Journal of Convex Analysis
Volume 28 (2021), No. 3, 803–818

Locally Convex Properties of Baire Type Function Spaces

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Received: July 26, 2019
Accepted: May 30, 2020

For an infinite Tychonoff space X, a nonzero countable ordinal α and a locally convex space E over the field $\mathbb{F}$ of real or complex numbers, we denote by $B_\alpha(X,E)$ the class of Baire-α functions from X to E. In terms of the space E we characterize the space $B_\alpha(X,E)$ satisfying various weak barrelledness conditions, (DF)-type properties, the Grothendieck property, or Dunford-Pettis type properties. We solve Banach-Mazur's separable quotient problem for $B_\alpha(X,E)$ in a strong form: $B_\alpha(X,E)$ contains a complemented subspace isomorphic to $\mathbb{F}^{\mathbb{N}}$. Applying our results to the case when X is metrizable and $E=\mathbb{R}$, we show that the space $B_\alpha(X) := B_\alpha(X,\mathbb{R})$ is Baire-like (and hence barrelled), has the Grothendieck property and the Dunford-Pettis property. Further, the space $B_\alpha(X)$ is (semi-)Montel iff it is (semi-)reflexive iff it is (quasi-)complete iff $B_\alpha(X)=\mathbb{R}^X$ (for $\alpha=1$ the last equality is equivalent to X of being a Q-space).

Keywords: Baire type function spaces, Baire-like, weak barrelledness, Grothendieck property, Dunford-Pettis property, (quasi-)(DF)-space, (semi-)reflexive, (semi-)Montel.

2010 Mathematics Subject Classification: Primary 46A03; secondary 46A08, 54C35.

1. Introduction

The study of locally convex properties in some classes of locally convex spaces is one of the main topics in the general Theory of Locally Convex Spaces. In this article we consider weak barrelledness conditions, (DF)-type properties, the Grothendieck property, Dunford-Pettis type properties, and the property of being a Baire-like space, a (semi-)reflexive space or a (semi-)Montel space (we recall their definitions before the corresponding theorems in Section 3). All of these properties are of highest importance and form the cornerstone in the study of locally convex spaces. They were examined for various classes of locally convex spaces, in particular for the spaces $C_k(X)$ and $C_p(X)$ of real-valued continuous functions on a Tychonoff (=completely regular and Hausdorff) space X, endowed with the compact-open topology and the pointwise topology, respectively. We refer the reader to the classical books [9], [21], [28] or [22] and the articles [12, 16, 23, 24]. However, there are other classes of function spaces which are of great importance and widely studied in Topology and Analysis. For example, the class of derivatives of differentiable functions, the class of measurable functions, the class of Baire one functions etc.

ISSN 0944-6532 / \$ 2.50

In this paper we study the aforementioned locally convex properties in the classes of Baire type functions. The spaces of Baire-α functions were defined and studied by René Baire in his PhD thesis [3]. Let X and Y be Tychonoff spaces. The *Baire class zero* $B_0(X,Y)$ is the space $C_p(X,Y)$ of all continuous functions from X to Y endowed with the topology of pointwise convergence. For any nonzero countable ordinal α, let $B_\alpha(X,Y)$ be the family of all functions $f: X \to Y$ which are pointwise limits of sequences from $\bigcup_{\beta<\alpha} B_\beta(X,Y)$. All the spaces $B_\alpha(X,Y)$ are endowed with the topology of pointwise convergence, inherited from the Tychonoff product Y^X. Functions in the family $B_1(X,Y)$ are called the *functions of the first Baire class*, they are pointwise limits of sequences of continuous functions. If $Y=\mathbb{R}$, set $B_\alpha(X) := B_\alpha(X,\mathbb{R})$.

The interest in the study of the classes of Baire type functions is explained not only by the classical results due to René Baire, Henri Lebesgue and others, but also by some deep results in the Banach space theory. Let us recall that the Odell-Rosenthal theorem [27] states that a separable Banach space E does not contain an isomorphic copy of ℓ_1 if and only if $\chi|_K$ belongs to $B_1(K)$ for every $\chi \in E''$, where $K = B_{E'}$ is the closed unit ball of the dual Banach space E' of E endowed with the weak-$*$ topology. For further results we refer the reader to the survey [2].

Strengthening a result of Rosenthal [30], Bourgain, Fremlin and Talagrand proved in [7] the following important theorem: *If X is a Polish space, then the space $B_1(X)$ is angelic.* In [32], Stegall generalized this theorem by showing that $B_1(X,Y)$ is angelic for every Polish spaces X, Y. In particular, if Y is a Banach space, then $B_1(X,Y)$ is angelic. This last result was generalized by Mercourakis and Stamati in [25, Theorem 1.8]. The angelicity of $B_1(X,Y)$ implies that every compact subset of $B_1(X,Y)$ is Fréchet-Urysohn and a subspace K of $B_1(X,Y)$ is compact if and only if K is sequentially compact if and only if it is countably compact. This result plays a fundamental role in Functional Analysis. For a Polish space X, the compact subsets of $B_1(X)$ are called *Rosenthal compact* and have been studied intensively also by Godefroy [18], Todorčević [33] and others. Various other topological properties of the classes of Baire type functions are studied for example in [6, 14, 29]. However, a little is known about locally convex properties of the classes of Baire type functions.

In this article we characterize those Tychonoff spaces X and locally convex spaces E, for which the spaces $B_\alpha(X,E)$ with $\alpha \geq 1$ have one of the above mentioned locally convex properties. We emphasize that our results give not only a characterization of some of the most important locally convex properties in Baire type function spaces, but also a new *method* for constructing locally convex spaces with those properties.

Applying the main results of the article, which are formulated and proved in Section 3, we immediately obtain the following theorem.

Theorem 1.1. *Let X be a Tychonoff space and let α be a nonzero countable ordinal. Then:*

(i) $B_\alpha(X)$ *is Baire-like and hence barrelled.*

(ii) $B_\alpha(X)$ *is a (DF)-space if and only if X is finite.*

(iii) $B_\alpha(X)$ *has the Grothendieck property.*

(iv) $B_\alpha(X)$ *has the Dunford-Pettis property.*

(v) *If X is infinite, then $B_\alpha(X)$ has a complemented subspace isomorphic to $\mathbb{R}^{\mathbb{N}}$.*

(vi) *If X has countable pseudocharacter (for example, X is metrizable), then the following assertions are equivalent:*
   (a) *$B_\alpha(X)$ is a Montel space;*
   (b) *$B_\alpha(X)$ is a semi-Montel space;*
   (c) *$B_\alpha(X)$ is a reflexive space;*
   (d) *$B_\alpha(X)$ is a semi-reflexive space;*
   (e) *$B_\alpha(X)$ is a complete space;*
   (f) *$B_\alpha(X)$ is a quasi-complete space;*
   (g) *$B_\alpha(X) = \mathbb{R}^X$.*

It is worth mentioning that, for a normal space X, we proved in [6] that $B_1(X) = \mathbb{R}^X$ if and only if X is a Q-space (i.e., every subset of X is of type F_σ in X).

