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On the Linear Structures Induced by the Four Order Isomorphisms Acting on $\mathbf{Cvx_0}(\mathbb{R}^n)$

Dan I. Florentin
Department of Mathematics, Cleveland State University, Cleveland, OH 44115-2214, U.S.A.
danflorentin@gmail.com

Alexander Segal
Afeka Academic College of Engineering, Tel Aviv, 69107, Israel
segalale@gmail.com

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It is known that the volume functional $\phi \mapsto \int e^{-\phi}$ satisfies certain concavity or convexity inequalities with respect to three of the four linear structures induced by the order isomorphisms acting on $\mathrm{Cvx}_0(\mathbb{R}^n)$. In this note we define the fourth linear structure on $\mathrm{Cvx}_0(\mathbb{R}^n)$ as the pullback of the standard linear structure under the $\mathcal{J}$ transform. We show that, interpolating with respect to this linear structure, no concavity or convexity inequalities hold, and prove that a quasi-convexity inequality is violated only by up to a factor of 2. We also establish all the order relations which the four different interpolations satisfy.

Keywords: Convexity, interpolation, order isomorphisms, duality, Legendre transform, $\mathcal{A}$-transform, $\mathcal{J}$-transform.

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1. Introduction

In this note we study properties of the volume functional $\mathrm{Vol}\,(\phi) := \int e^{-\phi}$, defined on $\mathrm{Cvx}_0(\mathbb{R}^n)$, the class of all lower semi continuous convex functions from $\mathbb{R}^n$ to $[0, \infty]$ which attain the minimal value of 0 at the origin. As a first example let us mention that the classical Hölder inequality implies that log(Vol) is convex with respect to the standard linear structure on $\mathrm{Cvx}_0(\mathbb{R}^n)$, namely pointswise addition:

$$\int_{\mathbb{R}^n} e^{-((1-\lambda)\varphi+\lambda\psi)} \leq \left(\int_{\mathbb{R}^n} e^{-\varphi}\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{\lambda}. \tag{1}$$

We thus say that Vol is log-convex, or 0-convex, with respect to pointwise addition. When $p \neq 0$, a functional $F : \mathrm{Cvx}_0(\mathbb{R}^n) \to \mathbb{R}$ is called p-convex, if for any $\lambda \in [0, 1]$ and $\varphi, \psi \in \mathrm{Cvx}_0(\mathbb{R}^n)$ one has:

$$F\left((1-\lambda)\varphi + \lambda\psi\right) \leq \left((1-\lambda)F(\varphi)^p + \lambda F(\psi)^p\right)^{\frac{1}{p}}.$$

Log-concavity and p-concavity are defined similarly. While Vol is not p-concave with respect to pointwise addition (for any p), it does satisfy certain p-concavity inequalities with respect to other linear structures defined on $\mathrm{Cvx}_0(\mathbb{R}^n)$, which we next describe.

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In [2], Artstein and Milman described all the order isomorphisms acting on the class of *geometric convex functions* $\mathrm{Cvx}_0(\mathbb{R}^n)$. They proved that up to linear terms, there exist only two order reversing isomorphisms, namely the Legendre transform $\mathcal{L}$ and the polarity transform $\mathcal{A}$, and two order preserving isomorphisms, namely the identity and the gauge transform $\mathcal{J}$. We refer the reader to [2] for the definitions and a detailed description of these transforms. Each of the transforms induces a linear structure on $\mathrm{Cvx}_0(\mathbb{R}^n)$. For example, the inf-convolution $\varphi\Box\psi$ of two geometric convex functions φ, ψ may be defined as the pullback of standard addition under the Legendre transform:

$$\varphi\Box\psi := \mathcal{L}^{-1}\left(\mathcal{L}\varphi + \mathcal{L}\psi\right).$$

The renowned Prékopa-Leindler inequality (see [4, 5]) states that the functional Vol is log-concave with respect to the linear structure $\Box$. Namely, for all $\varphi, \psi \in \mathrm{Cvx}_0(\mathbb{R}^n)$ and $\lambda \in (0,1)$:

$$\int_{\mathbb{R}^n} e^{-\varphi\Box_\lambda\psi} \geq \left(\int_{\mathbb{R}^n} e^{-\varphi}\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} e^{-\psi}\right)^{\lambda}, \tag{2}$$

where $\varphi\Box_\lambda\psi = \mathcal{L}^{-1}\left((1-\lambda)\mathcal{L}\varphi + \lambda\mathcal{L}\psi\right)$ denotes the average of φ and ψ with respect to $\Box$. The linear structure induced by the polarity transform $\mathcal{A}$ was defined in [1] by:

$$\varphi\boxdot\psi := \mathcal{A}^{-1}\left(\mathcal{A}\varphi + \mathcal{A}\psi\right),$$

and it was proven that Vol is (-1)-concave with respect to $\boxdot$, namely:

$$\int_{\mathbb{R}^n} e^{-\varphi\boxdot_\lambda\psi} \geq \left(\frac{1-\lambda}{\int_{\mathbb{R}^n} e^{-\varphi}} + \frac{\lambda}{\int_{\mathbb{R}^n} e^{-\psi}}\right)^{-1}. \tag{3}$$

