Journal of Convex Analysis
Volume 28 (2021), No. 3, 903–926

Second-Order Efficiency Conditions for Vector Equilibrium Problems with Constraints via Contingent Epiderivatives

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Received: July 16, 2018
Accepted: July 28, 2020

We establish second-order necessary and sufficient conditions for weakly efficient, Henig efficient, globally efficient and superefficient solutions of vector equilibrium problems with constraints in terms of contingent epiderivatives in Banach spaces. Firstly, some results on the existence and the uniqueness of second-order contingent epiderivatives with the functions defined on infinite-dimensional spaces are established. Secondly, under suitable assumptions, second-order necessary and sufficient conditions for weakly efficient solutions of such problems are derived. As applications, we provide the second-order necessary and sufficient efficiency conditions for Henig efficient, globally efficient and superefficient solutions of constrained vector equilibrium problems via contingent epiderivatives. Some illustrative examples are given as well.

Keywords: Vector equilibrium problem, second-order efficiency conditions, second-order contingent epiderivative, efficient solutions.

2010 Mathematics Subject Classification: 90C46, 90C29, 90C48, 49J52.

1. Introduction

The vector equilibrium problem plays an important role in nonlinear analysis; vector optimization problem and vector variational inequality problem are its special cases. Necessary and sufficient conditions for solutions of vector equilibrium problems have attracted attention in recent years. There are many papers deal with second-order and higher-order optimality conditions (see, e.g., [2, 3, 8, 9, 11, 12, 18, 19, 20, 21, 24, 25, 26], and the references therein). The notions of contingent epiderivatives and hypoderivatives have provided good calculus rules for establishing necessary and sufficient optimality conditions for vector optimization problems

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ISSN 0944-6532 / $ 2.50

(see, e.g., [1, 10, 12, 13, 14, 25, 26], and the references therein). For example, Rodríguez-Marín and Sama [22] have studied existence, uniqueness and properties of contingent epiderivatives, relationships between them and their applications. Jiménez, Novo and Sama [14] established necessary and sufficient optimality conditions for strict minimizers of a vector optimization problem in terms of contingent epiderivatives and hypoderivatives of scalar functions associated with positive polar cone of ordering cone. Especially, Li, Zhu and Teo [15] proposed the concept of new generalized second-order contingent epiderivatives and discussed relationships between second-order contingent epiderivatives. On using this new second-order contingent epiderivatives, a second-order sufficient and necessary optimality condition for set-valued optimization problems is derived. It is actually a generalization of the known corresponding results in the literature (see, e.g., [1, 11, 13, 15], and the references therein). Up to now, the second-order necessary and sufficient optimality conditions for vector equilibrium problems in terms of contingent epiderivatives and hypoderivatives are still open questions.

The purpose of this paper is to investigate the existence and the uniqueness of second-order contingent epiderivatives and establish the second-order necessary and sufficient optimality conditions for efficiency of constrained vector equilibrium problems in terms of contingent epiderivatives in Banach spaces.

The remainder of the paper is organized as follows. In Section 2, after some preliminaries and notations, existence, uniqueness and some properties of second-order contingent epiderivative are derived. In Section 3, we provide a second-order necessary and sufficient condition for weakly efficient solution of vector equilibrium problem with constraints (VEPC) in terms of contingent epiderivatives. The illustrative examples for obtained results in Section 2 and Section 3 also are provided. As applications, in Section 4, we derive a second-order necessary and sufficient optimality condition for Henig efficient, globally efficient and superefficient solutions to the (VEPC).

2. Preliminaries and definitions

Now let X, Y, Z and W be real Banach spaces, C be a nonempty subset in X, $Q \subset Y$ be a convex cone, which defines a partial order on Y. Let S be a solid convex cone in Z. Given a vector bifunction $F : X \times X \to Y$ such that $F(x, x) = 0$ for all $x \in X$, and the constraints functions $g : X \to Z,\ h : X \to W$. Let us consider the following vector equilibrium problem with constraints:

$$\text{(VEPC)} \qquad \text{Find } \overline{x} \in K \text{ such that } \ F(\overline{x}, y) \notin P \ \ (\forall y \in K),$$

where P is a convex cone in Y and K is the feasible set of problem (VEPC)

$$K = \{x \in C : \ g(x) \in -S,\ h(x) = 0\}.$$

In case that $\text{int}Q \neq \emptyset$, if $P = -\text{int}Q$, the vector $\overline{x} \in K$ is called a *weakly efficient solution* to the problem (VEPC). If $P = -Q \setminus \{0\}$, then the vector $\overline{x} \in K$ is said to be an *efficient solution* to the problem (VEPC).

Let Y^* be the topological dual space of Y. Denote the *dual cone* of the cone Q by

$$Q^+ := \{\xi \in Y^* \mid \langle \xi, q\rangle \geq 0\ \forall\ q \in Q\},$$

where $\langle\, . \,, . \,\rangle$ denotes the coupling between Y^* and Y.

The *quasi-interior* of Q^+ is defined as

$$Q^\sharp := \{\xi \in Y^* \mid \langle \xi, q\rangle > 0 \;\forall\, q \in Q \setminus \{0\}\}.$$

A nonempty convex subset B of Q is called a *base* of Q, if $0 \notin \mathrm{cl}(B)$ and $Q = \mathrm{cone}(B)$, where $\mathrm{cone}\,(B) = \{tb \,|\, b \in B,\ t \geq 0\}$ is the cone hull of B. Note that $Q^\sharp \neq \emptyset$ if and only if Q has a base B. Furthermore, a cone which has a base must be pointed. Let B be a base of cone Q. We set

$$Q^\Delta(B) = \{\xi \in Q^\sharp \,|\, \exists\, t > 0 \text{ such that } \langle \xi, b\rangle \geq t \;\forall\, b \in B\}.$$

Then $Q^\Delta(B) \neq \emptyset$. If B is a base of Q, one has $0 \notin \mathrm{cl}\, B$, and hence, by the separation theorem 3.6 [5], there exists $\xi \in Y^* \setminus \{0\}$ such that

$$r := \inf\{\langle \xi, b\rangle \,:\, b \in B\} > \langle \xi, 0\rangle = 0.$$

We set

$$V_B = \Big\{y \in Y \,|\, |\langle \xi, y\rangle| < \frac{r}{2}\Big\}.$$

The notion V_B will be used throughout this paper. It is well-known that for each convex neighborhood U of the origin in Y with $U \subset V_B$, $\mathrm{cone}(U + B)$ is a pointed convex cone such that the inclusion $Q \setminus \{0\} \subset \mathrm{int}\ \mathrm{cone}(U + B)]$ holds.

The following definitions are needed in the following

Definition 2.1. ([6, 7, 16]) A vector $\overline{x} \in K$ is called a *globally efficient solution* to the VEPC iff, there exists a pointed convex cone $H \subset Y$ with $Q \setminus \{0\} \subset \mathrm{int}H$ such that

$$F(\overline{x}, K) \cap \big(-H \setminus \{0\}\big) = \emptyset.$$

Definition 2.2. ([6, 7, 16]) A vector $\overline{x} \in K$ is called a *Henig efficient solution* to the VEPC iff, there exists some absolutely convex neighborhood U of 0 with $U \subset V_B$ such that

$$\mathrm{cone}\,\big(F(\overline{x}, K)\big) \cap \big(-\mathrm{int}[\mathrm{cone}(U + B)]\big) = \emptyset.$$

Definition 2.3. ([6, 7, 16]) A vector $\overline{x} \in K$ is called a *superefficient solution* to the VEPC iff, for each neighborhood V of the origin, there exists some neighborhood U of the origin such that

$$\mathrm{cone}\,\big(F(\overline{x}, K)\big) \cap (U - Q) \subset V.$$

Remark 2.4. The relationships among the three kind of efficient solutions are as follows.

(i) For the case that $\mathrm{int}Q \neq \emptyset$, if the vector $\overline{x}$ is a globally efficient solution (or efficient solution) of problem (VEPC) then the vector $\overline{x}$ is also a weakly efficient solution of problem (VEPC). In fact, it follows from the fact $Q \setminus \{0\} \subset \mathrm{int}H$ that $\mathrm{int}Q \subset \mathrm{int}H$, and so, $F(\overline{x}, K) \cap (-\mathrm{int}Q) = \emptyset$.

(ii) (see Long et al. [16]) If $\overline{x}$ is a superefficient solution of problem (VEPC), then $\overline{x}$ is also a Henig efficient solution of problem (VEPC).

(iii) (see Gong [6, 7]) If the base B is bounded and closed, then a Henig efficient solution of problem (VEPC) is also a superefficient solution of problem (VEPC).

Additionally, $\mathrm{int}(Q^+) = Q^\Delta(B)$, where $\mathrm{int}(Q^+)$ denotes the interior of the dual cone Q^+ of Q with respect to the strong topology $\beta(Y^*, Y)$, where the sets

$$\omega = \left\{ \bigcap_{i=1}^{n} \left\{ y^* \in Y^* : \sup_{a \in A_i} |\, \langle y^*, a_i \rangle \,| < \epsilon \right\} : \begin{array}{l} A_i \, (i = 1, 2, ..., n) \text{ are bounded sub-} \\ \text{sets of } Y, \ \epsilon > 0, \ n \geq 1, \ n \text{ is integer} \end{array} \right\}$$

form a base of neighborhoods of the origin of the dual space Y^* of Y with respect to $\beta(Y^*, Y)$.

For $M \subset X$, the closure and the interior of M are denoted by $\mathrm{cl}M, \mathrm{int}M$, respectively; $\mathbb{R}$ (resp., $\mathbb{N}$) is the set of the real (natural, respectively) numbers; one writes $t_n \to 0^+$ stands for the sequence of positive numbers $(t_n)_{n \geq 1}$ with limit 0. Given a set-valued mapping $F : X \rightrightarrows Y$, the *effective domain*, the *graph* and the *epigraph* of F are defined respectively by

$$\begin{aligned} \mathrm{dom}F &= \{x \in X \mid F(x) \neq \emptyset\}, \\ \mathrm{graph}F &= \{(x, y) \in X \times Y \mid y \in F(x)\}, \\ \mathrm{epi}F &= \{(x, y) \in X \times Y \ : \ x \in \mathrm{dom}F, \ y \in F(x) + Q\}. \end{aligned}$$

For the sake of brevity, one writes $F(M) = \bigcup_{x \in M} F(x)$, and the profile mapping $F_+ : X \rightrightarrows Y$ is defined for any $x \in \mathrm{dom}F$ as

$$F_+(x) = F(x) + Q.$$

For all $\overline{x} \in C$, let us denote by $H := h^{-1}(0) = \{x \in X : h(x) = 0\}$ and the mapping

$$(F(\overline{x}, \, .\,), g) : X \to Y \times Z$$

be defined as $(F(\overline{x}, \, .\,), g)(x) = (F(\overline{x}, \, x\,), g(x))$ for all $x \in X$.

