Journal of Convex Analysis
Volume 28 (2021), No. 3, 771–794

On Some Nonlinear Parabolic Equations with Nonmonotone Multivalued Terms

Mitsuharu Ôtani*
Department of Applied Physics, School of Science and Engineering
Waseda University, Tokyo, Japan
otani@waseda.jp

Vasile Staicu†
CIDMA – Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro, Aveiro, Portugal
vasile@ua.pt

Received: February 26, 2020
Accepted: April 26, 2020

We study the local and the global existence of solutions to a class of nonlinear parabolic initial-boundary value problems driven by the equation

$$\frac{\partial u\left(x,t\right)}{\partial t}-\triangle u\left(x,t\right)\in-\partial\Phi\left(u\left(x,t\right)\right)+G\left(x,t,u\left(x,t\right)\right),\left(x,t\right)\in Q_{T},$$

where $\partial\Phi$ denotes the subdifferential (in the sense of convex analysis) of a proper, convex and lower semicontinuous function $\Phi\colon\mathbb{R}\to[0,\infty]$, $\Omega\subseteq\mathbb{R}^{N}$ is a bounded open set, $T>0$, $Q_{T}:=\Omega\times[0,T]$, and $G\colon Q_{T}\times\mathbb{R}\to2^{\mathbb{R}}$ is a multivalued mapping whose growth order with respect to u is Sobolev sub-critical.

We prove two local existence results: one for the case where the multivalued mapping $u\mapsto G\left(\cdot,\cdot,u\right)$ is upper semicontinuous with closed convex values and the second one deals with the case when $u\mapsto G\left(\cdot,\cdot,u\right)$ is lower semicontinuous with closed (not necessarily convex) values. We also give two types of results concerning the global continuation of local solutions. One is for any large data and the other one for small data. Finally, we exemplify the applicability of our results.

Keywords: Nonlinear parabolic equations, nonmonotone multivalued terms, initial-boundary value problems, existence results, local solutions, small and large global solutions.

2010 Mathematics Subject Classification: 35K55, 35K61, 35B60, 35G25.

1. Introduction

In the present paper we deal with the following nonlinear parabolic inclusion

$$\frac{\partial u\left(x,t\right)}{\partial t}-\triangle u\left(x,t\right)\in-\partial\Phi\left(u\left(x,t\right)\right)+G\left(x,t,u\left(x,t\right)\right),\left(x,t\right)\in Q_{T}\tag{1}$$

*Partly supported by the Grant-in-Aid for Scientific Research, ♯15K13451, the Ministry of Education, Culture, Sports, Science and Technology, Japan.
†Partially supported by The Center for Research and Development in Mathematics and Applications (CIDMA) through the Portuguese Foundation for Science and Technology (FCT – Fundação para a Ciência e a Tecnologia), references UID/MAT/04106/2019(CIDMA) and UIDB/04106/2020.

ISSN 0944-6532 / \$ 2.50

coupled with initial-boundary conditions:

$$\begin{cases} u(x,t) = 0, & (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2}$$

Here Ω is assumed to be a bounded open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $T > 0$, $Q_T := \Omega \times [0,T]$, $\partial\Phi$ denotes the subdifferential (in the sense of convex analysis) of a proper, convex and lower semicontinuous function $\Phi\colon \mathbb{R} \to [0,\infty]$ and $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued mapping.

Such inclusions appear naturally in the study of parabolic problems with discontinuous nonlinearities which arise from simplified models in the description of porous medium combustion (see [12], [13]), chemical reactor theory (see [14]), and game theory (see [11] and [18] for details and their references). It is well known that such problems need not have a solution, and to guarantee the existence of a solution, we need to extend the discontinuous nonlinearity to a multivalued mapping by filling the jumps at the discontinuity points of the nonlinearity (see [25]).

The dynamic (parabolic) problem with discontinuities are not investigated so extensively as the corresponding stationary problem. However, several methods has been developed and involved during the last decades to study the existence of solutions to parabolic boundary value problems with discontinuous or multivalued nonlinearities: upper and lower solutions together with a generalized iteration method in [7], [6]; pseudomonotone operators, multivalued analysis together with truncation and penalization techniques and the method of upper lower solutions in [24]. Upper and lower solutions method was also involved in [8] and [9] to prove the existence and extremality of solutions for initial boundary-value problem to parabolic inclusions whose multivalued nonlinearity is characterized by Clarke's generalized gradient of some locally Lipschitz function, and also in [5].

We also refer to [3] where existence results were obtained for homogeneous Cauchy-Dirichlet problem of a class of parabolic equations with either Carathéodory or discontinuous nonlinear terms; and [16] where the existence of solutions to parabolic problems with discontinuous and nonmonotone nonlinearity was obtained by passing to a multivalued version by filling in the gaps at the discontinuity points. Many results have been obtained in a variational framework since the pioneering work of Chang [10] dealing with partial differential equations involving a discontinuous reaction term.

The goal of this paper is to adapt and improve the techniques and arguments developed in [23] for the stationary case, in order to obtain existence results for the parabolic inclusion (1) with the initial and boundary conditions (2) which generalize corresponding results given by many authors, especially given in [21], [22].

Our approach uses tools from the multivalued analysis, together with the theory of nonlinear operators of monotone type and methods from the theory of nonlinear evolution equations.

We prove two local existence results: one for the case where the multivalued mapping $u \mapsto G(\cdot,\cdot,u)$ is upper semicontinuous (u.s.c.) with closed convex values and the second one deals with the case when $u \mapsto G(\cdot,\cdot,u)$ is lower semicontinuous (l.s.c.) with closed (not necessarily convex) values.

For both cases, following the strategy in [21], we apply Schauder-Tikhonov-type fixed point theorems for the mapping $\mathcal{G}\colon h \mapsto G(x,t,u_h)$, where u_h is the unique solution of the equation (1) with $G(x,t,u)$ replaced by h. With the aid of results in [23], it is shown that $\mathcal{G}$ becomes u.s.c. or l.s.c. with respect to the weak topology of $\mathcal{H} := L^2(0,T;L^2(\Omega))$ according to the case where $u \mapsto G(\cdot,\cdot,u)$ is u.s.c. or l.s.c. respectively.

Another crucial step is to show that there exists $R > 0$ such that $\mathcal{G}$ maps the set $K_R := \{ h \in \mathcal{H};\ \|h\|_{\mathcal{H}} \leq R \}$ into itself. For this purpose, we rely on the arguments developed in [21, 23] based on some interpolation inequality.

Our advantage here lies in the fact that our treatment allows Sobolev sub-critical growth order of $G(x,t,u)$ with respect to u, which are not achieved in other papers.

We also discuss the extension of large or small local solutions along the line of arguments developed in [21].

The paper has the following structure. In Section 2, we recall some notations and basic definitions from the nonlinear operator theory and multivalued analysis used in the following sections. In Section 3 we prove some auxiliary results. Section 4 is devoted to local existence of solutions to problem $(1)-(2)$. We obtain two such existence results: one for the case where G is closed convex valued, upper semicontinuous with respect to the third variable (Section 4.1) and the other for the case where the multivalued mapping G is lower semicontinuous with closed (not necessarily convex) values (Section 4.2). In Section 5 we study the global existence of solutions, namely, the existence of *large global solutions* without assuming the smallness of the given data (Section 5.1) and the existence of *small global solutions* when the given data are sufficiently small (Section 5.2). In Section 6 we exemplify the applicability of our results.

2. Notations and preliminaries

For easy reference, in this section we recall some notations and basic definitions from the nonlinear operator theory and the multivalued analysis, that we shall use in the sequel. For further details we refer to [1], [2], [4] and [19].

Let X and Y be topological spaces and 2^Y be the family of all subsets of Y. A multivalued mapping $F\colon X \to 2^Y$ is said to be *upper semi-continuous* (u.s.c., for short) on X if for every $x_0 \in X$ and for each open subset $V \subset Y$ such that $F(x_0) \subset V$, there exists a neighborhood U of x_0 such that $F(x) \subset V$ for all $x \in U$. We say that the multivalued mapping $F\colon X \to 2^Y$ is *lower semi-continuous* (l.s.c., for short) on X if for each $x_0 \in X$ and for every $y_0 \in F(x_0)$ and any neighborhood V of y_0, there exists a neighborhood U of x_0 such that $F(x) \cap V \neq \varnothing$ for all $x \in U$.

Equivalently, a multivalued mapping $F\colon X \to 2^Y$ is *upper semicontinuous* on X if and only if, for every closed subset C of Y the set

$$F^-(C) := \{x \in X : F(x) \cap C \neq \varnothing\}$$

is closed in X, and *lower semicontinuous* on X if and only if

$$F^+(C) := \{x \in X : F(x) \subset C\}$$

is closed in X for each closed subset C of Y.

It is well known that if $F\colon X \to 2^Y$ is upper semicontinuous on X with closed values, then its *graph*

$$Gr(F) := \{(x, y) \in X \times Y : y \in F(x)\}$$

is closed in $X \times Y$. Conversely, if $F\colon X \to 2^Y$ has a closed graph and if for each $x \in X$, there exists a neighborhood U of x such that $F(U) := \bigcup_{x\in U} F(x)$ is precompact, then F is u.s.c. on X.

By a *selection* of a multivalued mapping $F\colon X \to 2^Y$ we mean a function $f\colon X \to Y$ such that

$$f(x) \in F(x) \ \text{ for all } x \in X.$$

Let (Ω, Σ) be a measurable space. We are particularly interested in the case where Σ is the σ-algebra $\mathcal{L}(Q_T)$ of Lebesgue measurable subsets of $Q_T = \Omega \times [0, T]$, with $T > 0$ and $\Omega \subseteq \mathbb{R}^N$ a given open set, as well as the case when $\Sigma = \mathcal{L}(Q_T) \otimes \mathcal{B}(X)$, where X is a Banach space and $\mathcal{L}(Q_T) \otimes \mathcal{B}(X)$ is the product σ-algebra on $Q_T \times X$ generated by sets of the form $A \times B$ with $A \in \mathcal{L}(Q_T)$ and $B \in \mathcal{B}(X)$ with $\mathcal{B}(X)$ being the Borel σ-algebra of X.

Let $(X, \|\cdot\|)$ be a separable Banach space. A multivalued mapping with closed values $\Psi\colon \Omega \to 2^X$ is said to be Σ-*measurable* (or simply, *measurable*) if for every open set $U \subset X$, we have

$$\Psi^-(U) := \{\omega \in \Omega : \Psi(\omega) \cap U \neq \varnothing\} \in \Sigma.$$

It is known that $\Psi\colon \Omega \to 2^X$ is measurable if and only if for every $x \in X$, the map

$$\omega \mapsto d(x, \Psi(\omega)) := \inf\{\|z - x\| : z \in \Psi(\omega)\}$$

is a measurable $\overline{\mathbb{R}^+} = \mathbb{R}^+ \cup \{\infty\}$-valued function (see [19], Corollary 19, p. 142).