Here, $\varphi\boxdot_\lambda\psi = \mathcal{A}^{-1}\left((1-\lambda)\mathcal{A}\varphi + \lambda\mathcal{A}\psi\right)$ denotes averaging the functions φ, ψ with respect to $\boxdot$. In an analogous way, we define here a fourth linear structure on $\mathrm{Cvx}_0(\mathbb{R}^n)$, denoted by $\boxtimes$, as the pullback of standard addition under the $\mathcal{J}$ transform, that is:

$$\varphi\boxtimes\psi := \mathcal{J}^{-1}\left(\mathcal{J}\varphi + \mathcal{J}\psi\right), \quad \text{and} \quad \varphi\boxtimes_\lambda\psi := \mathcal{J}^{-1}\left((1-\lambda)\mathcal{J}\varphi + \lambda\mathcal{J}\psi\right).$$

Our first result is a geometric description of $\boxtimes$ as the classical operation of radial harmonic sum, applied to the epi-graphs, namely:

Lemma 1.1. *Let $\varphi, \psi \in \mathrm{Cvx}_0(\mathbb{R}^n)$, and $\lambda \in [0,1]$. Then:*

$$\mathrm{epi}(\varphi\boxtimes_\lambda\psi) = \left(\ (1-\lambda)\,\mathrm{epi}(\varphi)^\circ + \lambda\,\mathrm{epi}(\psi)^\circ\ \right)^\circ, \tag{4}$$

where $\mathrm{epi}\,\phi = \{(x,z) \in \mathbb{R}^n \times \mathbb{R}^+ \,:\, \phi(x) < z\}$.

In light of the log-convexity of Vol with respect to the standard linear structure (Hölder's inequality (1)), and its log-concavity and (-1)-concavity with respect to $\Box$ and $\boxdot$ respectively (inequalities (2) and (3)), it is natural to ask whether Vol is p-convex, for some p, with respect to $\boxtimes$. In this note we provide a negative answer to this question, but show that Vol does possess some convexity property, i.e. a quasi-convexity inequality is satisfied up to a constant (in contrast to any type of concavity, which cannot be considered even in this weak sense, as shown in Corollary 3.1). More precisely, for any $p \in [-\infty, \infty]$ we use the notation

$$M_{p,\lambda}(a,b) = ((1-\lambda)a^p + \lambda b^p)^{\frac{1}{p}}$$

for the p-average of two non-negative numbers a, b, with weights $(1-\lambda)$, λ respectively, and define:

$$c_p := \sup_{\varphi,\psi,\lambda} \left\{ \frac{\int_{\mathbb{R}^n} e^{-\varphi \boxtimes_\lambda \psi}}{M_{p,\lambda}\left(\int_{\mathbb{R}^n} e^{-\varphi}, \int_{\mathbb{R}^n} e^{-\psi}\right)} \right\}.$$

Clearly c_p is decreasing in p, since $M_{p,\lambda}(a,b)$ is increasing. Theorem 1.2 states that $c_\infty > 1$ (which implies $c_p > 1$ for all p).

Theorem 1.2. *There exist* $\varphi, \psi \in \mathrm{Cvx}_0(\mathbb{R}^\mathrm{n})$ *and* $\lambda \in (0,1)$ *such that:*

$$\max\left\{\int_{\mathbb{R}^n} e^{-\varphi}, \int_{\mathbb{R}^n} e^{-\psi}\right\} < \int_{\mathbb{R}^n} e^{-\varphi \boxtimes_\lambda \psi}.$$

Thus the answer to the question "is Vol p-convex with respect to $\boxtimes$?" is negative (for any p). However, up to a constant the answer is positive. More precisely, $c_\infty \leq 2$.

