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Multimeasures with Values in Conjugate Banach Spaces and the Weak Radon-Nikodým Property

Kazimierz Musiał
Institute of Mathematics, Wroclaw University, 50-384 Wroclaw, Poland
kazimierz.musial@math.uni.wroc.pl

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I prove that for a Banach space X the conjugate space X^* has the WRNP if and only if for every complete probability space (Ω, Σ, μ), every μ-continuous multimeasure of σ-finite variation that takes as its values closed (closed bounded, weak*-compact) and convex subsets of X^* can be represented as a Pettis integral of a multifunction with closed bounded (closed bounded, weak* compact) and convex values. This generalizes the known characterization of conjugate Banach spaces with the weak Radon-Nikodým property via functions (cf. [18] or [21]). The main tool is a lifting of a multifunction (see section 3), that is Effros measurable with respect to the weak* open subsets of X^*.

Keywords: Multimeasures, multifunctions, weak Radon-Nikodým property, Pettis integral, lifting.

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1. Introduction

It is known that a Banach space X does not contain any isomorphic copy of ℓ_1 if and only if X^* has the weak Radon-Nikodým property (cf. [18] or [21]). In case of the Radon-Nikodým property one knows that X^* has the RNP if and only if each separable subspace of X has a separable conjugate (see [28]). To the best of my knowledge no characterization of X^* possessing the WRNP via multimeasures with values in X^* is known. I try to fill in that gap. In this article I prove that vector measures may be replaced by multimeasures with weak*-compact (closed, closed bounded) and convex values. More precisely, X^* has the WRNP if and only if for every complete probability space (Ω, Σ, μ) and every μ-continuous multimeasure $M\colon \Sigma \to cw^*k(X^*)$ (non-empty weak*-compact and convex subsets of X^*) of σ-finite variation there exists a Pettis integrable multifunction $\Gamma\colon \Omega \to cw^*k(X^*)$ such that $M(E) = (Pettis)\int_E \Gamma\, d\mu$. If M takes as its values closed convex sets or bounded closed convex sets, then Γ takes its values in the family of closed bounded and convex subsets of X^*. The characterization of X^* is not a simple consequence of the Rådström embedding of the hyperspace of closed bounded and convex subsets of X^* into a Banach space Y (usually it is $\ell_\infty(B(X))$), as somebody could expect, since the differential structure of Y is not too friendly. Consequently, the proof is independent of the Rådström embedding.

To achieve the characterization, I modify slightly Haydon's characterization [13] of Banach spaces not containing any isomorphic copy of ℓ_1 (see Proposition 5.8) and apply it to a lifted multifunction, that is Effros measurable with respect to the weak*-open subsets of X^*.

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A presentation of the general theory of Pettis integration of vector functions can be found in [29], [21] and [19]. For the general theory of Pettis integrable multifunctions I recommend [12], [22], [23] and [2].

2. Preliminaries

Throughout the paper (Ω, Σ, μ) is a complete probability space and $\mathcal{L}_\infty(\mu)$ is the space of all μ-measurable real valued functions. $L_\infty(\mu)$ is the ordinary quotient space $\mathcal{L}_\infty(\mu)$ and ρ is a lifting on $L_\infty(\mu)$ and on (Ω, Σ, μ) (formally these are two different liftings but each of them uniquely determines the other). $\overline{\int}_E$ and $\underline{\int}_E$ are the upper and lower integrals on the set E, respectively. X is a Banach space with its dual X^* and the closed unit ball of X is denoted by $B(X)$. If $A \subset X^*$, then $\overline{A}^*$ is the weak* closure of A and $\overline{A}^{\|\cdot\|}$ is its norm closure. If $A \subset X$ is nonempty, then we write $\|A\| := \sup\{\|x\| : x \in A\}$.

$c(X)$ denotes the collection of all nonempty closed convex subsets of X, $cb(X)$ is the collection of all bounded elements of $c(X)$, $cwk(X)$ denotes the family of all weakly compact elements of $cb(X)$ and $cw^*k(X^*)$ denotes the family of all non-empty convex and weak*-compact subsets of X^*. For every $C \in c(X)$ the *support function of* C is denoted by $s(\cdot, C)$ and defined on X^* by $s(x^*, C) = \sup\{\langle x^*, x\rangle : x \in C\}$, for each $x^* \in X^*$. Notice that we have always $s(x^*, C) \neq -\infty$.

A function $f \colon \Omega \to \mathbb{R}$ is called *quasi-integrable* (cf. [25]), if the integral $\int_\Omega f\,d\mu$ exists. A map $\mathit{\Gamma} \colon \Omega \to c(X)$ is called a *multifunction*. $\mathit{\Gamma}$ is said to be *scalarly measurable* (*scalarly integrable, scalarly quasi-integrable*) if for every $x^* \in X^*$, the map $s(x^*, \mathit{\Gamma}(\cdot))$ is measurable (integrable, quasi-integrable). A multifunction $\mathit{\Gamma} \colon \Omega \to c(X^*)$ is said to be *weak***-scalarly measurable* (*weak***-scalarly integrable, weak***-scalarly quasi-integrable*) if for every $x \in X$, the map $s(x, \mathit{\Gamma}(\cdot))$ is measurable (integrable, quasi-integrable). A multifunction $\mathit{\Gamma} \colon \Omega \to c(X^*)$ is *weak***-scalarly bounded* if there is a constant $K > 0$ such that for every $x \in X$

$$|s(x, \mathit{\Gamma})| \leq K\|x\| \qquad a.e.$$

A function $\gamma \colon \Omega \to X$ is called a *selection of* $\mathit{\Gamma}$, if for almost every $\omega \in \Omega$, one has $\gamma(\omega) \in \mathit{\Gamma}(\omega)$. A function $\gamma \colon \Omega \to X^*$ is said to be a *weak** *quasi-selection* of a multifunction $\mathit{\Gamma} \colon \Omega \to c(X^*)$ if γ is weak*-scalarly measurable and we have $\langle x, \gamma(\omega)\rangle \in \langle x, \mathit{\Gamma}(\omega)\rangle := \{\langle x, x^*\rangle \colon x^* \in \mathit{\Gamma}(\omega)\}$ for μ-almost every $\omega \in \Omega$ (the exceptional sets depend on x). The collection of all weak* quasi-selections of $\mathit{\Gamma}$ will be denoted by $\mathcal{W}^*\mathcal{QS}_{\mathit{\Gamma}}$. One can easily check that a weak*-scalarly measurable $\gamma \colon \Omega \to X^*$ is a weak* quasi-selection of $\mathit{\Gamma}$ if and only if for every $x \in X$

$$\langle x, \gamma\rangle \leq s(x, \mathit{\Gamma}) \quad a.e. \tag{1}$$

If moreover $\gamma(\omega) \in \mathit{\Gamma}(\omega)$ for almost every $\omega \in \Omega$, then γ called a weak* selection of $\mathit{\Gamma}$. We denote the set of all weak* selections of $\mathit{\Gamma}$ by $\mathcal{W}^*\mathcal{S}_{\mathit{\Gamma}}$.

A map $M \colon \Sigma \to c(X)$ is *additive*, if $M(A \cup B) = \overline{M(A) + M(B)}$ for each pair of disjoint elements of Σ. An additive map $M \colon \Sigma \to c(X)$ is called a *multimeasure* if $s(x^*, M(\cdot)) \colon \Sigma \to (-\infty, +\infty]$ is a measure, for every $x^* \in X^*$. $M \colon \Sigma \to c(X^*)$ is called a *weak** *multimeasure* if $s(x, M(\cdot)) \colon \Sigma \to (-\infty, +\infty]$ is a measure, for every $x \in X$.

$\bigcup M(\Sigma) := \bigcup_{E\in\Sigma} M(E)$ denotes the range of M in X or X^*.

One can easily see that if M is a weak* multimeasure (multimeasure), and if for $y \in X(X^*)$ the equality $s(y, M) = s(y, M)^+ - s(y, M)^-$ is the Jordan decomposition of $s(y, M)$, then

$$s(y, M(\Omega))^- < \infty. \tag{2}$$

If $M\colon \Sigma \to cb(X)(cb(X^*))$, then the measures $s(y, M)$ and their variations $|s(y, M)|$ are finite measures. If M is a point map, then we talk about weak* measure and measure, respectively. An additive map $M\colon \Sigma \to cb(X)$ is called an *h-multimeasure*, if it is countably additive in the Hausdorff metric of $cb(X)$. It is known that each multimeasure $M\colon \Sigma \to cwk(X)$ is an h-multimeasure.

If $m\colon \Sigma \to X^*$ is a weak* measure such that $m(A) \in M(A)$, for every $A \in \Sigma$, then m is called a *weak* selection* of M. $\mathcal{W}^*\mathcal{S}(M)$ will denote the set of all weak*-countably additive selections of M. Similarly, a vector measure $m\colon \Sigma \to X$ such that $m(A) \in M(A)$, for every $A \in \Sigma$, is called a *selection* of M. $\mathcal{S}(M)$ will denote the set of all countably additive selections of M.

If M is a weak* multimeasure or a multimeasure, then M is called absolutely continuous with respect to μ (we write then $M \ll \mu$), if $\mu(E) = 0$ yields $M(E) = \{0\}$. M is dominated by μ if there exists $\alpha > 0$ such that $|M(E)| \le \alpha\mu(E)$, for every $E \in \Sigma$. If $M\colon \Sigma \to cw^*k(X^*)$ is a weak* multimeasure, then the weak* semivariation of M on a set $E \in \Sigma$ is defined by $|||M|||^*(E) := \sup\{|s(x, M)|(E) : x \in B(X)\}$, where $|s(x, M)|(E)$ is the ordinary variation of the measure $s(x, M)$ on the set E.

If $M\colon \Sigma \to cb(X)$ is a multimeasure, then the semivariation of M on a set $E \in \Sigma$ is defined by $|||M|||(E) := \sup\{|s(x^*, M)|(E) : x^* \in B(X^*)\}$. If M is a weak* multimeasure or a multimeasure, then the variation of M is defined by

$$|M|(E) := \sup_{\mathcal{P}} \left\{ \sum_{A\in\mathcal{P}} \|M(A)\| : \mathcal{P} \text{ is a finite measurable partition of } E \right\}.$$

One can easily see that $|M|$ is a measure (cf. [23, page 863]) and, $M \ll \mu$ if and only if $|M| \ll \mu$ (that is $\mu(E) = 0 \Rightarrow |M|(E) = 0$). If $M \ll \mu$ is of σ-finite variation, then there exists a partition $(\Omega_n)_n$ of Ω such that $|M(E)| \le n\mu(E)$ (equivalently $|M|(E) \le n\mu(E)$) for every $E \in \Omega_n \cap \Sigma$, $n \in \mathbb{N}$ (it is a consequence of the classical Radon-Nikodým theorem).

3. Weak* multimeasures

We begin with a simple observation that makes a bridge between weak* multimeasures with norm closed and convex values and weak* multimeasures with weak*-compact values.

