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Maximum Principle for Abstract Convex Functions

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We prove a maximum principle for abstract fragmented convex functions on a compact space. In particular we give an answer to Question 5.6 posed previously by the authors [*On fragmented convex functions*, J. Math. Anal. Appl. 484 (2020) 123757].

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1. Introduction

We work within the framework of real vector spaces. If X is a compact convex subset of a (Hausdorff) locally convex space, it is well known that any semicontinuous convex function $f\colon X \to \mathbb{R}$ satisfies the *maximum principle*, i.e., $f \leq 0$ provided $f \leq 0$ on the set $\operatorname{ext} X$ of extreme points of X. Recently we proved that any fragmented bounded convex function on X satisfies the maximum principle (see below for the definition of fragmented functions). Nevertheless, we were not able to prove an analogous maximum principle for abstractly defined fragmented $\mathcal{H}$-convex functions defined with respect to a function space $\mathcal{H}$ on a compact space. The maximum principle for semicontinuous $\mathcal{H}$-convex functions is well known, see [4, Section 3.9] or [2, Theorem 2.2]. The aim of this paper is to prove this maximum principle for fragmented $\mathcal{S}$-convex functions defined with respect to a function cone $\mathcal{S}$ on a compact space. We even prove a monotone version of the Rainwater theorem for fragmented $\mathcal{S}$-convex functions (see Theorem 4.2) and reprove a variant of [6, Corollary 4(b)] for lower semicontinuous $\mathcal{S}$-convex functions (see Theorem 4.3).

1.1. Compact spaces and measures

Let K be a compact (Hausdorff) space and $\mathbb{C}(K)$ stand for the Banach space of all continuous functions on K endowed with the supremum norm. We write $\mathbb{M}(K)$ for the space of all (signed) Radon complete measures on K endowed with the weak*-topology given by the duality $(\mathbb{C}(K))^* = \mathbb{M}(K)$. Let $\mathbb{M}^1(K)$ denote the set of all probability Radon measures on K and ε_x stand for the Dirac measure at $x \in K$.

A function $f\colon K \to \mathbb{R}$ is *universally measurable* if it is μ-measurable for every $\mu \in \mathbb{M}^1(K)$. A set $A \subset K$ is *universally measurable* if A is μ-measurable for each

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$\mu \in \mathbb{M}^1(K)$. As a particular class of universally measurable functions we mention the space of all functions with the *point of continuity property* (see [4, Proposition A.118]). We recall that a function $f\colon K \to \mathbb{R}$ has the point of continuity property if $f|_F$ has a point of continuity for each nonempty closed set $F \subset K$. Equivalently, a function f on K has the point of continuity property if and only if it is *fragmented*, i.e., for any $\varepsilon > 0$ and nonempty closed set $F \subset K$ there exists a relatively nonempty open set $U \subset F$ such that $\operatorname{diam} f(U) < \varepsilon$ (see [4, Theorem A.121]). A set $F \subset K$ is called *resolvable* if the characteristic function χ_F is fragmented, where $\chi_F(x) = 1$ if $x \in F$ and $\chi_F(x) = 0$ otherwise.

1.2. Function cones

Throughout the paper we consider a *function cone* on a compact space K. By this we mean a cone $\mathcal{S} \subset \mathbb{C}(K)$ (i.e., $\lambda f \in \mathcal{S}$ whenever $\lambda \geq 0$ and $f \in \mathcal{S}$, and $f_1 + f_2 \in \mathcal{S}$ whenever $f_1, f_2 \in \mathcal{S}$) such that $\mathcal{S}$ separates points of K and contains constant functions. For a point $x \in K$ we denote by $\mathbb{M}_x(\mathcal{S})$ the set of all $\mathcal{S}$*-representing measures* for x, precisely

$$\mathbb{M}_x(\mathcal{S}) = \left\{\mu \in \mathbb{M}^1(K);\, s(x) \geq \mu(s), s \in \mathcal{S}\right\}.$$

Further, the *Choquet boundary* $\operatorname{Ch}_{\mathcal{S}} K$ is defined as

$$\operatorname{Ch}_{\mathcal{S}} K = \{x \in K;\, \mathbb{M}_x(\mathcal{S}) = \{\varepsilon_x\}\}.$$

By [1, Theorem I.5.13] or [4, Theorem 10.68], $\operatorname{Ch}_{\mathcal{S}} K$ is a nonempty Baire space, i.e., the intersection of a sequence of open dense sets in $\operatorname{Ch}_{\mathcal{S}} K$ is dense.

A bounded universally measurable function $f\colon K \to \mathbb{R}$ is termed $\mathcal{S}$*-convex* if we have $f(x) \leq \mu(f)$ for any $x \in K$ and $\mu \in \mathbb{M}_x(\mathcal{S})$. A function f on K is $\mathcal{S}$*-concave* if $-f$ is $\mathcal{S}$-convex. We write $\mathcal{K}^c(\mathcal{S})$ for the cone of all continuous $\mathcal{S}$-convex functions on K. Obviously, any function $s \in \mathcal{S}$ satisfies $-s \in \mathcal{K}^c(\mathcal{S})$.