A multivalued mapping $\Psi\colon \Omega \to 2^Y$ with nonempty values is said to be *graph measurable* if

$$Gr(\Psi) := \{(\omega, z) \in \Omega \times X : z \in \Psi(\omega)\} \in \Sigma \otimes \mathcal{B}(Y).$$

For multivalued mappings with closed values, the measurability implies the graph measurability, while the converse is true if Σ is complete.

Let $L^p(\Omega)$ be the Banach space of (equivalence classes of) measurable functions $u\colon \Omega \to \mathbb{R}$ such that $x \mapsto |u(x)|^p$ is Lebesgue integrable, endowed with the usual norm

$$\|u\|_{L^p} = \left(\int_\Omega |u(x)|^p\,dx\right)^{\frac{1}{p}}.$$

A set $K \subseteq L^p(\Omega, Y)$ is said to be *decomposable* if for all u, $v \in K$ and all $A \in \Sigma$ we have $u\chi_A + v\chi_{\Omega\setminus A} \in K$, where χ_A denotes the characteristic function of A defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise.

Clearly the set $\mathcal{S}^p_\Psi$ is decomposable.

In the remaining of this section we collect some definitions and properties concerning maximal monotone mappings.

Assume that H is a real Hilbert space with inner product $(\cdot,\cdot)_H$ and norm $\|\cdot\|_H$. Assume $A\colon H \to 2^H$ is a maximal monotone operator. The *minimal selection* (or *minimal section*) of A is the function $A^0\colon H \to H$ satisfying the following conditions:

$$A^0(x) \in A(x) \text{ and } \|A^0(x)\|_H = \inf\{\|\xi\|_H : \xi \in A(x)\} \ \forall x \in D(A).$$

Recall that, the graph of any maximal monotone operator is *demiclosed*, i.e., closed in $H \times H_w$, where H_w denotes the space H furnished with the weak topology.

Let $\varphi\colon H \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function. We say that φ is *proper* if its *effective domain*

$$D(\varphi) := \{x \in H : \varphi(x) < +\infty\}$$

is nonempty. The multivalued mapping $\partial\varphi\colon H \to 2^H$ defined by

$$\partial\varphi(x) = \{g \in H : \varphi(y) - \varphi(x) \geq (g, y - x)_H \text{ for all } y \in H\} \tag{3}$$

is called the *subdifferential* of φ (in the sense of convex analysis). It is known that the *subdifferential* $\partial\varphi$ of a proper lower semicontinuous convex function φ is a maximal monotone operator and

$$D(\partial\varphi) := \{x \in H : \partial\varphi(x) \neq \varnothing\} \subset D(\varphi).$$

We shall use $\partial^0\varphi$ instead of $(\partial\varphi)^0$ to denote the minimal selection of the maximal monotone operator $\partial\varphi$.

3. Auxiliary results

In this section we always assume that Ω is a bounded open subset of $\mathbb{R}^N$ with finite Lebesgue measure denoted by $|\Omega|$, $N \geq 1$, $\Phi\colon \mathbb{R} \to [0, \infty]$ is a proper convex and lower semicontinuous function, $T > 0$, $Q_T := \Omega \times [0, T]$, and $\mathcal{H} := L^2(0, T; L^2(\Omega))$.

Definition 3.1. Let $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ be a multivalued mapping. The multivalued mapping $\widetilde{G}\colon \mathcal{H} \to 2^{\mathcal{H}}$ defined by

$$\widetilde{G}(u) = \{g \in \mathcal{H} : g(x,t) \in G(x,t,u(x,t)) \text{ a.e. } (x,t) \in Q_T\} \tag{4}$$

is called the *realization* of G in $\mathcal{H}$.

Definition 3.2. We say that the realization $\widetilde{G}$ of G in $\mathcal{H}$ is *a.e.-demiclosed* if for any sequence $(u_n)_{n\in\mathbb{N}}$ of functions from Q_T into $\mathbb{R}$ which converges almost everywhere in Q_T to a function $u\colon Q_T \to \mathbb{R}$ and for any sequence $(g_n)_{n\in\mathbb{N}}$ of functions from Q_T into $\mathbb{R}$ such that

$$g_n(x,t) \in G(x,t,u_n(x,t)) \text{ for each } n \in \mathbb{N} \text{ and almost all } (x,t) \in Q_T,$$

which converges weakly in $\mathcal{H}$ to a function $g \in \mathcal{H}$, then one has $g \in \widetilde{G}(u)$, that is,

$$g(x,t) \in G(x,t,u(x,t)) \text{ for almost all } (x,t) \in Q_T.$$

Proposition 3.3. *Let $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ be a multivalued mapping with nonempty closed convex values such that :*

(i) *For almost all $(x,t) \in Q_T$, $G(x,t,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous,*

(ii) *For each $u \in \mathbb{R}$, $G(\cdot,\cdot,u) : Q_T \to 2^{\mathbb{R}}$ is $\mathcal{L}(Q_T)$-measurable.*

Then the realization $\widetilde{G}$ of G in $\mathcal{H}$ is a.e.-demiclosed.

Proof. Assume $(u_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are sequences of functions from Q_T into $\mathbb{R}$ such that $(u_n)_{n\in\mathbb{N}}$ converges almost everywhere in Q_T to $u\colon Q_T\to \mathbb{R}$, $(g_n)_{n\in\mathbb{N}}$ converges weakly in $\mathcal{H}$ to a function $g\in\mathcal{H}$ and

$$g_n(x,t)\in G(x,t,u_n(x,t)) \text{ for each } n\in\mathbb{N} \text{ and almost all } (x,t)\in Q_T. \qquad (5)$$

Since $(g_n)_{n\in\mathbb{N}}$ converges weakly in $\mathcal{H}$ to a function g, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ of convex combinations of $\left\{g_n,g_{n+1},\cdots,g_{k(n)}\right\}$, which converges strongly in $\mathcal{H}$ to the function g. Here we put, for each $n\in\mathbb{N}$,

$$h_n(x,t)=\sum_{i=n}^{k(n)} a_i(n)\,g_i(x,t) \qquad \text{for} \quad (x,t)\in Q_T,$$

where $a_i(n)\in[0,1]$ for $n\le i\le k(n)$ and $\sum_{i=n}^{k(n)} a_i(n)=1$. Then there exists a subsequence of $(h_n)_{n\in\mathbb{N}}$ denoted again by $(h_n)_{n\in\mathbb{N}}$, which converges to g almost everywhere in Q_T. Let

$$\begin{aligned} Q_0=\{\,(x,t)\in Q_T\,;\, u_n(x,t)\to u(x,t)\,,\ h_n(x,t)\to g(x,t)\,,\ G(x,t,\cdot) \text{ is u.s.c.}\\ \text{and} \ \ g_n(x,t)\in G(x,t,u_n(x,t)) \quad \text{for each} \quad n\in\mathbb{N}\,\}\,. \end{aligned}$$

Clearly $Q_T\backslash Q_0$ is a null set. Now let $(x_0,t_0)\in Q_0$ and let V be an open half-space in $\mathbb{R}$ such that $G(x_0,t_0,u(x_0,t_0))\subset V$.

By virtue of the upper semicontinuity of $G(x_0,t_0,\cdot)$ at $u(x_0,t_0)$, there exists a neighborhood U of $u(x_0,t_0)$ such that $G(x_0,t_0,u)\subset V$ for all $u\in U$. Since $(u_n(x_0,t_0))_{n\in\mathbb{N}}$ converges to $u(x_0,t_0)$ as $n\to\infty$, there exists $n_U\in\mathbb{N}$ such that $u_n(x_0,t_0)\in U$ for all $n\ge n_U$, whence follows that $G(x_0,t_0,u_n(x_0,t_0))\subset V$ for all $n\ge n_U$.

Here we note that by (5), $g_n(x_0,t_0)\in G(x_0,t_0,u_n(x_0,t_0))\subset V$ for all $n\ge n_U$ and

$$h_n(x_0,t_0)=\sum_{i=n}^{k(n)} a_i(n)\,g_i(x_0,t_0)\in co\left(\bigcup_{n\ge n_U} G(x_0,t_0,u_n(x_0,t_0))\right)\subset V, \qquad (6)$$

where *co* stands for the convex hull. Now, passing to the limit $n\to\infty$, in (6) we get $g(x_0,t_0)\in\overline{V}$. This is true for each open half-space V containing $G(x_0,t_0,u(x_0,t_0))$. Since $G(x_0,t_0,u(x_0,t_0))$ is a closed and convex set, we find

$$\begin{aligned} &G(x_0,t_0,u(x_0,t_0))\\ &\quad=\bigcap\left\{\,\overline{V};V \text{ open half-space in } \mathbb{R} \text{ with } G(x_0,t_0,u(x_0,t_0))\subset V\,\right\}. \end{aligned}$$

Thus, we conclude that

$$g(x_0,t_0)\in G(x_0,t_0,u(x_0,t_0)) \text{ for each } (x_0,t_0)\in Q_0,$$

which completes the proof. □

Assume now that $\Phi\colon\mathbb{R}\to[0,+\infty]$ is a proper lower semicontinuous convex function with $\Phi(0)=0$ and define the functional φ by

$$\varphi(u)=\begin{cases} \frac{1}{2}\displaystyle\int_\Omega \|\nabla u(x)\|_{L^2}^2\,dx+\int_\Omega \Phi(u(x))\,dx & \text{if} \ \ u\in D(\varphi)\,,\\ +\infty & \text{otherwise,}\end{cases} \qquad (7)$$

with domain $D(\varphi) = \left\{ u \in H_0^1(\Omega) \text{ and } \widetilde{\Phi}(u) := \int_\Omega \Phi(u(x))\, dx < +\infty \right\}.$

Then φ becomes a proper lower semicontinuous convex functional from $\mathcal{H}$ into $[0, +\infty]$ and its subdifferential is given by

$$\partial\varphi(u) = -\triangle u + \partial\Phi(u) \tag{8}$$

where $\triangle u$ is the Laplacian of $u \in H_0^1(\Omega)$, with domain

$$D(\partial\varphi) = \left\{ u \in D(\varphi) \ : \ \begin{array}{l} u \in H^2(\Omega) \cap H_0^1(\Omega), \text{ there exists } b \in L^2(\Omega) \\ \text{such that } b(x) \in \partial\Phi(u(x)) \text{ a.e. } x \in \Omega \end{array} \right\}.$$

Moreover, if $z = -\triangle u + b \in \partial\varphi(u)$ with $b \in \partial\Phi(u)$, then it holds that

$$\|z\|_{L^2}^2 \geq \|\triangle u\|_{L^2}^2 + \|b\|_{L^2}^2 \text{ and } (-\triangle u, b)_H \geq 0, \tag{9}$$

(see Lemma 1 in [23]). Thus we have the following standard result from Kōmura-Brézis theory (see Theorem 3.6 of H. Brézis [4]).

Proposition 3.4. *For any $h \in \mathcal{H}$ and $u_0 \in D(\varphi)$, the problem*

$$\left\{ \begin{array}{l} \dfrac{\partial u}{\partial t}(x,t) - \triangle u(x,t) \in -\partial\Phi(u(x,t)) + h(x,t), \ \ (x,t) \in Q_T, \\ u(x,t) = 0, \ \ (x,t) \in \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), \ \ x \in \Omega, \end{array} \right. \tag{E_h}$$

admits a unique solution $u \in C([0,T]; L^2(\Omega))$ satisfying $\frac{\partial u}{\partial t}, \ \triangle u \in L^2(0,T; L^2(\Omega))$.

