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An Application of the Generalised James' Weak Compactness Theorem

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We provide a short proof of following theorem, due to Delbaen and Orihuela and independently, Pérez-Aros and Thibault. Let A be a nonempty closed and bounded convex subset of a Banach space $(X, \|\cdot\|)$ and let W be a nonempty weakly compact subset of $(X, \|\cdot\|)$. If we have $x_0^* \in \{x^* \in X^* : \sup_{a\in A} x^*(a) < 0\}$ and $\operatorname{argmax}(y^*|_A) \neq \varnothing$ for each

$$y^* \in \{x^* \in X^* : \sup_{a\in A} x^*(a) < 0 \text{ and } \sup_{w\in W} |(x^* - x_0^*)(w)| < 1\},$$

then A is weakly compact.

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1. Introduction

Recently, several authors, building upon earlier results contained in [2, 5, 6, 7, 10, 11, 14], have independently derived the result stated in the abstract, see [3, 13]. In this paper we show how this result may also be obtained as a consequence of the convex analyst's version of the James Weak Compactness Theorem, given in [8], along with the aid of the Brøndsted-Rockafellar Theorem.

We shall begin with some basic notation and assumed terminology. In this short note all normed linear spaces will be over the field of real numbers (denoted $\mathbb{R}$). If X is a set and $f : X \to (-\infty, \infty]$ is a function, then $\operatorname{Dom}(f) := \{x \in X : f(x) < \infty\}$ and we say that f is a *proper* function if $\operatorname{Dom}(f) \neq \varnothing$. Furthermore, we define $\operatorname{argmax}(f) := \{x \in X : f(y) \leq f(x) \text{ for all } y \in X\}$.

We shall call a proper function $f : X \to (-\infty, \infty]$, defined on a vector space X, (over the real numbers) a *convex* function if, for each $x, y \in \operatorname{Dom}(f)$ and $0 < \lambda < 1$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ and we call a function $f : (X, \tau) \to (-\infty, \infty]$ from a topological space (X, τ) *lower semicontinuous* if for each $r \in \mathbb{R}$ the set $\{x \in X : f(x) \leq r\}$ is a closed subset of (X, τ). This is equivalent to saying that the *epigraph* of f $\operatorname{epi}(f) := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r\}$ is closed in $X \times \mathbb{R}$, (with the product topology).

In addition to these definitions we need the notion of an ε-subdifferential. Suppose that $f : X \to (-\infty, \infty]$ is a proper function on a normed linear space $(X, \|\cdot\|)$ and $x \in \operatorname{Dom}(f)$. Then for any $0 \leq \varepsilon$, we define the *ε-subdifferential* $\partial_\varepsilon f(x)$ by,

$$\partial_\varepsilon f(x) := \{x^* \in X^* : x^*(y - x) \leq f(y) - f(x) + \varepsilon \text{ for all } y \in \operatorname{Dom}(f)\}.$$

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In the special case when $\varepsilon = 0$ we simply call the ε-subdifferential the *subdifferential* and we denote $\partial_0 f(x)$ by $\partial f(x)$.
Finally, we define $\mathrm{Dom}(\partial_\varepsilon f) := \{x \in \mathrm{Dom}(f) : \partial_\varepsilon f(x) \neq \varnothing\}$.

Theorem 1.1. (Brøndsted-Rockafellar, [1, Lemma]) *Let $f : X \to (-\infty, \infty]$ be a proper convex lower semicontinuous function defined on a Banach space $(X, \|\cdot\|)$. Then given any point $x_0 \in \mathrm{Dom}(f)$, $0 < \varepsilon$, $0 < \lambda$ and any $x_0^* \in \partial_\varepsilon f(x_0)$, there exist $x \in \mathrm{Dom}(f)$ and $x^* \in X^*$ such that*

$$x^* \in \partial f(x), \quad \|x - x_0\| \leq \varepsilon/\lambda \quad \textit{and} \quad \|x^* - x_0^*\| \leq \lambda.$$

For any x in a normed linear space $(X, \|\cdot\|)$ we shall define $\widehat{x} \in X^{**}$ by, $\widehat{x}(x^*) := x^*(x)$ for all $x^* \in X^*$. Then, $x \mapsto \widehat{x}$, is a linear isometric embedding of $(X, \|\cdot\|)$ into $(X^{**}, \|\cdot\|^{**})$. In particular, if $(X, \|\cdot\|)$ is a Banach space, then $\widehat{X} := \{\widehat{x} : x \in X\}$ is a closed linear subspace of $(X^{**}, \|\cdot\|^{**})$.

