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On F-Convex Functions

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We present a geometric view on F-convex functions and we discus their continuity and differentiability properties. Relations between F-convex functions and some concepts of convexity are also investigated.

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Introduction

Let $I \subset \mathbb{R}$ be an interval and $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. Recall ([1]) that the function $f : I \to \mathbb{R}$ is F-*convex* if for every $x, y \in I$ and $t \in [0, 1]$ it satisfies the following inequality:

$$f\left(tx + (1-t)y\right) \leq tf(x) + (1-t)f(y) - t(1-t)F(x-y). \tag{1}$$

Such functions were defined in the context of strongly convex functions. If we take $F(x) = cx^2$ with $c > 0$ we get the definition of strongly convex functions introduced by Polyak (see [22]). But note that from F-convexity we can also obtain another concepts of convexity. Namely:

- for $F(x) = cx^2$ with $c \in \mathbb{R}$ we get the definition of *c-convex functions* introduced by Vial (see [25]);
- for $F(x) = -cx^2$ with $c > 0$ we get *weakly convex functions* investigated by Aussel, Daniilidis, and Thibault (see [4]);
- for $F(x) = -c|x|$ with $c > 0$ we get *c-convex functions* introduced by Luc, Ngai, and Théra (see [9], [12]);
- for $F(x) = -c|x|^p$ with $c > 0$ and $p > 0$ we get *approximately convex functions of order p* introduced by Nikodem and Páles (see [18])
- for $F(x) = -|x|\omega(|x|)$ with nondecreasing, upper-semicontinuous function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$ we obtain the definition of *semiconvex functions* introduced by Alberti, Ambrosio and Cannarsa (see [6]).

We can thus treat F-convexity as a generalization of presented convexities. Different kinds of convexity were investigated and developed by many authors. It appears that convexities have many interesting properties themselves, but they are also useful in different branches of mathematics, economics and engineering – examples can be found in the mentioned literature and for instance in [2, 3, 5, 7, 8, 10, 11, 13, 15, 16,

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17, 19, 20, 23, 24, 26]. We will discuss continuity and differentiability of F-convex functions from a geometric viewpoint.

1. Main results

1.1. Geometric view of F-convex functions

Let I be a nonempty and open subinterval of $\mathbb{R}$. It is well known that $f : I \to \mathbb{R}$ is a convex function if and only if for all $x_1, x_2 \in I$ ($x_1 \neq x_2$) the adequate part of its graph lies below the secant line passing through the points $(x_1, f(x_1))$, $(x_2, f(x_2))$, i.e.

$$f(x) \leq l(x), \quad x \in [x_1, x_2] \cup [x_2, x_1]\,,$$

where $l(x) = ax + b$. And the coefficients a, b are in the forms:

$$a = \frac{\begin{vmatrix} f(x_1) & 1 \\ f(x_2) & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}, \quad b = \frac{\begin{vmatrix} x_1 & f(x_1) \\ x_2 & f(x_2) \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}.$$

For strongly convex functions a similar result was obtained by Merentes and Nikodem in [16], namely $f : I \to \mathbb{R}$ is a strongly convex function with modulus c if and only if for all $x_1, x_2 \in I$ ($x_1 \neq x_2$) and the function $l(x) = cx^2 + ax + b$ passing through the points $(x_1, f(x_1))$, $(x_2, f(x_2))$ we have

$$f(x) \leq l(x), \quad x \in [x_1, x_2] \cup [x_2, x_1]\,.$$

In this case coefficients a and b have the following forms:

$$a = \frac{\begin{vmatrix} f(x_1) - cx_1^2 & 1 \\ f(x_2) - cx_2^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}, \quad b = \frac{\begin{vmatrix} x_1 & f(x_1) - cx_1^2 \\ x_2 & f(x_2) - cx_2^2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}.$$

An analogous result for F-convex functions looks as follows.