Theorem 1.3. *For any* $\varphi, \psi \in \mathrm{Cvx}_0(\mathbb{R}^\mathrm{n})$ *and* $\lambda \in [0,1]$*, we have:*

$$\int_{\mathbb{R}^n} e^{-\varphi \boxtimes_\lambda \psi} \leq 2 \cdot \max\left\{\int_{\mathbb{R}^n} e^{-\varphi}, \int_{\mathbb{R}^n} e^{-\psi}\right\}.$$

The paper is organized as follows. In Section 2 we prove Lemma 1.1, and prove pointwise order relations between $\boxtimes$ and the other three summations. In Section 3 we prove Theorems 1.2 and 1.3. In the Appendix we compute the asymptotics of the lower bound on c_∞ which was obtained in Theorem 1.2, and show that in fact $c_\infty > 1 + \frac{c}{\sqrt{n}}$.

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2. Order relations between the linear structures

In this section we provide a geometric description of $\boxtimes$, and prove four (of the possible six) order relations between the four linear structures $+, \square, \boxdot, \boxtimes$. Below, the term *convex body* refers to a closed, convex set $K \subset \mathbb{R}^n$ which contains the origin.

In [3], Firey defined the polar mean K_λ of convex bodies $K_0, K_1 \subset \mathbb{R}^n$, using Minkowski average and the duality map $K \mapsto K^\circ := \{y : \forall x \in K, \langle y, x\rangle \leq 1\}$:

$$K_\lambda = ((1-\lambda)K_0^\circ + \lambda K_1^\circ)^\circ.$$

The body K_λ is often called the *radial harmonic average* of K_0 and K_1, since:

$$\rho_{K_\lambda} = M_{-1,\lambda}\left(\rho_{K_0}, \rho_{K_1}\right),$$

where ρ_{K_i} is the radial function of K_i. As mentioned in the introduction, Lemma 1.1 states that the operation $\boxtimes_\lambda$ corresponds to radial harmonic average of epi-graphs.

Proof of Lemma 1.1. It is known that $\mathrm{epi}(\phi\Box_\lambda\eta) = (1-\lambda)\,\mathrm{epi}(\phi) + \lambda\,\mathrm{epi}(\eta)$, see e.g. [2]. Denoting the reflection about $\mathbb{R}^n \times \{0\}$ by $R(x,z) = (x,-z)$, it was shown in [2] that $\mathrm{epi}(\mathcal{A}\phi) = R(\mathrm{epi}(\phi)^\circ)$. Since $R(K^\circ) = (RK)^\circ$ and $\mathcal{J} = \mathcal{A}\circ\mathcal{L} = \mathcal{L}\circ\mathcal{A}$, we have:

$$\begin{aligned}
\mathrm{epi}(\mathcal{J}((1-\lambda)\mathcal{J}\varphi + \lambda\mathcal{J}\psi)) &= \mathrm{epi}(\mathcal{A}\mathcal{L}((1-\lambda)\mathcal{L}\mathcal{A}\varphi + \lambda\mathcal{L}\mathcal{A}\psi)) \\
&= \mathrm{epi}(\mathcal{A}((\mathcal{A}\varphi)\Box_\lambda(\mathcal{A}\psi))) = R(\mathrm{epi}((\mathcal{A}\varphi)\Box_\lambda(\mathcal{A}\psi))^\circ) \\
&= R((\ (1-\lambda)R(\mathrm{epi}(\varphi)^\circ) + \lambda R(\mathrm{epi}(\psi)^\circ)\ )^\circ) \\
&= (\ (1-\lambda)\,\mathrm{epi}(\varphi)^\circ + \lambda\,\mathrm{epi}(\psi)^\circ\ )^\circ. \qquad \Box
\end{aligned}$$

The four linear structures $+$, $\Box$, $\boxdot$, $\boxtimes$ offer four different ways to interpolate between two given functions $\varphi, \psi \in \mathrm{Cvx}_0(\mathbb{R}^n)$. We now turn to establish order relations between the four interpolations $\varphi\Box_\lambda\psi$, $\varphi\boxdot_\lambda\psi$, $\varphi\boxtimes_\lambda\psi$, and $\varphi +_\lambda \psi := (1-\lambda)\varphi + \lambda\psi$.

First, let us consider two basic examples, one where the averaged functions φ, ψ are convex indicator functions, and one where they are norms (or more precisely, the Minkowski functional of a convex body, defined by $\|x\|_K = \inf\{t > 0 : x \in tK\}$).