Proposition 1.2. *Let $f : X^* \to (-\infty, \infty]$ be a proper convex lower semicontinuous function defined on the dual of a Banach space $(X, \|\cdot\|)$. If $0 \leq \varepsilon < \infty$ and $\partial f(x^*) \subseteq \widehat{X}$ for all $x^* \in \mathrm{Dom}(\partial f)$, then $\partial_\varepsilon f(x^*) \subseteq \widehat{X}$ for all $x^* \in \mathrm{Dom}(f)$.*

Proof. This follows directly from the Brøndsted-Rockafellar Theorem. $\square$

In addition to the Brøndsted-Rockafellar Theorem we need the following convex analyst's version of the James Weak Compactness Theorem.

Theorem 1.3. (Generalised James' Weak Compactness Theorem, [8, Thm. 4.13]) *Let $(X, \|\cdot\|)$ be a Banach space and let U be a nonempty open convex subset of $(X^*, \|\cdot\|^*)$. If $f : U \to \mathbb{R}$ is a continuous convex function and $\partial f(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in U$, then $\partial f(x^*) \subseteq \widehat{X}$ for all $x^* \in U$.*

A more general version of Theorem 1.3 is given in [9, Theorem 2.13].

Corollary 1.4. *Let $f : X^* \to (-\infty, \infty]$ be a proper convex lower semicontinuous function defined on the dual of a Banach space $(X, \|\cdot\|)$. If $0 \leq \varepsilon < \infty$, $\mathrm{Dom}(f)$ is an open subset of $(X^*, \|\cdot\|^*)$ and $\partial f(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(f)$, then $\partial_\varepsilon f(x^*) \subseteq \widehat{X}$ for all $x^* \in \mathrm{Dom}(f)$.*

Proof. Since f is a proper convex function, $\mathrm{Dom}(f)$ is a nonempty convex set, and by assumption, also open. Therefore, by [12, Proposition 3.3], $f|_{\mathrm{Dom}\, f}$ is continuous. Hence, by Theorem 1.3, $\partial f(x^*) \subseteq \widehat{X}$ for all $x^* \in \mathrm{Dom}(f)$. The result then follows from Proposition 1.2. $\square$

2. Main results

Next, we need to recall some basic facts from convex analysis. Their proofs are included for completeness.

Lemma 2.1. *Let $\psi : X \to \mathbb{R}$ be a convex function defined on a normed linear space $(X, \|\cdot\|)$ and let $\varphi : \mathbb{R} \to (-\infty, \infty]$ be a proper increasing convex function on $\mathbb{R}$. If* (i) $x_0 \in X$, (ii) $\psi(x_0) \in \mathrm{Dom}(\varphi)$, (iii) $\mathrm{Dom}(\varphi)$ *is an open sub-interval of* $\mathbb{R}$ *and* (iv) φ *is differentiable on* $\mathrm{Dom}(\varphi)$, *then $(\varphi \circ \psi) : X \to (-\infty, \infty]$ is a proper convex function and $\varphi'(\psi(x_0)) \cdot \partial\psi(x_0) \subseteq \partial(\varphi \circ \psi)(x_0)$.*

Proof. Firstly we note that $\mathrm{Dom}(\varphi\circ\psi) = \psi^{-1}(\mathrm{Dom}(\varphi))$. Since $x_0 \in \psi^{-1}(\mathrm{Dom}(\varphi))$, $\varphi \circ \psi$ is a proper function. Let $x, y \in \mathrm{Dom}(\varphi \circ \psi)$ and let $0 < \lambda < 1$. Then, $\psi(\lambda x + (1-\lambda)y) \leq \lambda\psi(x) + (1-\lambda)\psi(y)$. Now because φ is increasing,

$$(\varphi \circ \psi)(\lambda x + (1-\lambda)y) = \varphi(\psi(\lambda x + (1-\lambda)y)) \leq \varphi(\lambda\psi(x) + (1-\lambda)\psi(y)).$$