2. Auxiliary results

All locally convex spaces E (lcs, for short) are assumed to be Hausdorff and the field $\mathbb{F}$ of E is either $\mathbb{R}$ or $\mathbb{C}$. For a set X, we denote by id_X the identity map from X to X.

For a dual pair (E,F) of locally convex spaces, we shall use the standard notations $\sigma(E,F), \mu(E,F)$ and $\beta(E,F)$ for the *weak topology*, the *Mackey topology* and the *strong topology* on E, respectively, see [9, 21, 28]. Define $E_w := \big(E, \sigma(E,F)\big)$, $E_\beta := \big(E, \beta(E,F)\big)$, and $E_\mu := \big(E, \mu(E,F)\big)$. The value of $f \in F$ on $e \in E$ is denoted by $\langle f|e\rangle$ (following the standard bracket notation widely used in Quantum Physics). The *polar* of a subset A of E is the set

$$A^\circ := \bigcap_{a\in A} \{f \in F : |\langle f|a\rangle| \le 1\}.$$

The dual space of E is denoted by E' and, for the dual pair (E,E'), set $E'' := (E'_\beta)'$. Also we denote the space $(E', \sigma(E',E''))$ by $E'_{w''}$.

If X is a set and E is an lcs, the sets of the form

$$[F;V] := \big\{f \in E^X : f(F) \subseteq V\big\},$$

where F is a finite subset of X and V is an open neighborhood of $0 \in E$, form a base of the pointwise topology or the Tychonoff product topology on E^X at zero function $\mathbf{0} \in E^X$. It is well known (see [21, Proposition 8.8.1]) that the dual space $(E^X)'$ of E^X can be identified with the linear space of all functions $\mu : X \to E'$ that have finite support $\mathrm{supp}(\mu) := \{x \in X : \mu(x) \neq 0\}$. Such functions μ will be called *measures* on X. The action $\langle \mu|f\rangle$ of a measure μ on a function $f : X \to E$ is defined by the formula

$$\langle \mu|f\rangle = \sum_{x\in\mathrm{supp}(\mu)} \langle \mu(x)|f(x)\rangle.$$

From now on (and till the end of this section), we assume that X is a Tychonoff space, E is a locally convex space and $L \subseteq E^X$ is a linear subspace such that

$C_p(X,E) \subseteq L$. The latter inclusion implies that L is dense in E^X. Taking into account that any continuous linear functional $f\colon L \to \mathbb{F}$ admits a unique continuous extension to E^X, we conclude that the dual space L' of L can be identified with the dual $(E^X)'$ of E^X. Therefore, elements of the dual space L' can be thought as finitely supported functions from X to E'.

For every subset $Z \subseteq X$ consider the *projector* $p_Z\colon L \to E^Z$ assigning to each function $f \in L$ its restriction $f|_Z$.

Lemma 2.1. *For any finite subset $Z \subseteq X$ there exists a linear continuous operator $i_Z : E^Z \to C_p(X,E) \subseteq L$ such that $p_Z \circ i_Z$ is the identity operator on E^Z.*

Proof. Using the Hausdorff property of X, find a disjoint family $\{U_z\}_{z\in Z}$ of open sets in X such that $z \in U_z$ for every $z \in Z$. Since the space X is Tychonoff, for every $z \in Z$ there exists a continuous function $\varphi_z : X \to [0,1]$ such that $\varphi_z(z) = 1$ and $\mathrm{supp}(\varphi_z) \subseteq U_z$. For each function $f : Z \to E$ consider the continuous function $i_Z f : X \to E$ defined by $i_Z f(x) = \sum_{z\in Z} \varphi_z(x) \cdot f(z)$. It is clear that the operator $i_Z : E^Z \to C_p(X,E)$ is linear, continuous and $p_Z \circ i_Z = \mathrm{id}_{E^Z}$. □

Now for a subset $Z \subseteq X$ and the restriction operator $p_Z : L \to E^Z$, consider the dual operator $p'_Z : (E^Z)' \to L'$. This operator assigns to each finitely supported measure $\mu : Z \to E'$ the measure $p'_Z\mu \in L'$ acting on a function $f \in L$ by the formula

$$\langle p'_Z\mu|f\rangle = \langle \mu|p_Z f\rangle,$$

which implies that $\mathrm{supp}(p'_Z\mu) \subseteq Z$ and $p'_Z\mu|_Z = \mu$. This description implies that the operator p'_Z is injective.

If the set $Z \subseteq X$ is finite, then by Lemma 2.1, there exists a linear continuous operator $i_Z : E^Z \to L$ such that $p_Z \circ i_Z = \mathrm{id}_{E^Z}$. Applying to the latter equality the duality functor, we obtain the equality $i'_Z \circ p'_Z = \mathrm{id}_{(E^Z)'}$. This observation and the continuity of the dual operators in the strong, Mackey and weak topologies [21, §8.6] imply the following useful fact.

Lemma 2.2. *For any finite subset $Z \subseteq X$, the dual operator $p'_Z : (E^Z)' \to L'$ is an embedding of*

(i) $(E^Z)'_w$ *into* L'_w*;*

(ii) $(E^Z)'_\mu$ *into* L'_μ*;*

(iii) $(E^Z)'_\beta$ *into* L'_β*;*

(iv) $(E^Z)'_{w''}$ *into* $L'_{w''}$*.*

If $Z = \{z\} \subseteq X$ is a singleton, then the projector $p_{\{z\}} : L \to E$ will be denoted by p_z and its dual by $p'_z : E' \to L'$.

Recall that a subset $A \subseteq E'$ is *equicontinuous* if $A \subseteq U^\circ$ for some neighborhood $U \subseteq E$ of zero.

Lemma 2.3. *Let Z be a finite subset of X. A subset $A \subseteq (E^Z)'$ is equicontinuous if and only if $p'_Z(A)$ is equicontinuous in L'.*

Proof. Since Z is finite, p_Z is a quotient map. Now the assertion follows from Theorems 8.11.2 and 8.13.4 of [26]. □

Let X and Y be topological spaces. Besides the classes $B_\alpha(X,Y)$ we shall consider also the *stable* Baire-α classes $B^{st}_\alpha(X,Y)$ of functions, which were introduced and studied by Császár and Laczkovich [8]. We say that a sequence $\{f_n\}_{n\in\mathbb{N}}\subseteq Y^X$ *stably converges* to a function $f\in Y^X$ if for every $x\in X$ the set $\{n\in\mathbb{N}: f_n(x)\neq f(x)\}$ is finite. Then the class $B^{st}_1(X,Y)$ is defined as the family of all functions from Y^X which are limits of stably convergent sequences of continuous functions. For every countable ordinal $\alpha>1$, denote by $B^{st}_\alpha(X,Y)\subseteq Y^X$ the family of all functions $f: X\to Y$ which are pointwise limits of function sequences from $\bigcup_{\beta<\alpha}B^{st}_\beta(X,Y)$. Functions in the family $B^{st}_\alpha(X,Y)$ are called the *functions of stable Baire-α class.* It is clear that

$$C_p(X,Y)\subseteq B^{st}_\alpha(X,Y)\subseteq B_\alpha(X,Y)$$

for every nonzero countable ordinal α. Note that the inclusion $B^{st}_1(X,Y)\subseteq B_1(X,Y)$ can be strict. If $Y=\mathbb{R}$, set $B^{st}_1(X):=B^{st}_1(X,\mathbb{R})$.

