

Convex Bodies with Affinely Equivalent Projections and Affine Bodies of Revolution

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We study affine bodies of revolution. This will allow us to prove that a convex body all whose orthogonal n -projections are affinely equivalent is an ellipsoid, provided $n \equiv 0, 1, 2 \pmod{4}$, $n > 1$, with the possible exemption of $n = 133$. Our proof uses convex geometry and topology of compact Lie groups.

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1. Introduction

The purpose of this paper is to prove the following Theorem.

Theorem 1.1. *Suppose $B \subset \mathbb{R}^N$ is a convex body, all of whose orthogonal projections onto n -dimensional linear subspaces, for some fixed integer n , $1 < n < N$, are affinely equivalent. If $n \equiv 0, 1, 2 \pmod{4}$, then B is an ellipsoid, with the possible exemption of $n = 133$.*

The analogue for sections is the following statement which is equivalent to the real Banach conjecture [1] stated in 1932.

Suppose $B \subset \mathbb{R}^N$ is a convex body containing the origin in its interior, all of whose sections through n -dimensional linear subspaces, for some fixed integer n , $1 < n < N$, are affinely equivalent. Then B is an ellipsoid,

Real Banach Isometric Conjecture. *Suppose V be a real Banach N -dimensional space all of whose n -dimensional subspaces, for some fixed integer n , $1 < n < N$, are isometrically isomorphic to each other. Then V is a Hilbert space.*

The conjecture was proved by Gromov [4] in 1967, for $n = \text{even}$ or $N > n + 1$ and by Montejano et.al. [2], for $n \equiv 1 \pmod{4}$ with the possible exception of $n = 133$. The history behind this conjecture can be read in [11]. It is also worthwhile to see [10] and the notes of Section 9 of [8]. Furthermore, for more about bodies having affinely equivalent or congruent sections or projections, see Problem 3.3, Note 3.2 and Problem 7.4, Note 7.2 of Richard Gardner's Book [3].

The reason for the strange exception $n \neq 133$ is because 133 is the dimension of the exceptional Lie group E_7 for which it was not possible to prove Theorem 1.6 of [2], crucial for the proof of our Theorem 2.4.

The proof of Theorem 1.1 combines two main ingredients: convex geometry and topology and geometry of transformation groups. The first part of the article consists of using topological methods to show that, under the hypotheses of Theorem 1.1, all orthogonal projections of B are affinely equivalent to a symmetric body of revolution. Section 3 is devoted to prove the following characterization of ellipsoids.

Theorem 1.2. *A symmetric convex body $B \subset \mathbb{C}^{n+1}$, $n \geq 4$, all of whose orthogonal projections onto hyperplanes are linearly equivalent to a fixed body of revolution, is an ellipsoid.*

2. Reducing the structure group of the tangent bundle of the sphere

During this section, we assume $B \subset \mathbb{R}^{n+1}$ is a convex body, all of whose orthogonal projections onto n -dimensional linear subspaces are linearly equivalent, $n \equiv 0, 1, 2 \pmod{4}$, $n > 1$ and $n \neq 133$. If this is the case, we shall prove that there is a symmetric convex body of revolution $K \subset \mathbb{R}^n$ with the property that every orthogonal projection of B is linearly equivalent to K .

The link to topology is via a beautiful idea that traces back to the work of Gromov [4]. It consists of the following key observation.

Lemma 2.1. *Let $B \subset \mathbb{R}^{n+1}$ be a symmetric convex body, all of whose orthogonal projections onto hyperplanes are linearly equivalent to some fixed symmetric convex body $K \subset \mathbb{R}^n$. Let $G_K := \{g \in SO(n) | g(K) = K\}$ be the group of symmetries of K . Then the structure group of the tangent bundle of S^n can be reduced to G_K .*