Finally, from the convexity of φ,

$$\begin{aligned}\varphi(\lambda\psi(x) + (1-\lambda)\psi(y)) &\leq \lambda\varphi(\psi(x)) + (1-\lambda)\varphi(\psi(y)) \\ &= \lambda(\varphi\circ\psi)(x) + (1-\lambda)(\varphi\circ\psi)(y).\end{aligned}$$

Let $x^* \in \partial\psi(x_0)$ and let $x \in \mathrm{Dom}(\varphi\circ\psi)$. Then $x^*(x - x_0) \leq \psi(x) - \psi(x_0)$. Since φ is an increasing function and φ is differentiable at $\psi(x_0) \in \mathrm{Dom}(\varphi)$, $0 \leq \varphi'(\psi(x_0))$.

Therefore, $$(\varphi'(\psi(x_0)) \cdot x^*)(x - x_0) \leq \varphi'(\psi(x_0))(\psi(x) - \psi(x_0)). \qquad (*)$$

Next, as $\varphi'(\psi(x_0)) \in \partial\varphi(\psi(x_0))$, or more precisely, the mapping $y \mapsto \varphi'(\psi(x_0))y$ is a member of $\partial\varphi(\psi(x_0))$, we have that

$$\varphi'(\psi(x_0))(\psi(x) - \psi(x_0)) \leq \varphi(\psi(x)) - \varphi(\psi(x_0)) = (\varphi\circ\psi)(x) - (\varphi\circ\psi)(x_0). \qquad (**)$$

By combining the inequalities $(*)$ and $(**)$ we obtain

$$(\varphi'(\psi(x_0)) \cdot x^*)(x - x_0) \leq (\varphi\circ\psi)(x) - (\varphi\circ\psi)(x_0).$$

Since $x \in \mathrm{Dom}(\varphi\circ\psi)$ was arbitrary, $\varphi'(\psi(x_0)) \cdot x^* \in \partial(\varphi\circ\psi)(x_0)$. This completes the proof. $\square$

Lemma 2.2. *Suppose that* $\varphi_j : X \to (-\infty, \infty]$, $1 \leq j \leq n$, *are proper convex functions on a normed linear space* $(X, \|\cdot\|)$ *and* $\{\varepsilon_j : 1 \leq j \leq n\}$ *are non-negative real numbers. If* $x_0 \in \bigcap_{1\leq j\leq n} \mathrm{Dom}(\partial_{\varepsilon_j}\varphi_j)$, *then* $\sum_{1\leq j\leq n} \partial_{\varepsilon_j}\varphi_j(x_0) \subseteq \partial_\varepsilon(\sum_{1\leq j\leq n}\varphi_j)(x_0)$, *where* $\varepsilon := \sum_{1\leq j\leq n}\varepsilon_j$.

Proof. Straightforward after recalling $\mathrm{Dom}(\sum_{1\leq j\leq n}\varphi_j) = \bigcap_{1\leq j\leq n}\mathrm{Dom}(\varphi_j)$. $\square$

To establish our main theorem we need one more ingredient, that is, the notion of the Fenchel conjugate.

Let $(X, \|\cdot\|)$ be a normed linear space and let $f : X \to (-\infty, \infty]$ be a proper function, then the *Fenchel conjugate* of f is the function $f^* : X^* \to (-\infty, \infty]$ defined by,

$$f^*(x^*) := \sup_{x\in X}(x^* - f)(x) = \sup_{x\in \mathrm{Dom}(f)}(x^* - f)(x).$$

It is well-known and easily verified that f^* is convex and lower semicontinuous on $(X^*, \|\cdot\|^*)$, see [8, Theorem 5.2 part (i)].