Proposition 3.1. *If the map $M\colon \Sigma \to cb(X^*)$ is a weak* multimeasure, then $M^*\colon \Sigma \to cw^*k(X^*)$ defined by $M^*(E) := \overline{M(E)}^*$ is also a weak* multimeasure.*

Proof. If $K \in cb(X^*)$ and $x \in X$, then $s(x, K) = s(x, \overline{K}^*)$. Consequently, if $(E_n)_n$ is a sequence of pairwise disjoint elements of Σ and $x \in X$, then

$$s\Big(x, M^*\Big(\bigcup_n E_n\Big)\Big) = s\Big(x, M\Big(\bigcup_n E_n\Big)\Big) = \sum_n s(x, M(E_n)) = \sum_n s(x, M^*(E_n)). \quad \square$$

Proposition 3.2. (see [23, Proposition 6.1 and 6.2])
If $M\colon \Sigma \to cb(X^*)$ is a weak multimeasure, then $|||M|||^*(\Omega) < \infty$.*
If $M\colon \Sigma \to cb(X)$ is a multimeasure, then $|||M|||(\Omega) < \infty$.
If $M\colon \Sigma \to cb(X^*)$ is a weak multimeasure (or $M\colon \Sigma \to cb(X)$ is a multimeasure) of finite variation, then it is an h-multimeasure.*

Proof. We apply [23, Proposition 6.1 and 6.2] and Proposition 3.1. □

Given a sequence of sets $W_n \in cw^*k(X^*)$, we say that the series $\sum_{n=1}^{\infty} W_n$ is weak* unconditionally convergent if every series $\sum_{n=1}^{\infty} w_n$, where $w_n \in W_n$, for all n, is weak* unconditionally convergent. We denote then by $\sum_{n=1}^{\infty} W_n$ the set of all sums $\sum_{n=1}^{\infty} w_n$, where $w_n \in W_n$, for all n and $\sum_{n=1}^{\infty} w_n$ is weak* unconditionally convergent. Following the proof of [5, Lemma 2.2], one can show that $\sum_{n=1}^{\infty} W_n \in cw^*k(X^*)$, for each weak* unconditionally convergent $\sum_{n=1}^{\infty} W_n$.

To shorten the formulae in the next proposition, we write w^*uc instead of weak* unconditionally convergent.

Proposition 3.3. *Let $M\colon \Sigma \to cw^*k(X^*)$ be a weak* multimeasure. Then for every sequence of pairwise disjoint sets $E_n \in \Sigma$*

$$M\Big(\bigcup_n E_n\Big) = \Big\{\sum_n x_n^* : \forall\, n\ x_n^* \in M(E_n)\ \&\ \sum_n x_n^* \textit{ is } w^*uc\Big\} = \sum_{n=1}^{\infty} M(E_n)\,.$$

Proof. Let $M\colon \Sigma \to cw^*k(X^*)$ be a weak* multimeasure and $E_n \in \Sigma$ be pairwise disjoint sets. If $x_n^* \in M(E_n)$, then for every $x \in X$

$$\begin{aligned}\sum_n |x_n^*(x)| &\le \sum_n |s(x, M(E_n))| + \sum_n |s(-x, M(E_n))| \\ &\le 2|||M|||^*(\Omega)\|x\| < \infty. \end{aligned} \tag{3}$$

It follows that the series $\sum_n x_n^*$ is weak* unconditionally Cauchy and so it is w*uc. If $x^* := \sum_n x_n^*$ then we have $x^*(x) \le \sum_n s(x, M(E_n)) = s(x, M(\bigcup_n E_n))$ and so $x^* \in M(\bigcup_n E_n)$. Thus,

$$\Big\{\sum_n x_n^* : \forall\, n\ x_n^* \in M(E_n)\ \&\ \sum_n x_n^* \textit{ is } w^*uc\Big\} \subseteq M\Big(\bigcup_n E_n\Big).$$

Take now an arbitrary $x^* \in M\Big(\bigcup_n E_n\Big)$. For every $n \in \mathbb{N}$ we have

$$M\Big(\bigcup_n E_n\Big) = \sum_{k=1}^{n} M(E_k) + M(\bigcup_{k=n+1}^{\infty} E_k)$$

and inductively we can define points $x_k^* \in M(E_k)$ and $y_n^* \in M(\bigcup_{k=n+1}^{\infty} E_k)$ such that $x^* = \sum_{k=1}^{n} x_k^* + y_n^*$ (in general the points are not uniquely determined), where $y_n^* \in M(\bigcup_{k=n+1}^{\infty} E_k)$.

It is my aim now to prove that $x^* = \sum_k x_k^*$ in the weak* topology. So let us fix $\varepsilon > 0$, $x \in X$ and n_0 such that $\sum_{k>n_0} [|s(-x, M(E_k))| + |s(x, M(E_k))|] < \varepsilon$.

If $n > n_0$, then
$$\Big|x^*(x) - \sum_{k=1}^{n} x_k^*(x)\Big| = |y_n^*(x)|$$
$$\leq \Big|s\Big(-x, M\Big(\bigcup_{k>n} E_k\Big)\Big)\Big| + \Big|s\Big(x, M\Big(\bigcup_{k>n} E_k\Big)\Big)\Big| < \varepsilon$$

Hence, $x^*(x) = \sum_{k=1}^{\infty} x_k^*(x)$ and, consequently
$$M\Big(\bigcup_n E_n\Big) \subseteq \Big\{\sum_n x_n^* : \forall\, n\; x_n^* \in M(E_n)\ \&\ \sum_n x_n^* \ is\ w^*uc\Big\}. \qquad \square$$

As an immediate consequence of the above result, we have the following theorem, generalizing an earlier achievement of Diestel and Faires [10, Corollary 1.2]:

Theorem 3.4. *For a fixed X the following conditions are equivalent:*
(i) $l_\infty \not\subseteq X^*$ *isomorphically;*
(ii) *for each (Ω, Σ) each weak* multimeasure $M\colon \Sigma \to cw^*k(X^*)$ is a multimeasure;*
(iii) *for each (Ω, Σ) each weak* multimeasure $M\colon \Sigma \to cw^*k(X^*)$ is an h-multimeasure.*

Proof. (i)⇒(iii) Assume that $l_\infty \not\subseteq X^*$. With the notation from the proof of Proposition 3.3 we have from (3)
$$\sum_{n=1}^{m} |x_n^*(x)| \leq \sum_n |s(x, M(E_n))| + \sum_n |s(-x, M(E_n))| \leq 2|||M|||^*(\Omega)\|x\| < \infty$$
for every $m \in \mathbb{N}$. Hence, if $\|x^{**}\| \leq 1$, then for every $m \in \mathbb{N}$
$$\sum_{n=1}^{m} |x_n^*(x^{**})| \leq 2|||M|||^*(\Omega).$$
But that means that the series $\sum_{n=1}^{\infty} x_n^*$ is weakly unconditionally Cauchy. Since $c_0 \not\subseteq X^*$, the series is weakly unconditionally convergent (see [1]) and hence it is unconditionally convergent. Consequently, M is an h-multimeasure (cf. [14, Proposition 8.4.7]).

(ii)⇒(i) follows from [10]. $\square$

The following result is a direct consequence of Theorem 3.4. It will be applied in the main theorem (Theorem 5.10), but not only.

Corollary 3.5. *If X does not contain any isomorphic copy of ℓ_1, then every weak* multimeasure $M\colon \Sigma \to cb(X^*)$ is a multimeasure.*

The following result generalizes an old result from [7].

Theorem 3.6. *If $M\colon \Sigma \to cw^*k(X^*)$ is a weak* multimeasure, then $\mathcal{W}^*\mathcal{S}(M) \neq \emptyset$ and*
$$M(E) = \{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\} \quad \text{for each } E \in \Sigma. \tag{4}$$

Proof. It is known that if x^* is an extreme point of $M(\Omega)$, then for each nontrivial measurable decomposition $\Omega = E \cup E^c$ there are uniquely determined extreme point y^*, z^* of $M(E)$ and $M(E^c)$, respectively such that $x^* = y^* + z^*$. Setting $m(E) = y^*$, $m(E^c) = z^*$ and $m(\Omega) = x^*$, we obtain an additive set function

$m\colon \Sigma \to X^*$. Since $-s(-x, M(E)) \le \langle x, m(E)\rangle \le s(x, M(E))$, for every $E \in \Sigma$ and $x \in X$, m is a weak* countably additive selection of M. The Krein-Milman theorem yields the inclusion $M(E) \subset \overline{\{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\}}^*$ for every $E \in \Sigma$.

In order to prove (4) let us fix $A \in \Sigma$, $W_A := \{m(A) : m \in \mathcal{W}^*\mathcal{S}(M)\}$ and let

$$\widetilde{W} := \big\{\langle x, m\rangle_{x\in X} \in ca(\Sigma)^X \colon m \in \mathcal{W}^*\mathcal{S}(M)\big\},$$

where $ca(\Sigma)$ is the space of real measures on Σ. We consider $ca(\Sigma)$ with the weak topology (= topology of pointwise convergence on Σ, (cf. [11, Theorem IV.9.5])) and we equip $ca(\Sigma)^X$ with the product topology. Then, let $S\colon \widetilde{W} \to X^*$ be defined by $S(\langle x, m\rangle_{x\in X}) = m(A)$. It is clear that $S(\widetilde{W}) = W_A$ and S is continuous if X^* is considered with the weak* topology. It is our aim to prove that $\widetilde{W}$ is compact. This will finish the proof of (4).

First we will prove that $\widetilde{W}$ is closed. So let $(m_\alpha)_\alpha$ be a net in $\widetilde{W}$ that is convergent to $(\nu_x)_{x\in X} \in ca(\Sigma)^X$. That is

$$\lim_\alpha \langle x, m_\alpha(E)\rangle = \nu_x(E) \qquad \text{for every } E \in \Sigma.$$

Since for each $E \in \Sigma$ the net $(m_\alpha)_\alpha$ is contained in the weak*-compact set $M(E)$, it is weak*-convergent to a point $\nu(E) \in M(E)$. Obviously ν is additive and as $|\langle x, \nu(E)\rangle| \le |s(x, M(E))| + |s(-x, M(E))|$, $\langle x, \nu\rangle$ is absolutely continuous with respect to $|s(x, M)| + |s(-x, M)|$ in the $\varepsilon - \delta$ sense. Consequently, $\langle x, \nu\rangle$ is a μ-continuous scalar measure, what proves that ν is a weak* measure. As it is a selection of M we have $\nu \in \widetilde{W}$.

Then, since for each $x \in X$ the set

$$\{\kappa \in ca(\Sigma)\colon -s(-x, M(E)) \le \kappa(E) \le s(x, M(E)) \qquad \text{for every } E \in \Sigma\}$$

is weakly compact in $ca(\Sigma)$ (cf. [11, Theorem IV.9.2]), the whole $\widetilde{W}$ is relatively compact in $ca(\Sigma)^X$. It follows that $\widetilde{W}$ is compact. □

Definition 3.7. Denote by $\mathcal{C}$ a non-empty subfamily of $c(X^*)$. A weak*-scalarly quasi-integrable multifunction $\Gamma\colon \Omega \to c(X^*)$ is *Gelfand integrable* in $\mathcal{C}$, if for each set $A \in \Sigma$ there exists a set $M_\Gamma^G(A) \in \mathcal{C}$ such that

$$s(x, M_\Gamma^G(A)) = \int_A s(x, \Gamma)\, d\mu\,, \text{ for every } x \in X. \tag{5}$$

It is clear that $(G)\int_A \Gamma\, d\mu := M_\Gamma^G(A)$ is a weak* multimeasure. $M_\Gamma^G(A)$ is called the *Gelfand integral* of Γ on A.