Finally, for all $\overline{x} \in K$, we denote $f(x)$ instead of $F(\overline{x}, x)$ for all $x \in X$.

To give some definitions concerning second-order contingent derivatives and epi-derivatives, the following tangent sets will be used.

Definition 2.5. ([1]) Let M be a subset of X and $\overline{x}, u \in X$.

(i) The *second-order tangent set* T^2, the *adjacent set* A^2, and *interior tangent set* IT^2 of M at $\overline{x}$ in the direction u is defined as

$$\begin{aligned} T^2(M, \overline{x}, u) &= \left\{ x \in X : \exists\, t_n \to 0^+, \exists x_n \to x \text{ s.t } \overline{x} + t_n u + \tfrac{1}{2} t_n^2 x_n \in M \ \forall\, n \right\}, \\ A^2(M, \overline{x}, u) &= \left\{ x \in X : \forall\, t_n \to 0^+, \exists x_n \to x \text{ s.t } \overline{x} + t_n u + \tfrac{1}{2} t_n^2 x_n \in M \ \forall\, n \right\}, \\ IT^2(M, \overline{x}, u) &= \left\{ x \in X : \begin{array}{l} \forall t_n \to 0^+, \ \forall x_n \to x \ \text{ s.t } \ \overline{x} + t_n u + \frac{1}{2} t_n^2 x_n \in M, \\ \text{for all sufficiently large } n \end{array} \right\}. \end{aligned}$$

(ii) $T^2(M, \overline{x}, 0)$ (resp., $A^2(M, \overline{x}, 0)$, $IT^2(M, \overline{x}, 0)$) are called the *tangent* (resp., *adjacent, interior tangent*) *cone* of M at $\overline{x}$, and are denoted by $T(M, \overline{x})$ (resp., $A(M, \overline{x})$, $IT(M, \overline{x})$).

It should be noted here that if $\overline{x} \notin \mathrm{cl}M$ then all the above tangent cones are empty, and if $u \notin T(M,\overline{x})$, then all the above second-order tangent sets are empty.

Proposition 2.6. ([8, 11, 12, 15]) *Let M be a subset of X and let $\overline{x}, u \in X$. Then*

(i) $IT^2(M,\overline{x},u) \subset A^2(M,\overline{x},u) \subset T^2(M,\overline{x},u) \subset cl\ cone[cone(M-\overline{x})-u]$;

(ii) $IT^2(M,\overline{x},u) = IT^2(intM,\overline{x},u)$, *and if* $u \in bd[cone(M-\overline{x})]$, *then* $0 \notin IT^2(M,\overline{x},u)$;

(iii) *If* $u \notin T(M,\overline{x})$, *then* $T^2(M,\overline{x},u) = \emptyset$. *Furthermore, if M is convex,* $intM \neq \emptyset$ *and* $u \in T(M,\overline{x})$, *one has the following:*

(iv) $IT(int M,\overline{x}) = int\ cone(M-\overline{x})$, *and consequently,* $0 \notin int\ cone(M-\overline{x})$ *for* $\overline{x} \notin intM$;

(v) *If* $A^2(M,\overline{x},u) \neq \emptyset$, *then* $IT^2(M,\overline{x},u) = int A^2(M,\overline{x},u)$ *and* $clIT^2(M,\overline{x},u) = A^2(M,\overline{x},u)$.

Definition 2.7. ([1, 17]) Let $f : X \to Y$ be a mapping, $\overline{x} \in X$ and $(u,v) \in X \times Y$.

(i) The *contingent derivative* of f (resp., f_+) at a point $\overline{x}$ is defined as

$$\mathrm{graph}\Big(D_c f(\overline{x})\Big) = T(\mathrm{graph}(f),(\overline{x},f(\overline{x})))$$
$$\Big(\text{resp., } \mathrm{graph}\Big(D_c(f_+)(\overline{x},f(\overline{x}))\Big) = T(\mathrm{epi}(f),(\overline{x},f(\overline{x}))\Big).$$

(ii) The *second-order contingent derivative* of f (resp., f_+) at a point $\overline{x}$ in the direction (u,v) is defined as

$$\mathrm{graph}\Big(D_c^2 f(\overline{x},f(\overline{x}),u,v)\Big) = T^2(\mathrm{graph}(f),(\overline{x},f(\overline{x})),(u,v))$$
$$(\text{resp., } \mathrm{graph}\Big(D_c^2(f_+)(\overline{x},f(\overline{x}),u,v)\Big) = T^2(\mathrm{epi}(f),(\overline{x},f(\overline{x})),(u,v))\Big).$$

(iii) The *contingent epiderivative* of f at a point $\overline{x}$ is defined as

$$\mathrm{epi}\Big(\underline{D}f(\overline{x})\Big) = T(\mathrm{epi}(f),(\overline{x},f(\overline{x}))).$$

(iv) The *second-order contingent epiderivative* of f at a point $\overline{x}$ in the direction (u,v) is defined as

$$\mathrm{epi}\Big(\underline{D}^2 f(\overline{x},f(\overline{x}),u,v)\Big) = T^2(\mathrm{epi}(f),(\overline{x},f(\overline{x})),(u,v)).$$

Definition 2.8. ([17]) Let M be a nonempty subset of Y.

(i) $y \in M$ is an *ideal minimal point* (resp., *ideal maximal point*) of M with respect to Q iff $y \preceq a$ (resp., $a \preceq y$) for any $a \in M$. The set of all ideal points is denoted by $\mathrm{IMin}(M|Q)$ (resp., $\mathrm{IMax}(M|Q)$), where $y \preceq x$ iff $x-y \in Q$ for all $x,y \in Y$.

(ii) $y \in M$ is an *efficient point* of M with respect to Q iff $a \preceq y$, for some $a \in M$, implies $y \preceq a$. The set of all efficient points of M is denoted by $\mathrm{Min}(M|Q)$.

Some characterizations on the existence of second-order contingent epiderivative can be stated as follows.

Proposition 2.9. *Let $k : X \to Y$, $\overline{x} \in X, u, v \in X \times Y$ and assume that the cone Q has a compact base B. Then for any $x \in dom(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v))$ the second-order contingent epiderivative $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)$ exists if and only if*

$$IMin(D_c^2 k(\overline{x}, k(\overline{x}), u, v)(x)|Q) \neq \emptyset$$

and
$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) = IMin\Big(D_c^2 k(\overline{x}, k(\overline{x}), u, v)(x)|Q\Big).$$

Proof. Take $x \in \mathrm{dom}(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v))$ such that $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)$ exists. Then, in a similar way as in the proof of Proposition 3.2 of [23], in which

$$\mathrm{Min}(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x)|Q) \text{ is replaced by } \mathrm{Min}(D_c(f+Q)(\overline{x})(u)|Q),$$

we deduce that
$$\mathrm{Min}(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x)|Q) \subset D_c^2 k(\overline{x}, k(\overline{x}), u, v)(x). \tag{1}$$

By virtue of the definition of second-order contingent derivatives of k and k_+ at $(\overline{x}, k(\overline{x}))$ in the direction (u, v), it follows that

$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) \in \mathrm{IMin}\Big(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x)|Q\Big). \tag{2}$$

Combining (2) with Proposition 2.2 in Luc ([17]) yields that

$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) \in \mathrm{Min}\Big(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x)|Q\Big). \tag{3}$$

It follows from (1) and (3) that

$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) \in D_c^2 k(\overline{x}, k(\overline{x}), u, v)(x). \tag{4}$$

On the other hand, it is obvious that

$$D_c^2 k(\overline{x}, k(\overline{x}), u, v)(x) \subset D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x) \subset \underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) + Q. \tag{5}$$

Making use of (4), (5) and Definition 2.8(i), one obtains the following relation

$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) \in \mathrm{IMin}\Big(D_c^2 k(\overline{x}, k(\overline{x}), u, v)(x)|Q\Big).$$

Since Q has a compact base B, the cone Q is pointed, and so the conclusion follows. The converse follows immediately and the proof is complete. $\square$

Example 2.10. Let X be a real Banach space with the norm $\|.\|_X$, and

$$Y := l^2 = \{x = (x_n)_{n\geq 1} \subset \mathbb{R} : \sum_{n=1}^{\infty} x_n^2 < +\infty\},$$

the Banach space with its usual norm $\|(x_n)_{n\geq 1}\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}.$

By $(e_n)_{n\in\mathbb{N}}$ we denote its standard unit base. Moreover, we consider in l^2 the natural ordering cone $Q := L_+^2 = \{x = (x_n)_{n\geq 1} \subset l^2 : x_1, x_2 \geq 0, x_i = 0 \ \forall\, i \geq 3\}$.