Proposition 2.3. ([8, Theorem 5.2 part (iv)]) *Suppose that* $f : X \to (-\infty, \infty]$ *is a proper function on a Banach space* $(X, \|\cdot\|)$ *and suppose that* $0 \leq \varepsilon$, $x^* \in \mathrm{Dom}(f^*)$, $x \in X$ *and* $f^*(x^*) - \varepsilon \leq (x^* - f)(x)$, *then* $\widehat{x} \in \partial_\varepsilon f^*(x^*)$.

Proof. Suppose that $0 \leq \varepsilon$, $x^* \in \mathrm{Dom}(f^*)$, $x \in X$ and $f^*(x^*) - \varepsilon \leq (x^* - f)(x)$. Let y^* be any element of $\mathrm{Dom}(f^*)$. Then,

$$\begin{aligned}\widehat{x}(y^* - x^*) &= y^*(x) - x^*(x) = [y^*(x) - f(x)] - [x^*(x) - f(x)] \\ &\leq [y^*(x) - f(x)] - [f^*(x^*) - \varepsilon] \leq f^*(y^*) - f^*(x^*) + \varepsilon.\end{aligned}$$

Since y^* was arbitrary, $\widehat{x} \in \partial_\varepsilon f^*(x^*)$. $\square$

Proposition 2.4. *Let $f : X \to (-\infty, \infty]$ be a proper function defined on a Banach space $(X, \|\cdot\|)$. If $x^* \in \operatorname{int}(\operatorname{Dom} f^*)$ and $0 < \varepsilon$, then $\{x \in X : f^*(x^*) - \varepsilon \leq (x^* - f)(x)\}$ is bounded in $(X, \|\cdot\|)$.*

Proof. Let $x^* \in \operatorname{int}(\operatorname{Dom} f^*)$ and $0 < \varepsilon$. Since $\operatorname{Dom}(f^*)$ is convex, $\operatorname{int}(\operatorname{Dom} f^*)$ is convex and nonempty as $x^* \in \operatorname{int}(\operatorname{Dom} f^*)$. Therefore, by [12, Proposition 3.3], $f^*|_{\operatorname{int}(\operatorname{Dom} f^*)}$ is continuous. Moreover, by [12, Proposition 1.11], $\partial f^*|_{\operatorname{int}(\operatorname{Dom} f^*)}$ is locally bounded on $\operatorname{int}(\operatorname{Dom} f^*)$. Thus, there exists a $0 < \delta$ and an $0 < M$ such that (i) $B[x^*, \delta] \subseteq \operatorname{int}(\operatorname{Dom} f^*)$ and (ii) $\|y^{**}\| \leq M$ for all $y^{**} \in \partial f^*(B[x^*, \delta])$. Let $\lambda := \varepsilon/\delta$. Then $\|x\| \leq \lambda + M$ for all $x \in \{y \in X : f^*(x^*) - \varepsilon \leq (x^* - f)(y)\}$. To see this, let $x \in \{y \in X : f^*(x^*) - \varepsilon \leq (x^* - f)(y)\}$. Then $\widehat{x} \in \partial_\varepsilon f^*(x^*)$. Thus, by the Brøndsted-Rockafellar Theorem, there exists $y^* \in B[x^*, \delta]$ and $y^{**} \in \partial f^*(y^*)$ such that $\|y^{**} - \widehat{x}\| \leq \lambda$. Hence, $\|x\| = \|\widehat{x}\| \leq \|\widehat{x} - y^{**}\| + \|y^{**}\| \leq \lambda + M$. □

Theorem 2.5. *Let $f : X \to (-\infty, \infty]$ be a proper function defined on a Banach space $(X, \|\cdot\|)$ and let $g : X^* \to (-\infty, \infty]$ be a proper lower semicontinuous convex function such that $\operatorname{Dom}(g)$ is an open subset of $\operatorname{Dom}(f^*)$ and $\partial g(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \operatorname{Dom}(g)$. If $\partial f^*(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \operatorname{Dom}(g)$, then for every $0 < \varepsilon$ and every $x_0^* \in \operatorname{int}(\operatorname{Dom} f^*)$, $\{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}$ is relatively weakly compact.*