The following lemma will be essentially used in the proofs of our main results.

Lemma 2.4. *If $B^{st}_1(X,E)\subseteq L$, then each $\sigma(L',L)$-bounded subset $\Phi\subseteq L'$ has finite support* $\mathrm{supp}(\Phi):=\bigcup_{\varphi\in\Phi}\mathrm{supp}(\varphi)$.

Proof. To derive a contradiction, assume that $\mathrm{supp}(\Phi)$ is infinite. Then, by Lemma 11.7.1 of [21], there is a one-to-one sequence $\{a_n\}_{n\in\mathbb{N}}$ in A and a sequence $\{V_n\}_{n\in\mathbb{N}}$ of open pairwise disjoint subsets of X such that $a_n\in V_n$ for every $n\in\mathbb{N}$. Separating the set $\mathbb{N}$ onto a disjoint union $\bigcup_{n\in\mathbb{N}}I_n$ of countably infinite subsets of $\mathbb{N}$ and setting $W_n:=\bigcup_{k\in I_n}V_k$, we can find a disjoint sequence $\{W_n\}_{n\in\mathbb{N}}$ of open subsets of X such that for every $n\in\mathbb{N}$ the intersection $W_n\cap\mathrm{supp}(\Phi)$ is infinite. Choose any point $x_1\in W_1\cap\mathrm{supp}(\Phi)$ and find a measure $\mu_1\in\Phi$ with $x_1\in\mathrm{supp}(\mu_1)$. Let $U_1\subseteq W_1$ be a neighborhood of x_1 such that $U_1\cap\mathrm{supp}(\mu_1)=\{x_1\}$. Proceeding by induction, for every natural number $n>1$, choose a point

$$x_n\in W_n\cap\big(\mathrm{supp}(\Phi)\setminus\textstyle\bigcup_{i=1}^{n-1}\mathrm{supp}(\mu_i)\big),$$

a functional $\mu_n\in\Phi$ with $x_n\in\mathrm{supp}(\mu_n)$ and a neighborhood $U_n\subseteq W_n$ of x_n with

$$U_n\cap\bigcup_{i=1}^{n}\mathrm{supp}(\mu_i)=\{x_n\}.$$

Set $h_0:=0\in C(X,E)$. For every $k\in\mathbb{N}$, choose a function $h_k\in C(X,E)$ such that $\mathrm{supp}(h_k)\subseteq U_k\subseteq W_k$, and

$$\langle\mu_k(x_k)|h_k(x_k)\rangle=k-\sum_{0\le i<k}\langle\mu_k|h_i\rangle. \tag{1}$$

Since $\mathrm{supp}(h_k)\cap\mathrm{supp}(\mu_k)\subseteq U_k\cap\mathrm{supp}(\mu_k)=\{x_k\}$ we have $\langle\mu_k|h_k\rangle=\langle\mu_k(x_k)|h_k(x_k)\rangle$.

Since the supports of the functions h_k are pairwise disjoint, the function sequence $\big\{\sum_{i=1}^k h_i\big\}_{k\in\mathbb{N}}$ stably converges to the function

$$h=\textstyle\sum_{k\in\mathbb{N}}h_k\in B^{st}_1(X,E)\subseteq L.$$

The choice of the neighborhoods U_i ensures that, for every pair $k,i\in\mathbb{N}$ such that $k<i$, we have

$$\mathrm{supp}(\mu_k)\cap\mathrm{supp}(h_i)\subseteq\mathrm{supp}(\mu_k)\cap U_i=\emptyset,\quad\text{so}\quad\langle\mu_k|h_i\rangle=0. \tag{2}$$

By (1) and (2),
$$\begin{aligned}\langle \mu_k | h \rangle &= \sum_{0 \le i < k} \langle \mu_k | h_i \rangle + \langle \mu_k | h_k \rangle + \sum_{k<i} \langle \mu_k | h_i \rangle \\ &= \sum_{0 \le i < k} \langle \mu_k | h_i \rangle + \langle \mu_k(x_k) | h_k(x_k) \rangle = k,\end{aligned}$$

which contradicts the $\sigma(L', L)$-boundedness of the set $\Phi \subseteq L'$. □

Finally, we establish some properties of the stable first Baire class $B_1^{st}(X, E)$.

Lemma 2.5. *If X is a countable Tychonoff space, then $B_1^{st}(X, E) = E^X$ for any lcs E.*

Proof. If X is finite, then the assertion is trivial. Assume that X is countably infinite and let $X = \{x_k\}_{k \in \mathbb{N}}$ be a one-to-one enumeration of X. Let $f : X \to E$ be an arbitrary function. Since X is zero-dimensional ([11, Corollary 6.2.8]), for every $n \in \mathbb{N}$ there are pairwise disjoint closed-and-open subsets $U_{1,n}, \dots, U_{n,n}$ of X such that $x_i \in U_{i,n}$ for every $i \in \{1, \dots, n\}$. Define $f_n : X \to E$ by

$$f_n(x) = f(x_i) \text{ if } x \in U_{i,n} \text{ for some } i \in \{1, \dots, n\}, \text{ and } f_n(x) = 0 \text{ otherwise.}$$

It is clear that all functions f_n are continuous and the sequence $\{f_n\}_{n \in \mathbb{N}}$ stably converges to f. Thus $B_1^{st}(X, E) = E^X$. □

Recall that a topological space X is said to have *countable pseudocharacter* if every singleton $\{x\} \subseteq X$ is the intersection of a countable family of open subsets of X.