We introduce now the following growth condition on G.

(GC) We say that a multivalued mapping $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ satisfies (**GC**), if there exist nonnegative numbers $k \in [0,1)$, q, C_q and a function $a(\cdot) \in L^1(Q_T)$ such that for a.e. $(x,t) \in Q_T$ and for all $u \in D(\partial\Phi)$ we have

$$|||G(x,t,u)|||^2 \leq |a(x,t)| + k\left|\partial^0\Phi(u)\right|^2 + C_q\,|u|^{2(q-1)}, \tag{10}$$

where $|||G(x,t,u)||| := \sup\{\ |\xi| \ ; \ \xi \in G(x,t,u)\ \}$ and $1 < q < 2^*$ with

$$2^* := \left\{ \begin{array}{ll} \infty & \text{if } N = 1 \text{ or } 2, \\ \frac{2N}{N-2} & \text{if } N \geq 3. \end{array} \right. \tag{11}$$

Here we introduce the multivalued mapping $\mathcal{G}_S$, $S \in (0,T]$ (which plays an important role in the later arguments) by

$$\mathcal{G}_S(h) := \left\{\ g \in L^2(Q_S) \ ; \ g(x,t) \in G(x,t,u_h(x,t)) \ \text{ a.e. } (x,t) \in Q_S \right\}, \tag{12}$$

with

$$|||\mathcal{G}_S(h)|||_{L^2(Q_S)} := \sup\{\ \|g(x,t)\|_{L^2(Q_S)} \ ; \ g(x,t) \in G(x,t,u_h(x,t))\ \}, \tag{13}$$

where u_h is a solution of (E_h). For $R > 0$, $S \in (0,T]$, we put

$$\mathbf{K}_R^S := \{\ h \in L^2(Q_S);\ \|h\|_{L^2(Q_S)}^2 \leq R^2\ \}. \tag{14}$$

Then one can show that $\mathcal{G}_S$ maps $\mathbf{K}_R^S$ into itself for a sufficiently small S.

Proposition 3.5. *Assume that* $\Phi\colon \mathbb{R} \to [0, +\infty]$ *is a proper lower semicontinuous convex function with* $\Phi(0) = 0$ *and* $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ *satisfies* $(\mathbf{GC})$. *Let* $u_0 \in D(\varphi)$ *and set*

$$R^2 := \max\left\{\frac{4}{1-k}\,\|a\|_{L^1(Q_T)}\,,\frac{8}{1-k}\varphi(u_0)\,,1\right\}. \tag{15}$$

Then there exists a positive number T_0 *depending on* k, $\|a\|_{L^1(Q_T)}$ *and* $\varphi(u_0)$ *such that* $\mathcal{G}_{T_0}(\cdot)$ *maps* $\mathbf{K}_R^{T_0}$ *into itself.*

To prove this proposition, we prepare the following lemma.

Lemma 3.6. *Let* $q \in (2, 2^*)$, *then for any* $\eta > 0$, *there exists* $K_q(\eta) > 0$ *such that*

$$\|u\|_{L^{2(q-1)}}^{2(q-1)} \le \eta\,\|\triangle u\|_{L^2}^2 + K_q(\eta)\,\|\nabla u\|_{L^2}^{\gamma} \quad \forall u \in H^2(\Omega) \cap H_0^1(\Omega),$$

$$\gamma = \begin{cases} 2(q-1) & \text{if } N \in \{1,2\} \text{ or } N \ge 3 \text{ and } q \le \frac{2(N-1)}{N-2} \\ 2 + \frac{4\,(q-2)}{2N-(N-2)\,q} & \text{if } N \ge 3 \text{ and } q > \frac{2(N-1)}{N-2}. \end{cases} \tag{16}$$

Proof. For the case where $N \in \{1, 2\}$, (16) obviously holds with $\eta = 0$, since $H_0^1(\Omega)$ is continuously embedded in $L^r(\Omega)$ for all $r \in (1, \infty)$.

For the case where $N \ge 3$ and $2(q-1) \le \frac{2N}{N-2}$, equivalently $q \le \frac{2(N-1)}{N-2}$, we can take $\eta = 0$ and $\gamma = 2(q-1)$, since $H_0^1(\Omega)$ is continuously embedded in $L^{2(q-1)}(\Omega)$.

As for the case $N \ge 3$ and $q > \frac{2(N-1)}{N-2}$, the following interpolation inequality holds (see, e.g., [20])

$$\|u\|_{L^{2(q-1)}} \le C_{GN}\,\|\triangle u\|_{L^2}^{\theta}\,\|\nabla u\|_{L^2}^{1-\theta} \;\; \forall u \in H^2(\Omega) \cap H_0^1(\Omega), \tag{17}$$

$$\frac{1}{2\,(q-1)} = \theta\left(\frac{1}{2} - \frac{2}{N}\right) + (1-\theta)\,\frac{N-2}{2N}, \tag{18}$$

where C_{GN} is the embedding constant. Hence we have

$$2\,(q-1)\,\theta = (N-2)\,q - 2N + 2 > 0,$$

so $2\,(q-1)\,\theta < 2$ holds if and only if $q < 2^*$. Then applying Young's inequality with the following Hölder conjugate exponents

$$(p,\ p') = \left(\frac{2}{(N-2)\,q - 2N + 2}, \frac{2}{2N - (N-2)\,q}\right),$$

we get
$$\|u\|_{2(q-1)}^{2(q-1)} \le \eta\,\|\triangle u\|_{L^2}^2 + K_q(\eta)\,\|\nabla u\|_{L^2}^{\gamma}\,,$$

with
$$K_q(\eta) := C_{GN}^{\frac{4(q-1)}{2N-(N-2)q}}\,(p')^{-1}\,p^{-\frac{p'}{p}}\,\eta^{-\frac{(N-2)\,q-2N+2}{2N-(N-2)\,q}}$$

and
$$\gamma = 2\,(q-1)\,(1-\theta)\,p' = 2 + \frac{4\,(q-2)}{2N-(N-2)\,q} > 2. \qquad \square$$

Proof of Proposition 3.5. For the case where $q \in (1, 2]$, there exists $\widetilde{q} \in (2, 2^*)$ such that $|u|^{2(q-1)} \le |u|^{2(\widetilde{q}-1)} + 1$, G satisfies $(\mathbf{GC})$ with q, $|a|$ replaced by $\widetilde{q}$, and $|a| + C_q \in L^1(Q_T)$.

Hence we have only to consider the case where $2 < q$.

Let $h \in \mathbf{K}_R^S$ and multiply (E_h) by $z_h(t) = -\triangle u_h(t) + b_h(t) \in \partial\varphi(u_h(t))$ with $b_h(t) \in \partial\Phi(u_h(t))$. We then get

$$\frac{d}{dt}\varphi(u_h(t)) + \|z_h(t)\|_{L^2}^2 \leq \|z_h(t)\|_{L^2}^2 \|h(t)\|_{L^2}^2 = \frac{1}{2}\|z_h(t)\|_{L^2}^2 + \frac{1}{2}\|h(t)\|_{L^2}^2$$

and

$$\sup_{0\leq t\leq S} 2\varphi(u_h(t)) + \int_0^S \|z_h(t)\|_{L^2}^2 \, dt \leq 2\varphi(u_0) + R^2. \tag{19}$$

Here since (9) assures that $\|\triangle u_h(t)\|_{L^2}^2 + \|b_h(t)\|_{L^2}^2 \leq \|z_h(t)\|_{L^2}^2$, we can obtain the following estimates

$$\int_0^S \left\|\partial^0\Phi(u_h)\right\|_{L^2}^2 dt \leq \int_0^S \|b_h(t)\|_{L^2}^2 dt \leq 2\,\varphi(u_0) + R^2, \tag{20}$$

$$\int_0^S \|\triangle u_h(t)\|_{L^2}^2 dt \leq 2\,\varphi(u_0) + R^2, \tag{21}$$

$$\sup_{0\leq t\leq S} \|\nabla u_h(t)\|_{L^2}^2 \leq 2 \sup_{0\leq t\leq S} \varphi(u_h(t)) \leq 2\,\varphi(u_0) + R^2. \tag{22}$$

Then in view of (15) and (20), we have $2\varphi(u_0) + R^2 \leq \left(\frac{1-k}{4} + 1\right) R^2$, and

$$\begin{cases} k \displaystyle\int_0^S \left\|\partial^0\Phi(u_h)\right\|_{L^2}^2 dt \leq k\left(2\,\varphi(u_0) + R^2\right) \leq k\left(1 + \frac{1-k}{4}\right) R^2, \\ \|a\|_{L^1(Q_S)} \leq \dfrac{1-k}{4} R^2. \end{cases} \tag{23}$$

Furthermore, by virtue of Lemma 3.6 with $\eta = \frac{1-k}{C_q(5-k)}$, (21), (22) and (23), we get

$$\begin{aligned} C_q \int_0^S \|u_h\|_{L^{2(q-1)}}^{2(q-1)} dt &\leq \eta\, C_q \int_0^S \|\triangle u_h(t)\|_{L^2}^2 dt + C_q\, K_q(\eta) \int_0^S \|\nabla u_h\|_{L^2}^\gamma dt \\ &\leq \frac{1-k}{4} R^2 + C_q\, K_q(\eta) \left(\frac{1-k}{4} + 1\right)^{\frac{\gamma}{2}} R^\gamma S. \end{aligned} \tag{24}$$

Therefore, (10) together with (23) and (24) gives

$$\begin{aligned} |||\mathcal{G}_S(h)|||_{L^2(Q_S)}^2 &= \int_{Q_S} |||G(x,t,u_h)|||^2 \, dx\, dt \\ &\leq \int_{Q_S} |a(x,t)|\, dxdt + k\int_{Q_S} \left|\partial^0\Phi(u_h)\right|^2 dxdt + C_q \int_{Q_S} |u|^{2(q-1)} dxdt \\ &\leq \left(\frac{1-k}{4} + k\left(1 + \frac{1-k}{4}\right) + \frac{1-k}{4} + C_q K_q(\eta)\left(\frac{1-k}{4} + 1\right)^{\frac{\gamma}{2}} R^{\gamma-2} S\right) R^2 \\ &\leq \left(k + \frac{3}{4}(1-k) + C_q K_q(\eta)\left(\frac{1-k}{4} + 1\right)^{\frac{\gamma}{2}} R^{\gamma-2} S\right) R^2. \end{aligned} \tag{25}$$

Here we take a (sufficiently small) $T_0 \in (0, T]$ such that

$$C_q\, K_q\,(\eta) \, \left(\frac{1-k}{4}+1\right)^{\frac{\gamma}{2}} R^{\gamma-2}\, T_0 = \frac{1-k}{4}. \tag{26}$$

Then, by (25) and (26) we easily see that $\mathcal{G}_{T_0}$ maps $\mathbf{K}_R^{T_0}$ into itself. $\square$

Remark 3.7. Supose that (**GC**) is satisfied with $k = 0$ (in (10)). Then it is clear that we can repeat the proof of Proposition 3.5 with $\Phi\,(u) \equiv 0$ and $\partial\Phi\,(u) = \{0\}$. Therefore the local existence results (Theorems 4.1 and 4.2 to be given later) hold true with $\Phi\,(u) \equiv 0$ and $\partial\Phi\,(u) = \{0\}$.