Denote by $\mathcal{D}$ a non-empty subfamily of $c(X)$. A scalarly quasi-integrable multifunction $\Gamma\colon \Omega \to c(X^*)$ is *Pettis integrable* in $\mathcal{D}$, if for each $A \in \Sigma$ there exists a set $M_\Gamma(A) \in \mathcal{D}$ such that

$$s(x^*, M_\Gamma(A)) = \int_A s(x^*, \Gamma)\, d\mu \quad \text{for every } x^* \in X^*. \tag{6}$$

$M_\Gamma\colon \Sigma \to c(X)$ is a multimeasure and $M_\Gamma(A)$ called the *Pettis integral of* Γ over A. We write $(P)\int_A \Gamma\, d\mu = M_\Gamma(A)$. □

As observed in [12], if $\Gamma\colon \Omega \to c(X)$ is Pettis integrable in $c(X)$, then

$$\int_\Omega s(x^*, \Gamma)^- d\mu < \infty, \quad \text{for every } x^* \in X^*.$$

A similar inequality holds true for Gelfand integral. $s(x^*, \Gamma)^-$ denotes the negative part of $s(x^*, \Gamma)$.

It has been proven in [3] and in [23, Theorem 6.7] that each weak*-scalarly integrable $\Gamma\colon \Omega \to cw^*k(X^*)$ is Gelfand integrable in $cw^*k(X^*)$. But following the proof of [23, Theorem 6.9] one can see that a more general result holds true:

Theorem 3.8. *Every weak*-scalarly integrable* $\Gamma\colon \Omega \to cb(X^*)$ *is Gelfand integrable in* $cw^*k(X^*)$.

In view of [23, Proposition 6.5] each Gelfand integral of a multifunction is a weak* multimeasure of σ-finite variation. It follows that each weak* measure being a selection of the Gelfand integral is of σ-finite variation. But then, it is a classical result (cf. [19, Theorem 11.1] or [21, Theorem 9.1]) that each such weak* measure has a weak*-integrable density. Consequently, in view of Theorem 3.6, we have the following fact

Corollary 3.9. *If* $\Gamma\colon \Omega \to cb(X^*)$ *is Gelfand integrable in* $cw^*k(X^*)$, *then* $\mathcal{W}^*\mathcal{QS}_\Gamma \neq \emptyset$.

Proof. Let $\Gamma^*(\cdot) := \overline{\Gamma(\cdot)}^*$. Since $s(x, \Gamma^*) = s(x, \Gamma)$ everywhere on Ω, Γ^* is Gelfand integrable in $cw^*k(X^*)$ and so, in view of Theorem 3.6, there exists $\gamma \in \mathcal{W}^*\mathcal{QS}_{\Gamma^*}$. That means, that for each $x \in X$ we have $\langle x, \gamma\rangle \leq s(x, \Gamma^*)$ a.e. It follows that $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$. □

In fact a more precise description is possible.

Proposition 3.10. [3, Theorem 4.5] *If* $\Gamma\colon \Omega \to cb(X^*)$ *is Gelfand integrable in* $cw^*k(X^*)$, *then*

$$M^G_{\Gamma^*}(E) = \left\{(G)\int_E \gamma\, d\mu : \gamma \in \mathcal{W}^*\mathcal{QS}_{\Gamma^*}\right\}. \tag{7}$$

If X^* has the WRNP, then still a more precise description is possible:

Proposition 3.11. *Let* X *be a Banach space with* X^* *possessing the WRNP and* $\Gamma\colon \Omega \to cw^*k(X^*)$ *be a weak*-scalarly bounded and weak*-scalarly measurable multifunction. Then, given a lifting* ρ *on* $L_\infty(\mu)$, *we have*

$$\forall\, E \in \Sigma\ M^G_\Gamma(E) = \left\{(P)\int_E \rho(\gamma)\, d\mu : \gamma \in W^*QS_\Gamma\right\}.$$

Proof. We know from Proposition 3.10 that

$$\forall\, E \in \Sigma\ M^G_\Gamma(E) = \left\{(G)\int_E \gamma\, d\mu : \gamma \in W^*QS_\Gamma\right\}.$$

If $\gamma \in W^*QS_\Gamma$, then $\rho(\gamma)$ is a Pettis integrable selection of $\rho\Gamma$, due to $\ell_1 \not\subseteq X$ (This fact can be found in the proof of [19, Theorem 12.1] but explicitely it was formulated in [20, Theorem 6.5]) and, $(G)\int_E \gamma\, d\mu = (G)\int_E \rho(\gamma)\, d\mu = (P)\int_E \rho(\gamma)\, d\mu$. □

Remark 3.12. One should notice that the value of Gelfand integral strongly depends on the family $\mathcal{C}$. Assume that X is not reflexive and define $\mathit{\Gamma}\colon \mathit{\Omega} \to cb(X^{**})$ by $\mathit{\Gamma}(\omega) = B(X)$. Then, let $\mathit{\Gamma}^*(\cdot) := \overline{\mathit{\Gamma}(\cdot)}^* \equiv B(X^{**})$. Both multifunctions are Gelfand integrable; $\mathit{\Gamma}$ in $cb(X^{**})$ and $\mathit{\Gamma}^*$ in $cw^*k(X^{**})$. So we have the equality

$$s(x^*, \mu(E)B(X^{**})) = s(x^*, M^G_{\mathit{\Gamma}^*}(E)) = \int_E s(x^*, \mathit{\Gamma}^*)\, d\mu$$

for all $E \in \mathit{\Sigma}$ and $x^* \in X^*$, and

$$s(x^*, \mu(E)B(X)) = s(x^*, M^G_{\mathit{\Gamma}}(E)) = \int_E s(x^*, \mathit{\Gamma})\, d\mu \quad \text{for all } E \in \mathit{\Sigma} \text{ and } x^* \in X^*.$$

But $s(x^*, B(X^{**})) = s(x^*, B(X))$ and $s(x^*, \mathit{\Gamma}^*) = s(x^*, \mathit{\Gamma})$. Thus, Gelfand integral depends on the range space. This does not happen in case of Pettis integral (due to Hahn-Banach theorem). □

4. Liftings of multifunctions

Lifting was already applied in the context of Radon-Nikodým theorem for support functions of multifunctions (cf. [26] or [9]). It is my aim to go a step ahead, to define a lifting of a multifunction and investigate its properties. Throughout this section ρ is an arbitrary but fixed lifting on $L_\infty(\mu)$. If $\gamma\colon \mathit{\Omega} \to X^*$ is a weak*-scalarly measurable and weak*-scalarly bounded function, then $\rho(\gamma)\colon \mathit{\Omega} \to X^*$ is defined by $\langle x, \rho(\gamma)(\omega)\rangle := \rho(\langle x, \gamma\rangle)(\omega)$ (see [15, ch. IV.5]).

4.1. Existence

Proposition 4.1. *If $\mathit{\Gamma}\colon \mathit{\Omega} \to cw^*k(X^*)$ is a weak*-scalarly bounded and weak*-scalarly measurable multifunction and if ρ is a lifting on $L_\infty(\mu)$, then there exists a weak*-scalarly bounded and weak*-scalarly measurable multifunction $\rho\mathit{\Gamma}\colon \mathit{\Omega} \to cw^*k(X^*)$ such that*

$$s(x, \rho\mathit{\Gamma}(\omega)) = \rho(s(x, \mathit{\Gamma}))(\omega) \quad \textit{for every } x \in X \textit{ and every } \omega \in \mathit{\Omega}.$$

Proof. Let $F\colon X \to \mathbb{R}$ be a non-negative and norm continuous sublinear functional and let

$$\nabla F := \{x^* \in X^* : \forall\, x \in X\ \langle x^*, x\rangle \le F(x)\}.$$

It is clear that $\nabla F \neq \emptyset$. It is known that ∇F is convex, weak*-compact and $F(x) = s(x, \nabla F)$ for every $x \in X$. According to Corollary 3.9 there exists a quasi-selection $\gamma \in W^*QS_{\mathit{\Gamma}}$. Let $G_\gamma := \mathit{\Gamma} - \gamma$. Denote for each $\omega \in \mathit{\Omega}$ by F_ω the function $x \longrightarrow \rho(s(x, G_\gamma))(\omega)$. F_ω is sublinear and non-negative. We will prove that each F_ω is norm-continuous. Assume that for each $x \in X$ one has $|s(x, G_\gamma)| \le N\|x\|$ a.e. If $x \in X$ is fixed and $\|x_n - x\| \to 0$, then

$$|s(x_n, G_\gamma) - s(x, G_\gamma)| \le |s(x_n - x, G_\gamma)| + |s(x - x_n, G_\gamma)| \le 2N\|x_n - x\| \text{ a.e.}$$

Hence $|\rho(s(x_n, G_\gamma)) - \rho(s(x, G_\gamma))| \le 2N\|x_n - x\|$ everywhere. In particular, each F_ω is norm-continuous. Consequently, there exists $\widehat{G_\gamma}(\omega) = \nabla F_\omega \in cw^*k(X^*)$ with $F_\omega(\bullet) = s(\bullet, \widehat{G_\gamma}(\omega))$.

We set $\rho G_\gamma(\omega) := \widehat{G_\gamma}(\omega)$ and call ρG_γ the lifting of G_γ. The lifting of $\mathit{\Gamma}$ is defined by $\rho\mathit{\Gamma} := \rho G_\gamma + \rho(\gamma)$. If $x \in X$, then $\rho(s(x, \mathit{\Gamma})) = \rho(s(x, G_\gamma)) + \rho(\gamma)$ and so $\rho\mathit{\Gamma}$ is independent of the choice of a quasi-selection of $\mathit{\Gamma}$. □

The next statement is quite obvious.

Proposition 4.2. *The lifting $\rho\Gamma$ satisfies the following maximality condition: If $\Delta\colon \Omega \to cw^*k(X^*)$ is such that*

$$s(x,\Delta(\omega)) \le \rho\big[s(x,\Delta)\big](\omega) \le \rho\big[s(x,\Gamma)\big](\omega)$$

for every $x \in X$ and $\omega \in \Omega$, then $\Delta(\omega) \subseteq \rho\Gamma(\omega)$ for every $\omega \in \Omega$.

Our aim now is a more precise description of $\rho\Gamma$.

Proposition 4.3. *If $\Gamma\colon \Omega \to cw^*k(X^*)$ is a weak*-scalarly bounded and weak*-scalarly measurable multifunction and ρ is a lifting on $L_\infty(\mu)$, then $\rho\Gamma$ is defined for every $\omega \in \Omega$ by*

$$\rho\Gamma(\omega) := \overline{conv}^*\left\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\right\}.$$

$\rho\Gamma$ is Gelfand integrable in $cw^*k(X^*)$ and

$$\langle x, \rho(\gamma)\rangle(\omega) \le s(x,\rho\Gamma(\omega)) = \rho[s(x,\rho\Gamma)](\omega) = \rho[s(x,\Gamma)](\omega),$$

for every $x \in X$, $\omega \in \Omega$ and $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$. Moreover, $M^G_\Gamma = M^G_{\rho\Gamma}$.