Given measures $\mu, \nu \in \mathbb{M}(K)$, we write $\mu \prec_{\mathcal{S}} \nu$ if $\mu(k) \leq \nu(k)$ for each $k \in \mathcal{K}^c(\mathcal{S})$. Then $\prec_{\mathcal{S}}$ determines a partial ordering on $\mathbb{M}(K)$ such that the cone

$$E^+ = \{\mu \in \mathbb{M}(K);\, 0 \prec_{\mathcal{S}} \mu\}$$

is closed and proper. (Indeed, if $\mu \prec_{\mathcal{S}} \nu$ and $\nu \prec_{\mathcal{S}} \mu$, then $\mu(k) = \nu(k)$ for each $k \in \mathcal{K}^c(\mathcal{S})$. Since $\mathcal{K}^c(\mathcal{S}) - \mathcal{K}^c(\mathcal{S})$ is dense in $\mathbb{C}(K)$ due to the Stone-Weierstrass theorem, $\mu = \nu$.)

If $f \in \mathbb{C}(K)$, its *upper envelope* f^* is defined as

$$f^*(x) = \inf\left\{s(x);\, s \geq f, s \in -\mathcal{K}^c(\mathcal{S})\right\}, \quad x \in K.$$

The f^* is a bounded upper semicontinuous $\mathcal{S}$-concave function. (Indeed, given $x \in K$ and $\mu \in \mathbb{M}_x(\mathcal{S})$, we choose for $\varepsilon > 0$ a function $s \in -\mathcal{K}^c(\mathcal{S})$ such that $f \leq s$ and $f^*(x) + \varepsilon \geq s(x)$. Then $\mu(f^*) \leq \mu(s) \leq s(x) \leq f^*(x) + \varepsilon$.) Thus if $\mu, \nu \in \mathbb{M}^1(K)$ and $\mu \prec_{\mathcal{S}} \nu$, then $\mu(f^*) \geq \nu(f^*)$, see [4, Proposition 7.16].

If $A \subset K$ is universally measurable, we say that it is $\mathcal{S}$*-convex* if $x \in A$ whenever $x \in K$ and $\mu \in \mathbb{M}_x(\mathcal{S})$ satisfies $\mu(A) = 1$. The set A is $\mathcal{S}$*-extremal*, provided $\mu(A) = 1$ whenever $x \in A$ and $\mu \in \mathbb{M}_x(\mathcal{S})$.

1.3. Compact convex sets

Let X be a compact convex set in a (Hausdorff) locally convex space. If $\mathfrak{A}(X)$ stands for the space of all continuous affine functions on X, then $\mathfrak{A}(X)$ is a function cone on X. Then $\mathbb{M}_x(\mathfrak{A}(X))$ denotes the set of all measures $\mu \in \mathbb{M}^1(X)$ with the property that their *barycenter* $r(\mu)$ is equal to x (we recall that the barycenter of a measure $\mu \in \mathbb{M}^1(X)$ is the uniquely defined point $r(\mu) \in X$ satisfying $\mu(a) = a(r(\mu))$, $a \in \mathfrak{A}(X)$). Hence $\mathrm{Ch}_{\mathfrak{A}(X)} X = \operatorname{ext} X$ (see [1, Corollary I.2.4]). Further, the cone $\mathcal{K}^c(\mathfrak{A}(X))$ coincides with the cone of all continuous convex functions on X.

A set $A \subset X$ is *extremal* if $\lambda x + (1-\lambda)y \in A$ for some $\lambda \in (0,1)$ and $x, y \in X$ imply $x, y \in A$. By [4, Propositions 2.80 and 2.92], a resolvable extremal set is $\mathfrak{A}(X)$-extremal and a resolvable convex set is $\mathfrak{A}(X)$-convex.

1.4. Ordered compact convex sets

Let X be a compact convex set in a locally convex space E that is ordered by a closed proper cone $E^+ \subset E$. We write $x \leq y$ for $x, y \in E$, if $y - x \in E^+$. A function $f\colon X \to \mathbb{R}$ is called *isotone* if $f(x) \leq f(y)$ whenever $x \leq y$, $x, y \in X$.

A set $A \subset X$ is called hereditary upwards if $x \in A$, $y \in X$ with $x \leq y$ implies $y \in A$.

Let $\mathfrak{L}(X)$ denotes the cone of all affine continuous isotone functions on X. Then the cone $-\mathfrak{L}(X) = \{-l;\, l \in \mathfrak{L}(X)\}$ is a function cone on X due to [1, Proposition I.6.1]. Let $X_{\max}$ denote the set of all elements in X maximal with respect to $\leq$. Then $X_{\max}$ is an extremal subset of X (see [1, Proposition I.6.2]) and for each $x \in X$ there exists $z \in X_{\max}$ such that $x \leq z$ (see again [1, Proposition I.6.2]). By [1, Theorem I.6.6] (see also [4, Theorem 7.54(d)]), $\mathrm{Ch}_{-\mathfrak{L}(X)} X = \operatorname{ext} X_{\max} = \operatorname{ext} X \cap X_{\max}$.