4. Local existence of solutions

4.1. The upper semicontinuous case

In this section, we give an existence result for problem (1) – (2) for the case where the multivalued mapping G has closed convex values, upper semicontinuous with respect to the third variable. Namely we assume the following condition on G :

($\mathbf{H}_G^1$) $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued mapping with closed convex values satisfying the following conditions:

(i) For almost all $(x,t) \in Q_T$, $G\,(x,t,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is upper semicontinuous,

(ii) For each $u \in \mathbb{R}$, $G\,(\cdot,\cdot,u) : Q_T \to 2^{\mathbb{R}}$ is $\mathcal{L}\,(Q_T)$-measurable,

(iii) The growth condition **(GC)** is satisfied.

Our result for this case can be stated as follows:

Theorem 4.1. *Assume that* $\Phi\colon \mathbb{R} \to [0,\infty]$ *is a proper lower semicontinuous convex function with* $\Phi\,(0) = 0$, *and* $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ *is a multivalued mapping with closed convex values satisfying* ($\mathbf{H}_G^1$). *Then for each* $u_0 \in D(\varphi) = H_0^1(\Omega) \cap D(\tilde{\Phi})$, *there exists* $T_0 = T_0(\varphi(u_0)) > 0$ *such that* (1)–(2) *admits a solution* u *on* $[0,T_0]$ *satisfying*

$$\begin{cases} u \in C([0,T_0]; H_0^1(\Omega)),\ \ \tilde{\Phi}(u(x,t)) \text{ is absolutely continuous on } [0,T_0], \\ u_t,\ \ \Delta u,\ \ \partial^0\Phi(u(t)) \in L^2(0,T_0; L^2(Q_{T_0})). \end{cases} \tag{27}$$

Proof. Let $\mathbf{K}_R^{T_0}$ be defined by (14) with $S = T_0$. Then obviously $\mathbf{K}_R^{T_0}$ is a compact subspace of the space $L^2\,(0,T_0; L^2(\Omega))_w$, the space $L^2(0,T_0;L^2(\Omega))$ endowed with the weak topology. Since by Proposition 3.5, $\mathcal{G}_{T_0}\,(\cdot)$ defined by (12) with $S = T_0$, maps $\mathbf{K}_R^{T_0}$ into itself for a sufficiently small T_0, in order to prove that $\mathcal{G}_{T_0}\colon \mathbf{K}_R^{T_0} \to 2^{\mathbf{K}_R^{T_0}}$ is upper semicontinuous, it suffices to show that its graph is closed in $\left(\mathbf{K}_R^{T_0}\right)_w \times \left(\mathbf{K}_R^{T_0}\right)_w$, as is stated in Section 2. This part is essentially done in Lemma 3.11 of [21] in an abstract form. However for the sake of completeness, we repeat the same idea reflecting our present specific setting.

Let $(h_n)_{n\in\mathbb{N}}$ converge weakly in $L^2(0,T_0;L^2(\Omega))$ to some $h \in \mathbf{K}_R^{T_0}$ and assume that $g_n \in \mathcal{G}_{T_0}\,(h_n)$ is such that $(g_n)_{n\in\mathbb{N}}$ converges weakly in $L^2(0,T_0;L^2(\Omega))$ to some $g \in L^2(0,T_0;L^2(\Omega))$, i.e., $g_n(,x,t) \in G\,(x,t,u_{h_n}\,(x,t))$ and $u_{h_n}\,(x,t)$ is a solution of the problem

$$\begin{cases} \frac{\partial}{\partial t} u_{h_n} - \triangle u_{h_n} \in -\partial\Phi\left(u_{h_n}\right) + h_n & (x,t) \in Q_{T_0}, \\ u_{h_n}\left(t\right)|_{\partial\Omega} = 0 \quad t \in (0,T_0), & u_{h_n}(0) = u_0. \end{cases} \tag{28}$$

Multiplying (28) by u_{h_n}, we get

$$\frac{1}{2}\frac{d}{dt}\|u_{h_n}(t)\|_{L^2}^2 + \|\nabla u_{h_n}(t)\|_{L^2}^2 + \tilde{\Phi}\left(u_{h_n}\left(x,t\right)\right) \le \|h_n\left(t\right)\|_{L^2}\|u_{h_n}(t)\|_{L^2}.$$

Since $(h_n)_{n\in\mathbb{N}}$ is a weak convergent sequence in $L^2(0,T_0;L^2(\Omega))$, $\|h_n\|_{L^2(0,T_0;L^2(\Omega))}$ is bounded. Hence we get

$$\sup_{0\le t\le T_0}\|u_{h_n}(t)\|_{L^2} \le C_0, \tag{29}$$

where C_0 is a constant depending on $\|u_0\|_{L^2}$ but not on n . Furthermore multiplying (28) by $z_n = -\triangle u_{h_n} + b_n$ with $b_n \in \partial\Phi\left(u_{h_n}\right)$, we have

$$\frac{d}{dt}\varphi(u_{h_n}(t)) + \|z_n(t)\|_{L^2}^2 \le \|h_n\|_{L^2}\|z_n(t)\|_{L^2}.$$

Then integrating this over $(0,t)$ with $t \in (0,T_0]$, we obtain

$$\sup_{0\le t\le T_0}\{\, \|\nabla u_{h_n}(t)\|_{L^2}^2 + \tilde{\Phi}(u(t))\,\} + \int_0^{T_0}\|z_n(t)\|_{L^2}^2\, dt \le C_1,$$

where C_1 is a constant depending on $\|\nabla u_0\|_{L^2}$ and $\tilde{\Phi}(u_0)$ but not n. Therefore by (9), we get

$$\|\triangle u_{h_n}\|_{L^2(Q_{T_0})}^2 + \|b_n\|_{L^2(Q_{T_0})}^2 \le \|z_n\|_{L^2(Q_{T_0})}^2 \le C_1 \tag{30}$$

whence follows the boundedness of $\|du_{h_n}(t)/dt\|_{L^2(Q_{T_0})}$.

Hence, by Ascoli's theorem, we find that there exists a subsequence of $(u_{h_n})_{n\in\mathbb{N}}$, denoted again by $(u_{h_n})_{n\in\mathbb{N}}$, such that

$$\begin{aligned} &u_{h_n} \to u \quad \text{strongly in } \; C([0,T_0];L^2(\Omega)) \;\text{ and a.e. in }\; Q_{T_0}, \\ &\frac{\partial}{\partial t}u_{h_n} \rightharpoonup \frac{\partial}{\partial t}u \quad \text{weakly in }\; L^2(Q_{T_0}), \\ &-\triangle u_{h_n} \rightharpoonup -\triangle u \quad \text{weakly in }\; L^2(Q_{T_0}), \\ &b_n \rightharpoonup b \in \partial\Phi\left(u\right) \quad \text{weakly in }\; L^2(Q_{T_0}). \end{aligned}$$

Passing to the limit in (28), we find that u gives a solution of (E_h).

Since $g_n \in \mathcal{G}_{T_0}\left(h_n\right) = G\left(\cdot, u_{h_n}\left(\cdot\right)\right) = \widetilde{G}\left(u_{h_n}\right)$, $u_{h_n} \to u$ a.e. on Q_S and $(g_n)_{n\in\mathbb{N}}$ converges weakly in $L^2(Q_S)$ to g, Proposition 3.3 assures that $g \in \widetilde{G}\left(u\right) = G\left(\cdot, u\left(\cdot\right)\right)$, which together with the fact that u is a solution of (E_h) implies $g \in \mathcal{G}\left(h\right)$. Note that the argument here does not depend on the choice of subsequence of $(u_{h_n})_{n\in\mathbb{N}}$.

Thus the graph of $\mathcal{G}_{T_0}\left(\cdot\right)$ is proved to be closed in $\left(\mathbf{K}_R^S\right)_w \times \left(\mathbf{K}_R^S\right)_w$, whence follows the upper semicontinuity of $\mathcal{G}_{T_0}\left(\cdot\right) : \mathbf{K}_R^{T_0} \to 2^{\mathbf{K}_R^{T_0}}$.

Then the Kakutani-Ky Fan Theorem implies that there exists $\overline{h} \in \mathbf{K}_R^{T_0}$, a fixed point of $\mathcal{G}_{T_0}(\cdot)$, that is, $u_{\overline{h}}$ satisfies

$$\begin{cases} \frac{\partial u_{\overline{h}}(x,t)}{\partial t} - \triangle u_{\overline{h}}(x,t) \in -\partial\Phi\left(u_{\overline{h}}(x,t)\right) + G\left(x,t,u_{\overline{h}}(x,t)\right), (x,t) \in Q_{T_0}, \\ u_{\overline{h}}|_{\partial\Omega} = 0 \quad t \in (0,T_0), \quad u_{\overline{h}}(x,0) = u_0 \quad x \in \Omega. \end{cases}$$

Therefore $u_{\overline{h}}$ gives a solution of (1)–(2). □

4.2. The lower semicontinuous case

In this section, we are concerned with problem (1)–(2) for the case where the multivalued mapping G is lower semicontinuous with closed (not necessarily convex) values. Namely, the multivalued mapping is assumed to satisfy the following condition:

($\mathbf{H}_G^2$) $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is a multivalued mapping with closed values such that:

(i) For almost all $(x,t) \in Q_T$, $G(x,t,\cdot) : \mathbb{R} \to 2^{\mathbb{R}}$ is lower semicontinuous,

(ii) $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ is $\mathcal{L}(Q_T) \otimes \mathcal{B}(\mathbb{R})$-measurable.

(iii) The growth condition **(GC)** is satisfied.

Then our result for this case is stated as follows:

Theorem 4.2. *Assume that* $\Phi\colon \mathbb{R} \to [0,\infty]$ *is a proper lower semicontinuous convex function with* $\Phi(0) = 0$, *and* $G\colon Q_T \times \mathbb{R} \to 2^{\mathbb{R}}$ *is a multivalued mapping with closed values satisfying* ($\mathbf{H}_G^2$). *Then for each* $u_0 \in D(\varphi) = H_0^1(\Omega) \cap D(\tilde{\Phi})$, *there exists* $T_0 = T_0(\varphi(u_0)) > 0$ *such that* (1) *admits a solution* u *on* $[0,T_0]$ *satisfying*

$$\begin{aligned} &u \in C([0,T_0]; H_0^1(\Omega)),\ \widetilde{\Phi}(u(x,t)) \text{ is absolutely continuous on } [0,T_0], \\ &u_t,\ \Delta u,\ \partial^0\Phi(u(x,t)) \in L^2(0,T_0; L^2(Q_{T_0})). \end{aligned} \tag{31}$$

For the proof of this result, the following fact plays a crucial role.

