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Geometric Hardy Inequalities on Starshaped Sets

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We present geometric Hardy inequalities on starshaped sets in Carnot groups. Also, we obtain geometric Hardy inequalities on half-spaces for general vector fields.

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1. Introduction

In 1998, Danielli and Garofalo [5] introduced the concept of starshapedness in the setting of Carnot groups (see also [6]). Their paper describes geometrical properties of starshaped and convex sets. The convexity in the Heisenberg group was studied by many authors such as Monti and Rickly [13] who proved the geodesic convexity, or by Danielli, Garofalo, and Nhieu [7] (see also [10]) who introduced the concept of horizontal convexity (H-convexity). Bardi and Dragoni [1], [2] generalised the concept of convexity to general vector fields and introduced the notion of $\mathcal{X}$-convexity which is a generalisation of the H-convexity. This analysis allows one introducing the notion of the distance to the boundary for starshaped sets. Consequently, by using the distance formula one can obtain geometric Hardy type inequalities.

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The main aim of this paper is to obtain geometric Hardy inequalities on starshaped sets in Carnot groups. Furthermore, we establish geometric Hardy inequalities on the half-spaces for general vector fields.

This paper is organised in the following way: Section 1: A brief overview of the sub-Riemannian manifolds, Grushin plane, Carnot groups, Heisenberg groups, and Engel groups. Section 2: Geometric Hardy inequalities on the starshaped sets in Carnot groups are established, with some examples. Section 3: Geometric Hardy inequalities on half-spaces for general vector fields, and some examples. Section 4: The proofs of the main results.

1.1. Sub-Riemannian manifolds

Let M be a smooth manifold of dimension n, with a given family of vector fields $\{X_k\}_{k=1}^N$, $n \geq N$, defined on M and satisfying the Hörmander condition. They induce in a natural way a sub-Riemannian metric $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on the associated space $\mathcal{H}_x = \operatorname{span}(X_1(x),\dots,X_N(x))$. The triple $(M,\mathcal{H},\langle\cdot,\cdot\rangle_{\mathcal{H}})$ is a sub-Riemannian manifold. Note that, unlike for Carnot groups, in general, it is not possible to define dilations, translations, the homogeneous norm and the distance in the setting of general sub-Riemannian manifolds.

Let us denote the operator of the sum of squares of vector fields by

$$\mathcal{L} := \sum_{k=1}^{N} X_k^2. \tag{1}$$

These operators have been studied by many authors, for instance, it is well-known since Hörmander's pioneering work [11], that if the commutators of the vector fields $\{X_k\}_{k=1}^N$ generate the Lie algebra, then the operator $\mathcal{L}$ is locally hypoelliptic. The p-version of the sum of squares of vector fields can be given by the formula

$$\mathcal{L}_p f := \nabla_X \cdot (|\nabla_X f|^{p-2}\nabla_X f), \quad \text{where} \quad \nabla_X := (X_1,\dots,X_N). \tag{2}$$

1.2. Grushin plane

One of the important examples of a sub-Riemannian manifold is the Grushin plane. The Grushin plane is the space $\mathbb{R}^2$ with vector fields

$$X_1 = \frac{\partial}{\partial x_1}, \text{ and } X_2 = x_1\frac{\partial}{\partial x_2}, \quad \text{for } x := (x_1,x_2) \in \mathbb{R}^2.$$

1.3. Carnot groups

Let $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ be a stratified Lie group (or a homogeneous Carnot group or just a Carnot group), with the dilation structure δ_λ and Jacobian generators $X_1,\dots,X_N$, so that N is the dimension of the first stratum of $\mathbb{G}$. Let us denote by Q the homogeneous dimension of $\mathbb{G}$. We refer to the recent books [9] and [18] for extensive discussions of stratified Lie groups and their properties.