Let the mapping $k : X \to l^2$ be given as

$$k(x) = \begin{cases} \sum_{i=1}^{n} \|x\|_X e_i, & \text{if } \|x\|_X = \frac{1}{n},\ n \in \mathbb{N}, \\ \|x\|_X^2 (e_1 + \frac{1}{2} e_2), & \text{if } \|x\|_X \notin \{\frac{1}{n} : n \in \mathbb{N}\}. \end{cases}$$

For $\overline{x} = 0$, $u \in X$, $v = 0 \in l^2$, we set

$$B = \{x = (x_n)_{n \geq 1} \subset L^2_+ \,:\, x_1 + x_2 = 1\}.$$

It can be seen that B is compact convex and $0 \notin \operatorname{cl} B$. For all $x = (x_n)_{n \geq 1} \in L^2_+$, there exists $t \geq 0$ and $b \in B$ such that $x = tb$, that is, $L^2_+ = \operatorname{cone}(B)$. Indeed, if $x_1 = x_2 = 0$, we can choose $t = 0$, otherwise, we will choose $t = x_1 + x_2 > 0$ and $b = (\frac{x_1}{t}, \frac{x_2}{t}, 0, 0, \ldots) \in B$. By definition, B is a base of $Q = L^2_+$. By direct calculation one obtains for every $x \in X$.

$$\operatorname{dom}(D^2_c k_+(\overline{x}, k(\overline{x}), u, v)) = X,\ D^2_c k(\overline{x}, k(\overline{x}), u, v)(x) = \{(2\|u\|^2_X, \|u\|^2_X, 0, 0, \ldots)\}$$

Therefore, for each $x \in X$, the value of the second-order contingent epiderivative $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)$ exists if and only if

$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) + L^2_+ = D^2_c k_+(\overline{x}, k(\overline{x}), u, v) + L^2_+,$$

which is equivalent to $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) = (2\|u\|^2_X, \|u\|^2_X, 0, 0, \ldots).$ □

Proposition 2.11. *Let $k : X \to Y$, $\overline{x} \in X, u, v \in X \times Y$, and $Q \subset Y$ be pointed. Then for each $x \in dom(D^2_c k_+(\overline{x}, k(\overline{x}), u, v))$, $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)$ exists if and only if*

$$IMin(D^2_c k_+(\overline{x}, k(\overline{x}), u, v)(x) | Q) \neq \emptyset$$

and $$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) = IMin(D^2_c k_+(\overline{x}, k(\overline{x}), u, v)(x) | Q).$$

Proof. It follows from (2) and the pointedness of the cone Q. □

Proposition 2.12. *Let $k : X \to Y$, $\overline{x} \in X$, and cone $Q \subset Y$ be pointed. If the second-order contingent epiderivative of k at $\overline{x}$ in the direction $(u, v) \in X \times Y$ exists, then it is unique.*

Proof. Due to the result of Proposition 2.11, the conclusion follows. □

Proposition 2.13. *Let $k : X \to Y$, $\overline{x} \in X$. If the second-order contingent epiderivative of k at $\overline{x}$ in the direction $(u, v) \in X \times Y$ exists, then Q is closed.*

Proof. It can be easily seen that, for every $x \in \operatorname{dom}(D^2_c k_+(\overline{x}, k(\overline{x}), u, v))$,

$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) \in \operatorname{IMin}(D^2_c k_+(\overline{x}, k(\overline{x}), u, v)(x) | Q),$$

which is equivalent to

$$Q = D^2_c k_+(\overline{x}, k(\overline{x}), u, v)(x) - \{\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)\}, \tag{6}$$

as was to be shown. □

Proposition 2.13 is illustrated by the following example.

Example 2.14. Let X, Y be two real Banach spaces, $\overline{x} := 0 \in X$, let $u \in X$ be arbitrary, $v := 0 \in Y$. Fix $y^0 \in Y$ and consider the mapping $k : X \to Y$ be defined by $k(x) = y^0 \|x\|^2_X$ for every $x \in X$.

By an easy computation, one obtains

$$D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x) = 2\|u\|_X^2 y^0 + \mathrm{cl}\, Q \;\; \forall\, x \in X.$$

Hence, for each $x \in X$, the second-order contingent epiderivative $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)$ exists if and only if

$$\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) + Q = D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x),$$

and so, $Q = 2\|u\|_X^2 y^0 - \underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x) + \mathrm{cl}\, Q$.

This helps us to conclude that Q is closed, as was to be shown. □

Proposition 2.15. *Let $k : X \to Y$, $\overline{x} \in X, u, v \in X \times Y$. Assume that Q is pointed and there exists $y \in Y$ such that $D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x) \subset y + Q$ for all $x \in L := dom(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v))$. Let Ψ be a function from L to Y defined by*

$$\Psi(x) = IMax(\{y \in Y \,:\, D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x) \subset y + Q\} | \, Q) \quad (x \in L).$$

Then the following conditions are equivalent:

(i) $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)$ *exists for every $x \in L$;*

(ii) $\Psi(x) \in D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x)$ *for every $x \in L$.*

Proof. Arguing similarly as in the proof of Theorem 5.6 [21, p. 755] with noticing that the vector x is replaced by u, the second-order contingent derivative

$$D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)(x) \;\text{ is replaced by }\; D_c F_+(\overline{x}, \overline{y})(u),$$

and $\quad \mathrm{dom}(D_c^2 k_+(\overline{x}, k(\overline{x}), u, v)) \;$ is replaced by $\; \pi_X T(\mathrm{epi} F, (\overline{x}, \overline{y}))$,

the desired conclusion follows. □

Example 2.16. Let the spaces X, Y and the vectors $\overline{x}, u, v, k$ be given as in Example 2.10. Consider the cone

$$Q := l_+^2 = \{x = (x_n)_{n \geq 1} : x_n \geq 0 \;\forall\, n \geq 1\}.$$

It is obvious that Q is pointed. We take $y^0 := (2\|u\|_X^2, \|u\|_X^2, 0, 0, \ldots) \in l^2$. Taking into account the obtained result in Example 2.10, we conclude that $L = X$ and

$$D_c^2 k_+(\overline{x}, k(\overline{x}, u, v))(x) = y^0 + l_+^2, \forall x \in L.$$

Making use of the result of Proposition 2.11 to deduce that $\underline{D}^2 k(\overline{x}, k(\overline{x}), u, v)(x)$ exists for all $x \in L$. On the other hand, it follows from Corollary 5.3 [22] that $\Psi(x) = y^0$ for all $x \in L$. Hence, condition (ii) is satified. □

Remark 2.17. If cone Q has a compact base B, then by taking account of Proposition 2.9, it follows that Proposition 2.15 is still hold where the profile mapping k_+ is replaced by k.

3. Second-order conditions for efficiency

In this section, we derive second-order necessary and sufficient conditions for weakly efficient solutions of the problem (VEPC) in terms of contingent epiderivatives. From now on, if not otherwise stated, for each $\overline{x} \in K$, we always assume that there exist $u \in IT(C, \overline{x}) \cap IT(H, \overline{x})$ and $(v, w) \in D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) \cap (-Q) \times (-S)$ such that the following conditions hold:

(A) $\underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, (v, w))(x)$ exists for all $x \in \mathrm{dom}\big(D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, v, w)\big)$;

(B) $(f, g)(K) \subset \underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) + Q \times S$ for all $x \in M$;

(C) The following regularity condition holds:

$$\Big\{z \in Z \,:\, (y, z) \in \mathrm{cone}\big((f, g)(K)\big)\Big\} + \mathrm{cone}(S + w) = Z.$$

It is worth mentioning here that when we study a nonsmooth vector equilibrium problem using the tool of second-order contingent epiderivatives, the existence of these epiderivatives must be guaranteed, thus condition (A) plays a key role for establishing optimality conditions for the efficient solution types of problem (VEPC) via the second-order contingent epiderivatives, while the condition (B) helps us to construct both necessary and sufficient second-order optimality conditions, it means that the assumption (B) is used to replace the cone-convexity of objective and constraint functions and so these condition cannot be deleted in all the statements. In fact, as we know in the vector optimization theory that if all the objective and constraint functions of vector equilibrium problems are cone-convex on a convex set C and if this condition is removed then the necessary optimality condition cannot be the sufficient optimality condition (see Gong [6, 7] for more details). Finally, the regularity condition (C) is used for obtaining the Kuhn-Tucker type necessary second-order optimality conditions from the Fritz John type necessary second-order optimality conditions, it means that for a pair $(\lambda, \eta) \in (Y \times Z)^*$, not both zero, under the regularity condition (C), it ensures that $\lambda \neq 0$. Notice that the regularity condition (C) can be viewed as one of the constraint qualifications in the vector optimization theory, and moreover, the assumptions (A)-(C) are not coincide with the existing one in the literature (see, e.g., [1, 3, 8, 9, 10, 11, 12, 15, 19, 22, 23, 24, 25, 26] and the references therein).

The following example illustrates the preceding hypothesis mentioned.

Example 3.1. Let $X = Y = Z = W = l^2$ be the space of sequences $(x_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ for which $\sum_{n=1}^{\infty} x_n^2$ is finite with the norm $\|(x_n)\| = \big(\sum_{n=1}^{\infty} x_n^2\big)^{\frac{1}{2}}$. By $(e_n)_{n\in\mathbb{N}}$ we denote its standard unit basis. Let C be a closed unit ball in the space l^2, $\overline{x} = 0 \in C$, the natural ordering cones on l^2 be

$$Q := l^2_+ = \{(x_n)_{n\in\mathbb{N}} \subset l^2 \,:\, x_n \geq 0 \;\forall\, n \geq 1\},$$

$$S := l^2_- = \{(x_n)_{n\in\mathbb{N}} \subset l^2 \,:\, x_n \leq 0 \;\forall\, n \geq 1\}.$$

Fixing $u_0 \in l^2 \setminus \{0\}$ with

$$\|u_0\| \geq \frac{\sqrt{2}}{2},\; i_0 \in \mathbb{N},\; x^0 = (x_n^0)_{n\in\mathbb{N}} \subset l^2,\; 0 < x_{i_0}^0 \leq x_i^0 \leq 2\|u_0\|^2 \;(\forall\, i \geq 1),$$

the mappings $f, g, h : l^2 \to l^2$ are defined respectively by

$$f(x) = \begin{cases} \sum_{i=1}^{n} \|x\|\, e_i, & \text{if } \|x\| = \frac{1}{n},\ n \in \mathbb{N}, \\ -\|x\|^2\, e_{i_0}, & \text{if } \|x\| \notin \{\frac{1}{n},\ n \in \mathbb{N}\}, \end{cases}$$

$$g(x) = \begin{cases} \sum_{i=1}^{\infty} \|x\|^2\, x_{i_0}^0\, x_i e_i, & \text{if } \|x\| = \frac{1}{n},\ n \in \mathbb{N}, \\ \sum_{i=1}^{\infty} \|x\|^2\, x_i^0 e_i, & \text{if } \|x\| \notin \{\frac{1}{n},\ n \in \mathbb{N}\}, \end{cases}$$

$$h(x) = 0 \text{ for all } x = (x_n)_{n \geq 1} \in l^2.$$

Then the feasible set of problem (VEPC) is of the form $K = K_1 \cup K_2$, where

$$\begin{aligned} I_n &= \{\tfrac{1}{n} : n \in \mathbb{N}\}, \\ K_1 &= \{x = (x_n)_{n\geq 1} \in C \ :\ x_n \geq 0\, (\forall\, n \in \mathbb{N}),\ \ \|x\| \in I_n\}, \\ K_2 &= \{x = (x_n)_{n\geq 1} \in C \ :\ \|x\| \notin I_n\}. \end{aligned}$$

By direct calculation we get $IT(C, 0) \cap IT(H, 0) = l^2$, and for all $u \in l^2$

$$D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) = \{(v, w) \in l^2 \times l^2 \ :\ v \in Q,\ w \in S\}.$$