Example 2.1. Let K, T be convex bodies such that $\varphi = 1^\infty_K$, $\psi = 1^\infty_T \in \mathrm{Cvx}_0(\mathbb{R}^n)$. Since $\mathcal{J}1^\infty_K = \|\cdot\|_K$, $\mathcal{L}1^\infty_K = h_K$, and $\mathcal{A}1^\infty_K = 1^\infty_{K^\circ}$, it may be easily checked that:

$$\varphi\boxdot_{\frac12}\psi = 1^\infty_{K\vee T}, \qquad \varphi\boxtimes_{\frac12}\psi = 1^\infty_{\left(\frac{K^\circ+T^\circ}{2}\right)^\circ},$$

$$\varphi\Box_{\frac12}\psi = 1^\infty_{\frac{K+T}{2}}, \qquad \varphi +_{\frac12} \psi = 1^\infty_{K\cap T},$$

where $K \vee T$ denotes the convex hull of K and T. Note that:

$$K \cap T \subseteq \left(\frac{K^\circ + T^\circ}{2}\right)^\circ \subseteq \frac{K+T}{2} \subseteq K \vee T. \tag{5}$$

Thus the functions $\varphi = 1^\infty_K$, $\psi = 1^\infty_T$ satisfy the following inequalities:

$$\varphi\boxdot_{\frac12}\psi \ \le\ \varphi\Box_{\frac12}\psi \ \le\ \varphi\boxtimes_{\frac12}\psi \ \le\ \varphi +_{\frac12} \psi.$$

Note that all the inclusions in (5) are strict if $K \neq T$. Thus for $K \neq T$ we get:

$$\varphi\Box_{\frac12}\psi \not\le \varphi\boxdot_{\frac12}\psi, \qquad \varphi +_{\frac12} \psi \not\le \varphi\boxtimes_{\frac12}\psi. \tag{6}$$

Example 2.2. Let K, T be convex bodies with $\varphi = \|\cdot\|_K$, $\psi = \|\cdot\|_T \in \mathrm{Cvx}_0(\mathbb{R}^n)$. Since $\mathcal{J}\|\cdot\|_K = 1^\infty_K$, $\mathcal{L}\|\cdot\|_K = 1^\infty_{K^\circ}$, and $\mathcal{A}\|\cdot\|_K = h_K$, it is easy to check that:

$$\varphi\Box_{\frac12}\psi = \|\cdot\|_{K\vee T}, \quad \varphi +_{\frac12} \psi = \|\cdot\|_{\left(\frac{K^\circ+T^\circ}{2}\right)^\circ},$$

$$\varphi\boxdot_{\frac12}\psi = \|\cdot\|_{\frac{K+T}{2}}, \qquad \varphi\boxtimes_{\frac12}\psi = \|\cdot\|_{K\cap T}.$$

By (5), the functions $\varphi = \|\cdot\|_K$, $\psi = \|\cdot\|_T$ satisfy the following inequalities:

$$\varphi\square_{\frac{1}{2}}\psi \;\leq\; \varphi\boxdot_{\frac{1}{2}}\psi \;\leq\; \varphi +_{\frac{1}{2}} \psi \;\leq\; \varphi\boxtimes_{\frac{1}{2}}\psi.$$

As before, assuming $K \neq T$ we have:

$$\varphi\boxdot_{\frac{1}{2}}\psi \not\leq \varphi\square_{\frac{1}{2}}\psi, \qquad\qquad \varphi\boxtimes_{\frac{1}{2}}\psi \not\leq \varphi +_{\frac{1}{2}} \psi. \tag{7}$$

From (6) and (7) we conclude that the functions $\varphi\boxdot_\lambda\psi$ and $\varphi\square_\lambda\psi$ are not comparable in general. Similarly the functions $\varphi +_\lambda \psi$, $\varphi\boxtimes_\lambda\psi$ are not comparable, in general. However, the remaining four pairs of functions do satisfy, in general, the order relations exhibited in the examples above. That is, interpolating with respect to $+, \boxtimes$ (induced by the two order preserving transforms) always yields larger functions compared to $\square, \boxdot$ (induced by the order reversing transforms). More precisely:

Lemma 2.3. *Let* $\varphi, \psi \in \mathrm{Cvx}_0(\mathbb{R}^n)$. *Then:*

$$\max\left\{\varphi\square_\lambda\psi,\, \varphi\boxdot_\lambda\psi\right\} \leq \min\left\{\varphi +_\lambda \psi,\, \varphi\boxtimes_\lambda\psi\right\}.$$

Proof. We need to prove the following four inequalities.