2. Auxiliary results for function cones

For the proof of our main theorems we need a couple of lemmas, which are straightforward generalizations of Lemmas 2.3 and 2.5 from [5].

Lemma 2.1. *Let $\mathcal{S}$ be a function cone on a compact space K and $x \in \mathrm{Ch}_{\mathcal{S}} K$ be given. Then for every neighbourhood U of x and $\varepsilon > 0$ there exists a neighbourhood V of x such that for every $y \in V$ and $\mu \in \mathbb{M}_y(\mathcal{S})$ it holds that $\mu(U) > 1 - \varepsilon$.*

Proof. Assuming the contrary, there is an open neighbourhood U of x and $\varepsilon > 0$ such that for each neighbourhood V of x there exists $y_V \in V$ and $\mu_V \in \mathbb{M}_y(\mathcal{S})$ satisfying $\mu_V(U) \leq 1 - \varepsilon$.

We denote the system of all neighbourhoods of x by $\mathcal{V}_x$ and we consider the family of measures

$$\{\mu_V;\, V \in \mathcal{V}_x\}$$

as a net in $\mathbb{M}^1(K)$ with the ordering given by $V \leq W$ if $W \subset V$. Since $\mathbb{M}^1(K)$ is compact, we may assume that the net converges to some measure $\nu \in \mathbb{M}^1(K)$. Then it is clear that the net $\{y_V;\, V \in \mathcal{V}_x\}$ with the natural ordering converges to x. Thus for every $s \in \mathcal{S}$ it holds that

$$\lim_{V \in \mathcal{V}_x} \mu_V(s) \leq \lim_{V \in \mathcal{V}_x} s(y_V) = s(x).$$

Thus
$$\nu(s) \leq s(x), \quad s \in \mathcal{S},$$
in other words, ν is an $\mathcal{S}$-representing measure for x. Since $x \in \operatorname{Ch}_{\mathcal{S}} K$, $\nu = \varepsilon_x$.

Now we find a function $g \in \mathcal{C}(K)$ satisfying $0 \leq g \leq 1$, $g(x) = 1$ and $g = 0$ on $X \setminus U$. Then we have that
$$\lim_{V \in \mathcal{V}_x} \mu_V(g) = \varepsilon_x(g) = g(x) = 1.$$
On the other hand, for each $V \in \mathcal{V}_x$ it holds that
$$|\mu_V(g)| = \left| \int_{X\setminus U} g d\mu_V + \int_U g d\mu_V \right| \leq 0 + \int_U |g| d\mu_V \leq \mu_V(U) \leq 1 - \varepsilon,$$
which gives the desired contradiction. □

Lemma 2.2. *Let $\mathcal{S}$ be a function cone on a compact space K and $f\colon K \to \mathbb{R}$ be a bounded $\mathcal{S}$-convex function such that $f|_{\overline{\operatorname{Ch}_{\mathcal{S}} K}}$ is upper semicontinuous at a point $x \in \operatorname{Ch}_{\mathcal{S}} K$. Then f is upper semicontinuous at x.*

Proof. Let $L \geq 0$ be a constant satisfying $|f| \leq L$. Let $\varepsilon > 0$ be given.

Using the upper semicontinuity of $f|_{\overline{\operatorname{Ch}_{\mathcal{S}} K}}$ we find a neighbourhood $U \subset K$ of x such that
$$f(y) < f(x) + \varepsilon, \quad y \in U \cap \overline{\operatorname{Ch}_{\mathcal{S}} K}.$$
By Lemma 2.1 there exists a neighbourhood V of x such that for each $y \in V$ and $\mu \in \mathbb{M}_y(\mathcal{S})$ it holds that $\mu(U) > 1 - \varepsilon$.

We choose arbitrary $y \in V$ and pick a measure $\mu_y \in \mathbb{M}_y(\mathcal{S})$ carried by $\overline{\operatorname{Ch}_{\mathcal{S}} K}$ (see [4, Corollary 7.25]). Now we have
$$\begin{aligned} f(y) \leq \mu_y(f) = \int_{\overline{\operatorname{Ch}_{\mathcal{S}} K}} f d\mu_y &= \int_{\overline{\operatorname{Ch}_{\mathcal{S}} K} \cap U} f d\mu_y + \int_{\overline{\operatorname{Ch}_{\mathcal{S}} K} \setminus U} f d\mu_y \\ &\leq f(x) + \varepsilon + \varepsilon L = f(x) + \varepsilon(1 + L). \end{aligned}$$
This finishes the proof. □

The following Lemma is an analogue of [1, Theorem I.3.8] and [4, Proposition 3.89].