Thus for any $u \in l^2$, we obtain the following result

$$D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) \cap (-Q) \times (-S) = \{(0, 0)\}.$$

Taking $u = u_0$, $v = 0 \in l^2$, $w = 0 \in l^2$. For all $x \in M = l^2$, it results that

$$\operatorname{dom}(D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, v, w)) = l^2,$$

$$D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) = \\ = \left\{ ((y_n)_n, (z_n)_n) : \begin{array}{l} y_{i_0} \geq -2\|u_0\|^2,\ y_i \geq 0\ (\forall\, i \neq i_0,\ i \geq 1), \\ z_i \leq 2\|u_0\|^2 x_i^0,\ \ \forall\, i \geq 1 \end{array} \right\},$$

$$(a_x, b_x) = ((0, 0, \ldots, 0, -2\|u_0\|^2, 0, \ldots),\ (2\|u_0\|^2 x_1^0, 2\|u_0\|^2 x_2^0, \ldots, 2\|u_0\|^2 x_{i_0}^0, \ldots)),$$

which means that the second-order contingent epiderivative of (f, g) at $(\overline{x}, u, (v, w))$ exists. We see that Assumption (B) be valid. In fact, for all $x \in M = l^2$, we have

$$\underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) + Q \times S = \{(a_x + q, b_x - s) \ :\ (q, s) \in l_+^2 \times l_+^2\}.$$

For all $(y, z) \in (f, g)(K)$, there exists $x = (x_i)_{i\geq 1} \in K$ such that $(y, z) = (f(x), g(x))$. If $\|x\| = \frac{1}{n}$ and $x_i \geq 0\ \forall\, i \geq 1$, then $(y, z) = (\sum_{i=1}^{n} \frac{1}{n} e_i, \sum_{i=1}^{\infty} \frac{1}{n^2} x_{i_0}^0 x_i e_i)$. Obviously,

$$\frac{1}{n^2} x_i x_{i_0}^0 \leq x_{i_0}^0 x_i \leq \|x\| x_{i_0}^0 \leq x_{i_0}^0 \leq 2\|u_0\|^2 x_i^0\ \ (\forall\, i \geq 1),$$

which yields that $(y, z) \in \underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) + Q \times S$.

If $\|x\| \notin \{\frac{1}{n} : n \in \mathbb{N}\}$, then $(y, z) = (-\|x\|^2 e_{i_0},\ \|x\|^2 x_{i_0}^0 e_{i_0}))$.

Evidently we have $\|x\|^2 \leq 2\|u_0\|^2$ and $\|x\|^2 x_{i_0}^0 \leq 2\|u_0\|^2 x_{i_0}^0$ $(\forall\, i \geq 1)$.

This helps us to conclude that

$$(y,z) \in \underline{D}^2(f,g)(\overline{x},(f,g)(\overline{x}),u,v,w)(x) + Q \times S,$$

such that Assumption (B) holds. Finally, we check that

$$\big\{z \in l^2 \,:\, (y,z) \in \operatorname{cone}\big((f,g)(K)\big)\big\} + \operatorname{cone}(l^2_-) = l^2.$$

From the fact $\operatorname{cone}(l^2_-) = l^2_- \subset l^2$ it follows that

$$\big\{z \in l^2 \,:\, (y,z) \in \operatorname{cone}\big((f,g)(K)\big)\big\} + \operatorname{cone}(l^2_-) \subset l^2.$$

Take an arbitrary $x = (x_n)_{n\geq 1} \in l^2$; then we have $x = \sum_{i=1}^{\infty} x_i e_i$. We set

$$I_+ = \{i \in \mathbb{N} : x_i > 0\} \text{ and } I_- = \{i \in \mathbb{N} \,:\, x_i \leq 0\},$$

and define
$$x = E + F := \sum_{i \in I_+} x_i e_i + \sum_{i \in I_-} x_i e_i.$$

Evidently, $F \in l^2_-$ and $E \in \{z \in l^2 \,:\, (y,z) \in \operatorname{cone}((f,g)(K))\}$, which proves the assumptions (A)–(C) hold. □

Until now, for the sake of brevity in the statements we set

$$M := \operatorname{dom}\big(D^2_c(f_+,g_+)(\overline{x},(f,g)(\overline{x}),u,v,w)\big),$$

and
$$(a_x,b_x) := \underline{D}^2(f,g)(\overline{x},(f,g)(\overline{x}),u,v,w)(x).$$

A necessary and sufficient optimality condition for weak efficiency to VEPC can be stated as follows.

Theorem 3.2. (Second-order necessary and sufficient optimality condition) *Let $\overline{x}$ be a feasible point of problem* (VEPC) *and assume that* $\operatorname{int}Q \neq \emptyset$. *Then, $\overline{x}$ is a weakly efficient solution to the problem* (VEPC) *if and only if for every $x \in M$, there exist $(\lambda,\eta) \in (Y^* \times Z^*) \setminus \{(0,0)\}$ satisfying*

$$\lambda \in Q^+ \setminus \{0\},\ \eta \in S^+ \ \textit{with}\ \langle \eta, w\rangle = 0; \tag{7}$$

$$\langle \lambda, f(\widetilde{x})\rangle + \langle \eta, g(\widetilde{x}))\rangle \geq \langle \lambda, a_x\rangle + \langle \eta, b_x\rangle \geq 0 \quad \forall \widetilde{x} \in K. \tag{8}$$

Proof. Let us take and fix a direction $u \in IT(C,\overline{x}) \cap IT(H,\overline{x})$ and a direction $(v,w) \in D_c(f_+,g_+)(\overline{x},(f,g)(\overline{x}))(u) \cap (-Q) \times (-S)$, which satisfy Assumptions (A), (B) and (C). It is plain that $u \in IT(C \cap H,\overline{x})$. We first suppose that $\overline{x} \in K$ is a weakly efficient solution of problem (VEPC). Let us show that

$$\underline{D}^2(f,g)(\overline{x},(f,g)(\overline{x}),u,v,w)(x) \notin (-\operatorname{int}Q) \times IT(-S,w),\ \ \forall x \in M. \tag{9}$$

In fact, if it was not so, there exist $x \in M$ such that $a_x \in -\operatorname{int}Q$ and $b_x \in IT(-S,w)$. Making use of Definition 2.2, we get $(x,a_x,b_x) \in T^2(\operatorname{epi}(f,g),(f,g)(\overline{x}),u,(v,w))$, because $0 \in Q \times S$ and $(x,a_x,b_x) \in \operatorname{epi}(\underline{D}^2(f,g)(\overline{x},(f,g)(\overline{x}),u,v,w))$. By virtue of the definition of second-order contingent set of (f,g) at $(\overline{x},u,(v,w))$, one finds sequences $t_n \to 0^+$ and $(x_n,a_n,b_n) \to (x,a_x,b_x)$ such that

$$(\overline{x},(f,g)(\overline{x})) + t_n(u,(v,w)) + \tfrac{1}{2}t_n^2(x_n,a_n,b_n) \in \operatorname{epi}(f,g) \quad \forall n \in N,$$

which yields $$f(\overline{x}+t_nu+\tfrac{1}{2}t_n^2x_n)\in f(\overline{x})+t_nv+\tfrac{1}{2}t_n^2a_n-Q, \tag{10}$$

and $$g(\overline{x}+t_nu+\tfrac{1}{2}t_n^2x_n)\in g(\overline{x})+t_nw+\tfrac{1}{2}t_n^2b_n-S. \tag{11}$$

Since $b_x\in IT(-S,w)$, for the preceding sequences t_n and b_n, there exists $N_1\in\mathbb{N}$ such that $w+\frac{1}{2}t_nb_n\in -S$ for all $n\geq N_1$, and so, $t_nw+\frac{1}{2}t_n^2b_n+g(\overline{x})-S\subset -S$ for all $n\geq N_1$. Consequently,

$$g(\overline{x}+t_nu+\tfrac{1}{2}t_n^2x_n)\in -S \quad \forall\, n\geq N_1. \tag{12}$$

Observing that $u+\frac{1}{2}t_nx_n\to u\in IT(C\cap H,\overline{x})$. Then for the preceding sequence t_n, there exists $N_2\in\mathbb{N}$ such that $\overline{x}+t_nu+\frac{1}{2}t_n^2x_n\in C\cap H$ for all $n\geq N_2$, which together with (12) implies that

$$\overline{x}+t_nu+\tfrac{1}{2}t_n^2x_n\in K \quad \forall\, n\geq \max\{N_1,N_2\}. \tag{13}$$

On the other hand, we have $Q+\mathrm{int}Q=\mathrm{int}Q$, $f(\overline{x})=0$, $v\in -Q$, $a_n\to a_x\in -\mathrm{int}Q$. Hence, there exists $N_3\in\mathbb{N}$ such that

$$f(\overline{x}+t_nu+\tfrac{1}{2}t_n^2x_n)\in -\mathrm{int}Q \quad (\forall\, n\geq \max\{N_1,N_2,N_3\}). \tag{14}$$

Combining (13) and (14) yields that $\overline{x}$ is not a weakly efficient solution of problem (VEPC), this is a contradiction. Applying a separation theorem of disjoint convex sets ([5], Theorem 3.6), one can find $(\lambda,\eta)\in(Y^*\times Z^*)\setminus\{(0,0)\}$ such that

$$\langle\lambda,a_x\rangle+\langle\lambda,b_x\rangle\geq\langle\lambda,p\rangle+\langle\eta,q\rangle \quad \forall\, p\in -\mathrm{int}Q,\ \forall\, q\in IT(-S,w).$$

Consequently, $\langle\lambda,a_x\rangle+\langle\lambda,b_x\rangle\geq 0,\ \lambda\in Q^+,\ \eta\in S^+,\ <\eta,w>=0$. Making use of Assumptions (A) and (B), we have $(f,g)(\widetilde{x})-(a_x,b_x)\in Q\times S$, $\forall\widetilde{x}\in K$. Therefore,

$$\big\langle\lambda,f(\widetilde{x})\big\rangle+\big\langle\eta,g(\widetilde{x})\big\rangle\geq\langle\lambda,a_x\rangle+\langle\eta,b_x\rangle\geq 0, \quad \forall\widetilde{x}\in K.$$

Let us see that $\lambda\neq 0$. In fact, if it was not so, then $\eta\neq 0$. It can be easily seen that $\eta\in[\mathrm{cone}(S+w)]^+$. By Assumption (C), for each $z_o\in Z$ there exist $s_w\in\mathrm{cone}(S+w)$, $x_o\in K$, $t\geq 0$, $(y,z)=t\big(f(x_o),g(x_o)\big)$ such that $z_o=s_w+z$. In other words, it follows from Assumption (B) that $z_o\in s_w+tb_x+S$, and hence, $\langle\eta,z_o\rangle\geq\langle\eta,s_w\rangle+t\,\langle\eta,b_x\rangle\geq 0$. Since $z_o\in Z$ is taken arbitrarily, it follows that $\eta=0$, this is a contradiction. Thus conditions (7) and (8) hold.