Lemma 4.3. *Assume* ($\mathbf{H}_G^2$) *and let* $S \in (0,T_0]$ *where* T_0 *is the positive number given by Proposition* 3.5. *Then the mapping* $\mathcal{G}_S$ *defined by* (12) *is lower semicontinuous from* $L^2(Q_S)_w$ *into* $L^2(Q_S)_w$, *where* $L^2(Q_S)_w$ *denotes the space* $L^2(Q_S)$ *endowed with the weak topology.*

Proof. To check the lower semicontinuity of $\mathcal{G}_S$, we have to show that

$$\mathcal{G}_S^+(C) := \left\{ h \in L^2(Q_S)\ ;\ \mathcal{G}_S(h) \subset C \right\}$$

is closed in $L^2(Q_S)_w$ for any subset C which is closed in $L^2(Q_S)_w$. To this end, let C be closed in $L^2(Q_S)_w$ and let $(h_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}_S^+(C)$ weakly convergent in $L^2(Q_S)$ to some $h_0 \in L^2(Q_S)$. Take any element $g_0 \in \mathcal{G}_S(h_0)$. Then $g_0(x,t) \in G(x,t,u_{h_0}(x,t))$ a.e. $(x,t) \in Q_S$ and u_{h_0} is a solution of problem (E_h) with $h = h_0$.

In order to prove that $g_0 \in C$, let $\rho \in L^2(Q_S)$ be a fixed function such that $\rho(x,t) > 0$ on Q_S and $\iint_{Q_S} \rho^2(x,t)\,dxdt = 1$. For each $n \in \mathbb{N}$ we define $\psi_n\colon Q_S \to \mathbb{R}$ by

$$\begin{aligned} \psi_n(x,t) &= d\left(g_0(x,t), G\left(x,t,u_{h_n}(x,t)\right)\right) + \frac{\rho(x,t)}{n} \\ &= \inf\left\{|g_0(x,t) - y|;\ y \in G(x,t,u_{h_n}(x,t))\right\} + \frac{\rho(x,t)}{n}. \end{aligned} \tag{32}$$

Since G is $\mathcal{L}(Q_S)\times\mathcal{B}(\mathbb{R})$-measurable (cf. (ii) of $(\mathbf{H}^2_G)$) and since u_{h_n} is also measurable, we have

$$Gr\,(G)\cap\left(Gr\,(u_{h_n})\times\mathbb{R}\right)\in\mathcal{L}(Q_S)\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\,.$$

Let $\Lambda_n(x,t):=G\left(x,t,u_{h_n}(x,t)\right).$ Since the map $\eta\colon Q_S\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ defined by $\eta(x,t,y):=(x,t,u_{h_n}(x,t),y)$ is obviously measurable, we get

$$Gr\,(\Lambda_n)=\eta^{-1}\left(Gr\,(G)\cap\left(Gr\,(u_{h_n})\times\mathbb{R}\right)\right)\in\mathcal{L}(Q_S)\otimes\mathcal{B}(\mathbb{R})\,,$$

which implies the measurability of $(x,t)\mapsto G\left(x,t,u_{h_n}(x,t)\right).$ Then by using Theorems 6.5 and 5.8 of [17], we can easily see that for each $n\in\mathbb{N}$, $\psi_n(\cdot)$ is a measurable function, hence the multivalued mapping

$$(x,t)\mapsto B\left[\,g_0(x,t)\,,\psi_n(x,t)\,\right]=\left\{\,y\in\mathbb{R}\,;\;|\,g_0(x,t)-y\,|\le\psi_n(x,t)\,\right\}$$

is also measurable (because it is the composition of the measurable map $(x,t)\mapsto(\,g_0(x,t)\,,\psi_n(x,t))$ and the Hausdorff continuous map $(\omega,r)\mapsto B\left[\omega,r\right]$). For each $n\in\mathbb{N}$, the multivalued mapping

$$(x,t)\mapsto\Gamma_n(x,t):=B\left[\,g_0(x,t)\,,\psi_n(x,t)\,\right]\cap G\left(x,t,u_{h_n}(x,t)\right)$$

is seen to be measurable and with nonempty values (see (iii) of Theorem 3.5 in [17]).

Then, by the Kuratowsky-Ryll-Nardzewski Theorem (see Theorem 5.1 in [17]), there exists a measurable map $g_n\colon Q_S\to\mathbb{R}$ such that $g_n(x,t)\in\Gamma_n(x,t)$ a.e. $(x,t)\in Q_S$.

Since $(h_n)_{n\in\mathbb{N}}$ is bounded in $L^2(Q_S)$, by virtue of the growth condition $(\mathbf{GC})$, we can show as in the proof of Proposition 3.5 that $|||\mathcal{G}_S(h_n)|||_{L^2(Q_S)}$ is bounded, whence follows the boundedness in $L^2(Q_S)$ of $(g_n)_{n\in\mathbb{N}}$. Then there exists a subsequence of $(g_n)_{n\in\mathbb{N}}$, again denoted by $(g_n)_{n\in\mathbb{N}}$, such that

$$(g_n)_{n\in\mathbb{N}}\ \text{converges weakly in}\ L^2(Q_S)\ \text{to some}\ g^0. \tag{33}$$

On the other hand, as in the proof of Theorem 4.1, we can find that

$$u_{h_n}\to u_{h_0}\ \text{strongly in}\ L^2(Q_S)\ \text{as}\ n\to\infty.$$

In particular, $\quad u_{h_{n_k}}(x,t)\to u_{h_0}(x,t)\ \text{as}\ k\to\infty\ \text{a.e.}\ (x,t)\in Q_S$

along some subsequence $\left(u_{h_{n_k}}\right)_{k\in\mathbb{N}}$ of $(u_{h_n})_{n\in\mathbb{N}}$.

Hence, recalling that the lower semicontinuity of $u\mapsto G(x,t,u)$ implies the upper semicontinuity of $u\mapsto d\left(v,G(x,t,u)\right)$ (see Proposition 2.26 of [19]), we easily see

$$\psi_n(x,t)\to 0\ \text{as}\ n\to\infty\ \text{a.e.}\ (x,t)\in Q_S,$$

hence $\quad g_n(x,t)\to g_0(x,t)\ \text{as}\ n\to\infty\ \text{a.e.}\ (x,t)\in Q_S. \quad (34)$

By Egorov's theorem, for any $\varepsilon>0$, there exists a closed set $A_\varepsilon\subseteq Q_S$ such that $(g_n)_{n\in\mathbb{N}}$ converges to g_0 uniformly on A_ε and $\mu\left(Q_S\backslash A_\varepsilon\right)<\varepsilon$.

Therefore we get, for any $\varphi \in C_0^\infty(Q_S)$,

$$\begin{aligned}&\left|\iint_{Q_S}\left(g^0(x,t)-g_0(x,t)\right)\varphi(x,t)\,dxdt\right|\\&=\left|\lim_{n\to\infty}\iint_{Q_S}(g_n(x,t)-g_0(x,t))\,\varphi(x,t)\,dxdt\right|\\&\le\lim_{n\to\infty}\iint_{A_\varepsilon}|g_n(x,t)-g_0(x,t)|\,|\varphi(x,t)|\,dxdt\\&+\overline{\lim_{n\to\infty}}\iint_{Q_S\setminus A_\varepsilon}|g_n(x,t)-g_0(x,t)|\,|\varphi(x,t)|\,dxdt\\&\le\left(\|g_n\|_{L^2(Q_S)}+\|g_0\|_{L^2(Q_S)}\right)\|\varphi\|_{L^2(Q_S\setminus A_\varepsilon)},\end{aligned}$$

whence follows $g^0 = g_0$. Thus we find that $g_n \to g_0$ weakly in $L^2(Q_S)$ as $n \to \infty$.

Recall that $g_n(x,t) \in G(x,t,u_{h_n}(x,t))$, hence $g_n \in \mathcal{G}_S(h_n) \subset C$ and C is assumed to be weakly closed in $L^2(Q_S)$. Then we find that $g_0 \in C$. Since this holds true for any $g_0 \in \mathcal{G}_S(h_0)$, we conclude that $h_0 \in \mathcal{G}_S^+(C)$, therefore $\mathcal{G}_S^+(C)$ is closed in the weak topology. □

Proof of Theorem 4.2. Let $\mathbf{K}_R^{T_0}$ be defined by (14) and let $\mathcal{G}_{T_0}(\cdot)$ be defined by (12) with $S = T_0$, where T_0 is a positive number given by Proposition 3.5.

Then obviously $\mathbf{K}_R^{T_0}$ is a compact subspace of the space in $L^2(0,T_0;L^2(\Omega))_w$, the space $L^2(0,T_0;L^2(\Omega))$ endowed with the weak topology, and $\mathcal{G}_{T_0}(\cdot)$ maps $\mathbf{K}_R^{T_0}$ into itself.

Since G is closed valued and satisfies the growth condition (**GC**), it follows that $\mathcal{G}_{T_0}(h)$ is a closed decomposable nonempty subset of $L^2(Q_{T_0})$. Moreover, by Lemma 4.3, $\mathcal{G}_{T_0}$ is lower semicontinuous from $L^2(Q_{T_0})_w$ into $L^2(Q_{T_0})_w$, hence by Fryszkowski's theorem [15] there exists a continuous function $g\colon \mathbf{K}_R^{T_0} \to \mathbf{K}_R^{T_0}$ such that $g(h) \in \mathcal{G}_{T_0}(h)$ for each $h \in K_R$.

Since $g\colon \mathbf{K}_R^{T_0} \to \mathbf{K}_R^{T_0}$ is continuous from $L^2(Q_{T_0})_w$ into $L^2(Q_{T_0})_w$, by Schauder-Tikhonov's fixed point theorem, there exists $\overline{h} \in \mathbf{K}_R^{T_0}$ a fixed point of g. Therefore $\overline{h} = g(\overline{h}) \in \mathcal{G}_{T_0}(\overline{h})$, hence $\overline{h}(x,t) \in G(x,t,u_{\overline{h}}(x,t))$ for a.e. $(x,t) \in Q_{T_0}$. Thus $u_{\overline{h}}$ is a desired solution of (1)–(2). □

5. Existence of global solutions

Before discussing the existence of global solutions, we prepare the following alternative lemma concerning the maximal existence time of local solutions.

Theorem 5.1. *Let $u(t)$ be a local solution of* (1) *given in Theorem* 4.1 *or Theorem* 4.2. *Let T_m be the maximal existence time of u, i.e.,*

$$T_m = \sup\left\{T_0 \in (0,T]: \begin{array}{l} u(t) \text{ can be continued up to } [0,T_0] \\ \text{as a solution of } (1) \text{ satisfying } (27).\end{array}\right\}$$

Then one of the following alternatives (i) *or* (ii) *holds:*

(i) $T_m = T$, (ii) $T_m < T$ *and* $\lim_{t\nearrow T_m}\varphi(u(t)) = +\infty$.