Theorem 1.1. *Let I be a nonempty and open subinterval of $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ is F-convex, then for all $x_1, x_2 \in I$ ($x_1 \neq x_2$) we have*

$$f(x) \leq l(x), \quad x \in [x_1, x_2] \cup [x_2, x_1]\,, \tag{2}$$

where $l(x) = \gamma x^2 + ax + b$ is a function passing through the points $(x_1, f(x_1))$, $(x_2, f(x_2))$ and $\gamma = \frac{F(x_1 - x_2)}{(x_1 - x_2)^2}$. Moreover, if $F(0) \leq 0$, then (2) implies that f is a F-convex function.

Proof. Fix different points $x_1, x_2 \in I$ and put $\gamma = (F(x_1 - x_2))/(x_1 - x_2)^2$. Let $l(x) = \gamma x^2 + ax + b$ be a function passing through the points $(x_1, f(x_1))$, $(x_2, f(x_2))$. Observe that for all $t \in [0, 1]$ we have

$$
\begin{aligned}
l\,(tx_1+(1-t)x_2) &= \gamma\,(tx_1+(1-t)x_2)^2 + a\,(tx_1+(1-t)x_2)+b \\
&= \gamma(t^2x_1^2+2t(1-t)x_1x_2+(1-t)^2x_2^2) + a\,(tx_1+(1-t)x_2)+b \\
&= t(\gamma x_1^2+ax_1+b)+(1-t)(\gamma x_2^2+ax_2+b)-\gamma t(1-t)(x_1^2-2x_1x_2+x_2^2) \\
&= tf(x_1)+(1-t)f(x_2)-\gamma t(1-t)(x_1-x_2)^2 \\
&= tf(x_1)+(1-t)f(x_2)-t(1-t)F(x_1-x_2).
\end{aligned}
$$

So for all $t \in [0,1]$ we get the equality

$$
l\,(tx_1+(1-t)x_2) = tf(x_1)+(1-t)f(x_2)-t(1-t)F(x_1-x_2). \tag{3}
$$

Therefore, for an F-convex function f and $x \in [x_1,x_2] \cup [x_2,x_1]$ we obtain

$$
\begin{aligned}
f(x) = f(tx_1+(1-t)x_2) &\le tf(x_1)+(1-t)f(x_2)-t(1-t)F(x_1-x_2) \\
&= l\,(tx_1+(1-t)x_2) = l(x),
\end{aligned}
$$

and this is inequality (2). Assuming now inequality (2) and using (3) we have

$$
\begin{aligned}
f(tx_1+(1-t)x_2) &\le l(tx_1+(1-t)x_2) \\
&= tf(x_1)+(1-t)f(x_2)-t(1-t)F(x_1-x_2)
\end{aligned} \tag{4}
$$

for all different $x_1, x_2 \in I$ and $t \in [0,1]$. So we have (1) for all different $x_1, x_2 \in I$. Under the condition $F(0) \le 0$, inequality (1) is also true for the same x_1, x_2. It means that f is an F-convex function. □

Remark 1.2. Using standard methods of linear algebra we conclude that we have exactly one function $l(x) = \gamma x^2 + ax + b$ passing through the points $(x_1, f(x_1))$, $(x_2, f(x_2))$ and

$$
a = \frac{\begin{vmatrix} f(x_1)-\gamma x_1^2 & 1 \\ f(x_2)-\gamma x_2^2 & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}, \quad b = \frac{\begin{vmatrix} x_1 & f(x_1)-\gamma x_1^2 \\ x_2 & f(x_2)-\gamma x_2^2 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 \\ x_2 & 1 \end{vmatrix}}, \tag{5}
$$

where γ depends on x_1 and x_2. □

This remark shows that regardless of a function F we have a similar situation as in case of strong convexity, namely each adequate part of the graph of f lies bellow the parabola. But, in contrast to strong convexity, for F-convexity the coefficient of x^2 does not need to be constant. The coefficient of x^2 is constant if the differences between arguments are the same, respectively. Now, using the identity

$$
|tx+(1-t)y|^2 = t|x|^2+(1-t)|y|^2-t(1-t)|x-y|^2, \quad x,y \in \mathbb{R}, \tag{6}
$$

we get the following corollary.