Lemma 2.3. *Let $\mathcal{S}$ be a function cone on a compact space K and*
$$M = \left\{ (\varepsilon_x, \lambda) \in \mathbb{M}^1(K) \times \mathbb{M}^1(K);\ \varepsilon_x \prec_{\mathcal{S}} \lambda, x \in K \right\}.$$
Then M is a compact subset of $\mathbb{M}^1(K) \times \mathbb{M}^1(K)$. Further, for $\mu, \nu \in \mathbb{M}^1(K)$ one has $\mu \prec_{\mathcal{S}} \nu$ if and only if there exists $\Lambda \in \mathbb{M}^1(M)$ such that $(\mu, \nu) = r(\Lambda)$.

Proof. First we prove that M is closed in $\mathbb{M}^1(K) \times \mathbb{M}^1(K)$. Obviously, the set
$$\begin{aligned} N &= \left\{ (\lambda_1, \lambda_2) \in \mathbb{M}^1(K) \times \mathbb{M}^1(K);\ \lambda_1 \prec_{\mathcal{S}} \lambda_2 \right\} \\ &= \bigcap_{s \in \mathcal{K}^c(\mathcal{S})} \left\{ (\lambda_1, \lambda_2) \in \mathbb{M}^1(K) \times \mathbb{M}^1(K);\ \lambda_1(s) \leq \lambda_2(s) \right\} \end{aligned}$$

is closed in $\mathbb{M}^1(K)\times\mathbb{M}^1(K)$, and thus

$$M = N \cap \big\{(\varepsilon_x,\lambda)\in\mathbb{M}^1(K)\times\mathbb{M}^1(K);\ x\in K\big\}$$

is closed as well.

Let $(\mu,\nu)\in\mathbb{M}^1(K)\times\mathbb{M}^1(K)$ be given and let us assume that a measure $\Lambda\in\mathbb{M}^1(M)$ with $r(\Lambda)=(\mu,\nu)$ exists. Let $f\in\mathcal{K}^c(\mathcal{S})$ be given. Then the functions $\varphi_1\colon(\lambda_1,\lambda_2)\mapsto\lambda_1(f)$ and $\varphi_2\colon(\lambda_1,\lambda_2)\mapsto\lambda_2(f)$ are affine continuous functions on $\mathbb{M}^1(K)\times\mathbb{M}^1(K)$ and thus we have

$$\begin{aligned}\mu(f)=\varphi_1(\mu,\nu)&=\int_{\mathbb{M}^1(K)\times\mathbb{M}^1(K)}\varphi_1(\lambda_1,\lambda_2)d\Lambda(\lambda_1,\lambda_2)=\int_M\varphi_1(\varepsilon_x,\lambda)d\Lambda(\varepsilon_x,\lambda)\\&=\int_M f(x)d\Lambda(\varepsilon_x,\lambda)\le\int_M\lambda(f)d\Lambda(\varepsilon_x,\lambda)=\int_M\varphi_2(\varepsilon_x,\lambda)d\Lambda(\varepsilon_x,\lambda)\\&=\varphi_2(\mu,\nu)=\nu(f).\end{aligned}$$

Hence $\mu\prec_{\mathcal{S}}\nu$.

Conversely, let $\mu\prec_{\mathcal{S}}\nu$. We aim to show that $(\mu,\nu)\in\overline{\operatorname{co}}M$. Assume that this is not the case. Then we use the Hahn-Banach theorem to find a continuous linear function $F\colon\mathbb{M}(K)\times\mathbb{M}(K)\to\mathbb{R}$ and $c\in\mathbb{R}$ such that

$$F(\varepsilon_z,\lambda)<c<F(\mu,\nu),\quad(\varepsilon_z,\lambda)\in M.$$

Then there exist continuous functions $f_1,f_2\in\mathbb{C}(K)$ such that

$$F(\lambda_1,\lambda_2)=\lambda_1(f_1)+\lambda_2(f_2),\quad(\lambda_1,\lambda_2)\in\mathbb{M}^1(K)\times\mathbb{M}^1(K).$$

The assumption $\mu\prec_{\mathcal{S}}\nu$ gives $\nu(f_2^*)\le\mu(f_2^*)$, see [4, Proposition 7.16] or Subsection 1.2. Let $x\in K$ be fixed. By [4, Lemma 7.10] we have

$$\begin{aligned}\sup\{\varepsilon_x(f_1)+\lambda(f_2);\ \varepsilon_x\prec_{\mathcal{S}}\lambda\}&=f_1(x)+\sup\{\lambda(f_2);\ \lambda\in\mathbb{M}_x(\mathcal{S})\}\\&=f_1(x)+f_2^*(x).\end{aligned}$$

Hence
$$\begin{aligned}\mu(f_1+f_2^*)&\ge\mu(f_1)+\nu(f_2^*)\ge\mu(f_1)+\nu(f_2)\\&>c\ge\sup\{\varepsilon_z(f_1)+\lambda(f_2);\ (\varepsilon_z,\lambda)\in M\}\\&\ge f_1(x)+f_2^*(x).\end{aligned}$$

Thus
$$\mu(f_1+f_2^*)>c\ge\sup_{x\in K}\left(f_1(x)+f_2^*(x)\right).$$

Since μ is a probability measure, we have a contradiction.