Proof. To verify the assertion, it suffices to check (ii). Suppose that $T_m < T$ and $\lim_{t \nearrow T_m} \varphi(u(t)) = +\infty$ does not hold, which implies

$$\liminf_{t \nearrow T_m} \varphi(u(t)) = c_0 < +\infty.$$

Therefore, there exists a sequence $(t_k)_{k\in\mathbb{N}}$ such that $t_k \nearrow T_m$ as $k \nearrow +\infty$ and

$$t_k \nearrow T_m \text{ as } k \nearrow +\infty \text{ and } \varphi(u(t_k)) \leq c_0 + 1 \ \forall k \in \mathbb{N}. \tag{35}$$

Then we repeat the same arguments as those in the proofs of Theorems 4.1 and 4.2, essentially the same arguments as in the proof of Proposition 3.5 with the initial time $t = 0$ and the initial data $\varphi(u_0)$ replaced by $t = t_k$ and $\varphi(u(t_k))$.

Recalling (15), (26) and (35), we can find the existence time $T_0 = \delta > 0$ independent of k, such that u can be continued up to $[t_k, t_k + \delta]$ as a solution of (1).

Since $t_k \nearrow T_m$ as $k \nearrow +\infty$, there exists $k_0 \in \mathbb{N}$ such that

$$0 < T_m - t_k < \frac{\delta}{2} \quad \text{for all} \quad k \geq k_0.$$

Especially $t \to u(t)$ can be continued up to $t_{k_0} + \delta > T_m + \frac{\delta}{2}$, which contradicts the definition of T_m. □

Thus Theorem 5.1 assures that in order to show that existence of global solutions, it suffices to establish the a priori bound for $\varphi(u(t))$.

5.1. Large global solutions

Here we discuss the existence of global solutions without assuming the smallness of given data. The following result says that (**GC**) with $q = 2$ yields a sufficient condition for the global continuation of local solutions.

Theorem 5.2. *Assume that growth condition* (**GC**) *is satisfied with* $q = 2$. *Then the local solutions of* (1) *given in Theorem* 4.1 *or Theorem* 4.2 *can be continued globally up to* $[0, T]$.

Proof. Let $u(t)$ be the local solution of (1)–(2) on $[0, S]$ given in Theorem 4.1 or Theorem 4.2, i.e., u satisfies

$$\frac{\partial u}{\partial t}(x,t) - \triangle u(x,t) + b(x,t) = g(x,t), \tag{36}$$

with $b(x,t) \in \partial\Phi(u(x,t))$, $g(x,t) \in G(x,t,u(x,t))$ for a.e. $(x,t) \in Q_S$.

Multiply (36) by $b(x,t) \in \partial\Phi(u(x,t))$ and we get

$$\frac{d}{dt}\widetilde{\Phi}(u(t)) + (-\triangle u, b)_{L^2} + \|b\|_{L^2}^2 = (g,b)_{L^2} \leq \|g\|_{L^2}\|b\|_{L^2} \leq \frac{1}{2}\|g\|_{L^2}^2 + \frac{1}{2}\|b\|_{L^2}^2.$$

Since (**GC**) with $q = 2$ we obtain

$$|g(x,t)|^2 \leq |a(x,t)| + k\left|\partial^0\Phi(u(x,t))\right|^2 + C_2|u(x,t)|^2$$

and (9) assures $(-\triangle u, b)_{L^2} \geq 0$. Therefore

$$\frac{d}{dt}\widetilde{\Phi}(u(t)) + \frac{1-k}{2}\|b(t)\|_{L^2}^2 \leq \frac{1}{2}\|a(t)\|_{L^1} + \frac{C_2}{2}\|u(t)\|_{L^2}^2. \tag{37}$$

Next multiply (36) by $-\triangle u(x,t)$. Then we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\triangle u(t)\|_{L^2}^2 + (-\triangle u, b)_{L^2}\\ &\leq \frac{1}{2}\|\triangle u(t)\|_{L^2}^2 + \frac{1}{2}\|a(t)\|_{L^1} + \frac{k}{2}\left\|\partial^0\Phi(u(t))\right\|_{L^2}^2 + \frac{C_2}{2}\|u(t)\|_{L^2}^2, \text{ i.e.,}\end{aligned}$$

$$\frac{d}{dt}\|\nabla u(t)\|_{L^2}^2 + \|\triangle u(t)\|_{L^2}^2 \leq \|a(t)\|_{L^1} + k\left\|\partial^0\Phi(u(t))\right\|_{L^2}^2 + C_2\|u(t)\|_{L^2}^2. \tag{38}$$

Then, multiplying (37) by $2/(1-k)$ and adding to (38), we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\|\nabla u(t)\|_{L^2}^2 + \frac{2}{1-k}\widetilde{\Phi}(u(t))\right)\\ &\leq \frac{2-k}{1-k}\left(\|a(t)\|_{L^1} + C_2\|u(t)\|_{L^2}^2\right) \leq \frac{2-k}{1-k}\left(\|a(t)\|_{L^1} + \frac{C_2}{\lambda_1}\|\nabla u(t)\|_{L^2}^2\right),\end{aligned} \tag{39}$$

where we used the Poincaré inequality

$$\lambda_1\|u\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 \text{ for all } u \in H_0^1(\Omega), \tag{40}$$

where λ_1 is the first eigenvalue of $-\triangle$ with the homogeneous Dirichlet boundary condition. Then applying Gronwall's inequality for $\|\nabla(u(t))\|_{L^2}^2 + \frac{2}{1-k}\widetilde{\Phi}(u(t))$, we obtain

$$\begin{aligned}&\|\nabla u(t)\|_{L^2}^2 + \frac{2}{1-k}\widetilde{\Phi}(u(t))\\ &\leq \left[\|\nabla u_0\|_{L^2}^2 + \frac{2}{1-k}\widetilde{\Phi}(u_0) + \frac{2-k}{1-k}\|a\|_{L^1(Q_T)}\right] e^{\frac{2-k}{1-k}\frac{C_2}{\lambda_1}T} \quad \forall t \in [0,S],\end{aligned}$$

which gives the uniform a priori bound for $\varphi(u(t))$ on $[0,S]$ for all $S \in [0,T]$. □

5.2. Small global solutions

In this subsection we are concerned with the existence of global solutions when the given data are taken sufficiently small. Our result on the existence of small global solutions is stated as follows.

Theorem 5.3. *Assume that growth condition* (**GC**) *with* $2 < q < 2^*$ *is satisfied. Then there exists a (sufficiently small) number* $r_0 > 0$ *such that if* $|||a|||_1 \leq r_0$ *and* $\varphi(u_0) \leq r_0$, *then local solutions given in Theorem 4.1 or Theorem 4.2 can be continued globally up to* $[0,T]$. *Here we put*

$$|||a|||_1 := \sup_{0\leq t<\infty}\int_t^{t+1}\|\widetilde{a}(s)\|_{L^1(\Omega)}\,ds,$$

$\|\widetilde{a}(t)\|_{L^1(\Omega)}$ *is the zero extension of* $\|a(t)\|_{L^1(\Omega)}$ *to* $[0,\infty)$.

To prove this result we prepare some lemmas.

Lemma 5.4. *Let* (**GC**) *with* $2 < q < 2^*$ *be satisfied. Then there exists a (sufficiently small) number* $\varepsilon_0 > 0$ *such that*

$$(-\triangle u + b - g, u)_{L^2} \geq \frac{1}{4}\|\nabla u\|_{L^2}^2 + \left(1-\sqrt{k}\right)\widetilde{\Phi}(u) - \frac{1}{2\lambda_1}\|a(t)\|_{L^1} \tag{41}$$

$$\geq \frac{\lambda_1}{4}\|u\|_{L^2}^2 - \frac{1}{2\lambda_1}\|a(t)\|_{L^1} \tag{42}$$

holds for all $b(x,t)\in\partial\Phi(u(x,t))$, $g(x,t)\in G(x,t,u(x,t))$ *and* u *satisfying* $\varphi(u) \leq \varepsilon_0$.

Proof. We first recall the embedding

$$K_q\|u\|_{L^q} \leq \|\nabla u\|_{L^2} \ \ \forall u \in H_0^1(\Omega), \ \forall q \in (1, 2^*). \tag{43}$$

By virtue of (**GC**) and (40) we get

$$\begin{aligned}(-g,u)_{L^2} &\geq -\int_\Omega \left(|a(x,t)|^{\frac{1}{2}} + C_q^{\frac{1}{2}}|u|^{q-1} + \sqrt{k}\left|\partial^0\Phi(u)\right|\right)|u|\,dx\\ &\geq -\sqrt{C_q}\|u\|_{L^q}^q - \frac{\lambda_1}{2}\|u\|_{L^2}^2 - \frac{1}{2\lambda_1}\|a(t)\|_{L^1} - \sqrt{k}\int_\Omega |b|\,|u|\,dx \qquad (44)\\ &\geq -\sqrt{C_q}\|u\|_{L^q}^q - \frac{1}{2}\|\nabla u\|_{L^2}^2 - \frac{1}{2\lambda_1}\|a(t)\|_{L^1} - \sqrt{k}(b,u)_{L^2},\end{aligned}$$

where we used the fact that $|\partial^0\Phi(u)| \leq |b|$ and $|b|\,|u| = b(x,t)\cdot u(x,t)$ a.e. Since $b \in \partial\Phi(u)$ and $\Phi(0) = 0$, we have

$$(-\triangle u + b, u)_{L^2} = \|\nabla u\|_{L^2}^2 + (b,u)_{L^2}, \quad (b,u)_{L^2} \geq \widetilde{\Phi}(u). \tag{45}$$

Hence, by (44), (45) and (43), we obtain

$$\begin{aligned}&(-\triangle u + b - g, u)_{L^2}\\ &\geq \frac{1}{2}\|\nabla u\|_{L^2}^2 + \left(1-\sqrt{k}\right)\widetilde{\Phi}(u) - \sqrt{C_q}\|u\|_{L^q}^q - \frac{1}{2\lambda_1}\|a(t)\|_{L^1}\\ &\geq \left(\frac{1}{2} - \sqrt{C_q}K_q^q\|\nabla u\|_{L^2}^{q-2}\right)\|\nabla u\|_{L^2}^2 + \left(1-\sqrt{k}\right)\widetilde{\Phi}(u) - \frac{1}{2\lambda_1}\|a(t)\|_{L^1}.\end{aligned}$$

Here we define $\varepsilon_0 > 0$ by $\varepsilon_0 = \frac{1}{2}\left[\frac{1}{4\sqrt{C_q}K_q^q}\right]^{\frac{2}{q-2}}$.

Then we easily see that

$$\frac{1}{2} - \sqrt{C_q}K_q^q\|\nabla u\|_{L^2}^{q-2} \geq \frac{1}{4} \ \text{ for all } u \text{ satisfying } \ \frac{1}{2}\|\nabla u_0\|_{L^2}^2 \leq \varepsilon_0.$$

Consequently we obtain

$$(-\triangle u + b - g, u)_{L^2} \geq \frac{1}{4}\|\nabla u\|_{L^2}^2 + \left(1-\sqrt{k}\right)\widetilde{\Phi}(u) - \frac{1}{2\lambda_1}\|a(t)\|_{L^1}, \text{ if } \varphi(u) \leq \varepsilon_0. \quad \square$$

The next lemma is essentially proved in Lemma 4.3 of [21].