Corollary 1.3. *If a function $f : I \to \mathbb{R}$ is F-convex, then for all $x_1, y_1 \in I$ $(x_1 \neq y_1)$ the function $g(\cdot) = f(\cdot) - \frac{F(x_1-y_1)}{(x_1-y_1)^2}(\cdot)^2$ satisfies the following inequality*

$$g(tx + (1-t)y) \leq tg(x) + (1-t)g(y) \tag{7}$$

for all $x, y \in I$ such that $x - y = x_1 - y_1$ and $t \in (0,1)$. Moreover, if $F(0) \leq 0$, then we get a converse implication.

Proof. Fix $x_1, y_1 \in I$ $(x_1 \neq y_1)$ and put $g(\cdot) = f(\cdot) - \frac{F(x_1-y_1)}{(x_1-y_1)^2}(\cdot)^2$. If f is an F-convex function, then multiplying (6) by $\frac{F(x_1-y_1)}{(x_1-y_1)^2}$ and subtracting it from both sides from inequality (1) we get (7). If g satisfies (7), then multiplying (6) by $\frac{F(x_1-y_1)}{(x_1-y_1)^2}$ and adding it to both sides of inequality (7) we get (1) for all different $x, y \in I$ – because x_1 and y_1 are arbitrarily chosen. Taking into consideration the condition $F(0) \leq 0$, inequality (1) is also true if $x = y$. $\square$

1.2. Continuity of F-convex functions

As we already know, convex functions defined on an open real interval must be convex. It appears that considering some conditions weaker than convexity, we can also obtain continuity – for example, c-convexity, approximate convexity and semiconvexity (in the sense mentioned in the Introduction) force continuity. For F-convexity, we have the following theorem.

Theorem 1.4. *Let I be a nonempty and open subinterval of $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ is F-convex, then it is continuous.*

The proof of this theorem is an immediate consequence of the following two lemmas, which say that F-convex functions are upper-semicontinuous and lower-semicontinuous, respectively.

Lemma 1.5. *Let I be a nonempty and open subinterval of $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ is F-convex, then it is upper-semicontinuous.*

Proof. Fix an arbitrary point $x_0 \in I$. We show that f is upper-semicontinuous at x_0. Take points y_0, z_0 of interval I such that $y_0 < x_0 < z_0$. From Theorem 1.1 we get the inequalities

$$f(x) \leq l(x) \text{ for } x \in [y_0, x_0], \quad \text{and} \quad f(x) \leq k(x) \text{ for } x \in [x_0, z_0],$$

where l and k are the parabolas passing through the points $(x_0, f(x_0))$, $(y_0, f(y_0))$ and $(x_0, f(x_0))$, $(z_0, f(z_0))$, respectively. Moreover, the functions l and k are continuous. The following inequality is thus true

$$\liminf_{x \to x_0} f(x) \leq f(x_0).$$

Which means that f is upper-semicontinuous at x_0, and consequently f is an upper-semicontinuous function. $\square$

Lemma 1.6. *Let I be a nonempty and open subinterval of $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If a function $f : I \to \mathbb{R}$ is F-convex, then it is lower-semicontinuous.*

Proof. Fix an arbitrary point $x_0 \in I$. We show that f is lower-semicontinuous at x_0. Let's take a point y_0 ($x_0 \neq y_0$) such that

$$x_0 + |x_0 - y_0| \in I \text{ and } x_0 - |x_0 - y_0| \in I. \tag{8}$$

Define a function $g : I \to \mathbb{R}$ by the formula

$$g(x) = f(x) - \frac{F(x_0 - y_0)}{(x_0 - y_0)^2} x^2.$$

Of course, f is lower-semicontinuous at x_0 if and only if g is lower-semicontinuous at x_0, and by Corollary 1.3 g satisfies the inequality

$$g\left(tx + (1-t)y\right) \leq tg(x) + (1-t)g(y) \tag{9}$$

for all $x, y \in I$ such that $x - y = x_0 - y_0$ and $t \in (0,1)$.