Proof. Define $\widetilde{\Gamma}\colon \Omega \to cw^*k(X^*)$ by $\widetilde{\Gamma}(\omega) = \overline{conv}^*\left\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\right\}$.
To prove the assertion we fix $x \in X$ and $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$. We have then $\langle x,\gamma\rangle \le s(x,\Gamma)$ a.e. and so $\rho(\langle x,\gamma\rangle) = \langle x,\rho(\gamma)\rangle \le \rho(s(x,\Gamma))$ everywhere. Thus, if $\omega \in \Omega$ is fixed,

$$\begin{aligned} s(x,\widetilde{\Gamma}(\omega)) &= s(x, conv\left\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\right\}) \\ &= \sup\left\{\rho(\langle x,\gamma\rangle)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\right\} \\ &\le \rho[s(x,\Gamma)](\omega) = s(x,\rho\Gamma(\omega)). \end{aligned} \tag{8}$$

The inequality $s(x,\widetilde{\Gamma}(\omega)) \le s(x,\rho\Gamma(\omega))$ valid for every $x \in X$ yields the inclusion $\widetilde{\Gamma}(\omega) \subset \rho\Gamma(\omega)$ for every $\omega \in \Omega$ and the equality (8) implies the measurability of $s(x,\widetilde{\Gamma})$ (see [16, Theorem III.3.3]). The boundedness of $\rho[s(x,\Gamma)]$ and the inclusion yield boundedness of $\widetilde{\Gamma}(\omega)$. Hence $\widetilde{\Gamma}(\omega)$ is weak*-compact and $\widetilde{\Gamma}$ is Gelfand integrable in $cw^*k(X^*)$.

Suppose now that there is $x \in X$ such that $s(x,\widetilde{\Gamma})$ and $\rho\big[s(x,\Gamma)\big]$ are not a.e. equal. Then there is $\alpha \in \mathbb{R}$ such that

$$\mu\{\omega \in \Omega : s(x,\widetilde{\Gamma})(\omega) < \alpha < \rho[s(x,\Gamma)](\omega)\} > 0.$$

Let $A := \{\omega \in \Omega : s(x,\widetilde{\Gamma})(\omega) < \alpha < \rho[s(x,\Gamma)](\omega)\}$. We have then

$$\forall\, \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\ \forall\, \omega \in A\ [\langle x, \rho(\gamma)(\omega)\rangle < \alpha < \rho[s(x,\Gamma)](\omega)]. \tag{9}$$

Hence, $\displaystyle\int_A \langle x,\gamma\rangle\, d\mu < \alpha\mu(A) < \int_A s(x,\Gamma)\, d\mu = \langle x, M^G_\Gamma(A)\rangle$ for every $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$.

However, the above inequalities contradict the equality (7). Therefore we have $\rho[s(x,\widetilde{\Gamma})] = \rho[s(x,\Gamma)]$ everywhere on Ω. This yields immediately the equality $\int_E s(x,\Gamma)\, d\mu = \int_E s(x,\widetilde{\Gamma})\, d\mu$ for every $E \in \Sigma$, and this is the required coincidence of M^G_Γ and $M_{\widetilde{\Gamma}}$ (we assume that both multiintegrals take values in $cw^*k(X^*)$).

From [16, Theorem III.3.3] follows

$$\begin{aligned} s(x,\widetilde{\Gamma})(\omega) &= \sup\{\rho(\langle x,\gamma\rangle)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\} \\ &\le \rho(\sup\{\rho(\langle x,\gamma\rangle)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\}) = \rho\left(s(x,\widetilde{\Gamma})\right)(\omega). \end{aligned}$$

Hence, for every $x \in X$ and $\omega \in \Omega$,

$$s(x,\widetilde{\Gamma}(\omega)) \le \rho\big[s(x,\widetilde{\Gamma})\big](\omega) = \rho\big[s(x,\Gamma)\big](\omega).$$

We are going to show yet that $\widetilde{\Gamma} = \rho\Gamma$. So suppose that there is $\overline{\omega} \in \Omega$ such that $\widetilde{\Gamma}(\overline{\omega}) \subsetneq \rho\Gamma(\overline{\omega})$. Then there exist a point $\overline{x} \in X$ and $\overline{x}^* \in \rho\Gamma(\overline{\omega}) \setminus \widetilde{\Gamma}(\overline{\omega})$ such that $s(\overline{x},\widetilde{\Gamma}(\overline{\omega})) < \langle \overline{x}^*, \overline{x}\rangle = s(\overline{x},\rho\Gamma(\overline{\omega}))$.

Let $D(\omega)\colon = \{x^* \in \rho\Gamma(\omega) : \langle x^*, \overline{x}\rangle = s(\overline{x},\rho\Gamma(\omega))\}$. Then $D\colon \Omega \to cw^*k(X^*)$ and each weak* quasi-selection of D is a weak* quasi-selection of $\rho\Gamma$.

Let $\gamma\colon \Omega \to X^*$ be a weak* quasi-selection of D. Then, according to (1), for each $x \in X$

$$s(\overline{x},\rho\Gamma) = -s(-\overline{x},D) \le \langle \overline{x},\gamma\rangle \le s(\overline{x},D) = s(\overline{x},\rho\Gamma) \quad \text{a.e.}$$

It follows that for each $x \in X$

$$\langle \overline{x},\rho(\gamma)\rangle = \rho(\langle \overline{x},\gamma\rangle) = \rho\big[s(\overline{x},\rho\Gamma)\big] = s(\overline{x},\rho\Gamma) \text{ everywhere.} \tag{10}$$

Applying (10), we have $s(\overline{x},\widetilde{\Gamma}(\overline{\omega})) < s(\overline{x},\rho\Gamma(\overline{\omega})) = \langle \overline{x},\rho(\gamma)\rangle(\overline{\omega})$, what contradicts the description of $\widetilde{\Gamma}$, because $\rho(\gamma) \in \mathcal{W}^*\mathcal{S}_{\widetilde{\Gamma}}$. This proves that $\rho\Gamma = \widetilde{\Gamma}$. □

Remark 4.4. Even if $M_\Gamma^G\colon \Sigma \to cwk(X^*)$ it may be impossible to find a $cwk(X^*)$-valued multifunction $\rho\Gamma$. A counterexample is presented in Example 5.2. □

Lemma 4.5. *If X is separable and $\Gamma\colon \Omega \to cw^*k(X^*)$ is scalarly bounded and scalarly measurable, then $\rho\Gamma$ is also scalarly measurable and $\Gamma = \rho\Gamma$ a.e.*

Proof. Let $\{x_n\colon n \in \mathbb{N}\}$ be a dense subset of $B(X)$. Since $\rho(s(x,\Gamma)) = s(x,\rho\Gamma)$ a.e. for each $x \in X$ separately, there exists $N \in \Sigma_0$ such that

$$s(x_n,\Gamma(\omega)) = s(x_n,\rho\Gamma(\omega)) \quad \text{for every } \omega \notin N \text{ and every } n \in \mathbb{N}.$$

Due to separability of X, we have $\Gamma(\omega) = \rho\Gamma(\omega)$ for every $\omega \notin N$. □

Remark 4.6. Besides $\rho\Gamma$, we will consider also Γ_ρ defined by

$$\Gamma_\rho(\omega) := \overline{conv}^{\|\cdot\|}\left\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\right\} = \overline{\left\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\right\}}^{\|\cdot\|}.$$

If $x \in X$, then we have always $s(x,\Gamma_\rho(\omega)) = s(x,\rho\Gamma(\omega))$. Consequently,

$$\|\Gamma_\rho(\omega)\| = \sup_{x\in B(X)} s(x,\Gamma_\rho(\omega)) = \sup_{x\in B(X)} s(x,\rho\Gamma(\omega)) = \|\rho\Gamma(\omega)\|. \qquad \square$$

It has been proven in [23, Proposition 6.5] that Gelfand integral of a Gelfand integrable multifunction with weak*-compact values is of σ-finite variation.

The next theorem is a more precise description of [23, Proposition 6.5] in case of weak*-scalarly bounded multifunctions.

Theorem 4.7. *Let $\varGamma\colon \varOmega \to cw^*k(X^*)$ be a weak*-scalarly bounded multifunction that is Gelfand integrable in $cw^*k(X^*)$. If ρ is a lifting on $L_\infty(\mu)$, then $M_\varGamma^G$ is of finite variation and*

$$|M_\varGamma^G|(E) = \int_E \|\rho(\varGamma)\|\,d\mu = \int_E \|\varGamma_\rho\|\,d\mu \quad \textit{for every } E \in \varSigma.$$

Proof. The measurability of $\|\rho(\varGamma)\|$ is a consequence of the following equalities:

$$\|\varGamma_\rho(\omega)\| = \|\rho\varGamma(\omega)\| = \sup_{x\in B(X)} s(x,\rho\varGamma(\omega)) = \sup_{x\in B(X)} \rho(s(x,\rho\varGamma))(\omega).$$

It follows from Theorem 3.6 that for each $x \in X$ and $E \in \varSigma$

$$s(x, M_\varGamma(E)) = \sup\{\langle x, m_{x,E}(E)\rangle : m_{x,E} \in \mathcal{W}^*S(M_\varGamma^G)\}.$$

Each such $m_{x,E}$ has a weak* density $\gamma_{x,E}$ that is a weak* quasi-selection of $\varGamma$. In particular $\rho(\gamma_{x,E})$ is a weak* selection of $\rho\varGamma$ and

$$\langle x, m_{x,E}(E)\rangle = \int_E \rho\langle x,\gamma_{x,E}\rangle\,d\mu \quad \text{for every } E\in\varSigma.$$

Now, if $E \in \varSigma$ is fixed, then

$$\begin{aligned}
\|M_\varGamma^G(E)\| &= \sup_{x\in B(X)} s(x, M_\varGamma^G(E)) = \sup_{x\in B(X)} \langle x, m_{x,E}(E)\rangle \\
&= \sup_{x\in B(X)} \int_E \rho\langle x,\gamma_{x,E}\rangle\,d\mu \le \sup_{x\in B(X)} \int_E \sup_{\gamma\in\mathcal{W}^*\mathcal{QS}(\rho\varGamma)} \rho\langle x,\gamma\rangle\,d\mu \\
&= \sup_{x\in B(X)} \int_E s(x,\rho\varGamma)\,d\mu \le \int_E \sup_{x\in B(X)} s(x,\rho\varGamma)\,d\mu = \int_E \|\rho\varGamma\|\,d\mu
\end{aligned}$$

A bit more calculations (see [23, Proposition 6.5]) shows that

$$|M_\varGamma^G|(E) = \int_E \|\rho(\varGamma)\|\,d\mu\,, \quad \text{for every } E\in\varSigma\,. \qquad \square$$

4.2. Measurability

Following the classical terminology, I call a multifunction $\varGamma\colon \varOmega \to c(X^*)$ weak*-open measurable (Effros measurable with respect to weak*-open sets), if for every weak*-open $U \subset X^*$, we have

$$\{\omega\in\varOmega : U\cap\varGamma(\omega)\neq\emptyset\}\in\varSigma\,.$$

The subsequent proposition will be applied in the proof of Theorem 5.10.

Proposition 4.8. *If $\Gamma\colon \Omega \to cw^*k(X^*)$ is weak*-scalarly bounded and weak*-scalarly measurable, then $\rho\Gamma$ and Γ_ρ are weak*-open measurable. Moreover,*

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap U \neq \emptyset\} \subset \rho\left(\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap U \neq \emptyset\}\right)$$

and
$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap U = \emptyset\} \supset \rho\left(\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap U = \emptyset\}\right).$$

If K is weak-closed, then $\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset K\} \in \Sigma$ and*

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset K\} \supset \rho\left(\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset K\}\right).$$

Analogous inclusions hold true in case of Γ_ρ.