Proof. Consider the symmetric ellipsoid E of minimal volume containing $B \cap \mathbb{R}^n$. For the existence of Löwner-Johns minimal ellipsoids, see Gruber [5]. By translation and dilatation of the principal axes of this ellipsoid, we obtain an affine isomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $f(E)$ is the unit ball of $\mathbb{R}^n$. Let $K = f(B \cap \mathbb{R}^n)$. Obviously, the structure group of the tangent bundle of S^n can be reduced to $\{g \in GL_n(\mathbb{R}) | g(K) = K\}$. See Section 3.1 of [2] for the completely analogous proof of Lemma 1.5, for sections instead of projections, as well as a brief reminder about structure groups of differentiable manifolds and their reductions. Obviously, K is affinely equivalent to every orthogonal projection of B . The next step is to observe that $\{g \in GL_n(\mathbb{R}) | g(K) = K\}$ is a subgroup of the orthogonal group $O(n)$. This is so because every linear homeomorphism that fixes K also fixes the unit ball. Furthermore, if the structure group of the tangent bundle of S^n can be reduced to $\{g \in O(n) | g(K) = K\}$, then it can be reduced to G_K . $\square$

Lemma 2.1 can be also interpreted through the notion of *fields of convex bodies* tangent to S^n as follows. See, for example, Mani [7] and Montejano [9]. For every $u \in \mathbb{S}^n$, let $u^\perp$ be the hyperplane subspace orthogonal to u and $F(u)$ be the unique n -dimensional ellipsoid of least volume containing the orthogonal projection $\pi_u(B)$ of B in the direction u . The affine transformation β_u which maps $F(u)$ onto the n -dimensional Euclidean unit ball in $u^\perp$ by translating and dilating its principal axes is a continuous function of u . Then all $\beta_u(\pi_u(B))$, for $u \in \mathbb{S}^n$ are congruent, so there is a complete turning of $\beta_{e_1}(\pi(B))$, where π is the orthogonal projection onto $\mathbb{R}^n$ and $e_1 = (0, \dots, 0, 1)$. By Lemma 2, of [9], $\beta_{e_1}(\pi(B))$, is centrally symmetric, and so is any other orthogonal projection of B . This implies the following result.

Corollary 2.2. *Let $B \subset \mathbb{R}^N$ be a convex body, all of whose orthogonal projections onto n -dimensional subspaces are affinely equivalent, $1 < n < N$. Then B is centrally symmetric.*

Proof. If $n + 1 = N$, the proof follows immediately from the above paragraph and Lemma 2 of [9]. The corollary follows by induction using the obvious fact that every orthogonal projection of a convex body is centrally symmetric if and only if this body is centrally symmetric. $\square$

As an immediate consequence of Aleksandrov Theorem (Theorem 2.11.1 of [8]), we have

Theorem 2.3. *Let $B \subset \mathbb{R}^N$ be a symmetric convex body, all of whose orthogonal projections onto n -dimensional subspaces are congruent, $1 < n < N$. Then B is a ball.*

Let us prove now the main result of this section

Theorem 2.4. *Suppose $B \subset \mathbb{R}^{n+1}$ is a convex body, all of whose orthogonal projections onto hyperplanes, for some fixed integer n , $1 < n < N$, are affinely equivalent. If $n \equiv 0, 1, 2 \pmod{4}$ and $n \neq 133$, then there exist a symmetric body of revolution K such that every orthogonal projection of B onto a hyperplane subspace is linearly equivalent to K .*

Proof. By the first part of this section, we now that there exist a symmetric convex body K such that every orthogonal projection of B onto a hyperplane is linearly equivalent to K and such that the structure group of S^n can be reduced to $G_K = \{g \in SO(n) \mid g(K) = K\}$.