Conversely, we fix $\overline{x}\in K$ and suppose that for all $x\in K$, there exist $(\lambda,\eta)\in(Y^*\times Z^*)\setminus\{(0,0)\}$ satisfying conditions (7) and (8). We claim that $\overline{x}$ is a weakly efficient solution of (VEPC). Indeed, if it was not so, there exist $x_0\in K$ such that $F(\overline{x},x_0)\in -\mathrm{int}Q$. Since $\eta\in S^+$ and $g(x_0)\in -S$, it holds that $\langle\eta,g(x_0)\rangle\leq 0$. This along with the fact $\lambda\in Q^+\setminus\{0\}$ yields that $\langle\lambda,f(x_0)\rangle<0$. So it leads to

$$\langle\lambda,f(x_0)\rangle+\langle\eta,g(x_0)\rangle<0,$$

which conflicts with the initial assumptions, we get the desired conclusion. □

Remark 3.3. The result obtained in Theorem 3.2 is new and these has not been considered before. The novelty relies in the assumptions (B) and (C), where the asumption (B) is used to replace the generalized convexity of the mapping (f, g) on a convex set C, the assumption (C) is known as the new constraint qualification which these was not considered in the literature, e.g., in [1]–[26] and the references therein. The Assumptions (A) and (C) cannot be dropped because Assumption (A) guarantees the existence result of second-order contingent epiderivatives and Assumption (C) corresponds to the constraint qualification. Assumption (B) is applied to replace the $(Q \times S)$-convexity of (f, g) on a convex set C and if this assumption is deleted then the second-order sufficient optimality condition may not be true (this result is well-known in any convex optimization literature). Additionally, note that problem (VEPC) studied in our paper is the general case, where we have as feasible set

$$K = \{x \in C \, : \, g(x) \in -S, \ h(x) = 0\}.$$

Because of the direction $u \in IT(H, \overline{x})$, our result obtained here is not depend on h. In the case $(u, v, w) = (0, 0, 0)$, it is not hard to see that $\mathrm{cone}(S + w) = S$ and

$$D_c^2(f, g)(\overline{x}, (f, g)(\overline{x}, u, v, w)(x) = D_c(f, g)(\overline{x})(x)$$

for all $x \in \mathrm{dom}(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x})))$. Consequently, $(a_x, b_x) = \underline{D}(f, g)(\overline{x})(x)$, and so, we immediately obtain the following first-order necessary and sufficient optimality condition:

Corollary 3.4. (First-order necessary and sufficient optimality condition) *Let $\overline{x}$ be a feasible point of problem (VEPC) and assume that $intQ \neq \emptyset$. Then, $\overline{x}$ is a weakly efficient solution to problem (VEPC) if and only if $\forall\, x \in dom(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x})))$, there exist $(\lambda, \eta) \in (Y^* \times Z^*) \setminus \{(0, 0)\}$ satisfying*

$$\lambda \in Q^+ \setminus \{0\}, \ \eta \in S^+ \ \ \textit{with} \ \ \langle \eta, w \rangle = 0; \tag{15}$$

$$\langle \lambda, f(\widetilde{x}) \rangle + \langle \eta, g(\widetilde{x}) \rangle \geq \langle \lambda, a_x \rangle + \langle \eta, b_x \rangle \geq 0 \ \ \forall \widetilde{x} \in K, \tag{16}$$

where $(a_x, b_x) = \underline{D}(f, g)(\overline{x})(x)$.

We mention that according to the well-known result in Jiménez and Novo [13], if Y and Z are finite-dimensional, and f (or g) is Hadamard differentiable at $\overline{x}$, then

$$D_c(f, g)(\overline{x})(x) = D_c f(\overline{x})(x) \times D_c g(\overline{x})(x) \ \ \forall x \in \mathrm{dom}(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))),$$

which leads to $a_x = \underline{D}f(\overline{x})(x)$ and $b_x = \underline{D}g(\overline{x})(x)$.
In this case, $\mathrm{dom}(D_c(f, g)(\overline{x})) = X$, and this along with the inclusion

$$\mathrm{dom}\Big(D_c(f, g)(\overline{x})\Big) \subset \mathrm{dom}\Big(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))\Big)$$

yields that

$$\mathrm{dom}\Big(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))\Big) = X.$$

Therefore, the statement of Corollary 3.4 is still hold if we replace the initial assumption $\forall\, x \in \mathrm{dom}(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x})))$ by $\forall\, x \in X$.

Arguing similarly as in the proof of Corollary 3.4, we obtain the following statement.

Corollary 3.5. (First-order necessary and sufficient optimality condition) *Let $\overline{x}$ be a feasible point of problem* (VEPC) *and assume that Y and Z are finite-dimensional spaces, f (or g) be Hadamard differentiable at $\overline{x}$ and $intQ \neq \emptyset$. Then, $\overline{x}$ is a weakly efficient solution to problem* (VEPC) *if and only if for every $x \in X$, there exist $(\lambda, \eta) \in (Y^* \times Z^*) \setminus \{(0,0)\}$ satisfying*

$$\lambda \in Q^+ \setminus \{0\},\ \eta \in S^+ \ \textit{with}\ \ \langle \eta, w\rangle = 0;\ \textit{and} \tag{17}$$

$$\langle \lambda, f(\widetilde{x})\rangle + \langle \eta, g(\widetilde{x}))\rangle \geq \langle \lambda, \underline{D}f(\overline{x})(x)\rangle + \langle \eta, \underline{D}g(\overline{x})(x)\rangle \geq 0 \ \ \forall \widetilde{x} \in K. \tag{18}$$

Theorem 3.2 is illustrated by the following example.

Example 3.6. We consider the problem (VEPC) is given as in Example 3.1. It can be seen that $Q^+ = l^2_+$ and $S^+ = l^2_-$. Taking $u = u_0, v = 0, w = 0$. For every $x \in M = l^2$ we choose $\lambda = (\lambda_i)_{i\geq 1} \in Q^+ \setminus \{0\}$, with $\lambda_{i_0} = 0$, $\lambda_i > 0$ $(\forall\, i \neq i_0)$, and $\eta = 0 \in S^+$. Then the relation (7) is obviously satisfied. The second-order contingent epiderivative of (f, g) at $(0,(0,0),u,(0,0))$ is (a_x, b_x), with (a_x, b_x) is taken as in Example 3.1. Therefore, $\langle \lambda, a_x\rangle + \langle \eta, b_x\rangle = 0$.

For any $\widetilde{x} = (\widetilde{x}_i)_{i\geq 1} \in K$, two cases can occur as follows:

Case 1. $\widetilde{x}_i \geq 0$ for all $i \geq 1$ and $\|\widetilde{x}\| = \frac{1}{n}$ for some $n \in \mathbb{N}$, one gets $f(\widetilde{x}) = \sum_{i=1}^{n} \|\widetilde{x}\| e_i$. Thus

$$\left\langle \lambda, f(\widetilde{x})\right\rangle + \left\langle \eta, g(\widetilde{x})\right\rangle = \sum_{i=1}^{n} \frac{1}{n}\lambda_i \geq 0.$$

Case 2. $\|\widetilde{x}\| \notin \{\frac{1}{n} : n \geq 1\}$, one has $f(\widetilde{x}) = -\|\widetilde{x}\|^2 e_{i_0}$, which implies that

$$\left\langle \lambda, f(\widetilde{x})\right\rangle + \left\langle \eta, g(\widetilde{x})\right\rangle = 0.$$

Hence, (8) holds. Using the result of Theorem 3.2 to ensure that $\overline{x} = 0$ being a weakly efficient solution to the (VEPC). □

In the case $h = 0$, we state the second-order sufficient optimality conditions for weakly efficient solutions of (VEPC) as follows.

Theorem 3.7. *Let us consider the problem* (VEPC) *with $h = 0$ and $\overline{x} \in K$. Suppose that for every $x \in M$, there exist $u \in IT(C, \overline{x})$ and*

$$(v, w) \in D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) \cap (-Q) \times (-S)$$

satisfying Assumption (A) *and the following two conditions:*

(i) $\underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) \notin (-intQ) \times IT(-S, w)$;

(ii) *For every $y \in K$, there exist $e_1 \in Q$, $e_2 \in IT(S, -w)$ such that*

$$\underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) \in IMin((f, g)(y) - (e_1, e_2) + P \,|\, Q \times S),$$

where the cone P in the product space $Y \times Z$, either $Q \times S$, or $-(Q \times S)$.

Then $\overline{x}$ is a weakly efficient solution to the problem (VEPC).

Proof. Assume the contrary, namely that $\overline{x}$ is not a weakly efficient solution to the problem(VEPC), which means that there exists $\widetilde{y} \in C$ such that $g(\widetilde{y}) \in -S$ and $f(\widetilde{y}) \in -\mathrm{int}Q$.