Proof. We shall first consider the case when $x_0^* \in \operatorname{Dom}(g) \subseteq \operatorname{int}(\operatorname{Dom} f^*)$. Let $0 < \varepsilon$ be an arbitrary positive real number. By Proposition 2.4 we know that $\{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}$ is bounded in $(X, \|\cdot\|)$. Let $h : X^* \to (-\infty, \infty]$ be defined by, $h := g + f^*$. Then h is a proper lower semicontinuous convex function and $\operatorname{Dom}(h) = \operatorname{Dom}(g) \cap \operatorname{Dom}(f^*) = \operatorname{Dom}(g)$. By Lemma 2.2, $\partial h(x^*) \cap \widehat{X} \neq \varnothing$ for each $x^* \in \operatorname{Dom}(h)$. Therefore, by Corollary 1.4, $\partial_\varepsilon h(x^*) \subseteq \widehat{X}$ for all $x^* \in \operatorname{Dom}(h)$. Now, by Lemma 2.2, $\partial g(x^*) + \partial_\varepsilon f^*(x^*) \subseteq \partial_\varepsilon h(x^*) \subseteq \widehat{X}$ for all $x^* \in \operatorname{Dom}(h)$. Since $\partial g(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \operatorname{Dom}(h)$, it follows that $\partial_\varepsilon f^*(x^*) \subseteq \widehat{X}$ for all $x^* \in \operatorname{Dom}(h)$. Here we used the fact that if A and B are subsets of a vector space V and S is a subspace of V such that $A + B \subseteq S$, $A \cap S \neq \varnothing$ and $B \cap S \neq \varnothing$, then $A \subseteq S$ and $B \subseteq S$.

On the other hand, by Proposition 2.3,

$$\{\widehat{x} \in \widehat{X} : f^*(x^*) - \varepsilon \leq (x^* - f)(x)\} \subseteq \partial_\varepsilon f^*(x^*) \quad \text{for each } x^* \in \operatorname{Dom}(f^*).$$

Since $\partial_\varepsilon f^*(x_0^*)$ is weak* closed it follows that

$$\overline{\{\widehat{x} \in \widehat{X} : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}}^{w^*} \subseteq \partial_\varepsilon f^*(x_0^*) \subseteq \widehat{X}.$$

Hence, $\{x \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(x)\}$ is relatively weakly compact.

We now consider the general case. Let x_0^* be any element of $\operatorname{int}(\operatorname{Dom} f^*)$ and let y_0^* be a fixed element of $\operatorname{Dom}(g)$. By Proposition 2.4,

$$M := \sup\{\|x\| : x \in \{y \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(y)\}\} < \infty.$$

Let ε' be any positive real number greater than $\|y_0^* - x_0^*\|M + \varepsilon + f^*(y_0^*) - f^*(x_0^*)$ and let $x \in \{y \in X : f^*(x_0^*) - \varepsilon \leq (x_0^* - f)(y)\}$. Then,

$$\begin{aligned} f^*(y_0^*) &= [f^*(x_0^*) - \varepsilon] + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\ &\le (x_0^* - f)(x) + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\ &= (y_0^* - f)(x) + (x_0^* - y_0^*)(x) + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\ &\le (y_0^* - f)(x) + \|x_0^* - y_0^*\|\|x\| + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\ &\le (y_0^* - f)(x) + \|x_0^* - y_0^*\|M + [\varepsilon + f^*(y_0^*) - f^*(x_0^*)] \\ &\le (y_0^* - f)(x) + \varepsilon'. \end{aligned}$$

Therefore, $x \in \{y \in X : f^*(y_0^*) - \varepsilon' \le (y_0^* - f)(y)\}$ and so

$$\{y \in X : f^*(x_0^*) - \varepsilon \le (x_0^* - f)(y)\} \subseteq \{y \in X : f^*(y_0^*) - \varepsilon' \le (y_0^* - f)(y)\}.$$

Hence, $\{y \in X : f^*(x_0^*) - \varepsilon \le (x_0^* - f)(y)\}$ is relatively weakly compact. □

We now consider an application of Theorem 2.5.