Hence $(\mu,\nu)\in\overline{\operatorname{co}}M$, which gives by the Krein-Milman theorem the desired measure $\Lambda\in\mathbb{M}^1(M)$ with $r(\Lambda)=(\mu,\nu)$. □

3. Maximum principle for isotone fragmented functions

Throughout this section we consider an ordered compact convex set together with the function cone $-\mathfrak{L}(X)=\{-l;\ l\in\mathfrak{L}(X)\}$, where $\mathfrak{L}(X)$ denotes the cone of all affine continuous isotone functions on X. The main result is contained in Theorem 3.2 which will provide applications to function cones.

Lemma 3.1. *Let X be an ordered compact convex set and $f\colon X \to \mathbb{R}$ be an isotone bounded fragmented affine function. Then f is $-\mathfrak{L}(X)$-convex.*

Proof. Let $x \in X$ and $\mu \in \mathbb{M}_x(-\mathfrak{L}(X))$ be given, i.e., $\varepsilon_x \prec_{-\mathfrak{L}(X)} \mu$. Then $x \le r(\mu)$. Indeed, for any $l \in \mathfrak{L}(X)$ we have $-l \in -\mathfrak{L}(X)$, and thus $l \in \mathcal{K}^c(-\mathfrak{L}(X))$. Hence $l(x) \le \mu(l) = l(r(\mu))$. By [1, Proposition I.6.1], $x \le r(\mu)$. Since f is fragmented and affine, $\mu(f) = f(r(\mu))$, see [4, Theorem 10.75]. Since f is isotone, we obtain $f(x) \le f(r(\mu)) = \mu(f)$. This means that f is $-\mathfrak{L}(X)$-convex. $\square$

If E is a vector space and $a, b \in E$ are distinct points, we denote by (a, b) the open segment between points a and b, i.e., $(a, b) = \{\lambda a + (1-\lambda)b;\ \lambda \in (0,1)\}$. Analogously, $[a, b] = \{\lambda a + (1-\lambda)b;\ \lambda \in [0,1]\}$ is the closed segment between a and b.

Theorem 3.2. *Let X be an ordered compact convex set and $\{f_n\}$ be a bounded sequence of isotone fragmented affine functions on X such that $\{f_n\}$ is nonincreasing and $\lim f_n(x) \le 0$ for each $x \in \operatorname{Ch}_{-\mathfrak{L}(X)} X$. Then the limit function $f = \lim f_n$ satisfies $f \le 0$ on X.*

Proof. Obviously, f is a well defined bounded isotone affine function on X. Assume that the set $H = \{x \in X;\ f(x) \ge \eta\}$ is nonempty for some $\eta > 0$. Then H is a nonempty convex set that is hereditary upwards since f is isotone. Further, $H \cap \operatorname{Ch}_{-\mathfrak{L}(X)} X = \emptyset$ due to the assumption. Thus $\overline{H}$ is an ordered compact convex set and we can consider the Choquet boundary $\operatorname{Ch}_{-\mathfrak{L}(X)|_{\overline{H}}} \overline{H}$ of the function cone $-\mathfrak{L}(X)|_{\overline{H}} = \{-l|_{\overline{H}};\ l \in \mathfrak{L}(X)\}$ and the set $F = \overline{\operatorname{Ch}_{-\mathfrak{L}(X)|_{\overline{H}}} \overline{H}}$.

Let
$$C_n = \{x \in F;\ f_n|_F \text{ is continuous at } x\}, \quad n \in \mathbb{N}.$$

Since each f_n is fragmented, C_n is a dense G_δ-set in F (the set of the points of continuity is always G_δ and it is dense by [4, Theorem A.121(iii)]). If $G \subset F$ is an open dense set in F, the set $G \cap \operatorname{Ch}_{-\mathfrak{L}(X)|_{\overline{H}}} \overline{H}$ is an open dense subset of $\operatorname{Ch}_{-\mathfrak{L}(X)|_{\overline{H}}} \overline{H}$. Since the latter set is a Baire space (see [1, Theorem I.5.13] or [4, Theorem 10.68]), the set
$$\bigcap_{n=1}^{\infty} \left(C_n \cap \operatorname{Ch}_{-\mathfrak{L}(X)|_{\overline{H}}} \overline{H}\right)$$
is nonempty. Let x in this intersection be chosen. Then each function $f_n|_F$ is continuous at x, and since f_n is $-\mathfrak{L}(X)$-convex (see Lemma 3.1), $f_n|_{\overline{H}}$ is upper semicontinuous at x (see Lemma 2.2). Since $x \in \overline{H}$ and $f_n \ge f \ge \eta$ on H, it follows that $f_n(x) \ge \eta$ for every $n \in \mathbb{N}$. Hence $f(x) \ge \eta$ and thus $x \in H$.