Proof. By Proposition 4.3 we have $\rho\Gamma(\omega) := \overline{conv}^* \{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\}$, for every $\omega \in \Omega$. Assume that $\{\omega \in \Omega : \rho\Gamma(\omega) \cap U \neq \emptyset\} \neq \emptyset$. It follows directly that if $\rho\Gamma(\omega) \cap U \neq \emptyset$, then $U \cap conv\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma\} \neq \emptyset$. Take $\gamma_1, \ldots, \gamma_n \in \mathcal{W}^*\mathcal{QS}_\Gamma$ in such a way that $\sum_{i=1}^n \alpha_i \rho(\gamma_i)(\omega) \in U$, for some non-negative α_i summing to one. Then we obtain $\rho(\gamma) := \sum_{i=1}^n \alpha_i \rho(\gamma_i) \in \mathcal{W}^*\mathcal{QS}_\Gamma$ and $\rho(\gamma)(\omega) \in U \cap \rho\Gamma(\omega)$. Consequently, $\rho\Gamma(\omega) \cap U \neq \emptyset$ if and only if there exists $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$ with $\rho(\gamma)(\omega) \in U$. Equivalently, if and only if

$$\{\omega\colon \rho\Gamma(\omega) \cap U \neq \emptyset\} = \bigcup_{\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma} \{\omega\colon \rho(\gamma)(\omega) \in U\} = \bigcup_{\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma} \rho(\gamma)^{-1}(U).$$

According to [16, p. 52] we have $\rho(\gamma)^{-1}(U) \subset \rho\left(\rho(\gamma)^{-1}(U)\right)$ for every $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$. Applying now Theorem III.3.3 of [16] we obtain the measurability of the set $\bigcup_{\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma} \rho(\gamma)^{-1}(U)$ and

$$\begin{aligned}\{\omega\colon \rho\Gamma(\omega) \cap U \neq \emptyset\} &= \bigcup_{\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma} \rho(\gamma)^{-1}(U) \\ &\subset \rho\Bigg(\bigcup_{\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma} \rho(\gamma)^{-1}(U)\Bigg) = \rho\left(\{\omega\colon \rho\Gamma(\omega) \cap U \neq \emptyset\}\right).\end{aligned}$$

If K is weak*-closed, then

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset K\} = \{\omega \in \Omega\colon \rho\Gamma(\omega) \cap K^c = \emptyset\} \in \Sigma.$$

In case of Γ_ρ we have $\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap U \neq \emptyset\} = \{\omega \in \Omega\colon \Gamma_\rho(\omega) \cap U \neq \emptyset\}$ and $\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap U = \emptyset\} = \{\omega \in \Omega\colon \Gamma_\rho(\omega) \cap U = \emptyset\}$. □

Let us call a multifunction $\Gamma\colon \Omega \to c(X^*)$ *weak*-closed measurable*, if for every weak*-closed $C \subset X^*$, we have

$$\{\omega \in \Omega\colon C \cap \Gamma(\omega) \neq \emptyset\} \in \Sigma.$$

In many situations the closed measurability of multifunctions with values in a topological spaces is stronger then the open measurability. I do not need this property of $\rho\Gamma$ in this article so I am not going to do any deeper investigation, but for the sake of completeness I will sketch its much longer proof. The proof needs several intermediate steps. I will mention a few of them without detailed proofs. One of the essential facts is the following well known property of liftings.

Lemma 4.9. *Assume that $f \in \mathcal{L}_\infty(\mu)$ and $\rho(f) \le f$ pointwise. Then for each $\alpha \in \mathbb{R}$*

$$\rho\{f \ge \alpha\} \subset \{f \ge \alpha\} \quad \textit{and} \quad \{f < \alpha\} \subset \rho\{f < \alpha\}. \tag{11}$$

If $f \le \rho(f)$ pointwise, then

$$\rho\{f \le \alpha\} \subset \{f \le \alpha\} \quad \textit{and} \quad \{f > \alpha\} \subset \rho\{f > \alpha\}.$$

Lemma 4.10. *Let $\Gamma\colon \Omega \to cw^*k(X^*)$ be weak*-scalarly measurable and weak*-scalarly bounded. If $a_1, \ldots, a_n \in \mathbb{R}$, $x_1, \ldots, x_n \in X$, then*

$$\Omega' := \left\{\omega \in \Omega\colon \bigcap_{j=1}^{n} \{x_j(x^*) \ge a_j\} \cap \rho\Gamma(\omega) \ne \emptyset\right\} \in \Sigma \tag{12}$$

and $\rho(\Omega') \subset \Omega'$.

The proof is similar to the proof of [2, Theorem 5.10]. In particular, it needs [6, Proposition I-24]. It is rather long.

Lemma 4.11. *Let $\Gamma\colon \Omega \to cw^*k(X^*)$ be weak*-scalarly measurable and weak*-scalarly bounded. Assume that $V = \bigcup_{i=1}^{m} V_i$, where each of the sets V_i is of the form $\bigcap_{j=1}^{m_i} \{|x_{ij}(x^*)| < a_j\}$. Then $\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset V\} \in \Sigma$ and*

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset V\} \subset \rho(\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset V\}).$$

With the help of Lemma 4.10 the proof is a routine calculation.

Lemma 4.12. *Let $\Gamma\colon \Omega \to cw^*k(X^*)$ be weak*-scalarly measurable and weak*-scalarly bounded. If U is weak*-open, then $\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U\} \in \Sigma$ and*

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U\} \subset \rho(\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U\}).$$

Proof. Let $U = \bigcup_{\alpha \in \mathbb{A}} V_\alpha$, where each V_α is of the form $\bigcap_{j=1}^{m_\alpha} \{|x_{\alpha j}(x^*)| < a_j\}$. If $\widetilde{\mathcal{V}}$ consists of all possible finite unions of elements of $\{V_\alpha\colon \alpha \in \mathbb{A}\}$, then we have $U = \bigcup_{\beta \in \mathbb{B}} U_\beta$, where each $U_\beta \in \widetilde{\mathcal{V}}$. Since the sets $G(\omega)$ are weak*-compact, we have

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U\} = \bigcup_{\beta \in \mathbb{B}} \{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U_\beta\}$$

From Lemma 4.11 we get $\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U_\beta\} \subset \rho(\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U_\beta\})$ and so [16, Theorem III.3.3] implies the measurability of the set $\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U\}$ and the inclusion $\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U\} \subset \rho(\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset U\})$. □

Proposition 4.13. *If $\Gamma\colon \Omega \to cw^*k(X^*)$ is weak*-scalarly bounded and weak*-scalarly measurable, then $\rho\Gamma$ is weak*-closed measurable.*
Moreover, for every weak-closed $C \subset X^*$,*

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap C = \emptyset\} \subset \rho\left(\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap C = \emptyset\}\right)$$

and

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap C \ne \emptyset\} \supset \rho\left(\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap C \ne \emptyset\}\right)$$

Proof. The result is an immediate consequence of Lemma 4.12 because

$$\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap C = \emptyset\} = \{\omega \in \Omega\colon \rho\Gamma(\omega) \subset C^c\}.$$

□

5. Weak* Radon-Nikodým Theorem

We can now prove a Radon-Nikodým theorem for multimeasures with values in conjugate Banach spaces in the language of Gelfand integral. One can find earlier approaches to that problem in [9], [26] and [4].

Theorem 5.1. *Let X be a Banach space and $M\colon \Sigma \to cw^*k(X^*)$ $[cb(X^*), c(X^*)]$ be a μ-continuous weak* multimeasure of σ-finite variation. Then, there exists a multifunction $\Gamma\colon \Omega \to cw^*k(X^*)$ $[cb(X^*)]$ such that*

$$s(x, M(E)) = \int_E s(x,\Gamma)\, d\mu \quad \text{for every } x \in X \text{ and } E \in \Sigma. \tag{13}$$

If M is dominated by μ and ρ is a lifting on $L_\infty(\mu)$, then one may take $\Gamma = \rho\Gamma$ in the weak case and $\Gamma = \Gamma_\rho$ in the other ones.*

Proof. Assume first that there is $\alpha > 0$ such that $|M(E)| \le \alpha\mu(E)$, for every $E \in \Sigma$ and that ρ is a lifting on $L_\infty(\mu)$.
Consider the first case: $M\colon \Sigma \to cw^*k(X^*)$. By Theorem 3.6

$$M(E) = \left\{ (G)\!\int_E \gamma\, d\mu : \gamma \text{ is a weak}^* \text{ density of } m \in \mathcal{W}^*\mathcal{S}(M) \right\}.$$

But γ and $\rho(\gamma)$ are weak*-equivalent and hence $\|\rho(\gamma)(\omega)\| \le \alpha$ for every $\omega \in \Omega$ and

$$M(E) = \left\{ (G)\!\int_E \rho(\gamma)\, d\mu : \gamma \text{ is a weak}^* \text{ density of } m \in \mathcal{W}^*\mathcal{S}(M) \right\}.$$

Define $\Gamma\colon \Omega \to X^*$ pointwise by

$$\Gamma(\omega) := \overline{conv}^* \left\{ \rho(\gamma)(\omega) : (G)\!\int \gamma\, d\mu \in \mathcal{W}^*\mathcal{S}(M) \right\}.$$

Each $\Gamma(\omega)$ is a bounded set. If $x \in X$, then

$$s(x, \Gamma(\omega)) = \sup \left\{ \langle x, \rho(\gamma)(\omega)\rangle : (G)\!\int \gamma\, d\mu \in \mathcal{W}^*\mathcal{S}(M) \right\} \le \alpha\|x\|\,.$$

It follows that $s(x,\Gamma)$ is a bounded measurable function. Consequently, there exists a weak* multimeasure $M_\Gamma^G\colon \Sigma \to cw^*k(X^*)$ such that

$$s(x, M_\Gamma^G(E)) = \int_E s(x,\Gamma)\, d\mu \quad \text{for every } x \in X \text{ and } E \in \Sigma\,. \tag{14}$$

and (due to Proposition 3.10)

$$M_\Gamma^G(E) = \left\{ (G)\!\int_E \rho(\gamma)\, d\mu : \gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma \right\}.$$

Clearly $M = M_\Gamma^G$. According to Proposition 4.3

$$\rho\Gamma(\omega) = \overline{conv}^*\{\rho(\gamma)(\omega)\colon \gamma \in W^*QS_\Gamma\}\,,\ \omega \in \Omega.$$

But $\gamma \in W^*QS_\Gamma$ if and only if $\int \gamma\, d\mu \in \mathcal{W}^*\mathcal{S}(M)$. That proves that in case if M is dominated by μ, the equality $\Gamma = \rho\Gamma$ is fulfilled.
Assume now that $M\colon \Sigma \to c(X^*)$. Since $|M|(\Omega) < \infty$ we see that $M\colon \Sigma \to cb(X^*)$.