If n is even and $n \neq 6$, Theorem 1A) of Leonard [6] implies that $G = SO(n)$ and therefore K is a ball. In particular, K is a body of revolution. If $n = 6$, then Theorem 1A) of Leonard [6] implies that $G = SU(3)$ or $U(3)$, but in both cases the action in S^5 is transitive, which implies again that K is a ball. If $n \equiv 1 \pmod{4}$ and $n \neq 133$, then Theorem 1.6 of [2] implies that K is a body of revolution. $\square$

3. Affine bodies of revolution

A *symmetric convex body* is a compact convex subset of a finite dimensional real vector space with a nonempty interior, invariant under $x \mapsto -x$. A *hyperplane* is a codimension 1 linear subspace. An *affine hyperplane* is the translation of a hyperplane by some vector. Two sets, each a subset of a vector space, are *linearly* (respectively, *affinely*) *equivalent* if they can be mapped to each other by a linear (respectively, affine) isomorphism between their ambient vector spaces. Given an affine k -dimensional plane Λ in $\mathbb{R}^n$, we denote by $\Lambda^\perp$ the corresponding $(n - k)$ -dimensional subspace orthogonal to Λ .

An *ellipsoid* is a subset of a vector space which is affinely equivalent to the unit ball in euclidean space. A convex body $K \subset \mathbb{R}^n$ is a *body of revolution* if it admits an *axis of revolution*, i.e., a 1-dimensional line L such that each section of K by an affine hyperplane Δ orthogonal to L is an $n - 1$ dimensional euclidean ball in Δ , centered at $A \cap L$ (possibly empty or just a point). If L is an axis of revolution of K

then, $L^\perp$, is the associated *hyperplane of revolution*. An *affine body of revolution* is a convex body affinely equivalent to a body of revolution. The images, under an affine equivalence, of an axis of revolution and its associated hyperplane of revolution of the body of revolution are an axis of revolution and associated hyperplane of revolution of the affine body of revolution (not necessarily perpendicular anymore). Clearly, an ellipsoid centered at the origin is an affine symmetric body of revolution and any hyperplane serves as a hyperplane of revolution.

The aim of this section is to prove the following theorem.

Theorem 3.1. *Let $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, be a symmetric convex body, all of whose orthogonal projections are symmetric affine bodies of revolution. Then, at least one of its projections is an ellipsoid.*

The first step is to prove that the projection of an affine body of revolution is an affine body of revolution. For that purpose, we need to prove first the following technical lemma.

Lemma 3.2. *Let $\Gamma \subset \mathbb{R}^n$, $n \geq 3$, be an affine hyperplane and let C be an $(n-1)$ -dimensional ball contained in Γ with center at x . Let H be a hyperplane subspace non parallel to Γ and let $\pi : \mathbb{R}^n \rightarrow H$ be the orthogonal projection. Then, $\pi(C) \subset H$ is an ellipsoid of revolution with axis of revolution the line L , where*

$$L = (x + (\Gamma^\perp + H^\perp)) \cap H.$$

Proof. Suppose without loss of generality the center x of the $(n-1)$ -ball C is the origin. Consider the linear isomorphism $f : \Gamma \rightarrow H$ given by $f(z) = \pi(z)$. Consequently, $\pi(C) = f(C)$ is an ellipsoid. Note that f is the identity in the hyperplane subspace $\Gamma \cap H$ of H . Therefore, $\pi(C) = f(C)$ is an ellipsoid of the revolution with axis $(\Gamma^\perp + H^\perp) \cap H$, as we wished. $\square$

Lemma 3.3. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be an affine body of revolution with axis of revolution the line L and let π be an orthogonal projection along the 1-dimensional subspace ℓ . Then $\pi(K)$ is an affine body of revolution. Moreover, if L is parallel to ℓ , then $\pi(K)$ is an ellipsoid and if not, $\pi(L)$ is an axis of revolution of $\pi(K)$.*

Proof. Let us first prove the case in which K is a body of revolution. We will prove that in this case, if L is parallel to ℓ , then $\pi(K)$ is ball and if not, $\pi(K)$ is a body of revolution with axis $\pi(L)$.