Lemma 2.6. *Let E be a locally convex space and let X be a Tychonoff space of countable pseudocharacter. Then every function $f : X \to E$ with finite support belongs to the function space $B_1^{st}(X, E)$.*

Proof. Let $S := \text{supp}(f)$ be the finite support of the function f. Since X is Hausdorff, there exists a disjoint family of open sets $\{U_0(x)\}_{x \in S}$ in X such that $x \in U_0(x)$ for every $x \in S$. Since the space X has countable pseudocharacter, for every $x \in S$, there exists a sequence $\{U_n(x)\}_{n \in \mathbb{N}}$ of open neighborhoods of x such that $\{x\} = \bigcap_{n \in \mathbb{N}} U_n(x)$ and $U_n(x) \subseteq U_{n-1}(x)$ for every $n \in \mathbb{N}$. Using the Tychonoff property of X, for every $x \in S$ and $n \in \mathbb{N}$, find a continuous function $f_{x,n} : X \to [0,1] \cdot f(x) \subseteq E$ such that $f_{x,n}(x) = f(x)$ and $\text{supp}(f_{x,n}) \subseteq U_n(x)$. Since the family $\{U_n(x)\}_{x \in S}$ is disjoint, the continuous function $f_n := \sum_{x \in S} f_{x,n}$ has the properties: $f_n|_S = f|_S$ and $\text{supp}(f_n) \subseteq \bigcup_{x \in S} U_n(x)$. As

$$\bigcap_{n \in \mathbb{N}} \text{supp}(f_n) = \bigcap_{n \in \mathbb{N}} \bigcup_{x \in S} U_n(x) = S = \text{supp}(f),$$

the function sequence $\{f_n\}_{n \in \mathbb{N}}$ stably converges to f showing that $f \in B_1^{st}(X, E)$. □

3. Main results

Let us recall that an lcs E is

- (*quasi*)*barrelled* if every $\sigma(E', E)$-bounded (respectively, $\beta(E', E)$-bounded) subset of E' is equicontinuous;

- $\aleph_0$-*(quasi)barrelled* if every $\sigma(E', E)$-bounded (respectively, $\beta(E', E)$-bounded) subset of E' which is the countable union of equicontinuous subsets is also equicontinuous;
- ℓ_∞-*(quasi)barrelled* if every $\sigma(E', E)$-bounded (respectively, $\beta(E', E)$-bounded) sequence is equicontinuous;
- c_0-*(quasi)barrelled* if every $\sigma(E', E)$-null (respectively, $\beta(E', E)$-null) sequence is equicontinuous.

It is well known (and easy to see) that the above barrelledness properties are preserved by finite products of locally convex spaces.

Theorem 3.1. *Let X be a non-empty Tychonoff space, E be a locally convex space and let L be a linear subspace of E^X such that $B_1^{st}(X, E) \subseteq L$. Then:*

(i) *L is(quasi)barrelled if and only if E is (quasi)barrelled;*

(ii) *L is c_0-(quasi)barrelled if and only if E is c_0-(quasi)barrelled;*

(iii) *L is ℓ_∞-(quasi)barrelled if and only if E is ℓ_∞-(quasi)barrelled;*

(iv) *L is $\aleph_0$-(quasi)barrelled if and only if E is $\aleph_0$-(quasi)barrelled.*

Proof. Since the proofs of the statements (i), (ii), (iii) exploit the same idea, we present a detail proof only for the c_0-quasibarrelledness. Assume that the space L is c_0-quasibarrelled. To show that E is c_0-quasibarelled, we have to prove that every $\beta(E', E)$-null sequence $S = \{\chi_n\}_{n\in\mathbb{N}}$ in E' is equicontinuous. Fix an arbitrary $z \in X$. By Lemma 2.2, the linear operator $p'_z : E'_\beta \to L'_\beta$ is continuous. Consequently, the sequence $S_L = \{p'_z(\chi_n)\}_{n\in\mathbb{N}} \subseteq L'$ is $\beta(L', L)$-null. Since L is c_0-quasibarrelled, the sequence S_L is equicontinuous. Thus, by Lemma 2.3, the sequence S is equicontinuous.

Conversely, assume that E is c_0-quasibarrelled. To show that L is also c_0-quasibarrelled, we have to prove that every $\beta(L', L)$-null sequence $S_L = \{\mu_k\}_{k\in\mathbb{N}}$ in L' is equicontinuous. Being $\beta(L', L)$-null, the set S_L is $\sigma(L', L)$-bounded. By Lemma 2.4, the set $Z := \operatorname{supp}(S_L)$ is finite and hence $S_L \subseteq p'_Z((E^Z)')$. By Lemma 2.2, the operator $p'_Z : (E_Z)'_\beta \to L'_\beta$ is a topological embedding. Consequently, the sequence $S := \{(p'_Z)^{-1}(\mu_k)\}_{k\in\mathbb{N}}$ is $\beta((E^Z)', E^Z)$-null in $(E^Z)'$. Since the space E^Z is c_0-quasibarrelled (being a finite power of the c_0-quasibarrelled space E), the sequence S is equicontinuous. Thus, by Lemma 2.3, the sequence $S_L = p'_Z(S)$ is equicontinuous, too.

(iv) Assume that L is $\aleph_0$-(quasi)barrelled and let $B = \bigcup_{k\in\mathbb{N}} B_k$ be a $\sigma(E', E)$-bounded (respectively, $\beta(E', E)$-bounded) subset of E', where all B_k are equicontinuous. We have to show that B is equicontinuous. Fix an arbitrary $z \in X$ and consider the embedding operator $p'_z : E' \to L'$. Lemmas 2.2 and 2.3 imply that, for every $k \in \mathbb{N}$, the set $p'_z(B_k) \subseteq L'$ is equicontinuous and the union $p'_z(B)$ is $\sigma(E', E)$-bounded (resp., $\beta(E', E)$-bounded) and hence equicontinuous by the $\aleph_0$-(quasi)barrelledness of L. Thus, by Lemma 2.3, the set B is equicontinuous.

Conversely, assume that E is $\aleph_0$-(quasi)barrelled and let $A = \bigcup_{k\in\mathbb{N}} A_k$ be a $\sigma(L', L)$-bounded (respectively, $\beta(L', L)$-bounded) subset of L', where all A_k are equicontinuous. We have to show that also A is equicontinuous. By Lemma 2.4, the set $Z := \operatorname{supp}(A)$ is finite and hence $A \subseteq p'_Z((E^Z)')$. By Lemma 2.2, the set

$B := (p'_Z)^{-1}(A)$ is $\sigma((E^Z)', E^Z)$-bounded (resp. $\beta((E^Z)', E^Z)$-bounded) and, by Lemma 2.3, all sets $B_k := (p'_Z)^{-1}(A_k)$, $k \in \mathbb{N}$, are equicontinuous. Since E is $\aleph_0$-(quasi)barrelled, the set $B = \bigcup_{k\in\mathbb{N}} B_k$ is equicontinuous. Thus, by Lemma 2.3, the set $A = p'_Z(B)$ is equicontinuous. □

Let E be a locally convex space. We shall say that E has a *fundamental bounded sequence* if there exists a sequence $\{B_n\}_{n\in\mathbb{N}}$ of bounded subsets of E (called *fundamental*) such that for every bounded set B of E there is an $n \in \mathbb{N}$ such that $B \subseteq B_n$. A family $\mathcal{B} = \{B_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of bounded sets of E is called a *fundamental bounded resolution* if $\mathcal{B}$ satisfies the following two conditions: (1) $B_\alpha \subseteq B_\beta$ for any $\alpha \le \beta$ in $\mathbb{N}^\mathbb{N}$, and (2) every bounded subset of E is contained in some B_α.