Define $M^*\colon \Sigma \to cw^*k(X^*)$ by $M^*(\cdot) := \overline{M(\cdot)}^*$ and let Γ^* be defined for M^* in the same way as Γ for M in the first part of the proof. Then, let $\Gamma^*_\rho\colon \Omega \to cb(X)$ be defined for Γ^*. We already know that

$$s(x, M^*(E)) = \int_E s(x, \rho\Gamma^*)\, d\mu \quad \text{for every } x \in X \text{ and } E \in \Sigma.$$

But if $x \in X$, $E \in \Sigma$ and $\omega \in \Omega$ then we obtain directly $s(x, M^*(E)) = s(x, M(E))$ and $s(X, \Gamma^*_\rho(\omega)) = s(x, \rho\Gamma^*(\omega))$. Consequently we get, as required,

$$s(x, M(E)) = \int_E s(x, \Gamma^*_\rho)\, d\mu \quad \text{for every } x \in X \text{ and } E \in \Sigma.$$

Let now M be arbitrary. Then one can decompose Ω into countably many pairwise disjoint sets Ω_n in such a way that $|M|(E \cap \Omega_n) \le n\mu(E \cap \Omega_n)$. According to the first part of the proof, for each $n \in \mathbb{N}$ there exists a multifunction $\Gamma_n : \Omega_n \to cw^*k(X^*)$ $[cb(X^*)]$ that is Gelfand integrable and

$$s(x, M(E \cap \Omega_n)) = \int_{E\cap\Omega_n} s(x, \Gamma_n)\, d\mu \quad \text{for every } x \in X \text{ and } E \in \Sigma. \tag{15}$$

Setting $\Gamma(\omega) := \sum_n \Gamma_n(\omega)\chi_{\Omega_n}(\omega)$, we get a Gelfand integrable multifunction. Indeed, if $x \in X$ and M has only bounded values, then

$$\int_\Omega |s(x, \Gamma)|\, d\mu = \sum_n \int_{\Omega_n} |s(x, \Gamma)|\, d\mu = \sum_n |s(x, M)|(\Omega_n) = |s(x, M)|(\Omega)) < \infty$$

and so Γ is Gelfand integrable in $cw^*k(X^*)$. It follows from (15) that Γ is a weak* density of M. Assume now that $M\colon \Sigma \to c(X^*)$ and fix $x \in X$. If $F := \{\omega \in \Omega : s(x, \Gamma(\omega)) < 0\}$ then, according to (2),

$$\int_F s(x, \Gamma)^-\, d\mu = \sum_{n=1}^\infty \int_{F\cap\Omega_n} s(x, \Gamma)^-\, d\mu = \sum_{n=1}^\infty s(x, M(F \cap \Omega_n)) = s(x, M(F)) < \infty.$$

This yields weak* quasi-integrability of Γ and the expected equality (13). □

Example 5.2. Assume in Theorem 5.1 that $M\colon \Sigma \to cwk(X^*)$. It is unknown when Γ can be taken $cwk(X^*)$-valued. The following example is copied from [8] and shows that in general the Radon-Nikodým derivative of a multimeasure with (weakly) compact values may fail to take (weakly) compact sets as its values.

Let (Ω, Σ, μ) be equal to $([0, 1], \mathcal{L}, \lambda)$, $X = l_1$ and $\langle e_n\rangle$ be the canonical basis of l_1. Let $\langle \alpha^k_n\rangle_{n,k\ge0}$ be a double sequence of reals such that

$$\forall\, k \sum_{n\ge0} |\alpha^k_n| = 1 \qquad \text{and} \qquad \sum_{k\ge0}\left(\sum_{n\ge0} |\alpha^k_n|^2\right)^{1/2} < \infty.$$

If $\langle r_n\rangle_{n\ge0}$ is the sequence of Rademacher functions, then set for each $k \ge 0$ and $\omega \in (0, 1)$ $\sigma_k(\omega) = \langle \alpha^k_n r_n(\omega)\rangle_{n\ge0} \in l_1$. If $\Gamma(\omega) = \overline{\text{conv}}\{\sigma_k(\omega) : k \ge 0\}$, then Γ takes its values in $cb(l_1)$ and $\Gamma(\omega) \notin cwk(l_1)$ a.e. Moreover, Γ is Pettis integrable in $cwk(l_1) = ck(l_1)$. Due to separability of l_1 the density of the integral is a.e. uniquely determined and so M^G_Γ does not have any $cwk(l_1)$-valued Radon-Nikodým density.

Another example of this type can be found in [8, Example 1]. □

In case of X^* possessing the WRNP Theorem 5.1 can be essentially strengthen. To achieve the result we need a few complementary facts. The first three need only a routine calculation.

Lemma 5.3. *Let $K, L \in cw^*k(X^*)$ be arbitrary. Then $conv(K \cup L) \in cw^*k(X^*)$ and each functional $x^{**} \in X^{**}$ that is weak*-continuous on K and on L is also weak*-continuous on $conv(K \cup L)$.*

Lemma 5.4. *If $M\colon \Sigma \to cw^*k(X^*)$ is a μ-continuous weak* multimeasure of σ-finite variation, then its range $\bigcup M(\Sigma)$ is weak* relatively compact.*

Proof. By the assumption, for every $x \in X$, the set $s(x, \bigcup M(\Sigma))$ is bounded. Hence $\bigcup M(\Sigma)$ is bounded in X^*. Consequently, it is weak* relatively compact. $\square$

Lemma 5.5. *If $M\colon \Sigma \to cw^*k(X^*)$ is a μ-dominated weak* multimeasure and $\rho\Gamma$ is its weak* density, then*

$$M(E) \subset \mu(E)\,\overline{conv}^*\left(\bigcup\{\rho\Gamma(\omega)\colon \omega \in E\}\right) \quad \textit{for every } E \in \Sigma. \tag{16}$$

Proof. Let $E \in \Sigma$ be fixed. We have $M(E) = \{m(E) : m \in \mathcal{W}^*\mathcal{S}(M)\}$ according to Theorem 3.6. Hence, it is enough to prove that if m is a selection of M, then $m(E) \in \mu(E)\,\overline{conv}^* \bigcup\{\rho\Gamma(\omega)\colon \omega \in E\}$. But is is known, and easily follows from Hahn-Banach theorem, that if γ is a Gelfand integrable density of m, then

$$m(E) \in \mu(E)\,\overline{conv}^*\{\rho(\gamma)(\omega) : \omega \in E\} \subset \overline{conv}^*\left(\bigcup\{\rho\Gamma(\omega)\colon \omega \in E\}\right). \qquad \square$$

To formulate further results we need to recall a few notations concerning lattices and set functions defined on them.

Definition 5.6. A non-void collection $\mathfrak{L} \subset \mathcal{P}(\Omega)$ is a *lattice*, if for each two elements $A, B \in \mathfrak{L}$, we have $A \cup B \in \mathfrak{L}$ and $A \cap B \in \mathfrak{L}$. $\mathfrak{L}$ is σ-compact if $\mathfrak{L} \ni A_n \downarrow A$ implies $A \neq \emptyset$, provided $A_n \neq \emptyset$, for every $n \in \mathbb{N}$.
A set function $\kappa\colon \mathfrak{L} \to [0,\infty)$ is *increasing*, if $\kappa(A) \geq \kappa(B)$, whenever $A \supset B$. $\kappa\colon \mathfrak{L} \to [0,\infty)$ is supermodular, if $\kappa(A \cup B) + \kappa(A \cap B) \geq \kappa(A) + \kappa(B)$, for all $A, B \in \mathfrak{L}$. κ is σ-smooth if $\lim_n \kappa(A_n) = \kappa(A)$ for each sequence $\mathfrak{L} \ni A_n \downarrow A \in \mathfrak{L}$. $\square$

We take now as $\mathfrak{L}$ the σ-compact lattice of all weak*-compact subsets of $B(X^*)$ and define (see Proposition 4.8) for each weak*-scalarly bounded and weak*-scalarly measurable $\Gamma\colon \Omega \to cw^*k(X^*)$ a set function $\kappa : \mathfrak{L} \to [0,1]$ by

$$\kappa(K) := \mu\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset K\} = \mu\{\omega \in \Omega\colon \rho\Gamma(\omega) \cap K^c = \emptyset\}.$$

One can easily check that since $\rho\Gamma(\omega) = \overline{\Gamma_\rho(\omega)}^*$, we have also

$$\kappa(K) = \mu\{\omega \in \Omega\colon \Gamma_\rho(\omega) \subset K\} = \mu\{\omega \in \Omega\colon \Gamma_\rho(\omega) \cap K^c = \emptyset\}.$$

Proposition 5.7. *κ is a non-decreasing supermodular and σ-smooth set function on $\mathfrak{L}$. Moreover, there exists an inner regular measure $\overline{\kappa}\colon \sigma(\mathfrak{L}) \to [0,1]$ (approximation by $\mathfrak{L}$) such that $\overline{\kappa}(K) \geq \kappa(K)$, for every $K \in \mathfrak{L}$.*

Proof. Take arbitrary $K, L \in \mathfrak{L}$. Then

$$\{\omega : \rho\Gamma(\omega) \subset K \cup L\} \supset \{\omega : \rho\Gamma(\omega) \subset K\} \cup \{\omega : \rho\Gamma(\omega) \subset L\}$$

and $$\{\omega : \rho\Gamma(\omega) \subset K \cap L\} = \{\omega : \rho\Gamma(\omega) \subset K\} \cap \{\omega : \rho\Gamma(\omega) \subset L\}$$

As a result, we have

$$\begin{aligned}\kappa(K \cup L) + \kappa(K \cap L) &= \mu\{\omega : \rho\Gamma(\omega) \subset K \cup L\} + \mu\{\omega : \rho\Gamma(\omega) \subset K \cap L\} \\ &\geq \mu(\{\omega : \rho\Gamma(\omega) \subset K\} \cup \{\omega : \rho\Gamma(\omega) \subset L\}) \\ &\qquad + \mu(\{\omega : \rho\Gamma(\omega) \subset K\} \cap \{\omega : \rho\Gamma(\omega) \subset L\}) \\ &= \mu(\{\omega : \rho\Gamma(\omega) \subset K\}) + \mu(\{\omega : \rho\Gamma(\omega) \subset L\}) = \kappa(K) + \kappa(L)\end{aligned}$$

and the equality before the last one is a consequence of the additivity of μ.