Suppose L is not parallel to ℓ . Let $P = \ell + L$ be the 2-dimensional subspace generated by ℓ and L . For every $x \in L$, let $C_x = (x + L^\perp) \cap K$, then C_x is either empty, the point x or a ball contained in $(x + L^\perp)$ and center at x . Therefore

$$K = \cup_{\{x \in L\}} C_x, \quad \text{and} \quad \pi(K) = \cup_{\{x \in L\}} \pi(C_x). \quad (1)$$

By Lemma 3.2, $\pi(C_x)$ is an ellipsoid of revolution with axis $\pi(L)$. Consequently, by (1), $\pi(K)$ is the union of ellipsoids of revolution, all to them with the same axis $\pi(L)$. Consequently, $\pi(K)$ is a body of revolution with axis $\pi(L)$.

For the proof of the general case of the lemma we may assume, after a linear isomorphism, that K is a body of revolution and $\pi : \mathbb{R}^n \rightarrow H$ is an affine projection

in the direction of ℓ onto the hyperplane subspace H (not necessarily orthogonal to ℓ). Suppose L is not parallel to ℓ .

Consider $K \oplus \ell = \{x \in \mathbb{R}^n \mid (x + \ell) \cap K \neq \emptyset\}$. By definition $\pi(K) = (K \oplus \ell) \cap H$ and by the first part of the proof, $K_1 = (K \oplus \ell) \cap \ell^\perp$ is a body of revolution with axis $L_1 = (L + \ell) \cap \ell^\perp$.

Let $f : \ell^\perp \rightarrow H$ be the linear map given by $f(z) = (z + \ell) \cap H$, for every $z \in \ell^\perp$. Then $f(K_1) = \pi(K)$ and $f(L_1) = \pi(L)$. Consequently $\pi(K)$ is an affine body of revolution with axis of revolution $\pi(L)$, as we wished. $\square$

Lemma 3.4. *Let $K \subset \mathbb{R}^n$, $n \geq 3$, be an affine body of revolution with hyperplane of revolution H and let π be an orthogonal projection along the line ℓ . Suppose that ℓ is parallel to H and $\pi(K)$ is an ellipsoid. Then K is an ellipsoid.*

Proof. Since K is an affine body of revolution with hyperplane of revolution H and $\ell \subset H$, the shadow boundary

$$S\partial(K, \ell) = \{x \in \text{bd}K \cap L' \mid L' \text{ is a tangent line of } K \text{ parallel to } \ell\}$$

has the following property: there is a hyperplane Γ such that

$$L \subset \Gamma \text{ and } \Gamma \cap \text{bd}K = S\partial(K, \ell).$$

This implies that $\pi(\Gamma \cap K) = \pi(K)$. Consequently, the section $\Gamma \cap K$ is an ellipsoid and by Lemma 2.5 of [2], K is an ellipsoid. $\square$

From now on, let $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, be a symmetric convex body, all of whose orthogonal projections onto hyperplanes are non-elliptical, affine bodies of revolution.

Remember that for every line $\ell \subset \mathbb{R}^{n+1}$, we denote by $\ell^\perp$ the hyperplane subspace of $\mathbb{R}^{n+1}$ orthogonal to ℓ . Furthermore, by Lemma 2.3 of [2], denote by L_ℓ the unique axis of revolution of the orthogonal projection of B onto $\ell^\perp$, by H_ℓ the corresponding $(n-1)$ -dimensional subspace of revolution of the orthogonal projection of B onto $\ell^\perp$ and finally denote by $N_\ell \subset \ell^\perp$ the 1-dimensional subspace orthogonal to H_ℓ . That is, $N_\ell = H_\ell^\perp \cap \ell^\perp$. Note that by the symmetry, both L_ℓ and H_ℓ contain the origin.

We claim that the assignments $\ell \rightarrow L_\ell$ and $\ell \rightarrow N_\ell$ are continuous functions of ℓ . The proof is completely analogous to the proof of Lemma 2.7 of [2].

Lemma 3.5. *Let $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, be a symmetric convex body, all of whose hyperplane projections are non-elliptical, affine bodies of revolution. Let P be the plane generated by two different lines ℓ_1 and ℓ_2 and suppose $N_{\ell_1} \subset \ell_2^\perp$, then*

$$\Pi_P(L_{\ell_1}) = \Pi_P(L_{\ell_2}),$$

where Π_P is the orthogonal projection along P .