Suppose that g is not lower-semicontinuous at x_0. Therefore, there is a sequence (x_n) of elements of I and $\epsilon > 0$ such that

$$\lim_{n\to+\infty} x_n = x_0, \quad x_n \neq x_0 \tag{10}$$

and

$$g(x_n) - \epsilon < g(x_0). \tag{11}$$

By (8) we can take sequences $(y_n) \subset I$ and $(t_n) \subset (0,1)$ such that

$$x_n - y_n = x_0 - y_0 \tag{12}$$

and

$$x_0 = t_n x_n + (1-t_n) y_n. \tag{13}$$

Observe that

$$\lim_{n\to+\infty} y_n = y_0 \text{ and } \lim_{n\to+\infty} t_n = 1. \tag{14}$$

From (11) the value $M := \sup\{g(x_n) : n \in \mathbb{N}\} < g(x_0)$. Therefore, applying (9), (12), (13) and (14) we get

$$g(x_0) \leq t_n g(x_n) + (1-t_n)g(y_n) \leq t_n M + (1-t_n) g(y_n)$$

and consequently

$$\frac{g(x_0) - t_n M}{1 - t_n} \leq g(y_n), \quad n \in \mathbb{N},$$

which yields

$$\lim_{n\to+\infty} g(y_n) = +\infty.$$

But this is impossible, because from Lemma 1.5 the function g is upper-semiconti-nuous. This contradiction ends the proof. □

1.3. Differentiability of F-convex functions

As we know, convex functions defined on an open interval are one-sided differentiable and their left-hand derivative is not greater than the right-hand derivative. For F-convex functions, we can obtain a similar result, namely

Theorem 1.7. *Let I be a nonempty and open subinterval of $\mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be a fixed function. If $\liminf_{x\to 0} \frac{F(x)}{x^2} > -\infty$, then every F-convex function $f : I \to \mathbb{R}$ has one-sided derivatives at each point $x \in I$ and $f'_-(x) \le f'_+(x)$.*

Proof. From the assumption, there is $r > 0$ such that

$$c := \inf_{0<|x|\le r} \frac{F(x)}{x^2} > -\infty.$$

The function f is F-convex, thus we get the following inequality

$$f\left(tx + (1-t)y\right) \le tf(x) + (1-t)f(y) - ct(1-t)(x-y)^2$$

for all $x, y \in I$ such that $0 < |x-y| \le r$ and $t \in (0,1)$. For $x = y$ the above inequality is also true. Using now identity (6) we conclude that the function g defined by formula

$$g(x) = f(x) - cx^2, \quad x \in I, \tag{15}$$

satisfies the inequality

$$g\left(tx + (1-t)y\right) \le tg(x) + (1-t)g(y)$$

for all $x, y \in I$ such that $|x-y| \le r$ and $t \in (0,1)$. Thus the function g is locally convex, which together with the result from [18] means that g is a convex function. Taking into consideration (15) we conclude that f has one-sided derivatives at each point $x \in I$ and $f'_-(x) \le f'_+(x)$. □

Observe that the condition $\liminf_{x\to 0}(F(x)/x^2) = -\infty$ does not imply the inequality $f'_- \le f'_+$. We consider an example.

Example 1.8. The function defined by the formula $f(x) = (|x|-1)^2$ $(x \in (-1,1))$ is F-convex with $F(x) = x^2 - 4|x|$.