The sub-Laplacian on $\mathbb{G}$ is given by
$$\mathcal{L} = \sum_{k=1}^{N} X_k^2. \tag{3}$$

We also recall that the standard Lebesgue measure dx on $\mathbb{R}^n$ is the Haar measure for $\mathbb{G}$ (see, e.g. [9, Proposition 1.6.6]). Each left invariant vector field X_k has an explicit form and satisfies the divergence theorem, see e.g. [9] for the derivation of exact formula: more precisely, we can express

$$X_k = \frac{\partial}{\partial x'_k} + \sum_{l=2}^{r} \sum_{m=1}^{N_l} a_{k,m}^{(l)}(x', ..., x^{(l-1)}) \frac{\partial}{\partial x_m^{(l)}}, \tag{4}$$

with $x = (x', x^{(2)}, \ldots, x^{(r)})$, where r is the step of $\mathbb{G}$ and $x^{(l)} = (x_1^{(l)}, \ldots, x_{N_l}^{(l)})$ are the variables in the l^{th} stratum, see also [9, Section 3.1.5] for a general presentation.

The horizontal divergence is defined by

$$\mathrm{div}_H f := \nabla_H \cdot f, \quad \text{where} \quad \nabla_H := (X_1, \ldots, X_N)$$

is the horizontal gradient. The p-sub-Laplacian has the form

$$\mathcal{L}_p f = \nabla_H \cdot (|\nabla_H f|^{p-2} \nabla_H f). \tag{5}$$

1.4. Heisenberg groups

Let $\mathbb{H}_1$ be the Heisenberg group, that is, the set $\mathbb{R}^3$ equipped with the group law

$$x \circ x' := (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + 2(x'_1 x_2 - x_1 x'_2)),$$

where $x := (x_1, x_2, x_3) \in \mathbb{R}^3$, and $x^{-1} = -x$ is the inverse element of x with respect to the group law. The dilation operation on the Heisenberg group with respect to the group law has the form

$$\delta_\lambda(x) := (\lambda x_1, \lambda x_2, \lambda^2 x_3) \text{ for } \lambda > 0.$$

The Lie algebra $\mathfrak{h}$ of the left-invariant vector fields on the Heisenberg group $\mathbb{H}_1$ is spanned by

$$X_1 := \frac{\partial}{\partial x_1} + 2x_2 \frac{\partial}{\partial x_3}, \quad \text{and} \quad X_2 := \frac{\partial}{\partial x_2} - 2x_1 \frac{\partial}{\partial x_3},$$

with their (non-zero) commutator $[X_1, X_2] = -4 \frac{\partial}{\partial x_3}$.

The horizontal gradient on $\mathbb{H}_1$ is given by $\nabla_H := (X_1, X_2)$, so that the sub-Laplacian on $\mathbb{H}_1$ is given by $\mathcal{L} := X_1^2 + X_2^2$.

The Heisenberg group is the most common example of a step 2 stratified group (Carnot group).

1.5. Engel groups

Let $\mathbb{E}$ be the Engel group, that is, the set $\mathbb{R}^4$ equipped with the group law

$$x \circ x' := (x_1 + x'_1, x_2 + x'_2, x_3 + x'_3 + P_3, x_4 + x'_4 + P_4),$$

where
$$P_3 = \frac{1}{2}(x_1x_2' - x_2x_1'),$$
$$P_4 = \frac{1}{2}(x_1x_3' - x_3x_1') + \frac{1}{12}(x_1^2x_2' - x_1x_1'(x_2 + x_2') + x_2x_1'^2).$$

Here $x := (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The vector fields have the following form

$$\begin{aligned} X_1 :&= \frac{\partial}{\partial x_1} - \frac{x_2}{2}\frac{\partial}{\partial x_3} - \left(\frac{x_3}{2} + \frac{x_1x_2}{12}\right)\frac{\partial}{\partial x_4}, \\ X_2 :&= \frac{\partial}{\partial x_2} + \frac{x_1}{2}\frac{\partial}{\partial x_3} + \frac{x_1^2}{12}\frac{\partial}{\partial x_4}, \\ X_3 :&= [X_1, X_2] = \frac{\partial}{\partial x_3} + \frac{x_1}{2}\frac{\partial}{\partial x_4}, \\ X_4 :&= [X_1, X_3] = \frac{\partial}{\partial x_4}. \end{aligned}$$

The Engel group is a well-known example of a step 3 stratified group (Carnot group), see [3, 4] for details on the Engel and Cartan groups.