Let us recall that an lcs (E, τ) is called

- a (DF)*-space* if it has a fundamental bounded sequence and is $\aleph_0$-quasibarrelled;
- an *almost* (DF)*-space* if it has a fundamental bounded sequence and is ℓ_∞-quasibarrelled;
- a (*df*)*-space* if it has a fundamental bounded sequence and is c_0-quasibarrelled;
- a *quasi-*(DF)*-space* if it has a fundamental bounded resolution and the dual E' admits a cover $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ such that $A_\alpha \subseteq A_\beta$ for any $\alpha \le \beta$ in $\mathbb{N}^\mathbb{N}$, and each sequence in any A_α is equicontinuous.

It is worth mentioning that (DF)-spaces were introduced by Grothendieck [20], (*df*)-spaces were defined by Jarchow [21], the classes of quasi-(DF)-spaces and almost (DF)-spaces were introduced in [12] and [15], respectively. Let us note that there are quasi-(DF)-spaces which are not (*df*)-spaces, and there exist (*df*)-spaces which are not quasi-(DF)-spaces, see [15].

Proposition 3.2. (i) *The classes of* (DF)*-spaces and of* (*df*)*-spaces are stable under taking Hausdorff quotients and finite products.*

(ii) *The class of quasi-*(DF)*-spaces is stable under taking subspaces and countable products.*

(iii) *The class of almost* (DF)*-spaces is stable under taking complemented subspaces and finite products.*

Proof. The statements (i) and (ii) are proved in Theorem 12.4.8 of [21] and Theorem 4.1 of [12], respectively.

(iii) Let F be a complemented subspace of an almost (DF)-space E, so $E = F \times G$ for some closed subspace G of E. If $S = \{B_n\}_{n\in\mathbb{N}}$ is a fundamental bounded sequence in E, it is clear that the projection $\{\mathrm{pr}_F(B_n)\}_{n\in\mathbb{N}}$ of S onto F is a fundamental bounded sequence in F. Since F is a (Hausdorff) quotient of E, the space F is ℓ_∞-quasibarrelled by Proposition 8.2.31 of [28]. Thus F is an almost (DF)-space. Since the class of ℓ_∞-quasibarrelled spaces is stable under taking finite products ([28, Proposition 8.2.31]), it easily follows that also the class of almost (DF)-spaces is stable under taking finite products. □

Theorem 3.3. *Let X be a non-empty Tychonoff space, E be a locally convex space and let L be a linear subspace of E^X containing $B_1^{st}(X, E)$. Then:*

(i) *L is a (DF)-space if and only if X is finite and E is a (DF)-space;*

(ii) *L is an almost (DF)-space if and only if X is finite and E is an almost (DF)-space;*

(iii) *L is a (df)-space if and only if X is finite and E is a (df)-space;*

(iv) *L is a quasi-(DF)-space if and only if X is countable and E is a quasi-(DF)-space.*

Proof. Let $\mathbb{F}$ be the field of E. Assume that the space L has a fundamental bounded resolution. Therefore the space $C_p(X,E)$ also has a fundamental bounded resolution. It is well known that E can be represented as a direct product $\mathbb{F}\times\tilde{E}$, where $\tilde{E}$ is a linear subspace of E. Then $C_p(X,E)=C_p(X,\mathbb{F})\times C_p(X,\tilde{E})$. Therefore $C_p(X,\mathbb{F})$ and hence also $C_p(X)$ have a fundamental bounded resolution as well, and hence X is countable by Theorem 3.3 of [12] (which states that for a Tychonoff space X, the space $C_p(X)$ has a fundamental bounded resolution if and only if X is countable). Thus, by Lemma 2.5, $B_1^{st}(X,E)=E^X$ and therefore $L=E^X$.

(i)–(iii) Assume that L is a (DF)-space (an almost (DF)-space or a (df)-space). Then, by the remark above, X is countable. Assuming that X is countably infinite we obtain that $L=E^X=\mathbb{F}^{\mathbb{N}}\times\tilde{E}^X$. Since $\mathbb{F}^{\mathbb{N}}$ does not have a fundamental bounded sequence (the space $\mathbb{F}^{\mathbb{N}}$ is not σ-compact by the Baire Category Theorem), we obtain that L is not a (DF)-space (an almost (DF)-space or a (df)-space). Therefore X must be finite. Since E is a direct summand of $L=E^X$, Proposition 3.2 implies that E is a (DF)-space (an almost (DF)-space or a (df)-space).

Conversely, if E is a (DF)-space (an almost (DF)-space or a (df)-space) and X is finite, then $L=E^X$ (Proposition 2.5) and it is a (DF)-space (an almost (DF)-space or a (df)-space) by Proposition 3.2.

(iv) Assume that L is a quasi-(DF)-space. Then, as we showed above, X is countable and $L=E^X$. Therefore E is a quasi-(DF)-space by Proposition 3.2. Conversely, if E is a quasi-(DF)-space and X is countable, then $L=E^X$ (Proposition 2.5) and hence L is a quasi-(DF)-space by Proposition 3.2. $\square$

Recall that a locally convex space E is said to have the *Grothendieck property* if every $\sigma(E',E)$-convergent sequence in the strong dual E'_β is $\sigma(E',E'')$-convergent. It is easy to see that the Grothendieck property is preserved by finite products.

Theorem 3.4. *Let X be a non-empty Tychonoff space, E be a locally convex space and let L be a linear subspace of E^X containing $B_1^{st}(X,E)$. Then L has the Grothendieck property if and only if E has the Grothendieck property.*

Proof. Assume that L has the Grothendieck property and let $\{\chi_n\}_{n\in\mathbb{N}}$ be a $\sigma(E',E)$-null sequence in E'. We have to show that $\{\chi_n\}_{n\in\mathbb{N}}$ is $\sigma(E',E'')$-null. Fix an arbitrary $z\in X$. By Lemma 2.2, the operator $p'_z:(E^Z)'_w\to L'_w$ is continuous. Consequently, the sequence $\{p'_z(\chi_n)\}_{n\in\mathbb{N}}$ is $\sigma(L',L)$-null in L' and hence $\sigma(L',L'')$-null by the Grothendieck property of L. Since the operator $p'_z:(E^Z)'_{w''}\to L'_{w''}$ is a topological embedding (see Lemma 2.2), the sequence $\{\chi_n\}_{n\in\mathbb{N}}$ is $\sigma(E',E'')$-null.