Taking $x \in \mathrm{dom}(D_c^2(f_+, g_+)(\overline{x}, (f,g)(\overline{x}), u, v, w))$, by the initial assumptions, we get that there exists $u \in IT(C, \overline{x})$ and $(v, w) \in D_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}))(u) \cap (-Q) \times (-S)$ such that

$$\underline{D}^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, v, w)(x) \notin (-\mathrm{int}\, Q) \times IT(-S, w). \tag{19}$$

Making use of Assumption A, we get

$$(a_x, b_x) \notin (-\mathrm{int}\, Q) \times IT(-S, w). \tag{20}$$

On the one hand, it follows from the hypothesis (ii) that there exist $e_1 \in Q$ and $e_2 \in IT(S, -w)$ such that

$$(a_x, b_x) \in \mathrm{IMin}\Big((f,g)(\widetilde{y}) - (e_1, e_2) \pm Q \times S \,|\, Q \times S\Big),$$

which is equivalent to

$$(a_x, b_x) \in (f,g)(\widetilde{y}) - (e_1, e_2) \pm Q \times S \subset (a_x, b_x) + Q \times S.$$

On the other hand, by directly calculating one obtains

$$(f,g)(\widetilde{y}) - (e_1, e_2) \in (a_x, b_x) + Q \times S,$$

which is equivalent to $a_x \in f(\widetilde{y}) - e_1 - Q,\ b_x \in g(\widetilde{y}) - e_2 - S.$

Since $Q + \mathrm{int}Q = \mathrm{int}Q,\ f(\widetilde{y}) \in -\mathrm{int}Q$, it follows that $a_x \in -\mathrm{int}Q$. Observe that

$$IT(S, w) = \mathrm{intcone}\,(S + w) \quad g(\widetilde{y}) \in -S \subset -S - w \subset -\mathrm{cone}\,(S + w)$$

and $\mathrm{cone}\,(S + w) + \mathrm{int}\,\mathrm{cone}(S + w) = \mathrm{int}\,\mathrm{cone}\,(S + w) = IT(S, -w)$, one obtains

$$b_x \in -IT(S, -w) = IT(-S, w).$$

Consequently, $(a_x, b_x) \in (-\mathrm{int}Q) \times IT(-S, w)$, which contradicts condition (20). Thus the vector $\overline{x} \in K$ being a weakly efficient solution to the problem (VEPC), the conclusion follows □

Example 3.8. Let X be a real Banach space with a norm $\|\,.\,\|_X$ and $Y = Z = l^2$ with l^2 be given as in Example 3.1. We take

$$Q = l^2_+,\ S = l^2_-,\ C = \{x \in X \,:\, \|x\|_X \le 1\} \ \text{ and } \ \overline{x} = 0 \in C.$$

and fix $u_0 \in X \setminus \{0\}$ with $\|u_0\|_X > \frac{\sqrt{2}}{2}$, $x^0 = (x^0_n)_{n\in\mathbb{N}} \subset l^2$, $0 < x^0_n < 1$ ($\forall\, n \ge 1,\ n \in \mathbb{N}$). We now define the mappings $f, g : X \to l^2$ respectively as

$$f(x) = \begin{cases} \sum_{i=1}^{n} \|x\|_X\, e_i, & \text{if } \|x\|_X = \frac{1}{n},\ n \in \mathbb{N}, \\ 0, & \text{if } \|x\|_X \notin \{\frac{1}{n},\ n \in \mathbb{N}\}. \end{cases}$$

$$g(x) = \begin{cases} \sum_{i=1}^{\infty} \|x\|_X\, x^0_i\, e_i, & \text{if } \|x\|_X = \frac{1}{n},\ n \in \mathbb{N}, \\ \sum_{i=1}^{\infty} \|x\|_X^2\, x^0_i\, e_i, & \text{if } \|x\|_X \notin \{\frac{1}{n},\ n \in \mathbb{N}\}. \end{cases}$$

Then, it is easy to see that $K = C$, $IT(C, \overline{x}) = X$, $D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) = Q \times S$ for every $u \in X$. Hence, $D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) \cap (-Q) \times (-S) = \{(0, 0)\}$ for all $u \in X$. Taking $u = u_0 \in X$, $v = 0 \in l^2$, $w = 0 \in l^2$, we obtain by direct calculation

$$\mathrm{dom}(D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, v, w)) = X.$$

Moreover, for every $x \in M$, we have

$$D_c^2(f_+, g_+)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) = \left\{ ((y_n)_n, (z_n)_n) : \begin{array}{l} \text{for all } i \in \mathbb{N},\ y_i \geq 0 \text{ and} \\ z_i \leq 2\|u_0\|_X^2 x_i^0 \end{array} \right\},$$

which yields that $(a_x, b_x) = ((0, 0, \ldots), (2\|u_0\|_X^2 x_1^0, 2\|u_0\|_X^2 x_2^0, \ldots))$ for any $x \in M$. This means that the second-order contingent epiderivative of (f, g) at $(0, u_0, (0, 0))$ exists. From this result we will be allowed to conclude that Assumption (i) is obviously fulfilled due to $a_x = (0, 0, \ldots) \notin -\mathrm{int} l_+^2$. Finally, for all $x \in K$, three cases can be happen as follows:

Case 1. If $\|x\|_X = 0$, we may take $(e_1, e_2) = (-a_x, -b_x)$, and then one obtains $e_1 \in l_+^2$ and $e_2 \in IT(l_-^2, 0) = \mathrm{int} l_-^2$. Moreover,

$$\mathrm{IMin}\big(((f, g)(0) - (e_1, e_2) + l_+^2 \times l_-^2) \bigm| l_+^2 \times l_-^2\big) = \{(a_x, b_x)\}.$$

Case 2. If $\|x\|_X = \frac{1}{n}$ for some $n \in \mathbb{N}$, we choose $(e_1, e_2) = (f(x) - a_x, g(x) - b_x)$. Consequently, $(e_1, e_2) \in l_+^2 \times \mathrm{int}\, l_-^2$ because

$$(e_1, e_2) = \Big(\textstyle\sum_{i=1}^n \frac{1}{n} e_i,\ \sum_{i=1}^\infty (\frac{1}{n} - 2\|u_0\|_X^2) x_i^0 e_i \Big) \text{ and } \frac{1}{n} - 2\|u_0\|_X^2 \leq 1 - 2\|u_0\|_X^2 < 0.$$

It is not difficult to check that

$$\mathrm{IMin}\big(((f, g)(x) - (e_1, e_2) + l_+^2 \times l_-^2) \big| l_+^2 \times l_-^2\big) = \{(a_x, b_x)\}.$$

Case 3. If $\|x\|_X \notin \bigcup_{n \geq 1} \{\frac{1}{n}\} \bigcup \{0\}$, by taking a vector pair (e_1, e_2) as in Case 2, we also obtain the desired conclusion. It is worth noting here that $e_1 = 0 \in l_+^2$ and

$$e_2 = \sum_{i=1}^\infty (\|x\|_X^2 - 2\|u_0\|_X^2) x_i^0 e_i \in IT(S, 0) = \mathrm{int}\, l_-^2$$

is due to $\|x\|_X^2 - 2\|u_0\|_X^2 \leq 1 - (\sqrt{2}\|u_0\|)^2 < 0$. □

Applying Theorem 3.7, we deduce that $\overline{x}$ is a weakly efficient solution of (VEPC).

Corollary 3.9. *Let us consider the problem* (VEPC) *in the sense $h = 0$ and $\overline{x} \in K$. Assume that $\underline{D}(f, g)(\overline{x})$ exists, for all $x \in \mathrm{dom}\big(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))\big)$:*

(i) $\underline{D}(f, g)(\overline{x})(x) \notin (-intQ) \times (-intS)$;

(ii) *For every $y \in K$, there exist $e_1 \in Q$, $e_2 \in IT(S, -w)$ such that*

$$\underline{D}(f, g)(\overline{x})(x) \in IMin\big((f, g)(y) - (e_1, e_2) + P \,|\, Q \times S\big),$$

where the cone P in the product space $Y \times Z$, either $Q \times S$, or $-(Q \times S)$.

Then $\overline{x}$ is a weakly efficient solution to the problem (VEPC).

Proof. By direct application of Theorem 3.7. □

Remark 3.10. If f (or g) is Hadamard differentiable at $\overline{x}$, then, by directly using Remark 3.3, the hypothesis $x \in \mathrm{dom}\big(D_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}))\big)$ may be dropped and replaced by $x \in X$. In this case, a_x and b_x are also given as in Remark 3.3. □

Corollary 3.11. *Under the assumptions of Theorem 3.7, assume that for every $x \in M$, there exist $u \in IT(C, \overline{x})$, and $(v, w) \in D_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}))(u) \cap (-Q) \times (-S)$ satisfying*

(i) $\underline{D}^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, v, w)(x) \notin (-int Q) \times IT(-S, w)$;

(ii) *For every $y \in K$, there exist $e_1 \in Q$, $e_2 \in IT(S, -w)$ such that $a_x = f(y) - e_1$ and $b_x = g(y) - e_2$.*

Then $\overline{x}$ is a weakly efficient solution to the problem (VEPC).

Proof. By direct application of Theorem 3.7, where P is replaced by $Q \times S$. □

Corollary 3.12. *Under the assumptions of Theorem 3.7, assume that for every $x \in M$, there exist $u \in IT(C, \overline{x})$ and $(v, w) \in D_c(f_+, g_+)(\overline{x}, (f,g)(\overline{x}))(u) \cap (-Q) \times (-S)$ satisfying*

(i) $\underline{D}^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, v, w)(x) \notin (-int Q) \times IT(-S, w)$;

(ii) $(f,g)(K) \subset \underline{D}^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, v, w)(x) + Q \times (int S)$.

Then $\overline{x}$ is a weakly efficient solution to the problem (VEPC).

Proof. By the initial assumptions, for each $y \in K$, there exist $e_1 \in Q$ and $e_2 \in \mathrm{int} S$ such that $a_x = f(y) - e_1$ and $b_x = g(y) - e_2$. By directly application Corollary 3.11, it suffices to prove that $e_2 \in IT(S, -w)$. But from the facts that $\mathrm{int}\, S \subset \mathrm{intcone}(S + w) = IT(S, -w)$ and $e_2 \in \mathrm{int}\, S$, this follows. □

Example 3.13. Let $X, Y, Z, C, Q, S, \overline{x}, u_0$ and x^0 be as in Example 3.6. The mappings $f, g : X \to l^2$ respectively defined as

$$f(x) = \begin{cases} \sum_{i=1}^{n} \|x\|_X \, e_i, & \text{if } \|x\|_X = \frac{1}{n},\ n \in \mathbb{N}, \\ 0, & \text{if } \|x\|_X \notin \{\frac{1}{n},\ n \in \mathbb{N}\}, \end{cases}$$

$$g(x) = \begin{cases} \sum_{i=1}^{\infty} \|x\|_X \, x_i^0 e_i, & \text{if } \|x\|_X = \frac{1}{n},\ n \in \mathbb{N}, \\ \sum_{i=1}^{\infty} \|x\|_X^2 \, x_i^0 e_i, & \text{if } \|x\|_X \notin \{\frac{1}{n},\ n \in \mathbb{N}\}. \end{cases}$$

By an argument similar as in Example 3.1, we get $v = 0 \in l^2$, $w = 0 \in l^2$.

Moreover, $\underline{D}^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, v, w)(x)$ exists for all $x \in M$.