Further we claim that $x \in X_{\max}$. Indeed, let $z \in X_{\max}$ be such that $x \le z$ (see [1, Proposition I.6.2] or [4, Proposition 7.45(a)]). Since f is isotone, $\eta \le f(x) \le f(z)$, i.e. $z \in H$. Then $\varepsilon_z \in \mathbb{M}_x(-\mathfrak{L}(X)|_{\overline{H}})$. Indeed, if $s \in -\mathfrak{L}(X)$, $s(x) \ge s(z) = \varepsilon_z(s)$, and thus $\varepsilon_z \in \mathbb{M}_x(-\mathfrak{L}(X)|_{\overline{H}})$. Since $x \in \operatorname{Ch}_{-\mathfrak{L}(X)|_{\overline{H}}} \overline{H}$, $x = z$. Thus $x \in X_{\max}$.

We also have $x \in \operatorname{ext} \overline{H}$. Indeed, if $x = \lambda x_1 + (1-\lambda)x_2$, where $\lambda \in (0,1)$ and $x_1, x_2 \in \overline{H}$, then for $l \in \mathfrak{L}(X)$ and $\mu = \lambda\varepsilon_{x_1} + (1-\lambda)\varepsilon_{x_2}$ we have
$$\varepsilon_x(l) = l(x) = \lambda l(x_1) + (1-\lambda)l(x_2) = (\lambda\varepsilon_{x_1} + (1-\lambda)\varepsilon_{x_2})(l) = \mu(l).$$

Hence $\mu \in \mathbb{M}_x(-\mathfrak{L}(X)|_{\overline{H}})$, and thus $\varepsilon_x = \mu$. Hence we obtain $x = x_1 = x_2$ and $x \in \operatorname{ext} \overline{H}$.

If $x \in \operatorname{ext} X$, we have $x \in \operatorname{ext} X \cap X_{\max} = \operatorname{Ch}_{-\mathfrak{L}(X)} X$ by [4, Theorem 7.54(d)], a contradiction.

Thus we can find distinct points $a, b \in X$ such that $x \in (a, b)$. Since $x \in \operatorname{ext} \overline{H}$, it cannot happen that both a, b are contained in H. Further, since f is affine and $f(x) \geq \eta$, exactly one of the points a, b is contained in H. So we can assume that $f(a) < \eta$ and $f(b) \geq \eta$. Since $x \in \operatorname{ext} \overline{H}$, we have $f(x) = \eta$ (otherwise we would find a segment $[a_1, b_1]$ inside (a, b) containing x in its interior such that $a_1, b_1 \in H$, contradicting thus the fact $x \in \operatorname{ext} \overline{H}$).

Let
$$t = \sup\{s \geq 0;\ a + s(b-a) \in X\}$$

and $c = a + t(b-a)$. Then $x \in (a, c)$ and $a, c \in X_{\max}$, because $X_{\max}$ is an extremal subset of X (see [1, Proposition I.6.2] or [4, Proposition 7.45(b)]). Since f is affine on the segment $[a, c]$, $f(c) > \eta$, hence $c \notin \operatorname{ext} X \cap X_{\max} = \operatorname{Ch}_{-\mathfrak{L}(X)} X$. Thus we can find points $c_1, c_2 \in X$ such that $c \in (c_1, c_2)$ and c_1, c_2 do not lie on the line emanating from a through b. By shifting c_1, c_2 closer to c on the segment $[c_1, c_2]$ we can achieve that $f(c_1) > \eta$ and $f(c_2) > \eta$. Then the points a, c_1, c_2 form a non degenerate triangle in X whose interior contains x. Let $d_1 \in (a, c_1)$ be the unique point satisfying $f(d_1) = \eta$

$$\text{(i.e., } d_1 = \lambda a + (1-\lambda)c_1, \text{ where } \lambda = \frac{f(c_1) - \eta}{f(c_1) - f(a)}.\text{)}$$

Let $d_2 \in (a, c_2)$ be the unique point arising by the intersection of the line emanating from d_1 through x and the segment $[a, c_2]$

$$\text{(i.e., } d_2 = \lambda a + (1-\lambda)c_2 \text{ for } \lambda = \frac{f(c_2) - \eta}{f(c_2) - f(a)}\text{).}$$

Then $f(d_2) = \eta$. By this construction we have obtained points $d_1, d_2 \in H$ distinct from x such that $x \in (d_1, d_2)$. This contradicts the fact that $x \in \operatorname{ext} \overline{H}$ and finishes the proof. □

Corollary 3.3. *Let X be an ordered compact convex set and f be a bounded isotone fragmented affine function on X such that $f \leq 0$ on $\operatorname{Ch}_{-\mathfrak{L}(X)} X$. Then $f \leq 0$ on X.*