Conversely, assume that E has the Grothendieck property. Fix an arbitrary $\sigma(L',L)$-null sequence $S_L=\{\mu_k\}_{k\in\mathbb{N}}$ in L'. By Lemma 2.4, the set $Z:=\mathrm{supp}(S_L)$ is finite and hence $S_L\subseteq p'_Z((E^Z)')$. By Lemma 2.2, the operator $p'_Z:(E^Z)'_w\to L'_w$ is a topological embedding, which implies that the sequence $S := \{(p'_Z)^{-1}(\mu_n)\}_{n\in\mathbb{N}}$ is $\sigma((E^Z)', E^Z)$-null and hence $\sigma((E^Z)', (E^Z)'')$-null by the Grothendieck property of the finite power E^Z of E. By Lemma 2.2, the operator $p'_Z : (E^Z)'_{w''} \to L'_{w''}$ is continuous, which implies that the sequence $S_L = p'_Z(S)$ is $\sigma(L', L'')$-null. $\square$

A subset A of an lcs E is said to be *complete* if every Cauchy net in A has a limit in A. Recall that an lcs E is called *quasi-complete* if every closed bounded subset of E is complete.

The Dunford-Pettis property (the (DP) property for short) was introduced by Grothendieck [19] (for comments, see [9, p.633–634]). An lcs E is said to have the *(DP) property* if every continuous linear operator T from E into a quasi-complete locally convex space F, which transforms bounded sets of E into relatively weakly compact subsets of F, also transforms absolutely convex weakly compact subsets of E into relatively compact subsets of F. Below we shall use the following characterization that can be found in Theorem 9.3.4 of [9].

Theorem 3.5. *An lcs E has the (DP) property if and only if every absolutely convex, equicontinuous, and $\sigma(E', E'')$-compact subset of E' is precompact in the Mackey topology $\mu(E', E)$.*

We recall that a subset A of an lcs E is called *precompact* if for every open neighborhood U of zero in E there exists a finite subset F of E such that $A \subseteq \bigcup_{f\in F}(f + U)$.

In [19, Proposition 2], Grothendieck proved that a Banach space E has the (DP) property if and only if for any weakly null sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ and $\{\chi_n\}_{n\in\mathbb{N}} \subseteq E'_\beta$ we have $\lim_{n\to\infty}\langle\chi_n|x_n\rangle = 0$. Albanese, Bonet and Ricker extended Grothendieck's theorem to strict (LF)-spaces, see [1, Corollary 3.4]. These results motivated the author of [13] to introduce the following "sequential" version of the (DP) property: a locally convex space E is said to have the *sequential Dunford-Pettis property* (*(sDP) property*) if for any weakly null sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq E$ and $\{\chi_n\}_{n\in\mathbb{N}} \subseteq E'_\beta$ we have $\lim_{n\to\infty}\langle\chi_n|x_n\rangle = 0$. It is easy to see that the (sequential) Dunford-Pettis property is preserved by finite products.

Theorem 3.6. *Let X be a non-empty Tychonoff space, E be a locally convex space and let L be a linear subspace of E^X such that $B_1^{st}(X, E) \subseteq L$. Then:*

(i) *L has the (DP)-property if and only if E has the (DP)-property;*

(ii) *L has the (sDP)-property if and only if E has the (sDP)-property.*

Proof. (i) Assume that L has the (DP)-property and let K be an arbitrary absolutely convex, equicontinuous, and $\sigma(E', E'')$-compact subset of E'. Fix an arbitrary point $z \in X$. Lemmas 2.2 and 2.3 imply that the absolutely convex subset $p'_z(K)$ of L' is equicontinuous and $\sigma(L', L'')$-compact in L'. Since L has the (DP)-property, Theorem 3.5 implies that $p'_x(K)$ is precompact in the Mackey topology $\mu(L', L)$ on L'. By Lemma 2.2, K is precompact in the Mackey topology $\mu(E', E)$ on E'. Thus, by Theorem 3.5, E has the (DP)-property.

Conversely, assume that E has the (DP)-property and let $\tilde{K}$ be an arbitrary absolutely convex, equicontinuous, and $\sigma(L', L'')$-compact subset of L'. Then $\tilde{K}$ is $\sigma(L', L)$-bounded and, by Lemma 2.4, the support $Z := \mathrm{supp}(\tilde{K})$ is finite. By

Lemma 2.2, the operator $p'_Z : (E^Z)'_w \to L'_w$ is a topological embedding, which implies that the absolutely convex set $K := (p'_Z)^{-1}(\tilde{K}) \subseteq (E^Z)'$ is equicontinuous and $\sigma(E', E)$-precompact. Since, by [9, 9.4.3(a)], E^Z has the (DP)-property, Theorem 3.5 implies that the set K is precompact in the Mackey topology $\mu\big((E^Z)', E^Z\big)$ on $(E^Z)'$. Now the continuity of the operator $p'_Z : (E^Z)'_\mu \to L'_\mu$ ensures that the set $\tilde{K} = p'_Z(K)$ is precompact in the Mackey topology of L'. Thus, by Theorem 3.5, L has the (DP)-property.

(ii) Assume that L has the (sDP)-property. Take any weakly null sequences $\{e_k\}_{k\in\mathbb{N}} \subseteq E$ and $\{\chi_k\}_{k\in\mathbb{N}} \subseteq E'_\beta$. Fix an arbitrary $z \in X$ and consider the continuous operator $i_z : E \to L$ assigning to each $e \in E$ the constant function $i_z(e) : X \to \{e\} \subseteq E$. It follows that $p_z \circ i_z = \mathrm{id}_E$ and $\{i_z(e_k)\}_{k\in\mathbb{N}}$ is a weakly null sequence in L. By Lemma 2.2, the operator $p'_z : E'_\beta \to L'_\beta$ is continuous and hence the sequence $\{p'_z(\chi_k)\}_{k\in\mathbb{N}}$ is weakly null in L'_β. Since L has the (sDP)-property, we have

$$\langle \chi_k | e_k \rangle = \langle \chi_k | p_z \circ i_z(e_k) \rangle = \langle p'_z(\chi_k) | i_z(e_k) \rangle \to 0 \quad \text{as } k \to \infty.$$

Thus E has the (sDP)-property.

Conversely, assume that E has the (sDP)-property. Take any weakly null sequences $\{f_k\}_{k\in\mathbb{N}} \subseteq L$ and $\Phi := \{\mu_k\}_{k\in\mathbb{N}} \subseteq L'_\beta$. By Lemma 2.4, the set $Z := \mathrm{supp}(\Phi)$ is finite. By Lemma 2.2, the linear opeartor $p'_Z : (E^Z)'_\beta \to L'_\beta$ is a topological embedding, which implies that the sequence $\{\eta_k\}_{k\in\mathbb{N}} = \{(p'_Z)^{-1}(\mu_k)\}_{k\in\mathbb{N}}$ is weakly null in $(E^Z)'_\beta$. By the continuity of the operator $p_Z : L \to E^Z$, the sequence $\{p_Z(f_k)\}_{k\in\mathbb{N}}$ is weakly null in E^Z. Since E^Z has the (sDP)-property, we obtain

$$\langle \mu_k | f_k \rangle = \langle p'_Z(\eta_k) | f_k \rangle = \langle \eta_k | p_Z(f_k) \rangle \to 0 \quad \text{as } k \to \infty.$$

Thus L has the (sDP)-property. □

In [31], Saxon defined an lcs E to be *Baire-like* if for any increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of closed, absolutely convex subsets of E covering E there is an $m \in \mathbb{N}$ such that A_m is a neighborhood of zero in E. It is clear that every Baire-like space is barrelled.