Lemma 5.5. *Let $f \in L^1(0,T)$ and $j(\cdot)$ be an absolutely continuous positive function on $[0,S]$ with $0 < S \leq T$ such that*

$$\frac{d}{dt} j(t) + \delta j(t) \leq |f(t)| \quad a.e.\ t \in [0,S], \tag{46}$$

where $\delta > 0$. Then we have

$$j(t) \leq e^{-\delta t} j(0) + \frac{1}{1-e^{-\delta}} \|f\|_1 \tag{47}$$

where $\|f\|_1 = \sup_{0 \leq t < \infty} \int_t^{t+1} |\widetilde{f}(s)|ds$ *and $\widetilde{f}(\cdot)$ is the zero extension of $f(\cdot)$ to $[0,\infty)$.*

Proof. To derive (47), it suffices to see that (46) gives

$$\begin{aligned} j(t) &\leq j(0)e^{-\delta t} + \int_0^t |f(s)|\, e^{-\delta(t-s)} ds \leq j(0)e^{-\delta t} + \int_0^\infty |\widetilde{f}(s)|\, e^{-\delta(t-s)} ds \\ &= j(0)e^{-\delta t} + \sum_{n=0}^\infty \int_n^{n+1} |\widetilde{f}(s)|\, e^{-\delta n} ds \leq j(0)e^{-\delta t} + \frac{1}{1-e^{-\delta}} \|f\|_1. \qquad \square \end{aligned}$$

Lemma 5.6. *Let* (**GC**) *with $2 < q < 2^*$ be satisfied, and let $u(t)$ be a local solution of* (1) *on $[0,S]$ with $S \in [0,T]$. Then there exist $\varepsilon_1 > 0$ and $N > 0$ independent of T such that $N\varepsilon_1 \leq \varepsilon_0$ and for any $r \in [0,\varepsilon_1]$, $\varphi(u_0) \leq r^2$ and $|||a|||_1 \leq r^2$ assure*

$$\varphi(u(t)) \leq Nr^2 \text{ for all } t \in [0,S]. \tag{48}$$

Here ε_0 is the number given in Lemma 5.4.

Proof. We first note that multiplication of (1) by $z(t) = -\triangle u(t) + b(t) \in \partial\varphi(u(t))$ with $b(t) \in \partial\Phi(u(t))$ and (**GC**) give

$$\begin{aligned} &\frac{d}{dt}\varphi(u(t)) + \|z(t)\|_{L^2}^2 \leq \frac{1}{2}\|z(t)\|_{L^2}^2 + \frac{1}{2}\|g(t)\|_{L^2}^2 \\ &\leq \frac{1}{2}\|z(t)\|_{L^2}^2 + \frac{1}{2}\|a(t)\|_{L^1} + \frac{1}{2}C_q\|u(t)\|_{L^{2(q-1)}}^{2(q-1)} + \frac{k}{2}\left\|\partial^0\Phi(u(t))\right\|_{L^2}^2, \end{aligned}$$

where $g(x,t) \in G(x,u,u(x,t))$ a.e. $(x,t) \in Q_{S.}$. Furthermore the argument for (24) with $\eta = (2C_q)^{-1}$ yields

$$\frac{1}{2}C_q\|u(t)\|_{L^{2(q-1)}}^{2(q-1)} \leq \frac{1}{4}\|\triangle u(t)\|_{L^2}^2 + \frac{1}{2}C_q K_q\left(\frac{1}{2C_q}\right)\|\nabla u(t)\|_{L^2}^\gamma.$$

Hence we obtain by (9)

$$\begin{aligned} &\frac{d}{dt}\varphi(u(t)) + \frac{1}{4}\|\triangle u(t)\|_{L^2}^2 + \frac{1-k}{2}\|b(t)\|_{L^2}^2 \\ &\leq \frac{1}{2}\|a(t)\|_{L^1} + \frac{1}{2}C_q K_q\left(\frac{1}{2C_q}\right)\|\nabla u(t)\|_{L^2}^\gamma. \end{aligned} \tag{49}$$

Here,
$$N := 2 + \frac{1}{1 - e^{-\frac{\lambda_1}{4 \cdot 2_k}}} \left[\frac{1}{2} + 2^{\frac{\gamma-2}{2}} C_q K_q \left(\frac{1}{2C_q} \right) \left(N_1 + \frac{1}{\lambda_1} \right) \frac{2_k}{2} \right], \tag{50}$$
$$N_1 := \frac{2}{\lambda_1} + \frac{1}{1 - e^{-\frac{\lambda_1}{2}}} \frac{1}{\lambda_1}, \qquad 2_k = \max \left\{ 2, \frac{1}{1 - \sqrt{k}} \right\}.$$

Suppose that (48) does not hold. Then from the continuity of $\varphi(u(t))$ with respect to $t \in [0, S]$, there exists $t_1 \in [0, S]$ such that

$$\varphi(u(t)) < Nr^2 \text{ for all } t \in [0, t_1) \text{ and } \varphi(u(t_1)) = Nr^2. \tag{51}$$

Since $\varphi(u(t)) \leq Nr^2 \leq \varepsilon_0$ for all $t \in [0, t_1]$, multiplying (1) by $u(t)$ we get by (41)

$$\frac{1}{2} \frac{d}{dt} \|u(t)\|_{L^2}^2 + \frac{1}{2_k} \varphi(u(t)) \leq \frac{1}{2\lambda_1} \|a(t)\|_{L^1} \text{ a.e. } t \in [0, t_1] \tag{52}$$

and by (42):
$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + \frac{\lambda_1}{2} \|u(t)\|_{L^2}^2 \leq \frac{1}{\lambda_1} \|a(t)\|_{L^1} \text{ a.e. } t \in [0, t_1]. \tag{53}$$

Then Lemma 5.5 together with (53) gives

$$\begin{aligned} \sup_{0 \leq t \leq t_1} \|u(t)\|_{L^2}^2 &\leq \|u_0\|_{L^2}^2 + \frac{1}{1 - e^{-\frac{\lambda_1}{2}}} \frac{1}{\lambda_1} |||a|||_1 \\ &\leq \left[\frac{2}{\lambda_1} + \frac{1}{1 - e^{-\frac{\lambda_1}{2}}} \frac{1}{\lambda_1} \right] r^2 = N_1 r^2. \end{aligned} \tag{54}$$

Moreover integrating (52) over $[t, t+1]$, we obtain

$$\sup_{0 \leq t < \infty} \int_t^{t+1} \widetilde{\varphi}(u(s))\, ds \leq \frac{2_k}{2\lambda_1} |||a|||_1 + \frac{2_k}{2} \sup_{0 \leq t \leq S} \|u(t)\|_{L^2}^2 \leq \frac{2_k}{2} \left(\frac{1}{\lambda_1} + N_1 \right) r^2, \tag{55}$$

where $\widetilde{\varphi}(u(s))$ denotes the zero extension of $\varphi(u(s))$ to $[0, \infty)$. Here we note that for any $z \in \partial\varphi(u)$, we get

$$\|z\|_{L^2} \|u\|_{L^2} \geq |(z, u)_{L^2}| \geq \|\nabla u\|_{L^2}^2 + \widetilde{\Phi}(u) \geq \varphi(u(t)) \geq \sqrt{\varphi(u)} \sqrt{\frac{\lambda_1}{2}} \|u\|_{L^2},$$

whence follows
$$\frac{\lambda_1}{2} \varphi(u) \leq \|z\|_{L^2}^2, \ \forall z \in \partial\varphi(u), \ \forall u \in D(\partial\varphi). \tag{56}$$

Hence, in view of (49), we have

$$\begin{aligned} &\frac{d}{dt} \varphi(u(t)) + \frac{\lambda_1}{4} \frac{1}{2_k} \varphi(u(t)) \\ &\leq \frac{1}{2} \|a(t)\|_{L^1} + \frac{1}{2} C_q K_q \left(\frac{1}{2C_q} \right) (2\varphi(u(t)))^{\frac{\gamma}{2}} \ \forall t \in [0, t_1). \end{aligned} \tag{57}$$

Since $\gamma > 2$, taking $Nr^2 \leq \varepsilon_0 \leq 1$ without loss of generality, we obtain by Lemma 5.5 together with (55) and (57):

$$\varphi(u(t_1)) \leq \varphi(u_0) + \frac{1}{1 - e^{-\frac{\lambda_1}{4 \cdot 2_k}}} \left[\frac{1}{2} |||a|||_1 + 2^{\frac{\gamma-2}{2}} C_q K_q \left(\frac{1}{2C_q} \right) \left(\frac{1}{\lambda_1} + N_1 \right) \frac{2_k}{2} r^2 \right]$$

$$= \left[1 + \frac{1}{1 - e^{-\frac{\lambda_1}{4 \cdot 2_k}}} \left\{ \frac{1}{2} + 2^{\frac{\gamma-2}{2}} C_q K_q \left(\frac{1}{2C_q} \right) \left(\frac{1}{\lambda_1} + N_1 \right) \frac{2_k}{2} \right\} \right] r^2 < Nr^2,$$

which leads to a contradiction. □

Now we are ready to give a proof for Theorem 5.3.

Proof of Theorem 5.3. By virtue of Theorem 5.1, to assure the existence of a global solution, it suffices to give a priori bound on $\varphi(u(t))$. Indeed due to Lemma 5.6, we obtain the a priori bound for $\varphi(u(t))$ independent of T. □

6. Examples

In this section we exemplify the applicability of our results.

6.1. The case where $D(\Phi) = \mathbb{R}^1$

As a typical example, let $\beta(\cdot) = \partial\Phi(\cdot)$ be a maximal monotone graph in $\mathbb{R}^1 \times \mathbb{R}^1$ such that $\Phi(0) = 0$ and

$$\left|\partial^0 \Phi(u)\right| \geq |u|^{p-1} - C_1 \text{ for all } u \in D(\Phi) = \mathbb{R}^1, \ 1 < p < \infty, \ C_1 \geq 0. \tag{58}$$

We also introduce a class of continuous functions $\mathcal{C}_{p,q}$ by the following: $f \in \mathcal{C}_{p,q}$ if and only if $f\colon \mathbb{R}^1 \to \mathbb{R}^1$ is continuous and satisfy

$$|f(u)|^2 \leq C_0 + k|u|^{2(p-1)} + C_q |u|^{2(q-1)}, \ \forall u \in \mathbb{R}^1, \tag{59}$$

where $1 < q < \infty$, $C_0 \geq 0$, $C_q \geq 0$ and $k \in [0, 1)$ are constants.