2. Hardy inequalities on starshaped sets

In order to present the results on the starshaped domains, we recall the definition of starshaped sets in a Carnot group $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_t)$ and related arguments.

Definition 2.1. (Starshapedness [5]) Let $\Omega \subset \mathbb{G}$ be a C^1 domain containing the identity e. Then Ω is *starshaped with respect to* e if for every $x \in \partial\Omega$ one has

$$\langle Z(x), n(x)\rangle \geq 0, \tag{6}$$

where n is the Riemannian outer normal to $\partial\Omega$. When the strict inequality holds, then Ω is said to be strictly starshaped with respect to e. $\square$

Here the vector fields Z are the infinitesimal generators of this group automorphism. This vector fields Z takes the form

$$Z = \sum_{i=1}^{N} x_i' \frac{\partial}{\partial x_i'} + 2\sum_{l=1}^{N_2} x_{2,l}\frac{\partial}{\partial x_{2,l}} + \cdots + r\sum_{l=1}^{N_r} x_{r,l}\frac{\partial}{\partial x_{r,l}}. \tag{7}$$

Then for $x' \in \mathbb{R}^N$ and $x^{(i)} \in \mathbb{R}^{N_i}$ with $i = 2, \ldots, r$ we have

$$Z(x) = (x', 2x^{(2)}, \cdots, rx^{(r)}), \tag{8}$$

and

$$\begin{aligned} \langle Z(x), n(x)\rangle &= x'n' + 2x^{(2)}n^{(2)} + \ldots + rx^{(r)}n^{(r)} \\ &= x_1'n_1' + \cdots + x_N'n_N' + 2(x_{2,1}n_{2,1} + \cdots + x_{2,N_2}n_{2,N_2}) \\ &\quad + \cdots + r(x_{r,1}n_{r,1} + \cdots + x_{r,N_r}n_{r,N_r}), \end{aligned}$$

since $n(x) := (n', n^{(2)}, \ldots, n^{(r)})$ with $n' \in \mathbb{R}^N$ and $n^{(i)} \in \mathbb{R}^{N_i}$, $i = 2, \ldots, r$.

Based on the above arguments we now present the geometric Hardy inequalities on the starshaped sets for the sub-Laplacians.

Theorem 2.2. *Let Ω be a starshaped set on a Carnot group. Then for every $\gamma \in \mathbb{R}$ and $p > 1$ we have the following Hardy inequality*

$$\int_\Omega |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_\Omega \frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p} |f(x)|^p dx \qquad (9)$$

$$+ \gamma \int_\Omega \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}} |f(x)|^p dx,$$

for every function $f \in C_0^\infty(\Omega)$.

Corollary 2.3. *Let $\mathbb{H}^*$ be a starshaped set on the Heisenberg group $\mathbb{H}_1$. Then for $p > 1$, we have the following Hardy inequality*

$$\int_{\mathbb{H}^*} |\nabla_H f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^*} \frac{|(n_1 + 4x_2 n_3, n_2 - 4x_1 n_3)|^p}{|x_1 n_1 + x_2 n_2 + 2x_3 n_3|^p} |f(x)|^p dx, \qquad (10)$$

for every function $f \in C_0^\infty(\mathbb{H}^*)$.

Remark 2.4. Note that in the case
$\mathbb{H}^* := \{\langle Z(x), n(x)\rangle > 0,\ \forall x \in \partial\mathbb{H}^*,\quad Z(x) := (x_1, x_2, 2x_3)\} = \{x \in \mathbb{H}_1 \cong \mathbb{R}^3 : x_3 > 0\}$
with $n(x) := (0, 0, 1)$, and $p = 2$, we have the inequality

$$\int_{\mathbb{H}^*} |\nabla_H f(x)|^2 dx \geq \int_{\mathbb{H}^*} \frac{|x_1|^2 + |x_2|^2}{x_3^2} |f(x)|^2 dx.$$