Let A be a nonempty bounded subset of a normed linear space $(X, \|\cdot\|)$, then the *support function* for the set A is, $\sigma_A : X^* \to \mathbb{R}$, which is defined by

$$\sigma_A(x^*) := \sup\{x^*(a) : a \in A\} = \sup\{\widehat{a}(x^*) : a \in A\} \qquad \text{for all } x^* \in X^*.$$

Let $K := \sup\{\|\widehat{a}\| : a \in A\} = \sup\{\|a\| : a \in A\} < \infty$, (since the set A is bounded). Therefore, the functions $\{\widehat{a} : a \in A\}$ are pointwise bounded and K-Lipschitz on $(X^*, \|\cdot\|^*)$. As σ_A is the pointwise supremum of the family $\{\widehat{a} : a \in A\}$ of pointwise bounded linear K-Lipschitz functions, σ_A is itself convex and K-Lipschitz. [Recall the general fact that the pointwise supremum of a family of convex functions is convex and the pointwise supremum of a pointwise bounded above family of K-Lipschitz functions is again K-Lipschitz.]

Lemma 2.6. *Let A be a nonempty bounded subset of a normed linear space $(X, \|\cdot\|)$. If $x^* \in X^*$ and $x \in \operatorname{argmax}(x^*|_A)$, then $\widehat{x} \in \partial\sigma_A(x^*)$.*

Proof. Suppose that $x^* \in X^*$ and $x \in \operatorname{argmax}(x^*|_A)$, i.e., $x^*(x) = \sigma_A(x^*)$.
Let $y^* \in X^*$. Then, since $x \in A$,

$$\widehat{x}(y^* - x^*) = (y^* - x^*)(x) = y^*(x) - x^*(x) = y^*(x) - \sigma_A(x^*) \le \sigma_A(y^*) - \sigma_A(x^*).$$

Since $y^* \in X^*$ was arbitrary, $\widehat{x} \in \partial\sigma_A(x^*)$. □

Theorem 2.7. *Let A be a nonempty closed and bounded convex subset of a Banach space $(X, \|\cdot\|)$ and let W be a nonempty weakly compact subset of $(X, \|\cdot\|)$. If $x_0^* \in \{x^* \in X^* : \sup_{a\in A} x^*(a) < 0\}$ and $\operatorname{argmax}(y^*|_A) \neq \varnothing$ for each $y^* \in \{x^* \in X^* : \sup_{a\in A} x^*(a) < 0$ and $\sup_{w\in W} |(x^* - x_0^*)(w)| < 1\}$, then A is weakly compact.*

Proof. We shall apply Theorem 2.5. Let f denote the indicator function of the set A, i.e., $f(x) := 0$ if $x \in A$ and $f(x) := \infty$ otherwise. Then $f^* = \sigma_A$ and $\operatorname{int}(\operatorname{Dom} f^*) = \operatorname{Dom}(f^*) = X^*$.

Let us firstly recall that all weakly compact sets are norm bounded. This is a standard application of the uniform boundedness theorem, [4, page 66]. Secondly, by replacing the weakly compact set W by the weakly compact set $W \cup -W$, and relabelling, we may assume that the set

$$\{x^* \in X^* : \sup_{a\in A} x^*(a) < 0 \text{ and } \sup_{w\in W} |(x^* - x_0^*)(w)| < 1\}$$
$$= \{x^* \in X^* : \sigma_A(x^*) < 0 \text{ and } \sigma_W(x^* - x_0^*) < 1\}.$$

For each $\alpha \in \mathbb{R}$, let $\varphi_\alpha : \mathbb{R} \to (0, \infty]$ be defined by,

$$\varphi_\alpha(x) := \begin{cases} \frac{1}{\alpha - x} & \text{if } x < \alpha \\ \infty & \text{otherwise.} \end{cases}$$

Then each φ_α is a proper increasing lower semicontinuous convex function on $\mathbb{R}$ with $\mathrm{Dom}(\varphi_\alpha) = (-\infty, \alpha)$. Furthermore, each φ_α is differentiable at each point of $\mathrm{Dom}(\varphi_\alpha)$.