In the following we consider the case where

$$G(x, t, u) = G_0(u) + f_e(x, t), \text{ with } f_e \in L^2(Q_T). \tag{60}$$

6.1.1. The upper semicontinuous case

Take f_1^+, f_1^-, f_2^+, $f_2^- \in \mathcal{C}_{p,q}$, such that

$$f_1^-(u) < f_2^-(u) \ \forall u \in (-\infty, 0],$$
$$f_1^+(u) < f_2^+(u) \ \forall u \in [0, +\infty),$$
$$f_2^-(0) < f_1^+(0),$$

and define
$$G_0(u) = \begin{cases} \left[f_1^-(u), f_2^-(u)\right] & \text{if } u < 0, \\ \left[f_1^+(u), f_2^+(u)\right] & \text{if } u > 0, \\ \left[f_1^-(0), f_2^+(0)\right] & \text{if } u = 0. \end{cases} \tag{61}$$

Then it is easy to see that $G(x,t,u) = G_0(u) + f_e(x,t)$ is a closed convex and upper semicontinuous multivalued function satisfying (i) and (ii) of $(\mathbf{H}_G^1)$. We here note that there is no continuous selection of $G_0(\cdot)$.

Since f_1^+, f_1^-, f_2^+, $f_2^- \in \mathcal{C}_{p,q}$, by virtue of (58) and (59), there exist $\widetilde{C}_1 > 0$, $\widetilde{C}_q > 0$ and $\widetilde{k} \in [0,1)$ such that

$$|||G(x,t,u)|||^2 \le \widetilde{C}_1 \left(1 + |f_e(x,t)|^2\right) + \widetilde{k} \left|\partial^0 \Phi(u)\right|^2 + \widetilde{C}_q |u|^{2(q-1)} \ \forall u \in \mathbb{R}^1. \quad (62)$$

The existence of local and global solutions of our equation with (58), (60) and (61) falls into the following two cases:

(I) The case where $q \le 2$ or $2 < q < p < \infty$.

By Young's inequality, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|u|^{2(q-1)} \le \varepsilon |u|^{2(p-1)} + C_\varepsilon \left(|u|^2 + 1\right) \text{ for all } u \in \mathbb{R}^1.$$

Then (62) is again satisfied with $q = 2$ (without assuming $q < 2^*$). Therefore, by Theorem 4.1 and Theorem 5.2 our problem (1)–(2) admits a global solution for all $u_0 \in D(\varphi) = H_0^1(\Omega) \cap D(\widetilde{\Phi})$ and $f_e \in L^2(Q_T)$.

(II) The case where $2 < q < 2^*$.

Even for the case $p \le q$, (62) with $2 < q < 2^*$ implies (**GC**) with $2 < q < 2^*$**.** Then Theorem 5.3 assures that our problem (1)–(2) admits a global solution if $\varphi(u_0)$ and $\left|\left|\left| \, |f_e(\cdot,\cdot)|^2 \right|\right|\right|_1$ are sufficiently small.

6.1.2. The lower semicontinuous case

Take f^+, $f^- \in \mathcal{C}_{p,q}$ and $-\infty \le r_0 < r_1 \le +\infty$ such that

$$f^-(u) < f^+(u) \ \forall u \in (r_0, r_1)$$

and define

$$G_0(u) = \begin{cases} \{f^+(u)\} & \text{if } \ r_1 \le u (\text{when } r_1 < +\infty) \\ \{f^-(u)\} & \text{if } \ u \le r_0 (\text{when } -\infty < r_0) \\ [f^-(u), f^+(u)] \cap \mathbb{Q}_n & \text{if } \ u \in (r_0, r_1) \end{cases} \quad (63)$$

where $\mathbb{Q}_n := \{q \in \mathbb{Q} : 10^n q \in \mathbb{Z}\}$ with n sufficiently large so that, for all $u \in (r_0, r_1)$,

$$[f^-(u), f^+(u)] \cap \mathbb{Q}_n \ne \varnothing.$$

Then it is easy to see $G_0(\cdot)$ is lower semicontinuous and closed but not convex valued and there is no continuous selection of $G_0(\cdot)$.

Since $f^\pm \in \mathcal{C}_{p,q}$ implies (62), $G(x,t,u) = G_0(u) + f_e(x,t)$ satisfies $(\mathbf{H}_G^2)$. Hence for the existence of local and global solutions, we again have the same two classifications.

(I) The case where $q \le 2$ or $2 < q < p < \infty$.

Theorems 4.1 and Theorem 5.2 assure the existence of global solutions for problem (1)–(2) for all $u_0 \in D(\varphi)$ and $f_e \in L^2(Q_T)$.

(II) The case where $2 < q < 2^*$.

Theorem 4.1 assures the existence of a local solution for all $u_0 \in D(\varphi)$, whereas $f_e \in L^2(Q_T)$ and Theorem 5.3 assure the existence of global solutions to problem (1)–(2), if $\varphi(u_0)$ and $\left|\left|\left| |f_e(\cdot,\cdot)|^2 \right|\right|\right|_1$ are taken sufficiently small.

Remark 6.1. Let $p \leq q$ and $2 < q < 2^*$. Then (59) is satisfied with C_q and C_0 replaced by suitable ones. Then as is stated in Remark 3.7, the assertion in (II) above holds true with $\Phi(u) \equiv 0$ and $\partial\Phi(u) = \{0\}$.

6.2. The case where $D(\Phi)$ is precompact

Here we consider the case where $D(\Phi)$ is precompact, i.e.,

$$D(\Phi) = \left\{u \in \mathbb{R}^1 : \Phi(u) < +\infty\right\} \subset [a,b] \text{ with } -\infty < a < b < +\infty.$$

Typical examples are given by:

1.
$$\Phi(u) = I_{[a,b]}(u) = \begin{cases} 0 & \text{if } u \in [a,b], \\ +\infty & \text{otherwise,} \end{cases} \tag{64}$$

and
$$\partial\Phi(u) = \partial I_{[a,b]}(u) = \begin{cases} \{0\} & \text{if } u \in (a,b), \\ (-\infty,0] & \text{if } u = a, \\ [0,+\infty) & \text{if } u = b, \\ \varnothing & \text{if } u \notin [a,b]; \end{cases}$$

2.
$$\Phi(u) = \Phi_h(u) = \begin{cases} h(u) & \text{if } u \in (a,b), \\ +\infty & \text{otherwise,} \end{cases} \tag{65}$$

where $h \in C^1((a,b);\mathbb{R}^1)$ is convex and satisfies

$$\lim_{u\to a+0} h(u) = \lim_{u \to b-0} h(u) = +\infty.$$

Then we have $\partial\Phi(u) = \partial\Phi_h(u) = \begin{cases} h'(u) & \text{if } u \in (a,b), \\ \varnothing & \text{if } u \notin (a,b). \end{cases}$

We again define $G_0(u)$ by (61) or (63) and $G(x,t,u)$ by(60). Then, since we have $D\left(\partial I_{[a,b]}\right) = [a,b]$ and $D(\partial\Phi_h) = (a,b)$, we can verify **(GC)** with $k = 0$, $C_q = 0$ under the condition that $f_i^{\pm}$, $f^{\pm} \in C(\mathbb{R}^1;\mathbb{R}^1)$. Hence, Theorem 5.2 assures the existence of global solutions of (1) for every $u_0 \in D(\varphi)$ and $f_e \in L^2(Q_T)$.

References

[1] J. P. Aubin, A. Cellina: *Differential Inclusions*, Springer, Berlin (1984).

[2] V. Barbu: *Nonlinear Semigroups and Differential Equations in Banach Spaces*, Noordhoff, Leyden (1976).

[3] G. Barletta: *Parabolic equations with discontinuous nonlinearities*, Bull. Australian Math. Soc. 63 (2001) 219–228.

[4] H. Brézis: *Opérateurs Maximaux Monotones et Semigroupes de Contractions dans les Espaces de Hilbert*, North-Holland, Amsterdam (1973).

[5] T. Cardinali, N. S. Papageorgiou: *Periodic problems and problems with discontinuities for nonlinear parabolic equations*, Czechoslovak Math. J. 50 (2000) 467–497.

[6] S. Carl, C. Grossmann, C. V. Pao: *Existence and monotone iterations for parabolic differential inclusions*, Comm. Appl. Nonlinear Analysis 3 (1996) 1–24.

[7] S. Carl, S. Heikkila: *On a parabolic boundary value problem with discontinuous nonlinearity*, Nonlinear Analysis 15 (1990) 1091–1095.

[8] S. Carl, D. Motreanu: *Extremal solutions of quasilinear parabolic inclusions with generalized Clarke's gradient*, J. Differential Equations 191 (2003) 206–233.

[9] S. Carl, D. Motreanu: *Extremality in solving general quasilinear parabolic inclusions*, J. Optimization Theory Appl. 123 (2004) 463–477.

[10] K. C. Chang: *Variational methods for nondifferentiable functionals and their applications to partial differential equations*, J. Math. Analysis Appl 80 (1981) 102–129.

[11] H. Deguchi: *On weak solutions of parabolic initial value problems with discontinuous nonlinearities*, Nonlinear Analysis 63 (2005) 1–9.

[12] E. Feireisl: *A note on uniqueness for parabolic problems with discontinuous nonlinearities*, Nonlinear Analysis 19 (1991) 1053–1056.

[13] E. Feireisl, J. Norbury: *Some existence, uniqueness and nonuniqueness theorems for solutions of parabolic equations with discontinuous nonlinearities*, Proc. Royal Soc. Edinburgh, Sec. A Math. 119(1-2) (1991) 1–17.

[14] B. A. Fleishman, T. J. Mahar: *A step function model in chemical reactor theory: Multiplicity and stability of solutions*, Nonlinear Analysis 15 (1981) 645–654.

[15] A. Fryszkowski: *Continuous selections for a class of nonconvex multivalued maps*, Studia Math. 76 (1981) 163–174.

[16] N. Halidias, N. S. Papageorgiou: *Existence of solutions for nonlinear parabolic problems*, Archivum Mathematicum (Brno) 35 (1999) 255–274.

[17] C. J. Himmelberg: *Measurable relations*, Fund. Math. 87 (1975) 53–72.

[18] J. Hofbauer, P. L. Simon: *An existence theorem for parabolic equation on $\mathbb{R}^N$ with discontinuous nonlinearities*, Electron. J. Qual. Theory Diff. Equations 15 (1981) 645–654.

[19] S. Hu, N. S. Papageorgiou: *Handbook of Multivalued Analysis. Volume I: Theory*, Kluwer, Dordrecht (1997).

[20] L. Nirenberg: *On elliptic partial differential equations*, Ann. Scuola Normale Sup. Pisa, Classe Scienze 13 (1959) 115–162.

[21] M. Ôtani: *Non-monotone perturbations for nonlinear parabolic equations associated with subdifferential operators, Cauchy problems*, J. Differential Equations 46 (1982) 268–299.

[22] M. Ôtani: *Non-monotone perturbations for nonlinear parabolic equations associated with subdifferential operators, periodic problems*, J. Differential Equations 54 (1984) 248–273.

[23] M. Ôtani, V. Staicu: *Existence results for quasilinear elliptic equations with multivalued nonlinear terms*, Set-Valued Var. Analysis 22 (2014) 859–877.

[24] N. S. Papageorgiou: *On the existence of solutions for nonlinear parabolic problems with nonmonotone discontnuities*, J. Math. Analysis Appl. 205 (1997) 434–453.

[25] J. Rauch: *Discontinuous semilinear differential equations and multiple valued maps*, Proc. Amer. Math. Soc. 64 (1977) 277–282.