Proof. It is enough to consider a sequence $\{f_n\}$ in Theorem 3.2 consisting of the single function f. □

Corollary 3.4. *Let X be an ordered compact convex set and $\{f_n\}$ be a bounded sequence of isotone fragmented affine functions on X such that $\{f_n\}$ is nondecreasing and $\lim f_n(x) \leq 0$ for each $x \in \operatorname{Ch}_{-\mathfrak{L}(X)} X$. Then the limit function $f = \lim f_n$ satisfies $f \leq 0$ on X.*

Proof. We apply Corollary 3.3 to every f_n to infer that $f_n \leq 0$ on X. Thus $f = \lim f_n$ is nonpositive on X. □

4. $\mathcal{S}$-convex fragmented functions

Throughout this section we consider a function cone $\mathcal{S}$ on a compact space K.

Lemma 4.1. *Let $\mathcal{S}$ be a function cone on a compact space K and let $f\colon K \to \mathbb{R}$ be a bounded $\mathcal{S}$-convex fragmented function. Let $X = \mathbb{M}^1(K)$ be the compact convex set ordered by $\prec_{\mathcal{S}}$. Then the function $\widetilde{f}\colon X \to \mathbb{R}$ defined as $\widetilde{f}(\mu) = \mu(f)$ is affine, bounded, fragmented and isotone on X.*

Proof. By [4, Proposition A.118 and Theorem A.121], f is universally measurable, and thus $\widetilde{f}$ is well defined. Obviously, $\widetilde{f}$ is affine and bounded. By [3, Lemma 3.2], $\widetilde{f}$ is fragmented.

It remains to show that $\widetilde{f}$ is isotone. Let $\mu, \nu \in X$ with $\mu \prec_{\mathcal{S}} \nu$ be given. By Lemma 2.3, there exists a measure $\Lambda \in \mathbb{M}^1(M) \subset \mathbb{M}^1(X \times X)$ such that we have $(\mu, \nu) = r(\Lambda)$, where

$$M = \{(\varepsilon_x, \lambda) \in X \times X;\ \varepsilon_x \prec_{\mathcal{S}} \lambda, x \in K\}\,.$$

By [4, Proposition 3.90], for any pair (f_1, f_2) of bounded universally measurable functions on K it holds

$$\mu(f_1) + \nu(f_2) = \int_M (\varepsilon_x(f_1) + \lambda(f_2))\, \Lambda(\varepsilon_x, \lambda).$$

We use this equality for pairs $(f, 0)$ and $(0, f)$ and obtain due to the $\mathcal{S}$-convexity of f the inequality

$$\mu(f) = \int_M \varepsilon_x(f)\Lambda(\varepsilon_x, \lambda) \le \int_M \lambda(f)\Lambda(\varepsilon_x, \lambda) = \nu(f).$$

Hence $\widetilde{f}(\mu) \le \widetilde{f}(\nu)$ and $\widetilde{f}$ is isotone on X. □

Here we come to the main result of this section.

Theorem 4.2. *Let $\mathcal{S}$ be a function cone on a compact space K.*

(a) *Let $\{f_n\}$ be a monotone bounded sequence of fragmented $\mathcal{S}$-convex functions such that $\lim f_n(x) \le 0$ for $x \in \operatorname{Ch}_{\mathcal{S}} K$. Then the limit function $f = \lim f_n$ satisfies $f \le 0$ on K.*

(b) *Let $f\colon K \to \mathbb{R}$ be a fragmented $\mathcal{S}$-convex bounded function satisfying $f \le 0$ on $\operatorname{Ch}_{\mathcal{S}} K$. Then $f \le 0$ on K.*

Proof. (a) Let $\{f_n\}$ be a monotone bounded sequence of fragmented $\mathcal{S}$-convex functions such that $\lim f_n(x) \le 0$ for $x \in \operatorname{Ch}_{\mathcal{S}} K$. First we assume that $\{f_n\}$ is nonincreasing. Let $X = \mathbb{M}^1(K)$ endowed with the ordering $\prec_{\mathcal{S}}$ and $\widetilde{f_n}\colon X \to \mathbb{R}$ be defined as in Lemma 4.1. Then $\{\widetilde{f_n}\}$ is a bounded nonincreasing sequence of fragmented isotone affine functions on X. Let $\mu \in \operatorname{Ch}_{-\mathfrak{L}(X)} X$ be given. Then $\mu \in \operatorname{ext} X \cap X_{\max}$, and thus there exists $x \in K$ such that $\mu = \varepsilon_x$. Since $\mu \in X_{\max}$, we have $x \in \operatorname{Ch}_{\mathcal{S}} K$ (indeed, if $\nu \in \mathbb{M}_x(\mathcal{S})$, then $\varepsilon_x \prec_{\mathcal{S}} \nu$, and thus $\nu = \varepsilon_x$ by the maximality of $\mu = \varepsilon_x$). Thus $\lim \widetilde{f_n}(\mu) = \lim f_n(x) \le 0$. From Theorem 3.2 it follows that $\lim \widetilde{f_n} \le 0$ on X. In particular, $\lim f_n \le 0$ on K.