For fixed $x \in M$, it is plain that $a_x = 0 \notin -\mathrm{int} l^2_+$ and $b_x = \sum_{i=1}^{\infty} 2\|u_0\|_X^2 x_i^0 e_i$, which yields that the hypothesis (i) in Corollary 3.12 holds. For all $x \in M = X$, we obtain the following result

$$\underline{D}^2(f,g)(\overline{x}, (f,g)(\overline{x}), u_0, 0, 0)(x) + Q \times \mathrm{int}(S) = \{(q, b_x + s) \,:\, (q,s) \in l^2_+ \times \mathrm{int} l^2_-\}.$$

Finally, for any $(y, z) \in (f,g)(K)$ there exists $x \in K$ such that $(y,z) = (f(x), g(x))$.

Two cases can occur as follows:

Case 1. If $\|x\|_X = \frac{1}{n}$ and $x_i \geq 0\ \forall\, i \geq 1$, then $(y,z) = (\sum_{i=1}^{n} \frac{1}{n} e_i, \sum_{i=1}^{\infty} \frac{1}{n} x_i^0 e_i)$.

Because of $\frac{1}{n} x_i^0 \leq x_i^0 < 2\|u_0\|_X^2 x_i^0$ for all $i \geq 1$ we obtain

$$(y,z) \in \underline{D}^2(f,g)(\overline{x}, (f,g)(\overline{x}), u, v, w)(x) + l^2_+ \times (\mathrm{int}\, l^2_-).$$

Case 2. If $\|x\|_X \notin \{\frac{1}{n} : n \in \mathbb{N}\}$, then $(y, z) = (0, \sum_{i=1}^{\infty} \|x\|_X^2 x_i^0 e_i)$. We then obviously obtain $\|x\|_X^2 x_i^0 \leq x_i^0 < 2\|u_0\|^2 x_i^0$ for all $i \geq 1$.
A consequence is $(y, z) \in \underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u_0, 0, 0)(x) + l_+^2 \times \text{int} l_-^2$, which implies that all the hypothesis (ii) in Corollary 3.12 are fulfilled. Thus, the vector $\overline{x} = 0$ is a weakly efficient solution to the problem (VEPC).

Remark 3.14. If the regularity condition (C) holds then the assumption (i) is equivalent to the fact that there exist $\lambda \in Q^+ \setminus \{0\}$ and $\eta \in S^+$ with $\langle \eta, w \rangle = 0$ such that

$$\langle \lambda, a_x \rangle + \langle \eta, b_x \rangle \geq 0.$$

This plays an important role when we are working with solutions of their types of vector equilibrium problem with constraints (VEPC). □

4. Applications

By applying the results obtained in Section 3, we provide here the second-order necessary and sufficient optimality conditions for Henig efficient, globally efficient and superefficient solutions of constrained vector equilibrium problem (VEPC).

Theorem 4.1. *Let $\overline{x}$ be a feasible point for problem* (VEPC) *and B be a base of Q. Assume that Assumptions* (A), (B) *and* (C) *are fulfilled. Then $\overline{x}$ is a Henig efficient (resp., globally efficient, superefficient if, in addition, B compact) solution to problem* (VEPC) *if and only if for every $x \in M$, there exist $(\lambda, \eta) \in Y^* \times Z^*$ such that*

$$\lambda \in Q^{\Delta}(B) \ \ (resp., \ Q^{\sharp}, \ \ int(Q^+)); \tag{21}$$

$$\eta \in S^+ \ \ with \ \ \langle \eta, w \rangle = 0; \tag{22}$$

$$\big\langle \lambda, f(\widetilde{x}) \big\rangle + \big\langle \eta, g(\widetilde{x})) \big\rangle \geq \langle \lambda, a_x \rangle + \langle \eta, b_x \rangle \geq 0 \ \ \forall \, \widetilde{x} \in K. \tag{23}$$

Proof. Case 1. We assume that the vector $\overline{x}$ is a Henig efficient solution of problem (VEPC), which means that there exists some absolutely convex neighborhood U of 0 with $U \subset V_B$ such that

$$\text{cone}(f(K)) \cap (-\text{int}[\text{cone}(U + B)]) = \emptyset,$$

which is equivalent to $f(K) \cap (-\text{int}[\text{cone}(U + B)]) = \emptyset$, where cone$(U + B)$ is a pointed and convex cone. According to Theorem 3.2, for every $x \in M$, there exist $\lambda \in [\text{cone}(U + B)]^+ \setminus \{0\}$ and $\eta \in S^+$ with $\langle \eta, w \rangle = 0$ such that (23) holds. Making use of Lemma 2.1 (i) [16], we get $\lambda \in Q^{\Delta}(B)$. In the converse case, we may assume that for every $x \in M$, there exist $\lambda \in Q^{\Delta}(B)$ and $\eta \in S^+$ with $\langle \eta, w \rangle = 0$ satisfying (23). Because of $\lambda \in Q^{\Delta}(B)$, there exists a positive real number t such that

$$\langle \lambda, b \rangle \geq t \quad \forall \, b \in B.$$

We set $$U = \{y \in Y \, : \, | \, \langle \lambda, y \rangle \, | \, < \, min\{\frac{t}{2}, \frac{r}{2}\}\} \cap V_B.$$

Then U is an open absolutely convex neighborhood of the origin in Y with $U \subset V_B$.

We have $$cone(f(K)) \cap (-\text{int}[\text{cone}(U + B)]) = \emptyset,$$

which is equivalent to (see [6, 7]) $\text{cone}(f(K)) \cap (U - B) = \emptyset. \quad (24)$

Indeed, if (24) was not so, there would exist $r \geq 0$, $\widetilde{x} \in K$ and there exist $u \in U$, $b \in B$ such that $rf(\widetilde{x}) = u - b$. Therefore,

$$\big\langle \lambda, rf(\widetilde{x})\big\rangle = \langle \lambda, u\rangle - \langle \lambda, b\rangle \leq \langle \lambda, u\rangle - t < 0. \tag{25}$$

If $r = 0$, then $u = b$. But, $t \leq \langle \lambda, b\rangle = |\langle \lambda, u\rangle| < t$, a contradiction. From here we have $r > 0$. It follows from (25) that

$$\big\langle \lambda, f(\widetilde{x})\big\rangle + \big\langle \eta, g(\widetilde{x})\big\rangle \; < \; \langle \eta, g(\widetilde{x})\rangle \leq 0,$$

which contradicting the initial assumptions. Thus $\overline{x}$ is a Henig efficient solution to the problem (VEPC).

Case 2. See Gong [6, 7] or Long et al. [16].

Case 3. Suppose $\overline{x}$ is a globally efficient solution of problem (VEPC), which means that there exists a pointed convex cone $H \subset Y$ with $Q \setminus \{0\} \subset \mathrm{int}H$ such that

$$f(K) \cap (-H \setminus \{0\}) = \emptyset.$$

Hence, $f(K) \cap (-\mathrm{int}H) = \emptyset$ (see Definition 1.1). Taking account of Theorem 3.2, we deduce that there exist $\lambda \in H^+ \setminus \{0\}$ and $\eta \in S^+$ with $\langle \eta, w\rangle = 0$ satisfying (23). In order to have $\lambda \in Q^\sharp$, we take any $y \in Q \setminus \{0\}$, and have $\langle \lambda, y\rangle > 0$. By the definition of $Q^\sharp$, we infer that $\lambda \in Q^\sharp$. In the converse case, for every $x \in M$, there exist $\lambda \in Q^\sharp$ and $\eta \in S^+$ with $\langle \eta, w\rangle = 0$ such that (23) holds. We set

$$H = \{y \in Y \; : < \lambda, y >> 0\} \cup \{0\}.$$

By Gong [6], the set H is a pointed and convex cone satisfying $Q \setminus \{0\} \subset \mathrm{int}H$. By Definition 2.1, we only need to show that

$$f(K) \cap (-H \setminus \{0\}) = \emptyset. \tag{26}$$

In fact, if (26) was false, there would exist $\hat{x} \in K$ such that $f(\hat{x}) \in -H \setminus \{0\}$. Since $\lambda \in Q^\sharp$, it holds that $\langle \lambda, f(\hat{x})\rangle < 0$. Then,

$$\langle \lambda, f(\hat{x})\rangle + \langle \eta, g(\hat{x})\rangle \; < \; \langle \eta, g(\hat{x})\rangle \; \leq 0.$$

This contradicts the inequality (23), which completes the proof. □

Example 4.2. Let $X = Y = Z = \mathbb{R}$, $C = [-1, 1]$, $Q = S = \mathbb{R}_+$, $\overline{x} = 0$, and the mappings $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are defined respectively as

$$f(x) = x^4 \;\; (\forall\, x \in \mathbb{R}) \;\; \text{and} \;\; g(x) = -x^2 \;\; (\forall\, x \in \mathbb{R}).$$

It can easily be seen that the mappings $f_+, g_+ : \mathbb{R} \rightrightarrows \mathbb{R}$ are given respectively as

$$f_+(x) = \{y \in \mathbb{R} \, : \, y \geq x^4 \;\; (\forall\, x \in \mathbb{R}) \;\; \text{and} \;\; g_+(x) = \{y \in \mathbb{R} \, : \, y + x^2 \geq 0\} \;\; (\forall\, x \in \mathbb{R}).$$

By direct calculation we obtain $IT(C, 0) = \mathbb{R}$, and

$$D_c(f_+, g_+)(0, (0, 0))(1) \cap (-Q) \times (-S) = \{(0, 0)\}.$$

Taking $u = 1$ and $v = w = 0$, it results that

$$\mathrm{dom}\Big(D_c^2(f_+, g_+)(0, (0, 0), 1, 0, 0)\Big) = \mathbb{R}.$$

Moreover, for all $x \in \mathbb{R}$, $D_c^2(f_+, g_+)(0, (0, 0), 1, 0, 0)(x) = \mathbb{R}_+ \times [-2, +\infty)$. Observe that $\mathbb{R}_+^2$ has a compact base B such that $Q^\Delta(B) = Q^\sharp = \mathrm{int}(Q^+) = \mathrm{int}(\mathbb{R}_+^2)$.

By virtue of Proposition 2.9, we have $\underline{D}^2(f, g)(0, (0, 0), 1, 0, 0)(x) = (0, -2)$ for all $x \in \mathbb{R}$, which yields that $(a_x, b_x) = (0, -2)$, i.e., the assumption (A) is fulfilled.