Proof. We shall first prove that $\ell_1^\perp \cap \ell_2^\perp \cap B$ is a non-elliptical affine body of revolution. For that purpose, first note that $\ell_1^\perp \cap \ell_2^\perp$ is orthogonal to P . Furthermore, $N_{\ell_1} \subset \ell_2^\perp$ implies that $N_{\ell_1} \subset \ell_1^\perp \cap \ell_2^\perp$ and hence by Lemma 3.4, if $\ell_1^\perp \cap \ell_2^\perp \cap B$ is an ellipsoid, then so is $\ell_1^\perp \cap B$, contradicting our hypothesis. If this is the case, by Lemma 2.3 of [2], since $n-1 \geq 3$, we conclude that $\ell_1^\perp \cap \ell_2^\perp \cap B$ has only one axis of revolution Δ . Thus, by Lemma 3.3, considering the orthogonal projection of

$\ell_1^\perp$ onto $\ell_1^\perp \cap \ell_2^\perp$ along $P \cap \ell_1^\perp$, we have that $\Delta = \Pi_P(L_{\ell_1})$, but on the other hand, considering the orthogonal projection of $\ell_2^\perp$ onto $\ell_1^\perp \cap \ell_2^\perp$ along $P \cap \ell_2^\perp$, we have that $\Delta = \Pi_P(L_{\ell_2})$. $\square$

Before giving the proof of Theorem 3.1, we need a technical topological lemma

Lemma 3.6. *Let $\phi, \psi : \mathbb{RP}^n \rightarrow \mathbb{RP}^n$ two continuous maps, both homotopic to the identity, $n \geq 4$. Then $\phi^{-1}(\mathbb{RP}^{n-2}) \cap \psi^{-1}(\mathbb{RP}^{n-2}) \neq \emptyset$.*

Proof. The closed set $\phi^{-1}(\mathbb{RP}^{n-2})$ can be thought as the carrier of $\sigma^2 \in H^2(\mathbb{RP}^n)$, where $\sigma \in H^1(\mathbb{RP}^n)$ is the generator of the cohomology ring $H^*(\mathbb{RP}^n)$. Similarly, $\psi^{-1}(\mathbb{RP}^{n-2})$ can be thought as the carrier of $\sigma^2 \in H^2(\mathbb{RP}^n)$. Since $n \geq 4$, σ^4 is not zero and consequently $\phi^{-1}(\mathbb{RP}^{n-1}) \cap \psi^{-1}(\mathbb{RP}^{n-2}) \neq \emptyset$. $\square$

Proof of Theorem 3.1 Suppose not, suppose that B is a symmetric convex body all of whose orthogonal projection onto hyperplanes sections are non-elliptical, affine bodies of revolution. For every line $\ell \subset \mathbb{R}^{n+1}$, let H_ℓ , L_ℓ and N_ℓ as in the paragraphs before Lemma 3.5.

Given the line $\ell_2 \subset \mathbb{R}^{n+1}$, our next purpose is to show that there exist a line $\ell_1 \subset \mathbb{R}^{n+1}$ such that

1. $N_{\ell_1} \subset \ell_2^\perp \cap L_{\ell_2}^\perp$, and
2. $L_{\ell_1} \subset \ell_2^\perp \cap L_{\ell_2}^\perp$.

For that purpose, note that the continuous assignments $\ell \rightarrow L_\ell$ and $\ell \rightarrow N_\ell$ can be thought as continuous maps from $\mathbb{RP}^n$ into itself, moreover, since ℓ is orthogonal to L_ℓ and also orthogonal to N_ℓ , the assignments $\ell \rightarrow L_\ell$ and $\ell \rightarrow N_\ell$ can be thought as continuous maps from $\mathbb{RP}^n$ into itself, which are homotopic to the identity. By Lemma 3.6, there exist a line $\ell_1 \subset \mathbb{R}^{n+1}$ such that $N_{\ell_1} \subset \ell_2^\perp \cap L_{\ell_2}^\perp$ and $L_{\ell_1} \subset \ell_2^\perp \cap L_{\ell_2}^\perp$.