To prove smoothness of κ let $\mathfrak{L} \ni K_n \downarrow K \in \mathfrak{L}$. Then,

$$\begin{aligned}\kappa(K) = \kappa\Big(\bigcap_n K_n\Big) &= \mu\Big\{\omega : \Big(\bigcap_n K_n\Big)^c \cap \rho\Gamma(\omega) = \emptyset\Big\} \\ &= \mu\Big\{\omega : \bigcup_n \rho\Gamma(\omega) \cap K_n^c = \emptyset\Big\} = \mu\Big(\bigcup_n \{\omega : \rho\Gamma(\omega) \cap K_n^c = \emptyset\}\Big) \\ &= \lim_n \mu\{\omega : \rho\Gamma(\omega) \cap K_n^c = \emptyset\} = \lim_n \kappa(K_n).\end{aligned}$$

The existence of an inner regular measure $\overline{\kappa}$ dominating κ on $\mathfrak{L}$ follows from [17, Theorem 5] and [17, Proposition 7], since $\mathfrak{L}$ is σ-compact. $\square$

In order to formulate our main result we need yet to recall a result of Haydon [13]. I formulate it in the form presented by Stegall [27]:

Proposition 5.8. *For a Banach space X the following are equivalent:*

(h1) $\ell_1 \not\subseteq X$;

(h2) *for every $x^{**} \in X^{**}$ and every positive finite regular measure ν on the weak*-Borel subsets of $B(X^*)$ there exists a sequence $(x_n)_n \subset X$ such that we have $\|x_n\| \leq \|x^{**}\|$ for every $n \in \mathbb{N}$ and $x_n \longrightarrow x^{**}$ ν-a.e.*

(h3) *for every $x^{**} \in X^{**}$, every positive finite regular measure ν on the weak*-Borel subsets of $B(X^*)$ and every $\varepsilon > 0$, there exists $B(X^*) \supset K \in cw^*k(X^*)$ such that x^{**} is weak*-continuous on K and $\nu(K) \geq (1-\varepsilon)\nu(B(X^*))$.*

More detailed analysis of the proof in [27] shows that applying Lemma 5.3, one can formulate the above result in a formally stronger shape:

Proposition 5.9. *For a Banach space X the following are equivalent:*

(H1) $\ell_1 \not\subseteq X$;

(H2) *for every $x^{**} \in X^{**}$ and every positive finite regular measure ν on the weak*-Borel subsets of $B(X^*)$ there exist an non-decreasing sequence of weak*-compact and convex sets $K_m \subset B(X^*)$ and a sequence $(x_n)_n \subset X$ with the following properties:*

(i) *x^{**} is weak*-continuous on each K_m and $\nu(\bigcup_m K_m) = \nu(B(X^*))$;*

(ii) *$\|x_n\| \leq \|x^{**}\|$ for every $n \in \mathbb{N}$ and $x_n \longrightarrow x^{**}$ uniformly on each K_m.*

Theorem 5.10. *Let X be a Banach space with X^* possessing the WRNP. Assume that $M\colon \Sigma \to cw^*k(X^*)$ ($M\colon \Sigma \to cb(X^*)$) is a μ-continuous weak* multimeasure of σ-finite variation or $M\colon \Sigma \to c(X^*)$ is a μ-continuous multimeasure of σ-finite variation. Then there exists a Pettis integrable multifunction $\Gamma\colon \Omega \to cw^*k(X^*)$ ($\Gamma\colon \Omega \to cb(X^*)$) such that*

$$s(x^{**}, M(E)) = \int_E s(x^{**}, \Gamma)\, d\mu \quad \textit{for every } E \in \Sigma.$$

Proof. Assume first that μ is non-atomic and $|M(E)| \le \alpha\mu(E)$ for every $E \in \Sigma$ (to simplify the notation, we assume that $\alpha = 1$). Then also M is non-atomic. Assume first that $M\colon \Sigma \to cw^*k(X^*)$.

(I) According to Theorem 5.1 and Theorem 4.7 there exists a bounded weak*-scalarly measurable multifunction $\Gamma = \rho\Gamma\colon \Omega \to cw^*k(B(X^*))$ satisfying the equality

$$s(x, M(E)) = \int_E s(x, \rho\Gamma)\, d\mu \quad \text{for every } x \in X \text{ and every } \ E \in \Sigma\,. \tag{17}$$

Since M is non-atomic, the weak* density $\rho\Gamma$ is not constant on any set of positive measure.

Let x^{**} be fixed. If $\mathfrak{L}$ is the lattice of all weak*-compact subsets of $B(X^*)$, then we define $\kappa_1\colon \mathcal{L} \to [0,1]$ by

$$\kappa_1(K) = \mu\{\omega \in \Omega\colon \rho\Gamma(\omega) \subset K\}.$$

According to Proposition 5.7 there exists a regular probability measure $\overline{\kappa}_1$ on the weak*-Borel subsets of $B(X^*)$ such that $\overline{\kappa}_1(K) \ge \kappa_1(K)$ for every $K \in \mathcal{L}$. Since for each $F \in \Sigma$, we have

$$\kappa_1\Bigg(\overline{\bigcup_{\omega\in F} \rho\Gamma(\omega)}^{\,*}\Bigg) \ge \mu(F),$$

the measure $\overline{\kappa}_1$ is non-atomic. By Proposition 5.9 there exists a non-decreasing sequence of weak*-compact and convex sets $K^1_m \subset B(X^*)$ such that x^{**} is weak*-continuous on each K^1_m and $\overline{\kappa}_1(\bigcup_m K^1_m) = \mu(\Omega) = 1$. Moreover, there exists a sequence $(x^1_n)_n \subset X$ such that $\|x^1_n\| \le \|x^{**}\|$ for every $n \in \mathbb{N}$ and $x^1_n \longrightarrow x^{**}$ uniformly on each K^1_m. For simplicity, let $\rho\Gamma^-(K) := \{\omega \in \Omega\colon \rho\Gamma(\omega) \subset K\}$, whenever $K \subset X^*$. We may have either

$$\mu\Bigg(\bigcup_m \rho\Gamma^-(K^1_m)\Bigg) = 1 \quad \text{or} \quad \mu\Bigg(\bigcup_m \rho\Gamma^-(K^1_m)\Bigg) < 1\,.$$

In case of equality we finish the construction.

Set $\Omega_1 := \bigcup_m \rho\Gamma^-(K^1_m)$ $\Sigma_1 := \Sigma \cap \Omega_1\,; \mu_1 := \mu|\Sigma_1$ and assume that $\mu(\Omega_1) < 1$. We will use the transfinite induction. Assume that for an ordinal $\delta < \omega_1$ the following objects have already been defined:

(A) a transfinite sequence $(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha)_{\alpha<\delta}$ of measure spaces such that:

$$\forall\, \alpha < \delta\ \Omega_\alpha \subset \Omega\,, \quad \Sigma_\alpha = \Sigma \cap \Omega_\alpha\,, \quad \mu_\alpha = \mu|\Sigma_\alpha$$

$$\forall\, \alpha, \beta < \delta \text{ if } \alpha \ne \beta, \text{then } \Omega_\alpha \cap \Omega_\beta = \emptyset\,.$$

(B) For every $\alpha < \delta$ we have an increasing sequence $(K_m^\alpha)_m$ of weak*-compact and convex subsets of $B(X^*)$ with the two properties: $\Omega_\alpha = \bigcup_m \rho\Gamma^-(K_m^\alpha)$ and $\mu(\Omega_\alpha) < 1 - \sum_{\beta<\alpha} \mu(\Omega_\beta)$.

(C) For every $\alpha < \delta$ we have a sequence $(x_n^\alpha)_n \subset X$ such that $\|x_n^\alpha\| \le \|x^{**}\|$, x^{**} is weak*-continuous on every set K_m^α and $x_n^\alpha \longrightarrow x^{**}$ uniformly on every K_m^α.

If $1 = \sum_{\alpha<\delta} \mu(\Omega_\alpha)$, we finish the construction. Otherwise, we set $\kappa_\delta : \mathfrak{L} \to [0,1]$ by

$$\kappa_\delta(K) := \mu\Big\{\omega \in \Omega \setminus \bigcup_{\alpha<\delta} \Omega_\alpha \colon \rho\Gamma(\omega) \subset K\Big\}.$$

According to Proposition 5.7 there exists a regular measure $\overline{\kappa}_\delta$ on the weak*-Borel subsets of $B(X^*)$ such that $\overline{\kappa}_\delta(K) \ge \kappa_\delta(K)$ for every $K \in \mathcal{L}$. In the same way as at the beginning, we observe that $\overline{\kappa}_\delta$ is non-atomic. By Proposition 5.9 there exists a non-decreasing sequence of weak*-compact and convex sets $K_m^\delta \subset B(X^*)$ such that x^{**} is weak*-continuous on each K_m^δ and $\overline{\kappa}_\delta(\bigcup_m K_m^\delta) = \mu(\Omega \setminus \bigcup_{\alpha<\delta} \Omega_\alpha)$. Moreover, there exists a sequence $(x_n^\delta)_n \subset X$ such that $\|x_n^\delta\| \le \|x^{**}\|$ for every $n \in \mathbb{N}$ and $x_n^\delta \longrightarrow x^{**}$ uniformly on each K_m^δ.

We set $\Omega_\delta := \bigcup_m \rho\Gamma^-(K_m^\delta)$. If $\sum_{\alpha\le\delta} \mu(\Omega_\alpha) = 1$, we finish the construction, otherwise we continue as above.

Let $\delta < \omega_1$ be the first ordinal such that $\sum_{\alpha<\delta} \mu(\Omega_\alpha) = 1$. Since μ is a finite measure, such a δ exists. Now, we enumerate the set $\{(\Omega_\alpha, \Sigma_\alpha, \mu_\alpha) \colon \alpha < \delta\}$ as $(\overline{\Omega}_k, \overline{\Sigma}_k, \overline{\mu}_k)_{k\in\mathbb{N}}$. For each $k \in \mathbb{N}$ we denote by $(\overline{K}_m^k)_m$ and $(\overline{x}_n^k)_n$ the corresponding non-decreasing sequence $(K_m^\alpha)_m$ and the sequence $(x_n^\alpha)_n$. If $k \in \mathbb{N}$ is fixed, then $x_n^k \longrightarrow x^{**}$ uniformly on every $\overline{K}_m^k$ as $n \to \infty$. Consequently, if $\omega \in \rho\Gamma^-(\overline{K}_m^k)$, then $s(x_n^k, \rho\Gamma(\omega)) \longrightarrow s(x^{**}, \rho\Gamma(\omega))$. It follows that $s(x_n^k, \rho\Gamma(\omega)) \longrightarrow s(x^{**}, \rho\Gamma(\omega))$ at every point of $\overline{\Omega}_k$. In particular $s(x^{**}, \rho\Gamma)$ is a measurable and bounded function on $\overline{\Omega}_k$.

(II) But according to Lemma 5.5 $M(E \cap \rho\Gamma^-(\overline{K}_m^k)) \subset \mu(E)\overline{K}_m^k$ for every m, k and so the sequence $(x_n^k)_n$ is uniformly convergent on $M(E \cap \rho\Gamma^-(\overline{K}_m^k))$ to x^{**}. Applying (17), we have for each $E \in \Sigma$

$$s(x^{**}, M(E \cap \rho\Gamma^-(\overline{K}_m^k))) = \lim_n s(x_n^k, M(E \cap \rho\Gamma^-(\overline{K}_m^k)))$$
$$= \lim_n \int\limits_{E\cap\rho\Gamma^-(\overline{K}_m^k)} s(x_n^k, \rho\Gamma)\, d\mu = \int\limits_{E\cap\rho\Gamma^-(\overline{K}_m^k)} \lim_n s(x_n^k, \rho\Gamma)\, d\mu = \int\limits_{E\cap\rho\Gamma^-(\overline{K}_m^k)} s(x^{**}, \rho\Gamma)\, d\mu\,.$$

Since by the assumption the variation of $s(x^{**}, M)$ is finite and countably additive, also $s(x^{**}, M)$ is a finite measure. Hence, if $E \in \overline{\Sigma}_k$, then

$$\begin{aligned} s(x^{**}, M(E)) &= \lim_m s(x^{**}, M(E \cap \rho\Gamma^-(\overline{K}_m^k))) \\ &= \lim_m \int_{E\cap\rho\Gamma^-(\overline{K}_m^k)} s(x^{**}, \rho\Gamma)\, d\mu = \int_E s(x^{**}, \rho\Gamma)\, d\mu\,. \end{aligned}$$

Let now $E \in \Sigma$. Then

$$s(x^{**}, M(E)) = \sum_k s(x^{**}, M(E \cap \overline{\Omega}_k))$$
$$= \sum_k \int_{E \cap \overline{\Omega}_k} s(x^{**}, \rho\Gamma)\, d\mu = \int_E s(x^{**}, \rho\Gamma)\, d\mu\,.$$

(III) Assume now that $M\colon \Sigma \to cb(X^*)$ and set $M^*(E) := \overline{M(E)}^*$ for every $E \in \Sigma$. We apply the whole procedure from (I) to M^*. We may do it because the supermodular set function κ defined in Def. 5.6 is the same for $\rho\Gamma$ and Γ_ρ. With the same notation as above, we get $s(x_n^k, \Gamma_\rho(\omega)) \longrightarrow s(x^{**}, \Gamma_\rho(\omega))$ at every point of $\overline{\Omega}_k$, because $\Gamma_\rho(\omega) \subset \rho\Gamma(\omega)$ for every ω. In particular $s(x^{**}, \Gamma_\rho)$ is a measurable and bounded function on $\overline{\Omega}_k$.