We now consider three auxiliary functions. $g_A : X^* \to (0, \infty]$, $g_W : X^* \to (0, \infty]$ and $g : X^* \to (-\infty, \infty]$ defined by, $g_A := \varphi_0 \circ \sigma_A$, $g_W(x^*) := (\varphi_1 \circ \sigma_W)(x^* - x_0^*)$ for all $x^* \in X^*$ and $g := g_A + g_W$. By Lemma 2.1, g_A and g_W are proper convex lower semicontinuous functions with $\mathrm{Dom}(g_A) = \{x^* \in X^* : \sigma_A(x^*) < 0\}$ and $\mathrm{Dom}(g_W) = \{x^* \in X^* : \sigma_W(x^* - x_0^*) < 1\}$. Therefore, g is a proper lower semicontinuous convex function with

$$x_0^* \in \{x^* \in X^* : \sigma_A(x^*) < 0 \text{ and } \sigma_W(x^* - x_0^*) < 1\} = \mathrm{Dom}(g_A) \cap \mathrm{Dom}(g_W) = \mathrm{Dom}(g).$$

Note that $\mathrm{Dom}(g)$ is an open subset of $(X^*, \|\cdot\|^*)$, as both σ_A and σ_W are continuous functions. Next, by assumption and Lemma 2.6, we have $\partial\sigma_A(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(g)$ and so by Lemma 2.1, $\partial g_A(x^*) \cap \widehat{X} \neq \varnothing$ for all $x^* \in \mathrm{Dom}(g)$. Furthermore, since W is weakly compact, $(x^* - x_0^*)$ attains it maximum value over W for every $x^* \in X^*$, and so by Lemma 2.6 and Lemma 2.1, $\partial g_W(x^*) \cap \widehat{X} \neq \varnothing$ for each $x^* \in \mathrm{Dom}(g)$. Therefore, from Lemma 2.2, $\partial g(x^*) \cap \widehat{X} \neq \varnothing$, for each $x^* \in \mathrm{Dom}(g)$.

Now apply Lemma 2.6 and Theorem 2.5 to deduce that $\{x \in X : f^*(0) - 1 \leq -f(x)\}$ is relatively weakly compact. Since $f^*(0) = 0$,

$$\{x \in X : f^*(0) - 1 \leq -f(x)\} = \{x \in X : f(x) \leq 1\} = A.$$

This completes the proof. $\square$

It is interesting to see that all three approaches to the proof of Theorem 2.7 given in [3, 13] and the present paper, are distinct. In the paper [3] the authors revisit and refine the proof of James' Theorem. Whereas, in [13], the authors reapply and extend an idea from [6, 7], to reduce the problem to an application of the classical James' Theorem. In this paper we showed that Theorem 2.7 follows from [8, Theorem 4.13].

Next we recall the definition of a slice of a nonempty set. Suppose that A is a nonempty subset of a normed linear space $(X, \|\cdot\|)$, $x^* \in X^*$, $\sup_{a\in A} x^*(a) < \infty$ and $0 \leq \varepsilon$, then a *slice* of A is the set

$$S(A, x^*, \varepsilon) := \{x \in A : (\sup\{x^*(a) : a \in A\} - \varepsilon) \leq x^*(x)\}.$$

We end this paper with the following simple result that may be of some independent interest.

Proposition 2.8. *Let A be a nonempty closed and convex subset of a Banach space $(X, \|\cdot\|)$ and suppose that $0 < \varepsilon < \infty$, $x^* \in X^*$ and $\sup_{a\in A} x^*(a) < \infty$. If $S(A, x^*, \varepsilon)$ is weakly compact, then $\overline{\widehat{A}}^{w^*} \subseteq \widehat{X}$.*

Proof. Let A be a nonempty closed and convex subset of X. Suppose that $x^* \in X^*$, $\sup_{a\in A} x^*(a) < \infty$, $0 < \varepsilon$ and $S(A, x^*, \varepsilon)$ is weakly compact. We shall use the fact that if C is a weakly compact subset of $(X, \|\cdot\|)$, then $\overline{\widehat{C}}^{w^*} = \widehat{C} \subseteq \widehat{X}$. For more details see, [8, Remark 2.25].