1. $\varphi\square_\lambda\psi \leq \varphi +_\lambda \psi$
2. $\varphi\square_\lambda\psi \leq \varphi\boxtimes_\lambda\psi$
3. $\varphi\boxdot_\lambda\psi \leq \varphi +_\lambda \psi$
4. $\varphi\boxdot_\lambda\psi \leq \varphi\boxtimes_\lambda\psi$

The first inequality follows by letting $x = y = z$ in the definition of the inf-convolution:

$$(\varphi\square_\lambda\psi)(z) = \inf_{z=(1-\lambda)x+\lambda y}\{(1-\lambda)\varphi(x) + \lambda\psi(y)\} \leq (1-\lambda)\varphi(z) + \lambda\psi(z). \tag{8}$$

For the second inequality, recall that duality is a "convex operation", in the following sense (see [3]). If K and T are convex bodies and $0 < \lambda < 1$, then:

$$((1-\lambda)K + \lambda T)^\circ \subseteq (1-\lambda)K^\circ + \lambda T^\circ. \tag{9}$$

Combining the above with Lemma 1.1 we get:

$$\begin{aligned} \mathrm{epi}(\varphi\boxtimes_\lambda\psi) &= \left(\ (1-\lambda)\,\mathrm{epi}(\varphi)^\circ + \lambda\,\mathrm{epi}(\psi)^\circ\ \right)^\circ \\ &\subseteq (1-\lambda)\,\mathrm{epi}(\varphi) + \lambda\,\mathrm{epi}(\psi) \ \ = \mathrm{epi}(\varphi\square_\lambda\psi), \end{aligned}$$

which is equivalent to:
$$\varphi\square_\lambda\psi \leq \varphi\boxtimes_\lambda\psi. \tag{10}$$

The third and fourth inequalities we need to prove, are in fact equivalent to the second and first inequalities, respectively. To see this, first note that:

$$\mathcal{J}\left((\mathcal{J}\varphi)\square_\lambda(\mathcal{J}\psi)\right) = \varphi\boxdot_\lambda\psi, \tag{11}$$

$$\mathcal{J}\left((\mathcal{J}\varphi)\boxtimes_\lambda(\mathcal{J}\psi)\right) = \varphi +_\lambda \psi, \tag{12}$$

$$\mathcal{J}\left((\mathcal{J}\varphi) +_\lambda (\mathcal{J}\psi)\right) = \varphi\boxtimes_\lambda\psi. \tag{13}$$

Since $\mathcal{J}$ is order preserving, (8) implies that:

$$\varphi \boxdot_\lambda \psi = \mathcal{J}\left((\mathcal{J}\varphi)\Box_\lambda(\mathcal{J}\psi)\right) \le \mathcal{J}\left((\mathcal{J}\varphi) +_\lambda (\mathcal{J}\psi)\right) = \varphi \boxtimes_\lambda \psi.$$

Similarly, (10) implies that:

$$\varphi \boxdot_\lambda \psi = \mathcal{J}\left((\mathcal{J}\varphi)\Box_\lambda(\mathcal{J}\psi)\right) \le \mathcal{J}\left((\mathcal{J}\varphi)\boxtimes_\lambda(\mathcal{J}\psi)\right) = \varphi +_\lambda \psi. \qquad \square$$

3. Convexity and concavity properties of Vol

We begin this section by noting that the geometric Example 2.2 demonstrates that for any p, Vol is not p-concave with respect to $\boxtimes$, i.e. Vol is not even quasi-concave. Moreover, we show that in contrast to Theorem 1.3, a concavity inequality does not even hold up to a constant. More precisely,

Corollary 3.1. *The functional* Vol *is not quasi-concave with respect to* $\boxtimes$.

Moreover:
$$\inf_{\varphi,\psi,\lambda}\left\{\frac{\mathrm{Vol}(\varphi\boxtimes_\lambda\psi)}{\min\left\{\mathrm{Vol}(\varphi),\ \mathrm{Vol}(\psi)\right\}}\right\} = 0.$$

Proof. We may choose K, T to be boxes of unit volume, with intersection of arbitrarily small volume (we denote the volume of sets in $\mathbb{R}^n$ by Vol_n). Since we have $\mathrm{Vol}(\|\cdot\|_L) = n!\,\mathrm{Vol}_n(L)$ for every convex body L, the functions φ, ψ of Example 2.2 satisfy:

$$\frac{\mathrm{Vol}(\varphi\boxtimes_{\frac{1}{2}}\psi)}{\min\left\{\mathrm{Vol}(\varphi),\ \mathrm{Vol}(\psi)\right\}} = \frac{\mathrm{Vol}_n(K\cap T)}{\min\left\{\mathrm{Vol}_n(K),\ \mathrm{Vol}_n(T)\right\}} = \mathrm{Vol}_n(K\cap T),$$

which as mentioned above, can be made arbitrarily small. $\square$

We now turn to prove the main results of the paper. Recall that in [1], the measure ν on $\mathbb{R}^n \times \mathbb{R}^+$ was defined by $d\nu = n(z)dxdz$ for $n(z) = e^{-\frac{1}{z}}\left(\frac{1}{z}\right)^{n+2}$, and it was shown that for any $\phi \in \mathrm{Cvx}_0(\mathbb{R}^n)$, one has:

$$\nu\left(\mathrm{epi}\,\phi\right) = \int_{\mathbb{R}^n} e^{-\mathcal{J}\phi}. \tag{14}$$

We shall rely on this fact in the proof of Theorem 1.2, when constructing a counter example to quasi-convexity of Vol.