Proof of Corollary 2.3. We begin the proof of Corollary 2.3 by a simple computation such as

$$\begin{aligned}
\langle Z(x), n(x)\rangle &= x_1 n_1 + x_2 n_2 + 2x_3 n_3, \\
\nabla_H \langle Z(x), n(x)\rangle &= (n_1 + 4x_2 n_3, n_2 - 4x_1 n_3), \\
|\nabla_H \langle Z(x), n(x)\rangle|^p &= \left((n_1 + 4x_2 n_3)^2 + (n_2 - 4x_1 n_3)^2\right)^{p/2},
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{L}_p \langle Z(x), n(x)\rangle = \nabla_H \cdot (|\nabla_H \langle Z(x), n(x)\rangle|^{p-2} \nabla_H \langle Z(x), n(x)\rangle) \\
&= X_1(|\nabla_H \langle Z(x), n(x)\rangle|^{p-2}(n_1 + 4x_2 n_3)) + X_2(|\nabla_H \langle Z(x), n(x)\rangle|^{p-2}(n_2 - 4x_1 n_3)) \\
&= -4(p-2)|\nabla_H \langle Z(x), n(x)\rangle|^{p-4}(n_1 + 4x_2 n_3)(n_2 - 4x_1 n_3) n_3 \\
&\qquad + 4(p-2)|\nabla_H \langle Z(x), n(x)\rangle|^{p-4}(n_2 - 4x_1 n_3)(n_1 + 2x_4 n_3) n_3 \\
&= 0.
\end{aligned}$$

Plugging the above expressions into inequality (9) and maximising with respect to γ, we arrive at inequality (10) which completes the proof of Corollary 2.3. $\square$

Corollary 2.5. *Let $\mathbb{E}^*$ be a starshaped set on the Engel group $\mathbb{E}$. Then for every function $f \in C_0^\infty(\mathbb{E}^*)$ and for $\gamma \in \mathbb{R}$ and $p = 2$, we have*

$$\int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx \geq -\left(|\gamma|^2 + \gamma\right) \int_{\mathbb{E}^*} \frac{|\nabla_H \langle Z(x), n(x)\rangle|^2}{\langle Z(x), n(x)\rangle^2} |f(x)|^2 dx \qquad (11)$$
$$+ \frac{\gamma}{2} \int_{\mathbb{E}^*} \frac{x_2 n_4}{\langle Z(x), n(x)\rangle} |f(x)|^2 dx.$$

Proof of Corollary 2.5. As in the previous case, a straightforward calculation gives

$$\begin{aligned}
\langle Z(x), n(x)\rangle &= x_1 n_1 + x_2 n_2 + 2x_3 n_3 + 3x_4 n_4, \\
\nabla_H \langle Z(x), n(x)\rangle &= \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4}, n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4}\right), \\
|\nabla_H \langle Z(x), n(x)\rangle|^2 &= \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4}\right)^2 + \left(n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4}\right)^2,
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}\langle Z(x), n(x)\rangle =& \nabla_H \cdot \nabla_H \langle Z(x), n(x)\rangle \\
=& X_1 \left(n_1 - x_2 n_3 - \frac{3x_3 n_4}{2} - \frac{x_1 x_2 n_4}{4}\right) + X_2 \left(n_2 + x_1 n_3 + \frac{x_1^2 n_4}{4}\right) \\
=& \frac{x_2 n_4}{2}.
\end{aligned}$$

Plugging the above expressions into inequality (9), we have

$$\int_{\mathbb{E}^*} |\nabla_H f(x)|^2 dx \geq -\left(|\gamma|^2 + \gamma\right) \int_{\mathbb{E}^*} \frac{|\nabla_H \langle Z(x), n(x)\rangle|^2}{\langle Z(x), n(x)\rangle^2} |f(x)|^2 dx$$
$$+ \frac{\gamma}{2} \int_{\mathbb{E}^*} \frac{x_2 n_4}{\langle Z(x), n(x)\rangle} |f(x)|^2 dx,$$

that completes the proof of Corollary 2.5. □

3. Hardy inequalities on half-spaces for general vector fields

Let us define the half-space of a sub-Riemannian manifold by

$$\Omega^+ := \{x \in \mathbb{R}^n : \langle x, n(x)\rangle > d\},$$

where $n = n(x) \in \mathbb{R}^n$ is the Riemannian outer unit normal to $\partial\Omega^+$ and $d \in \mathbb{R}$. The Euclidean distance to the boundary $\partial\Omega^+$ is denoted by $dist(x, \partial\Omega^+)$ and defined by

$$dist(x, \partial\Omega^+) := \langle x, n(x)\rangle - d.$$

Then we have:

Theorem 3.1. *Let M be a sub-Riemannian manifold, let $\Omega^+ \subset M$ be a half-space and let $X_1, \ldots, X_N$ be the general vector fields. Then for every $\gamma \in \mathbb{R}$ and every $p > 1$, we have the following Hardy inequality*

$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega^+} \frac{|\nabla_X dist(x, \partial\Omega^+)|^p}{dist(x, \partial\Omega^+)^p} |f(x)|^p dx \tag{12}$$
$$+ \gamma \int_{\Omega^+} \frac{\mathcal{L}_p(dist(x, \partial\Omega^+))}{dist(x, \partial\Omega^+)^{p-1}} |f(x)|^p dx,$$

for every function $f \in C_0^\infty(\Omega^+)$.

Note that inequality (12) was obtained for the Carnot groups by the authors in [16], but here we extend it to general sub-Riemannian manifolds.

Let us give examples of these results for the Heisenberg group (step 2), the Engel group (step 3), and the Grushin plane, which does not have a group structure, but serves as an important example of the sub-Riemannian geometry.

Corollary 3.2. *Let Ω^+ be a half-space in the Grushin plane G. Then for every function $f \in C_0^\infty(\Omega^+)$ and every $p > 1$, we have the following Hardy inequality*

$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_{\Omega^+} \frac{(n_1^2 + x_1^2 n_2^2)^{p/2}}{(x_1 n_1 + x_2 n_2 - d)^p} |f(x)|^p dx \tag{13}$$
$$+ (p-2)\gamma \int_{\Omega^+} \frac{|\nabla_X dist(x, \partial\Omega^+)|^{p-4} n_1 n_2^2 x_1}{(x_1 n_1 + x_2 n_2 - d)^{p-1}} |f(x)|^p dx.$$

If one of the cases $n(x) = (1, 0)$ or $n(x) = (0, 1)$ holds, then we have

$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega^+} \frac{(n_1^2 + x_1^2 n_2^2)^{p/2}}{(x_1 n_1 + x_2 n_2 - d)^p} |f(x)|^p dx, \tag{14}$$

where $dist(x, \partial\Omega^+) = \langle x, n(x)\rangle - d$ and $d \in \mathbb{R}$.

Remark 3.3. Note that, with ∇_X the Grushin gradient, we have:

- If $\Omega^+ = \{x \in \mathbb{R}^2 : x_1 > d\}$ with $n(x) = (1, 0)$, then we have
$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega^+} \frac{|f(x)|^p}{(x_1 - d)^p} dx.$$
- If $\Omega^+ := \{x \in \mathbb{R}^2 : x_2 > d\}$ with $n(x) = (0, 1)$, then we have
$$\int_{\Omega^+} |\nabla_X f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p \int_{\Omega^+} \frac{|x_1|^p}{(x_2 - d)^p} |f(x)|^p dx.$$

Proof of Corollary 3.2. A simple calculation shows

$$dist(x, \partial\Omega^+) = x_1 n_1 + x_2 n_2 - d,$$
$$\nabla_X dist(x, \partial\Omega^+) = (n_1, x_1 n_2),$$
$$|\nabla_X dist(x, \partial\Omega^+)|^p = (n_1^2 + x_1^2 n_2^2)^{p/2},$$

and
$$\begin{aligned}\mathcal{L}_p dist(x,\partial\Omega^+) &= \nabla_X\cdot(|\nabla_X dist(x,\partial\Omega^+)|^{p-2}\nabla_X dist(x,\partial\Omega^+))\\ &= \frac{\partial}{\partial x_1}((n_1^2+x_1^2n_2^2)^{\frac{p-2}{2}}n_1)+x_1\frac{\partial}{\partial x_2}((n_1^2+x_1^2n_2^2)^{\frac{p-2}{2}}x_1n_2)\\ &= (p-2)|\nabla_X dist(x,\partial\Omega^+)|^{p-4}n_1n_2^2x_1.\end{aligned}$$