Theorem 3.7. *Let X be a Tychonoff space, E be a locally convex space and let L be a linear subspace of E^X such that $B_1^{st}(X, E) \subseteq L$. Then L is Baire-like if and only if E is Baire-like.*

Proof. Assume that L is Baire-like. Fix a point $z \in X$ and consider the projection $p_z : L \to E$ which is a continuous quotient operator. By Theorem 2.11 of [31], the space E is Baire-like.

Conversely, assume that E is Baire-like. Let $\{A_n\}_{n\in\mathbb{N}}$ be an increasing sequence of closed, absolutely convex subsets of L covering L. We have to show that there is an $m \in \mathbb{N}$ such that A_m is a neighborhood of zero in L. For each $n \in \mathbb{N}$, let B_n be the closure of A_n in E^X. Since E being Baire-like is barrelled, the space L is also barrelled by Theorem 3.1. As L is dense in E^X, we can apply Valdivia's Proposition 2.13 in [22] to get that $E^X = \bigcup_{n\in\mathbb{N}} B_n$. Since, by Theorem 2.9 of [31], the space E^X is Baire-like and the sets B_n are closed and absolutely convex, there exists an $m \in \mathbb{N}$ such that B_m contains an open neighborhood U of zero in E^X. Then the density of L in E^X and the closeness of A_m in L imply that $U \cap L \subseteq A_m$. Therefore A_m is a neighborhood of zero in L. Thus L is Baire-like. □

In particular, the space $B_1\big([0,1]\big)$ is Baire-like, but it is not Baire (see [6]). Note also that the space $C_p\big([0,1]\big)$ is not Baire-like (it is not even barrelled).

Let E be a locally convex space. Denote by $\psi_E : E \to E''$ the canonical evaluation inclusion defined by $\langle \psi_E(e)|\chi\rangle := \langle \chi|e\rangle$ for all $e \in E$ and $\chi \in E'$. Recall that an lcs E is called

- *semi-reflexive* if ψ_E is surjective,
- *reflexive* if ψ_E is a topological isomorphism,
- *semi-Montel* if every bounded subset of E is relatively compact,
- *Montel* if E is semi-Montel and reflexive.

Since every semi-Montel space is quasi-complete ([21, Proposition 11.5.2]) in the next theorem we consider also some completeness properties. Let us recall (see [21, Theorem 3.6.6]) that for every Tychonoff space X, the space $C_p(X)$ is complete if and only if it is quasi-complete if and only if $C_p(X) = \mathbb{R}^X$ if and only if X is discrete. But the situation for Baire type function spaces essentially differs from the C_p-case. Indeed, if X is any countable Tychonoff space then, by Lemma 2.5, $B_1^{st}(X) = \mathbb{R}^X$ is complete. Below we also generalize this result.

Theorem 3.8. *Let X be a Tychonoff space of countable pseudocharacter, E be a locally convex space and let L be a linear subspace of E^X containing $B_1^{st}(X,E)$. Then:*

(i) *L is (semi-)reflexive if and only if $L = E^X$ and E is (semi-)reflexive.*

(ii) *L is (semi-)Montel if and only if $L = E^X$ and E is (semi-)Montel.*

(iii) *L is (quasi-)complete if and only if $L = E^X$ and E is (quasi-)complete.*

Proof. (i) First we consider the semi-reflexive case. Assume that L is semi-reflexive. We shall use the following criterion, see [21, Proposition 11.4.1]: L is semi-reflexive if and only if every bounded subset of L is contained in a $\sigma(L,L')$-compact subset of L. To prove that $L = E^X$, let $g : X \to E$ be an arbitrary function. We have to show that $g \in L$. Consider the following subset of L:

$$B_g := \{f \in L : f(x) \in \{0, g(x)\} \text{ for every } x \in X\}.$$

It is clear that B_g is a bounded subset of L. Lemma 2.6 implies that B_g is a dense subset of the product $K := \prod_{x\in X}\{0, g(x)\} \subseteq (E_w)^X$. Therefore the weak closure K_g of B_g in L_w being compact must coincide with K. Therefore $g \in K_g \subseteq L$. Thus $L = E^X$. Since E is a closed subspace of $L = E^X$, the space E is also semi-reflexive, see [21, Proposition 11.4.5].

Now we assume that L is reflexive. Then $L = E^X$ and, by [21, Proposition 11.4.2], E^X is barrelled and so is the space E. Since E is also semi-reflexive by [21, Proposition 11.4.5], Proposition 11.4.2 of [21] implies that E is reflexive.

Conversely, assume that $L = E^X$ and E is (semi-)reflexive. Then L is also (semi-)-reflexive by [21, Proposition 11.4.5].

(ii) Assume that L is (semi-)Montel. Then L is (semi-)reflexive. Hence, by (i), $L = E^X$ and E is (semi-)reflexive. By [21, Proposition 11.5.4], the equality $L = E^X$ implies that E is semi-Montel. Finally, if L is Montel and hence L is barrelled, Proposition 11.3.1 of [21] implies that E is barrelled. Being also semi-Montel, E is a Montel space.

Conversely, assume that $L = E^X$ and E is (semi-)Montel. Then L is also (semi-)-reflexive by [21, Proposition 11.5.4].

(iii) Assume that L is (quasi-)complete. Let $g \in E^X$ be an arbitrary function. We have to show that $g \in L$. For every nonempty finite subset F of X, Lemma 2.6 implies that there is a function $g_F \in B_1^{st}(X,E) \subseteq L$ such that $\mathrm{supp}(g_F) \subseteq F$ and $f_F|_F = g|_F$. It is clear that the net $\{g_F\}$ is contained in the closed and bounded subset

$$B_g := L \cap \prod_{x\in X}\{0, g(x)\}$$

of L. Therefore the net $\{g_F\}$ converges to some $h \in B_g$. Since $\{g_F\}$ converges to g in the compact space $\prod_{x\in X}\{0,g(x)\}$, we obtain that $h = g$. Thus $g \in L$ and $L = E^X$. As L is (quasi-)complete, the space E is also (quasi-)complete by [21, Proposition 3.2.6].

The converse assertion immediately follows from Proposition 3.2.6 of [21]. □

The famous problem of Banach and Mazur asks if every (real) Banach space has a separable infinite-dimensional quotient space. This question has an affirmative answer for some important classes of Banach spaces including reflexive Banach spaces and weakly compactly-generated Banach spaces, but the general question remains open (however, it is known that every Banach space has the tubby torus group $\mathbb{T}^{\mathbb{N}}$ as a quotient group, see [17]). It is natural to consider the Banach-Mazur problem for other classes of locally convex spaces. We note here a result of Eidelheit [10] who proved that every non-normable Fréchet (real) space has the countable Tychonoff product $\mathbb{R}^{\mathbb{N}}$ of the reals as a quotient locally convex space.