On the other hand, it is not hard to check that the assumptions (B) and (C) also hold. We also mention that $\mathbb{R}_+$ has a compact base B_0 and

$$Q^\Delta(B_0) = Q^\sharp = \mathrm{int}(Q^+) = \mathrm{int}(\mathbb{R}_+) = (0, +\infty).$$

By taking $\lambda = 1$, $\eta = 0$, it follows that (23) holds. Theorems 3.2 and 4.1 imply that the vector $\overline{x}$ being one of the efficient solutions of problem (VEPC).

Corollary 4.3. *Let $\overline{x}$ be a feasible point for problem* (VEPC) *and the convex set B be a base of Q. Assume that all the assumptions of Corollary* 3.9 *are fulfilled. Then $\overline{x}$ is a Henig efficient (resp., globally efficient, superefficient if, in addition, B compact) solution to the problem (VEPC) if and only if for every $x \in dom\big(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))\big)$, there exist $(\lambda, \eta) \in (Y^* \times Z^*) \setminus \{(0, 0)\}$ satisfying conditions* (21)*,* (22) *and* (23) *in Theorem* 4.1*.*

Proof. This is a direct consequence of Theorem 4.1. □

Theorem 4.4. *Let $\overline{x}$ be a feasible point for problem* (VEPC) *and the set B a base of the cone Q. Assume that $h = 0$, Assumption* (A) *holds, and for every $x \in M$, there exist $u \in IT(C, \overline{x})$ and $(v, w) \in D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) \cap (-Q) \times (-S)$, such that*

(i) *there exist $\lambda \in Q^\Delta(B)$ (resp., $\lambda \in Q^\sharp$), $\eta \in S^+$ with $\langle \eta, w \rangle = 0$ with*

$$\langle \lambda, a_x \rangle + \langle \eta, b_x \rangle \geq 0;$$

(ii) *for all $y \in K$, there exist $e_1 \in Q$, $e_2 \in IT(S, -w)$ such that*

$$\underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) \in IMin\Big((f, g)(y) - (e_1, e_2) + P \,|\, Q \times S\Big),$$

where the cone P in $Y \times Z$ is either $Q \times S$, or $-(Q \times S)$.

Then $\overline{x}$ is a Henig (resp., globally) efficient solution to the problem (VEPC)*.*

Proof. We first assume that $\lambda \in Q^\Delta(B)$, and $\overline{x} \in K$ is not a Henig efficient solution to the problem (VEPC), which means that for all absolutely convex neighborhoods U of the origin in Y with $U \subset V_B$,

$$f(K) \cap (-\mathrm{intcone}(U + B)) \neq \emptyset. \tag{27}$$

We take an absolutely open convex neighborhood U_0 of the origin in Y with $U_0 \subset V_B$ such that $\lambda \in [\mathrm{cone}(U_0 + B)]^+ \setminus \{0\}$ (see Gong [6]). It follows from (27) that there exists $y_o \in K$ such that $f(y_o) \in -\mathrm{intcone}(U_0 + B)$. Making use of the proof of Theorem 3.7, we obtain that $a_x \in -\mathrm{intcone}(U_0 + B) - Q,\ b_x \in IT(-S, w)$.

From the fact that $Q \setminus \{0\} \subset \text{intcone}(U_0 + B)$, we conclude $a_x \in -\text{intcone}(U_0 + B)$, and this leads to

$$\langle \lambda, a_x \rangle + \langle \eta, b_x \rangle < 0,$$

a contradiction. Hence, $\overline{x}$ is a Henig efficient solution to the problem (VEPC).

We now suppose that $\lambda \in Q^\sharp$ and $\overline{x} \in K$ is not a globally efficient solution of problem (VEPC), which means that for all pointed convex cones $H \subset Y$ with $Q \setminus \{0\} \subset \text{int}H$, $f(K) \cap (-H \setminus \{0\}) \neq \emptyset$. We define H as in the proof of Theorem 4.1. Then there exists $\hat{y} \in K$ such that $f(\hat{y}) \in -H \setminus \{0\}$, which means that $\langle \lambda, f(\hat{y}) \rangle < 0$. This yields

$$\langle \lambda, f(\hat{y}) \rangle + \langle \eta, g(\hat{y}) \rangle < 0,$$

which contradicts assumption (i). From here we will be allowed to conclude that the vector $\overline{x}$ being a globally efficient solution of problem (VEPC). □

Corollary 4.5. *Let $\overline{x}$ be a feasible point for problem* (VEPC)*, $h = 0$ and let the set B be a base of the cone Q. Suppose that all the assumptions of Corollary* 3.9 *are fulfilled and for every $x \in dom\big(D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))\big)$,*

(i) *There exist $\lambda \in Q^\Delta(B)$ (resp., $\lambda \in Q^\sharp$), $\eta \in S^+$ with $\langle \eta, w \rangle = 0$ such that*

$$\langle \lambda, a_x \rangle + \langle \eta, b_x \rangle \geq 0;$$

(ii) *For all $y \in K$, there exist $e_1 \in Q$, $e_2 \in IT(S, -w)$ such that*

$$\underline{D}(f, g)(\overline{x})(x) \in IMin\big((f, g)(y) - (e_1, e_2) + P \,|\, Q \times S\big),$$

where the cone P in $Y \times Z$ is either $Q \times S$, or $-(Q \times S)$.

Then $\overline{x}$ is a Henig (resp., globally) efficient solution to the problem (VEPC).

Proof. The desired conclusion is an immediate consequence Theorem 4.4. □

Corollary 4.6. *Let $\overline{x}$ be a feasible point for problem* (VEPC) *and B a base of Q. Assume that $h = 0$ and Assumption* (A) *holds. Suppose also that for every $x \in M$, there exist $u \in IT(C, \overline{x})$ and $(v, w) \in D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) \cap (-Q) \times (-S)$,*

(i) *There exist $\lambda \in Q^\Delta(B)$ (resp., $\lambda \in Q^\sharp$), $\eta \in S^+$ with $\langle \eta, w \rangle = 0$ such that*

$$\langle \lambda, a_x \rangle + \langle \eta, b_x \rangle \geq 0;$$

(ii) *For all $y \in K$, there exist $e_1 \in Q$, $e_2 \in IT(S, -w)$ such that $a_x = f(y) - e_1$ and $b_x = g(y) - e_2$.*

Then $\overline{x}$ is a Henig (resp., globally) efficient solution to the problem (VEPC).

Proof. In the same way as in Corollary 3.11 the conclusion follows. □

Corollary 4.7. *Let $\overline{x}$ be a feasible point for problem* (VEPC) *and let the set B be a base of the cone Q. Assume that $h = 0$ and Assumption* (A) *holds. Suppose, in addition, that for every $x \in M$, there exist $u \in IT(C, \overline{x})$ and*

$$(v, w) \in D_c(f_+, g_+)(\overline{x}, (f, g)(\overline{x}))(u) \cap (-Q) \times (-S),$$

such that

(i) *there exist* $\lambda \in Q^{\Delta}(B)$ *(resp.,* $\lambda \in Q^{\sharp}$*),* $\eta \in S^{+}$ *with* $\langle \eta, w \rangle = 0$ *with*

$$\langle \lambda, a_x \rangle + \langle \eta, b_x \rangle \geq 0;$$

(ii) $(f, g)(K) \subset \underline{D}^2(f, g)(\overline{x}, (f, g)(\overline{x}), u, v, w)(x) + Q \times intS.$

Then $\overline{x}$ *is a Henig (resp., globally) efficient solution to the problem* (VEPC).

Proof. In the same way as in Corollary 3.12 and Corollary 4.6, the conclusion follows. □

Remark 4.8. If all the assumptions in Corollary 3.9 are fulfilled, then the statement of Corollary 4.7 is still true if the condition (ii) is removed and replaced by

$$(f, g)(K) \subset \underline{D}(f, g)(\overline{x})(x) + Q \times \mathrm{int}S.$$

Additionally, if the B is compact of Q and $\lambda \in \mathrm{int}(Q^{+})$, then we obtain the conclusions in Theorem 4.4, Corollaries 4.3, 4.5, 4.6 and 4.7 that $\overline{x}$ is a superefficient solution of problem (VEPC). □

We close this section by making some comparisons between the results obtained in the paper and the existing one in the literature.

Remark 4.9. So far as we known, there is no results in the literature on second-order necessary and sufficient optimality conditions for efficient solutions of vector equilibrium problems involving equality, generalized inequality and set constraints in terms of contingent epiderivatives in Banach spaces. Indeed, we used at least one of the Assumptions (A), (B) and (C) for obtaining second-order necessary and sufficient optimization conditions to such problems, while Li, Zhu and Teo [15] introduced a generalized second-order composed contingent epiderivative for a set-valued mapping and the author used this concept for establishing second-order optimality conditions for set-valued optimization problems with only a set constraint with generalized convex objective function; Su [26] established the new second-order optimality conditions for vector equilibrium problems with set and generalized inequality constraints in terms of contingent derivatives with 2-stable functions in real finite dimension spaces; Su [25] gave the second-order efficiency conditions for $C^{1,1}-$vector equilibrium problems in terms of contingent derivatives and its application to $C^{1,1}-$vector variational inequality problem and $C^{1,1}-$vector optimization problem with constraints; Su [24] only studied some basic properties of the second-order contingent epiderivative of a single-valued map from a variational perspective and its application to obtaining the second-order efficiency conditions for weakly efficient solutions of vector equilibrium problems with constraints.

5. Conclusions

In this paper, we have shown some important properties of second-order contingent epiderivatives for an arbitrary function. We introduced Assumptions (A), (B) and (C), which are illustrated by an example. Under these assumptions, we established the second-order necessary and sufficient conditions for weakly efficient, Henig efficient, globally efficient and superefficient solutions of vector equilibrium problem with constraints (VEPC) in terms of second-order contingent epiderivatives. In case that there are no Assumptions (A), (B) and (C), we only obtain second-order sufficient conditions for those solutions of problem (VEPC) via second-order contingent epiderivatives for the class of arbitrary functions. Some illustrative examples are also given. Our result obtained in this paper are new and, moreover, these result has not been considered before. In the future, these second-order necessary and sufficient optimality conditions may be used to formulate second-order optimality conditions for efficient solutions in vector equilibrium problems with complementarity, equilibrium, mixed constraints, etc.

Acknowledgements. The authors would like to express many thanks to two anonymous reviewers for their valuable comments and suggestions which improved the contents of the paper in its present form. Furthermore, the authors acknowledge the editors for sending our manuscript to reviewers.

Funding. The first and second authors were supposed by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant Number 101.01-2017.301.

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