Note now that both L_{ℓ_1} and N_{ℓ_1} are contained in $\ell_1^\perp \cap \ell_2^\perp$. If P is the plane orthogonal to $\ell_1^\perp \cap \ell_2^\perp$, then N_{ℓ_1} is orthogonal to P and thus $P \cap \ell_1^\perp \subset H_{\ell_1}$. By Lemma 3.5, $\Pi_P(L_{\ell_1}) = \Pi_P(L_{\ell_2})$, where Π_P is the orthogonal projection along P .

On the other hand, $L_{\ell_1} \subset \ell_1^\perp \cap \ell_2^\perp$ implies that $\Pi_P(L_{\ell_1}) = L_{\ell_1} = \Pi_P(L_{\ell_2})$. Finally, since L_{ℓ_1} is orthogonal to L_{ℓ_2} in $\ell_2^\perp$, then $\Pi_P(L_{\ell_2})$ is orthogonal to L_{ℓ_1} , contradicting the fact that $L_{\ell_1} = \Pi_P(L_{\ell_2})$. $\square$

Remark 3.7. We know that the projection or section of a convex body of revolution B is again a convex body of revolution. The converse of this result, as far as we know, is an open problem. Let us state a somewhat more precise question:

Let $B \subset \mathbb{R}^{n+1}$, $n \geq 4$, be a convex body containing the origin in its interior. Suppose every hyperplane section of B (projection onto a hyperplane) is a body of revolution, is B necessarily a body of revolution?

Note that our Theorem 3.1 points in that direction.

4. The proof of Theorem 1.1

The proof of Theorem 1.2 follows immediately from Theorem 3.1, because by Theorem 2.12.5 of [8], a convex body all whose hyperplane projections are ellipsoids is an ellipsoid.

The codimension 1 case of Theorem 1.1 follows directly from Corollary 2.2, Theorem 2.4 and Theorem 1.2. The rest of the proof follows from the fact that a convex body all whose hyperplane projections are ellipsoids is an ellipsoid.

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References

- [1] S. Banach: *Théorie des Opérations Linéaires*, Monografie Matematyczne, Warszawa-Lwow 1932, see also the English translation, *Theory of Linear Operations*, Volume 38, Elsevier, Amsterdam (1987).
- [2] G. Bor, L. Hernández Lamonedá, V. Jiménez-Desantiago: *On the isoperimetric conjecture of Banach*, arXiv: 1905.05878 (2019).
- [3] R. Gardner: *Geometric Tomography*, Encyclopedia of Mathematics and its Applications Volume 58, 2nd ed., Cambridge University Press, New York (2006).
- [4] M. L. Gromov: *A geometrical conjecture of Banach*, Mathematics of the USSR - Izvestiya 1.5 (1967) 1055.
- [5] P. M. Gruber: *Minimal ellipsoids and their duals*, Rend. Circ. Mat. Palermo (2) 37(1) (1988) 35–64.
- [6] P. Leonard: *G-structures on spheres*, Trans. Amer. Math. Soc. 157 (1971) 311–327.
- [7] P. Mani: *Fields of planar bodies tangent to spheres*, Monatshefte Math. 74 (1970) 145–149.
- [8] H. Martini, L. Montejano, D. Oliveros: *Bodies of Constant Width. An Introduction to Convex Geometry with Applications*, Birkhäuser, Basel (2019).
- [9] L. Montejano: *Convex bodies with homothetic sections*, Bull. London Math. Soc. 23 (1991) 381–386.
- [10] A. Pelczyński: *On some problems of Banach*, Russian Math. Surveys 28(6) (1973) 67–75.
- [11] V. Soltan: *Characteristic properties of ellipsoids and convex quadrics*, Aequationes Math. 93 (2019) 371–413.