Proof. This can be seen using a geometric view of F-convex functions. For all $x \in (-1,1)$ the value of $\frac{F(x)}{x^2}$ is negative, therefore inequality (2) is true if and only if for all $x_1, x_2 \in (-1,1)$ such that $x_2 < 0 < x_1$ there is $f(0) \le l(0)$. We calculate $l(0)$ which is equal b in (5).

$$\begin{aligned}
l(0) &= \frac{x_1(f(x_2) - \gamma x_2^2) - x_2(f(x_1) + \gamma x_1^2)}{x_1 - x_2} \\
&= \frac{x_1 f(x_2) - x_2 f(x_1) + \gamma x_1 x_2 (x_1 - x_2)}{x_1 - x_2} \\
&= \frac{\left(x_1(|x_2|-1)^2 - x_2(|x_1|-1)^2 + \frac{(x_1-x_2)^2 - 4(x_1-x_2)}{(x_1-x_2)^2} x_1 x_2 (x_1 - x_2)\right)}{x_1 - x_2} \\
&= \frac{x_1(x_2^2 + 2x_2 + 1) - x_2(x_1^2 - 2x_1 + 1) + (x_1 - x_2 - 4)x_1 x_2}{x_1 - x_2} = 1.
\end{aligned}$$

Hence, f is F-convex, but we have $\liminf_{x\to 0}(F(x)/x^2) = -\infty$ and, moreover, $2 = f'_-(0) \ge f'_+(0) = -2$. □

Taking into consideration the proof of Theorem 1.7 we can deduce the following corollaries.

Corollary 1.9. *If*

$$-\infty < \liminf_{x\to 0} \frac{F(x)}{x^2} < +\infty, \tag{16}$$

then each F-convex function $f : I \to \mathbb{R}$ is c-convex (in the sense of Vial).

Proof. Observe that there exists a decreasing and tending to zero sequence (r_n) such that $c_n := \inf_{0<|x|\le r_n} \frac{F(x)}{x^2} > -\infty$. The sequence (c_n) is non-decreasing and bounded, thus it has a limit c. From the proof of Theorem 1.7 we conclude that the functions

$$g_n(x) = f(x) - c_n x^2, \quad x \in I,$$

are convex for each $n \in \mathbb{N}$. Thus the function $g(x) = f(x) - cx^2$ is convex and it means that f is c-convex. □

This result generalizes the results obtained by Nikodem and Páles in [18]. Namely, they showed that if a function is approximately convex of order $p > 2$, then it must be convex. We obtain it from Corollary 1.9 fixing $F(x) = -cx^p$. It is also related to Rolewicz's result [24], stating that if a function $f : I \to \mathbb{R}$ is α-paraconvex, that is

$$f\left(tx + (1-t)y\right) \le tf(x) + (1-t)f(y) + c\alpha\left(|x-y|\right) \tag{17}$$

for all $x, y \in I$ and $t \in (0,1)$, where $\alpha : [0,+\infty) \to [0,+\infty)$ is a fixed nondecreasing function and $c > 0$, must be convex if $\lim_{x\to 0^+}(\alpha(x)/x^2) = 0$ and f is absolutely continuous. Of course, in our result we have F-convexity which is a stronger condition than (17), provided F is a nondecreasing and even function. But we do not need absolute continuity of f.

Based on the results presented in [14] we can conclude that there is no F-convex function if $\liminf_{x\to 0}(F(x)/x^2) = +\infty$. This can also be concluded from our investigation in a different way.

Corollary 1.10. *If* $\liminf_{x\to 0} \frac{F(x)}{x^2} = +\infty$*, then an F-convex function $f : I \to \mathbb{R}$ does not exist.*

Proof. If there was an F-convex function f, than the function $f(\cdot) - c(\cdot)^2$ would be convex with arbitrarily large c, which is impossible. □

Remark 1.11. If $\liminf_{x\to 0} \frac{F(x)}{x^2} = -\infty$, then an F-convex function $f : I \to \mathbb{R}$ does not need to be c-convex (see Example 1.8).