Theorem 3.9. *Let X be an infinite Tychonoff space, $\mathbb{F}$ be the field of real or complex numbers and let $L \subseteq \mathbb{F}^X$ be a linear subspace of $\mathbb{F}^X$ containing $B_1^{st}(X,\mathbb{F})$. Then L contains a complemented subspace isomorphic to $\mathbb{F}^{\mathbb{N}}$.*

Proof. Being infinite and Tychonoff, the space X contains a disjoint sequence $\{U_n\}_{n\in\mathbb{N}}$ of nonempty open sets. For every $n \in \mathbb{N}$, choose a continuous function $f_n : X \to [0,1] \subseteq \mathbb{F}$ such that $\mathrm{supp}(f_n) := f_n^{-1}\big((0,1]\big) \subseteq U_n$ and $f_n(x_n) = 1$ for some point $x_n \in U_n$. Consider the continuous function $\delta : X \to \mathbb{F}^{\mathbb{N}}$, $\delta : x \mapsto \big(f_n(x)\big)_{n\in\mathbb{N}}$, and observe that $\delta(x_n)$ coincides with the characteristic function e_n of the singleton $\{n\} \subseteq \mathbb{N}$ for every $n \in \mathbb{N}$. It follows that the sequence $\{e_n\}_{n\in\mathbb{N}}$ is linearly independent and discrete in the linear topological space $\mathbb{F}^{\mathbb{N}}$. For every $n \in \mathbb{N}$, consider the open set

$$O_n := \big\{(z_i) \in \mathbb{F}^{\mathbb{N}} : |z_n - 1| < \tfrac{1}{3} \text{ and } |z_i| < \tfrac{1}{3} \text{ for all } 0 \le i < n\big\}.$$

It is clear that $\{O_n\}_{n\in\mathbb{N}}$ is a disjoint family of open sets in $\mathbb{F}^{\mathbb{N}}$. For every $n \in \mathbb{N}$ we can choose a continuous function $\chi_n : \mathbb{F}^{\mathbb{N}} \to [0,1] \subseteq \mathbb{F}$ such that we have $\mathrm{supp}(\chi_n) := \chi_n^{-1}\big((0,1]\big) \subseteq O_n$ and $\chi_n(e_n) = 1$. Set $Y := \delta(X) \subseteq \mathbb{F}^{\mathbb{N}}$ and consider the continuous linear map

$$T : \mathbb{F}^{\mathbb{N}} \to B_1^{st}(Y,\mathbb{F}), \ \ T : z \mapsto \sum_{n\in\mathbb{N}} z(n)\cdot \chi_n|_Y \ .$$

This series stably converges to the function $T(z)$ since for every $y \in Y$ the series $\sum_{n\in\mathbb{N}} z(n) \cdot \chi_n(y)$ contains at most one non-zero summand. The continuity of T follows from the disjointness of the family $\{O_n\}_{n\in\mathbb{N}}$.

The continuous surjective map $\delta : X \to Y$ induces a continuous linear operator

$$\delta^* : B_1^{st}(Y, \mathbb{F}) \to B_1^{st}(X, \mathbb{F}) \subseteq L, \quad \delta^* : f \mapsto f \circ \delta.$$

Then the composition $\delta^* \circ T : \mathbb{F}^{\mathbb{N}} \to L$ is a continuous linear operator.

Analogously, the function $s : \mathbb{N} \to X$, $s : n \mapsto x_n$, induces a linear continuous operator

$$s^* : L \to \mathbb{F}^{\mathbb{N}}, \quad s^* : f \mapsto f \circ s = \big(f(x_n)\big)_{n\in\mathbb{N}}.$$

By the construction, $\delta(x_k) = e_k$ and $\chi_n(e_k) = \delta_{nk}$ for every $n, k \in \mathbb{N}$ (here δ_{nk} is the Kronecker symbol), and hence

$$\begin{aligned} s^* \circ \delta^* \circ T(z) &= s^* \circ \delta^* \big(\sum_{n\in\mathbb{N}} z_n \cdot \chi_n \big) = s^* \big(\sum_{n\in\mathbb{N}} z_n \cdot (\chi_n \circ \delta) \big) \\ &= \Big(\sum_{n\in\mathbb{N}} z_n \cdot \chi_n\big(\delta(x_k)\big) \Big)_{k\in\mathbb{N}} = (z_k)_{k\in\mathbb{N}}, \end{aligned}$$

for every sequence $z = (z_n) \in \mathbb{F}^{\mathbb{N}}$. This means that the map $\delta^* \circ T \colon \mathbb{F}^{\mathbb{N}} \to L$ is a right inverse to s^*. Thus $\mathbb{F}^{\mathbb{N}}$ is isomorphic to the complemented subspace $\delta^* \circ T(\mathbb{F}^{\mathbb{N}})$ of L, as desired. $\square$

The next theorem gives in particular the positive answer to the Banach-Mazur problem for function spaces $B_\alpha^{st}(X, E)$ and $B_\alpha(X, E)$.

Theorem 3.10. *Let X be an infinite Tychonoff space and let E be a locally convex space over the field $\mathbb{F}$ of real or complex numbers. Then for every countable ordinal $\alpha \geq 1$, the spaces $B_\alpha^{st}(X, E)$ and $B_\alpha(X, E)$ contain a complemented subspace isomorphic to $\mathbb{F}^{\mathbb{N}}$.*

Proof. It is well known that E can be represented as a direct sum $\mathbb{F} \times \tilde{E}$, where $\tilde{E}$ is a linear subspace of E. Then $C_p(X, E) = C_p(X, \mathbb{F}) \times C_p(X, \tilde{E})$. Hence $B_\alpha^{st}(X, E) = B_\alpha^{st}(X, \mathbb{F}) \times B_\alpha^{st}(X, \tilde{E})$ and $B_\alpha(X, E) = B_\alpha(X, \mathbb{F}) \times B_\alpha(X, \tilde{E})$. Now Theorem 3.9 applies. $\square$

Remark 3.11. It is easy to see that the proofs of Theorems 3.1, 3.6, 3.7 and 3.9 still valid for smaller than $B_\alpha^{st}(X, E)$ classes of functions. For example, they hold for the class of all functions $f : X \to E$ whose support $\mathrm{supp}(f)$ is functionally open in X and $f|_{\mathrm{supp}(f)}$ is continuous. In fact, besides $B_1^{st}(X, E)$ there many other interesting classes of discontinuous functions, for example, weakly discontinuous, resolvable functions, constructible-measurable, etc. For more information about such function classes, see [4, 5]. $\square$

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