Proof of Theorem 1.2. Let us define the (decreasing) function $G : [0,\infty] \to [0, n!]$ by $G(t) = \int_t^\infty n(z)dz = \int_0^{\frac{1}{t}} s^n e^{-s} ds$. Since $\mathcal{J}(\mathcal{J}\phi) = \phi$, we may rewrite (14) as:

$$\int_{\mathbb{R}^n} e^{-\phi} = \nu\left(\mathrm{epi}\,\mathcal{J}\phi\right) = \int_{\mathbb{R}^n} dx \int_{(\mathcal{J}\phi)(x)}^\infty n(z)dz = \int_{\mathbb{R}^n} G\left(\left(\mathcal{J}\phi\right)(x)\right)dx. \tag{15}$$

Since $\mathcal{J}\left(\varphi\boxtimes_\lambda\psi\right) = (1-\lambda)\mathcal{J}\varphi + \lambda\mathcal{J}\psi$, the problem boils down to prove the convexity of G. Differentiating twice we get $G''(t) = \frac{(n+2)e^{-\frac{1}{t}}}{t^{n+4}}\left(t - \frac{1}{n+2}\right)$, thus G is concave on the interval $\left[0, \frac{1}{n+2}\right]$. Therefore we shall choose φ, ψ such that $\mathcal{J}\varphi, \mathcal{J}\psi$ attain values in $\left[0, \frac{1}{n+2}\right]$, to get:

$$\int_{\mathbb{R}^n} e^{-\varphi \boxtimes_\lambda \psi} \geq (1-\lambda) \int_{\mathbb{R}^n} e^{-\varphi} + \lambda \int_{\mathbb{R}^n} e^{-\psi}.$$

If we can also choose φ, ψ with equal volumes $\int_{\mathbb{R}^n} e^{-\varphi} = \int_{\mathbb{R}^n} e^{-\psi}$, and such that the above inequality is strict, the theorem will be proven. The first condition is satisfied by choosing $\psi(x) = \varphi(-x)$. For the second condition, it suffices that $\mathcal{J}\varphi, \mathcal{J}\psi$ be supported on the same set, attain values in $\left[0, \frac{1}{n+2}\right]$, and such that $\mathcal{J}\varphi \neq \mathcal{J}\psi$ on a set of positive measure. For example, we may choose them to be piecewise linear on the cube. More precisely, let us denote the unit ball of l_∞^n by $C = [-1,1]^n$, and the half-space $\{x \in \mathbb{R}^n : x_1 \leq n+2\}$ by H, so that $\|x\|_H = \max\left\{0, \frac{x_1}{n+2}\right\}$. We define φ by setting $\mathcal{J}\varphi = \max\{1_C^\infty, \|\cdot\|_H\}$, that is:

$$\mathcal{J}\varphi(x) = \left\{ \begin{array}{ll} \max\left\{0, \frac{x_1}{n+2}\right\} & : x \in C \\ +\infty & : x \notin C \end{array} \right\}.$$

By (15) we have:

$$\begin{aligned}
\mathrm{Vol}(\varphi \boxtimes_\lambda \psi) &= \int_C G\left(\left(\mathcal{J}\left(\varphi \boxtimes_\lambda \psi\right)\right)(x)\right) dx \\
&= \int_C G\left((1-\lambda)\left(\mathcal{J}\varphi\right)(x) + \lambda\left(\mathcal{J}\psi\right)(x)\right) dx \\
&> (1-\lambda) \int_C G\left(\left(\mathcal{J}\varphi\right)(x)\right) dx + \lambda \int_C G\left(\left(\mathcal{J}\psi\right)(x)\right) dx \\
&= (1-\lambda)\mathrm{Vol}(\varphi) + \lambda \mathrm{Vol}(\psi) = \max\{\mathrm{Vol}(\varphi), \mathrm{Vol}(\psi)\},
\end{aligned}$$

which completes the proof. □

Remark 3.2. In the Appendix we show that in fact the functions φ, ψ satisfy:

$$\frac{\mathrm{Vol}\left(\varphi \boxtimes_{\frac{1}{2}} \psi\right)}{\max\{\mathrm{Vol}(\varphi), \mathrm{Vol}(\psi)\}} = 1 + \frac{1}{\sqrt{8\pi n}} + O\left(\frac{1}{n}\right), \tag{16}$$

which implies that $c_\infty > 1 + \frac{c}{\sqrt{n}}$ for some $c > 0$.