Our first goal is to show that $W \cap \overline{\widehat{A}}^{w^*} \subseteq \overline{\{\widehat{a} : a \in S(A, x^*, \varepsilon)\}}^{w^*}$, where

$W := (\widehat{x^*})^{-1}(s - \varepsilon, \infty) = \{x^{**} \in X^{**} : s - \varepsilon < x^{**}(x^*)\}$ and $s := \sup\{x^*(a) : a \in A\}$.

To this end, let $x^{**} \in W \cap \overline{\widehat{A}}^{w^*}$ and let V be an arbitrary weak* open neighbourhood of x^{**} in X^{**}. We will show that $V \cap \{\widehat{a} : a \in S(A, x^*, \varepsilon)\} \neq \varnothing$; from which it will follow that $x^{**} \in \overline{\{\widehat{a} : a \in S(A, x^*, \varepsilon)\}}^{w^*}$.

Now, $W \cap V$ is also a weak* open neighbourhood of x^{**} in X^{**}. Therefore we have $(V \cap W) \cap \widehat{A} \neq \varnothing$, since $x^{**} \in \overline{\widehat{A}}^{w^*}$. Choose $a_0 \in A$ so that $\widehat{a_0} \in (V \cap W)$. In particular, $\widehat{a_0} \in W$ and so $(s - \varepsilon) < \widehat{a_0}(x^*) = x^*(a_0)$. Therefore, $a_0 \in S(A, x^*, \varepsilon)$ and so $\widehat{a_0} \in V \cap \{\widehat{a} : a \in S(A, x^*, \varepsilon)\}$; which shows that $V \cap \{\widehat{a} : a \in S(A, x^*, \varepsilon)\} \neq \varnothing$.

Since V was an arbitrary weak* open neighbourhood of x^{**} we obtain in consequence that $x^{**} \in \overline{\{\widehat{a} : a \in S(A, x^*, \varepsilon)\}}^{w^*}$ and since x^{**} was an arbitrary element of $W \cap \overline{\widehat{A}}^{w^*}$ we get that $W \cap \overline{\widehat{A}}^{w^*} \subseteq \overline{\{\widehat{a} : a \in S(A, x^*, \varepsilon)\}}^{w^*}$.

Next, because $S(A, x^*, \varepsilon)$ is weakly compact, $\overline{\{\widehat{a} : a \in S(A, x^*, \varepsilon)\}}^{w^*} \subseteq \widehat{X}$. Thus, $W \cap \overline{\widehat{A}}^{w^*} \subseteq \widehat{X}$.

We now use this to show that $\overline{\widehat{A}}^{w^*} \subseteq \widehat{X}$. To achieve this goal we shall consider an arbitrary element $x^{**} \in \overline{\widehat{A}}^{w^*}$. Let $a \in S(A, x^*, \varepsilon/2) \neq \varnothing$. Then,

$$(s - \varepsilon) < (s - \varepsilon/2) \leq x^*(a) = \widehat{a}(x^*).$$

Therefore, $\widehat{a} \in W$. Since (X^{**}, weak^*) is a linear topological space and W is weak* open we have, by the continuity of the mapping, $\lambda \mapsto \lambda x^{**} + (1 - \lambda)\widehat{a}$, the existence of a $0 < \delta < 1$ such that $\delta x^{**} + (1 - \delta)\widehat{a} \in W$. Furthermore, from the convexity of $\overline{\widehat{A}}^{w^*}$, $\delta x^{**} + (1 - \delta)\widehat{a} \in \overline{\widehat{A}}^{w^*}$ and so $z^{**} := \delta x^{**} + (1 - \delta)\widehat{a} \in W \cap \overline{\widehat{A}}^{w^*} \subseteq \widehat{X}$. By rearranging this expression and using the fact that $\widehat{X}$ is a vector subspace, we have that $x^{**} = \delta^{-1}(z^{**} - (1 - \delta)\widehat{a}) \in \widehat{X}$. Since x^{**} was an arbitrary element of $\overline{\widehat{A}}^{w^*}$ we have that $\overline{\widehat{A}}^{w^*} \subseteq \widehat{X}$. $\square$

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