If $\{f_n\}$ is nondecreasing, we apply Corollary 3.4 instead to obtain $\lim f_n \leq 0$ on K.

(b) This is a direct consequence of (a). $\square$

The following result is an abstract variant of [6, Corollary 4(b)].

Theorem 4.3. *Let $\mathcal{S}$ be a function cone on a compact space K and $\{f_n\}$ be a bounded sequence of lower semicontinuous $\mathcal{S}$-convex functions with the property $\limsup f_n(x) \leq 0$ for every $x \in \mathrm{Ch}_{\mathcal{S}} K$. Then $\limsup f_n \leq 0$ on K.*

Proof. Let $L \geq 0$ be a constant satisfying $|f_n| \leq L$, $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ we define $g_n = \sup_{k \geq n} f_k$, . Then each g_n is a lower semicontinuous $\mathcal{S}$-convex function bounded by L, such that $\{g_n\}$ form a nonincreasing sequence with the property $\lim g_n(x) = \limsup f_n(x) \leq 0$ for every $x \in \mathrm{Ch}_{\mathcal{S}} K$. By Theorem 4.2 we obtain

$$\limsup f_n(x) = \lim g_n(x) \leq 0, \quad x \in K. \qquad \square$$

As it is well known, any closed nonempty $\mathcal{S}$-extremal set in K intersects the Choquet boundary $\mathrm{Ch}_{\mathcal{S}} K$. The following result shows that the same holds for any nonempty resolvable $\mathcal{S}$-extremal set.

Corollary 4.4. *Let $\mathcal{S}$ be a function cone on a compact space K and $F \subset K$ be a nonempty $\mathcal{S}$-extremal resolvable set. Then $F \cap \mathrm{Ch}_{\mathcal{S}} K \neq \emptyset$.*

Proof. Let F be a nonempty resolvable $\mathcal{S}$-extremal subset of K. Then $f = \chi_F$ is a fragmented bounded $\mathcal{S}$-convex function. Indeed, let $x \in K$ and $\mu \in \mathbb{M}_x(\mathcal{S})$ be given. If $x \notin F$, then clearly $f(x) \leq \mu(f)$. If $x \in F$, $\mu(F) = 1$ due to the $\mathcal{S}$-extremality of F, and thus $1 = f(x) \leq \mu(f) = \mu(F) = 1$.

Assuming that $F \cap \mathrm{Ch}_{\mathcal{S}} K = \emptyset$, $f \leq 0$ on $\mathrm{Ch}_{\mathcal{S}} K$. By Theorem 4.2, $f \leq 0$ on K which contradicts that $F \neq \emptyset$. $\square$

From Theorem 4.2 we obtain a positive answer to [5, Question 5.6].

We recall that a *function space* $\mathcal{H} \subset \mathbb{C}(K)$ is a vector subspace of $\mathbb{C}(K)$ separating points of K and containing constant functions. Thus any function space is a function cone. As for the function cones, we denote for a point $x \in K$ by $\mathbb{M}_x(\mathcal{H})$ the set of all *$\mathcal{H}$-representing measures* for x, precisely

$$\mathbb{M}_x(\mathcal{H}) = \left\{\mu \in \mathbb{M}^1(K);\, h(x) = \mu(h), h \in \mathcal{H}\right\}.$$

Further, the *Choquet boundary* $\mathrm{Ch}_{\mathcal{H}} K$ is defined as

$$\mathrm{Ch}_{\mathcal{H}} K = \{x \in K;\, \mathbb{M}_x(\mathcal{H}) = \{\varepsilon_x\}\}.$$

A bounded universally measurable function $k\colon K \to \mathbb{R}$ is termed *$\mathcal{H}$-convex* if we have $k(x) \leq \mu(k)$ for any $x \in K$ and $\mu \in \mathbb{M}_x(\mathcal{H})$.

Theorem 4.5. *Let K be a compact space and $\mathcal{H} \subset \mathbb{C}(K)$ be a function space. Let $f\colon K \to \mathbb{R}$ be a fragmented $\mathcal{H}$-convex bounded function satisfying $f \leq 0$ on $\mathrm{Ch}_{\mathcal{H}} K$. Then $f \leq 0$ on K.*

Proof. By [4, Proposition 7.9], if we consider $\mathcal{H}$ as a function cone $\mathcal{S}$, we obtain $\mathbb{M}_x(\mathcal{H}) = \mathbb{M}_x(\mathcal{S})$, $x \in K$, and $\mathrm{Ch}_{\mathcal{S}} K = \mathrm{Ch}_{\mathcal{H}} K$. Thus our function f is fragmented, bounded, $\mathcal{S}$-convex and nonpositive on $\mathrm{Ch}_{\mathcal{S}} K$, which implies $f \leq 0$ on K by Theorem 4.2(b). $\square$

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