Proof of Theorem 1.3. Recall that the function $G(t) = \int_t^\infty n(z)dz$ is decreasing on $[0, \infty]$, therefore on every interval (bounded or unbounded), G attains its maximum at the lower endpoint. In particular $G((1-\lambda)a + \lambda b) \leq G(a) + G(b)$. Thus we may use (15) again, to get:

$$\begin{aligned}
\mathrm{Vol}(\varphi \boxtimes_\lambda \psi) &= \int_{\mathbb{R}^n} G\left(\left(\mathcal{J}\left(\varphi \boxtimes_\lambda \psi\right)\right)(x)\right) dx \\
&= \int_{\mathbb{R}^n} G\left((1-\lambda)\left(\mathcal{J}\varphi\right)(x) + \lambda\left(\mathcal{J}\psi\right)(x)\right) dx \\
&\leq \int_{\mathbb{R}^n} G\left(\left(\mathcal{J}\varphi\right)(x)\right) dx + \int_{\mathbb{R}^n} G\left(\left(\mathcal{J}\psi\right)(x)\right) dx \\
&= \mathrm{Vol}(\varphi) + \mathrm{Vol}(\psi) \leq 2 \cdot \max\{\mathrm{Vol}(\varphi), \mathrm{Vol}(\psi)\}.
\end{aligned}$$

□

Appendix

We now turn to prove equation (16), which as mentioned in Remark 3.2, implies that for some $c > 0$ we have $c_\infty > 1 + \frac{c}{\sqrt{n}}$. We need to compute the volumes of the functions φ, ψ, $\varphi \boxtimes_\lambda \psi$ defined in the proof of Theorem 1.2. Recall our notation for the cube $C = [-1,1]^n$ and the half-space $H = \{x \in \mathbb{R}^n \,:\, x_1 \le n+2\}$, and note that $S_\lambda := ((1-\lambda)H^\circ + \lambda(-H)^\circ)^\circ$ is the slab $\left\{x \in \mathbb{R}^n \,:\, -\frac{n+2}{\lambda} \le x_1 \le \frac{n+2}{1-\lambda}\right\}$.

$$\mathcal{J}\left(\varphi \boxtimes_\lambda \psi\right)(x) = \left\{ \begin{array}{ll} \max\{\lambda(-x_1), (1-\lambda)x_1\} & : x \in C \\ +\infty & : x \notin C \end{array} \right\} = \max\left\{1_C^\infty, \|\cdot\|_{S_\lambda}\right\}.$$

Since $\mathcal{J}$ interchanges with taking a maximum (see [2]), we have:

$$\varphi = \mathcal{J}\left(\max\{1_C^\infty, \|\cdot\|_H\}\right) = \max\{\mathcal{J}1_C^\infty, \mathcal{J}\|\cdot\|_H\} = \max\{\|\cdot\|_C,\, 1_H^\infty\},$$

and similarly $\varphi \boxtimes_\lambda \psi = \max\left\{\|\cdot\|_C,\, 1_{S_\lambda}^\infty\right\}$. In order to compute the volumes of φ and $\varphi \boxtimes_\lambda \psi$ we require their level sets. At this point we choose $\lambda = \frac{1}{2}$, which maximizes $\mathrm{Vol}\left(\varphi \boxtimes_\lambda \psi\right)$, since $\lambda \mapsto \mathrm{Vol}\left(\varphi \boxtimes_\lambda \psi\right)$ is concave and symmetric about $\lambda = \frac{1}{2}$. We get:

$$L_z(\varphi) = (zC) \cap H = \left\{ \begin{array}{ll} zC & : z \le n+2 \\ [-z, n+2] \times [-z, z]^{n-1} & : n+2 \le z \end{array} \right\},$$

$$L_z(\varphi \boxtimes_{\frac{1}{2}} \psi) = (zC) \cap S_{\frac{1}{2}} = \left\{ \begin{array}{ll} zC & : z \le 2(n+2) \\ [-2(n+2), 2(n+2)] \times [-z, z]^{n-1} & : 2(n+2) \le z \end{array} \right\}.$$

