21)

$$\liminf_{\epsilon\to 0} \frac{1}{2}\int_{\omega_\epsilon} A\nabla v_\epsilon : \nabla v_\epsilon \geq \frac{1}{2}\int_\omega \bar{\sigma}^\circ(\nabla_S(v))\nabla_S(v),$$

i.e., $F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon)$. Hence,

$$F_{hom} = \Gamma - \lim\ F_\epsilon,$$

in the strong $L^2(\Omega)$ topology, and thus

$$J_{hom} = \Gamma - \lim\ J_\epsilon,$$

in the strong $L^2(\Omega)$ topology. □

Moreover, we have the following result.

Theorem 6.2. *J_ϵ Mosco converges to J_{hom} in the $L^2(\Omega)$ topology.*

Proof. As Mosco convergence is stable under continuous perturbations, it suffices to prove the following:

limsup inequality: For each v in $L^2(\Omega)$, there exists a sequence $\{v_\epsilon\}_{\epsilon>0}$ belonging to $L^2(\Omega)$ such that $v_\epsilon \to v$ in the strong topology of $L^2(\Omega)$ and

$$\limsup_{\epsilon\to 0} F_\epsilon(v_\epsilon) \leq F_{hom}(v), \tag{22}$$

liminf inequality: For each v in $L^2(\Omega)$ and sequence $\{v_\epsilon\}_{\epsilon>0}$ in $L^2(\Omega)$ such that $v_\epsilon \rightharpoonup v$ in the weak topology of $L^2(\Omega)$, it holds that

$$F_{hom}(v) \leq \liminf_{\epsilon\to 0} F_\epsilon(v_\epsilon). \tag{23}$$

The proof of the *limsup inequality* was discussed in Theorem 6.1. The proof of the *liminf inequality* also holds even if we only assume that $v_\epsilon \rightharpoonup v$ weakly in $L^2(\Omega)$. Indeed, (19) still holds. Moreover, we can still show that $v = u$ a.e., so that $v \in H^1(\Omega)$. To see this, observe that

$$v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'}) \rightharpoonup v - u, \quad \text{weakly in } L^2(\Omega).$$

By the uniform boundedness principle,

$$\|v-u\|_{L^2(\Omega)} \leq \liminf_{\epsilon'\to 0} \|v_{\epsilon'} - Q_{\epsilon'}(v_{\epsilon'})\|_{L^2(\Omega)} = 0.$$

The rest of the proof then follows similarly to that of Theorem 6.1. □

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