In [18] the authors proved that convexity property is localizable in the following sense: a function $f : I \to \mathbb{R}$ is convex if and only if for each point $x \in I$, there exists a neighbourhood U of x such that f is convex on $U \cap I$. Using this result and taking into consideration Corollary 1.9 and its proof we get:

Corollary 1.12. *Let a function $F : \mathbb{R} \to \mathbb{R}$ satisfies* (16)*. If a function $f : I \to \mathbb{R}$ is locally F-convex, then it is c-convex.*

Proof. Let U_x denote the neighbourhood of x such that f is F-convex on $U_x \cap I$. Each function g_n defined in the proof of Corollary 1.9 is convex on the interval $U_x \cap \left(x - \frac{r_n}{2}, x - \frac{r_n}{2}\right)$. This means that g_n are locally convex and consequently convex. Therefore, as in the proof of Corollary 1.9, f is c-convex. $\square$

1.4. An F-convex function as a sum of two functions

From the results presented in [21] we conclude that for each function $f : I \to \mathbb{R}$ of the form $f = g + h$ where $g : I \to \mathbb{R}$ is convex and $h : I \to \mathbb{R}$ is $\frac{1}{2}$-Lipschitz, i.e.

$$|h(x) - h(y)| \le \frac{1}{2}\,|x - y|\,, \quad x, y \in I,$$

the following inequality holds

$$f\,(tx + (1-t)y) \le tf(x) + (1-t)f(y) + t(1-t)|x - y|$$

for all $x, y, \in I$ and $t \in (0,1)$. It means that f is F-convex with $F(x) = -|x|$. In Theorem 1.13 we propose a generalization of this result. Observe that for a function $F : \mathbb{R} \to \mathbb{R}$ with the property

$$F(tx) \ge tF(x), \quad x \in \mathbb{R} \text{ and } t \in (0,1), \tag{18}$$

and each function $f : I \to \mathbb{R}$ of the form $f = g + h$, where $g : I \to \mathbb{R}$ is convex and $h : I \to \mathbb{R}$ satisfies the condition

$$2\,(h(x) - h(y)) \le -F(x - y), \tag{19}$$

we have

$$\begin{aligned} f(tx + (1-t)y) &= g\,(tx + (1-t)y) + h\,(tx + (1-t)y) \\ &\le tg(x) + (1-t)g(y) + t\,(h\,(tx + (1-t)y) - h(x)) + th(x) \\ &\qquad + (1-t)\,(h\,(tx + (1-t)y) - h(y)) + (1-t)h(y) \\ &\le tf(x) + (1-t)f(y) - \tfrac{1}{2}tF\,((1-t)(x-y)) - \tfrac{1}{2}(1-t)F\,(t(x-y)) \\ &\le tf(x) + (1-t)f(y) - \tfrac{1}{2}t(1-t)F\,((x-y)) - \tfrac{1}{2}(1-t)tF\,(x-y) \\ &= tf(x) + (1-t)f(y) - t(1-t)F\,(x-y)\,. \end{aligned}$$

In view of the above we get:

Theorem 1.13. *Let $F : \mathbb{R} \to \mathbb{R}$ be a fixed function satisfying* (18)*. Then for each function $f : I \to \mathbb{R}$ of the form $f = g + h$ where $g : I \to \mathbb{R}$ is convex and $h : I \to \mathbb{R}$ satisfies condition* (19) *the function f is F-convex.*

Corollary 1.14. *If a function $f : I \to \mathbb{R}$ is of the form $f = g + h$ where $g : I \to \mathbb{R}$ is convex and $h : I \to \mathbb{R}$ satisfies the Hölder condition with an exponent $p \in (0,1]$ and constant $L = \frac{1}{2}$, then f is an approximately convex functions of order p. For $p = 1$ we get the result obtained by Páles* [21]*.*

At the end we present a few simple observations connected with condition (19).

Observation 1.15. (1) If $F(x) = -|x|$, then condition (19) becomes the upper estimation of Lipschitz condition.

(2) If $p > 0$ and $F(x) = -|x|^p$, then condition (19) becomes the upper estimation of Hölder condition.

(3) If $F(x - y) > 0$ and $F(y - x) > 0$ for some different $x, y \in \mathbb{R}$, then a function h such that x and y belong to its domain and (19) holds does not exist.

(4) If $\lim_{x\to 0}(F(x)/x) = 0$ and a differentiable function h satisfies (19), than h is a constant function.

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