With the (volumes of the) level sets known, we may use Fubini to get:

$$\begin{aligned} \mathrm{Vol}(\varphi) = \mathrm{Vol}(\psi) &= \int_{\mathrm{epi}\,\varphi} d\mu = \int_0^\infty \mathrm{Vol}_n\left(L_z(\varphi)\right) e^{-z} dz = \\ &= 2^n \int_0^{n+2} z^n e^{-z} dz + 2^{n-1} \int_{n+2}^\infty (n+2+z) z^{n-1} e^{-z} dz = \\ &= 2^n \int_0^\infty z^n e^{-z} dz + 2^{n-1} \int_{n+2}^\infty (n+2-z) z^{n-1} e^{-z} dz = \\ &= 2^n n! - 2^{n-1} \int_{n+2}^\infty (z-(n+2)) z^{n-1} e^{-z} dz = \\ &= 2^n n! \left(1 - \frac{\int_{n+2}^\infty \left(\frac{z-(n+2)}{2}\right) z^{n-1} e^{-z} dz}{n!}\right) \equiv 2^n n! \left(1 - R(n)\right). \end{aligned}$$

As for $\varphi \boxtimes_{\frac{1}{2}} \psi$, we have:

$$\begin{aligned}
\mathrm{Vol}(\varphi \boxtimes_{\frac{1}{2}} \psi) &= \int_{\mathrm{epi}\left(\varphi \boxtimes_{\frac{1}{2}} \psi\right)} d\mu = \int_0^\infty \mathrm{Vol}_n\left(L_z(\varphi \boxtimes_{\frac{1}{2}} \psi)\right) e^{-z} dz = \\
&= 2^n \int_0^{2(n+2)} z^n e^{-z} dz + 2^n \int_{2(n+2)}^\infty 2(n+2) z^{n-1} e^{-z} dz = \\
&= 2^n \int_0^\infty z^n e^{-z} dz - 2^n \int_{2(n+2)}^\infty (z - 2(n+2)) z^{n-1} e^{-z} dz = \\
&= 2^n n! \left(1 - \frac{\int_{2(n+2)}^\infty (z-2(n+2)) z^{n-1} e^{-z} dz}{n!}\right) \\
&\equiv 2^n n! \left(1 - r(n)\right).
\end{aligned}$$

Note that $\int_{n+2}^\infty z^n e^{-z} dz < n!$, and by Taylor's expansion, $\frac{\left(1+\frac{2}{n}\right)^{n+1}}{e^2} = 1 + O\left(\frac{1}{n}\right)$. We use these estimates together with Stirling's formula $\frac{n!}{\sqrt{2\pi n}} = \left(\frac{n}{e}\right)^n \cdot \left(1 + O\left(\frac{1}{n}\right)\right)$ to obtain a lower bound on $R(n)$. After integration by parts we get:

$$\begin{aligned}
R(n) &= \frac{1}{n!}\left(\frac{\left(1+\frac{2}{n}\right)^{n+1}}{2e^2}\left(\frac{n}{e}\right)^n - \frac{1}{n}\int_{n+2}^\infty z^n e^{-z} dz\right) \\
&= \frac{\left(1 + O\left(\frac{1}{n}\right)\right)}{\sqrt{8\pi n}} - \frac{1}{n}\frac{\int_{n+2}^\infty z^n e^{-z} dz}{n!} \\
&> \frac{1}{\sqrt{8\pi n}} - \frac{1}{n} + O\left(\frac{1}{n^{\frac{3}{2}}}\right) = \frac{1}{\sqrt{8\pi n}} + O\left(\frac{1}{n}\right).
\end{aligned}$$

To obtain an upper bound on $r(n)$, we begin again with integration by parts:

$$\begin{aligned}
r(n) &= \frac{1}{n!}\left(\frac{\left(1+\frac{2}{n}\right)^{n+1}}{e^3}\left(\frac{n}{e}\right)^n\left(\frac{2}{e}\right)^{n+1} - \left(1+\frac{4}{n}\right)\int_{2(n+2)}^\infty z^n e^{-z} dz\right) \\
&< \frac{\left(1 + O\left(\frac{1}{n}\right)\right)}{e\sqrt{2\pi n}}\left(\frac{2}{e}\right)^{n+1} < \left(\frac{2}{e}\right)^n.
\end{aligned}$$

To sum up, we have

$$\begin{aligned}
\frac{\mathrm{Vol}\left(\varphi \boxtimes_{\frac{1}{2}} \psi\right)}{\max\left\{\mathrm{Vol}\left(\varphi\right), \mathrm{Vol}\left(\psi\right)\right\}} = \frac{1 - r(n)}{1 - R(n)} &> \frac{1 - \left(\frac{2}{e}\right)^n}{1 - \frac{1}{\sqrt{8\pi n}} + O\left(\frac{1}{n}\right)} \\
&= \left(1 - \left(\frac{2}{e}\right)^n\right) \cdot \left(1 + \frac{1}{\sqrt{8\pi n}} + O\left(\frac{1}{n}\right)\right) \\
&= \left(1 + \frac{1}{\sqrt{8\pi n}} + O\left(\frac{1}{n}\right)\right).
\end{aligned}$$

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