Moreover, we have always $M(E) \subset M^*(E)$ and so it follows from (II) that for every m, k the sequence $(x_n^k)_n$ is uniformy convergent on $M(E \cap \Gamma_\rho^-(\overline{K}_m^k))$ to x^{**}. The rest of the proof imitates (II).

Let now M be arbitrary.

(IV) We can decompose Ω into countably many pairwise disjoint sets Ω_n in such a way that $|M|(E \cap \Omega_n) \le n\mu(E \cap \Omega_n)$. According to the first part of the proof, for each $n \in \mathbb{N}$ there exists a multifunction $\Gamma_n\colon \Omega_n \to cw^*k(X^*)(cb(X^*))$ such that

$$s(x^{**}, M(E \cap \Omega_n)) = \int_{E \cap \Omega_n} s(x^{**}, \Gamma_n)\, d\mu \quad \text{for every } x^{**} \in X^{**} \text{ and } E \in \Sigma.$$

Let $\Gamma = \sum_n \Gamma_n \chi_{\Omega_n}$. Since M is a multimeasure in case of $M\colon \Sigma \to cw^*k(X^*)[cb(X^*)]$ (Cor. 3.5) we have

$$\int_\Omega |s(x^{**}, \Gamma)|\, d\mu = \sum_{n=1}^{\infty} \int_{\Omega_n} |s(x^{**}, \Gamma_n)|\, d\mu$$
$$= \sum_{n=1}^{\infty} |s(x^{**}, M)|(\Omega_n) = |s(x^{**}, M)|(\Omega) < \infty\,,$$

and so the multifunction Γ is scalarly integrable. Consequently,

$$s(x^{**}, M(E)) = \sum_{n=1}^{\infty} s(x^{**}, M(E \cap \Omega_n)) = \sum_{n=1}^{\infty} \int_{E \cap \Omega_n} s(x^{**}, \Gamma)\, d\mu = \int_E s(x^{**}, \Gamma)\, d\mu\,.$$

If $M\colon \Sigma \to c(X^*)$, we argue similarly to the corresponding part of the proof of Theorem 5.1. We skip the proof in case of an atomic μ. □

Corollary 5.11. *Let $\Omega = B(X^*)$ and let Σ be the completion of the σ-algebra of weak*-Borel subsets of $B(X^*)$ with respect to a regular probability measure μ. Assume that X^* has the WRNP and $M\colon \Sigma \to cwk(B(X^*))$ $[cb(B(X^*))]$ is a μ-continuous multimeasure of σ-finite variation. Then, there exists a multifunction $\Gamma\colon \Omega \to cw^*k(X^*)$ $[cb(X^*)]$ that is Pettis integrable in $cwk(X^*)[cb(X^*)]$ and*

$$s(x^{**}, M(E)) = \int_E s(x^{**}, \Gamma)\, d\mu \quad \textit{for every } x^{**} \in X^{**} \textit{ and } E \in \Sigma.$$

6. Separable Banach spaces

In case of a separable X and multimeasures with weakly compact values I offer also another proof of Theorem 5.10. Instead of the result of Haydon [13] I apply a result of Odell and Rosenthal [24] in the proof and the whole proof is different.

Theorem 6.1. *Let X be a separable Banach space with X^* possessing the WRNP and let $M\colon \Sigma \to cwk(X^*)$ be a μ-continuous multimeasure of σ-finite variation. Then there exists a multifunction $\Gamma\colon \Omega \to cw^*k(X^*)$ that is Pettis integrable and*

$$s(x^{**}, M(E)) = \int_E s(x^{**}, \Gamma)\, d\mu = \quad \textit{for every } x^{**} \in X^{**} \textit{ and } E \in \Sigma.$$

Proof.

Claim 1. $s(x^{**}, M(E)) \le \int_E s(x^{**}, \Gamma)\, d\mu$ *for every $x^{**} \in X^{**}$ and every $E \in \Sigma$.*

Proof. Assume first that there is $\alpha > 0$ such that $|M(E)| \le \alpha\mu(E)$ for every $E \in \Sigma$. Let ρ be a lifting on $L_\infty(\mu)$. Exactly as in the proof of Theorem 5.1 we obtain the equality

$$M(E) = \left\{ (G)\int_E \rho(\gamma)\, d\mu : \gamma \text{ is a weak}^* \text{ density of } m \in \mathcal{W}^*\mathcal{S}(M) \right\}.$$

Moreover, since X^* has the WRNP, each function $\rho(\gamma)$ above is Pettis integrable.

Define $\Gamma\colon \Omega \to X^*$ by the equality

$$\Gamma(\omega) := \overline{conv\,\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{S}(\Gamma)\}}^* = \overline{\{\rho(\gamma)(\omega) : \gamma \in \mathcal{W}^*\mathcal{S}(\Gamma)\}}^*.$$

Obviously, each set $\Gamma(\omega) = \rho\Gamma(\omega)$ is bounded and $|s(x, \Gamma)| \le \alpha$ everywhere, for each $x \in B(X)$. Moreover, each x^{**} is a weak* limit of a sequence from X. Since $s(\cdot, \Gamma)$ is weak* lower semicontinuous on X^{**}, also $|s(x^{**}, \Gamma)| \le \alpha$ everywhere, for each $x^{**} \in B(X^{**})$.

Given $x^{**} \in B(X^{**})$, we have $x^{**}\rho(\gamma)(\omega) \le s(x^{**}, \rho\Gamma(\omega))$ for each $\omega \in \Omega$ and each $\gamma \in \mathcal{W}^*\mathcal{QS}_\Gamma$. Since $l_1 \not\subseteq X$, each $\rho(\gamma)$ is Pettis integrable. It follows that

$$\left\langle x^{**}, (P)\int_E \rho(\gamma)\, d\mu \right\rangle = \int_E x^{**}\rho(\gamma)\, d\mu \le \underline{\int}_E s(x^{**}, \rho\Gamma)\, d\mu$$

and so (Proposition 3.11)

$$s(x^{**}, M(E)) = \sup_\gamma \left\langle x^{**}, (P)\int_E \rho(\gamma)\, d\mu \right\rangle \le \underline{\int}_E s(x^{**}, \rho\Gamma)\, d\mu \tag{18}$$

for every $E \in \Sigma$ and every $x^{**} \in X^{**}$.

Let now M be arbitrary. Then one can decompose Ω into countably many pairwise disjoint sets Ω_n in such a way that $|M|(E\cap\Omega_n) \le n\mu(E\cap\Omega_n)$. According to the first part of the proof, for each $n \in \mathbb{N}$ there exists a multifunction $\Gamma_n\colon \Omega_n \to cw^*k(X^*)$ such that

$$s(x^{**}, M(E \cap \Omega_n)) \le \underline{\int}_{E\cap\Omega_n} s(x^{**}, \Gamma_n)\, d\mu \quad \text{for every } x^{**} \in X^{**} \text{ and } E \in \Sigma.$$

Consequently,

$$\begin{aligned} s(x^{**}, M(E)) &= \sum_{n=1}^{\infty} s(x^{**}, M(E \cap \Omega_n)) \\ &\le \sum_{n=1}^{\infty} \underline{\int}_{E\cap\Omega_n} s(x^{**}, \Gamma_n)\, d\mu \le \underline{\int}_{E} s(x^{**}, \Gamma)\, d\mu\,, \end{aligned}$$

where $\Gamma = \sum_n \Gamma_n \chi_{\Omega_n}$. □

Claim 2. $\overline{\int}_E s(x^{**}, \Gamma)\, d\mu \le s(x^{**}, M(E))$ *for every* $x^{**} \in X^{**}$ *and every* $E \in \Sigma$.

Proof. Let $x^{**} \in B(X^{**})$ be fixed. Since $l_1 \not\subseteq X$ there is a sequence of points $x_n \in B(X)$ such that $x_n \to x^{**}$ in the topology $\sigma(X^{**}, X^*)$ ([24]). Then, there is a net of points $y_\alpha \in \operatorname{conv}\{x_n : n \in \mathbb{N}\}$ such that $y_\alpha \to x^{**}$ in the Mackey topology $\tau(X^{**}, X^*)$. Since the set $\bigcup M(\Sigma)$ is weakly relatively compact, we may choose a sequence of points $y_n \in \{y_\alpha : \alpha \in \mathbb{A}\}$, where $\mathbb{A}$ is the directed set ordering the net, in such a way that $y_n \to x^{**}$ uniformly on $\bigcup M(\Sigma)$. Moreover, we may do it in such a way that $y_n \in \operatorname{conv}\{x_i : i \ge n\}\,, n \in \mathbb{N}$. The last property guarantees that $y_n \overset{\sigma(X^{**},X^*)}{\longrightarrow} x^{**}$.

By the absolute continuity assumption there exists a function $g \in L_1(\mu)$ such that $s(x^{**}, M(E)) = \int_E g\, d\mu$, for every $E \in \Sigma$. Hence,

$$\int_E g\, d\mu = s(x^{**}, M(E)) = \lim_n s(y_n, M(E)) = \lim_n \int_E s(y_n, \Gamma)\, d\mu$$

and the convergence is uniform on Σ. Consequently,

$$\lim_n \int_\Omega |s(y_n, \Gamma) - g|\, d\mu = 0\,.$$

Without loss of generality, we may assume that $s(y_n, \Gamma) \to g$ almost everywhere. Applying the weak* lower semicontinuity of each $s(\cdot, \Gamma(\omega))$, we obtain

$$s(x^{**}, \Gamma) \le \liminf_n s(y_n, \Gamma) = g \quad \text{a.e.}$$

and so
$$\overline{\int}_E s(x^{**}, \Gamma)\, d\mu \le \int_E g\, d\mu = s(x^{**}, M(E))\,.$$
□

Connecting it with Claim 1, we have

$$\overline{\int}_E s(x^{**}, \Gamma)\, d\mu \le s(x^{**}, M(E)) \le \underline{\int}_E s(x^{**}, \Gamma)\, d\mu \le \overline{\int}_E s(x^{**}, \Gamma)\, d\mu\,,$$

which yield the measurability of all support functions $s(x^{**}, \Gamma)$ and the equality

$$s(x^{**}, M(E)) = \int_E s(x^{**}, \Gamma)\, d\mu \qquad \text{for every } E \in \Sigma \text{ and every } x^{**} \in X^{**}\,.$$
□

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