Plugging the above expressions into inequality (12), we arrive at

$$\begin{aligned}\int_{\Omega^+}|\nabla_X f(x)|^p dx \geq &-(p-1)(|\gamma|^{\frac{p}{p-1}}+\gamma)\int_{\Omega^+}\frac{(n_1^2+x_1^2n_2^2)^{p/2}}{(x_1n_1+x_2n_2-d)^p}|f(x)|^p dx\\ &+(p-2)\gamma\int_{\Omega^+}\frac{|\nabla_X dist(x,\partial\Omega^+)|^{p-4}n_1n_2^2x_1}{(x_1n_1+x_2n_2-d)^{p-1}}|f(x)|^p dx,\end{aligned}$$

that gives inequality (13). If one of the cases $n(x)=(1,0)$ or $n(x)=(0,1)$ holds, then the last term of the above inequality vanishes, so we get

$$\int_{\Omega^+}|\nabla_X f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}}+\gamma)\int_{\Omega^+}\frac{(n_1^2+x_1^2n_2^2)^{p/2}}{(x_1n_1+x_2n_2-d)^p}|f(x)|^p dx. \quad (15)$$

Then, we maximise the above inequality by differentiating with respect to γ, so that we obtain

$$\frac{p}{p-1}|\gamma|^{\frac{1}{p-1}}+1=0, \quad \text{and hence} \quad \gamma=-\left(\frac{p-1}{p}\right)^{p-1}.$$

By putting the value of γ into inequality (15), we complete the proof. □

Corollary 3.4. *Let Ω^+ be a half-space on the Heisenberg group. Then for every function $f\in C_0^\infty(\Omega^+)$ and $p>1$, we have*

$$\int_{\Omega^+}|\nabla_H f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p\int_{\Omega^+}\frac{|(n_1+2x_2n_3, n_2-2x_1n_3)|^p}{dist(x,\partial\Omega^+)^p}|f(x)|^p dx, \quad (16)$$

where $dist(x,\partial\Omega^+)=\langle x,n(x)\rangle-d$ and $d\in\mathbb{R}$.

Remark 3.5. Note that if we choose $n(x)=(0,0,1)$, $p=2$ and $d=0$ in inequality (16), then we get

$$\int_{\Omega^+}|\nabla_H f(x)|^2 dx \geq \int_{\Omega^+}\frac{|x_1|^2+|x_2|^2}{x_3^2}|f(x)|^2 dx. \quad (17)$$

The Hardy inequality of the form (17) in the half-space on the Heisenberg group was shown by Luan and Young [12]. □

Proof of Corollary 3.4. In order to prove Corollary 3.4, we calculate

$$\begin{aligned}dist(x,\partial\Omega^+) &= x_1n_1+x_2n_2+x_3n_3-d,\\ \nabla_X dist(x,\partial\Omega^+) &= (n_1+2x_2n_3, n_2-2x_1n_3),\\ |\nabla_X dist(x,\partial\Omega^+)|^p &= ((n_1+2x_2n_3)^2+(n_2-2x_1n_3)^2)^{p/2}.\end{aligned}$$

Then we compute

$$\begin{aligned}
&\mathcal{L}_p dist(x,\partial\Omega^+) = \nabla_H\cdot(|\nabla_H dist(x,\partial\Omega^+)|^{p-2}\nabla_H dist(x,\partial\Omega^+)) \\
&= X_1((n_1+2x_2n_3)^2+(n_2-2x_1n_3)^2)^{\frac{p-2}{2}}(n_1+2x_2n_3) \\
&\quad + X_2((n_1+2x_2n_3)^2+(n_2-2x_1n_3)^2)^{\frac{p-2}{2}}(n_2-2x_1n_3) \\
&= -2(p-2)|\nabla_H dist(x,\partial\Omega^+)|^{p-4}(n_1+2x_2n_3)(n_2-2x_1n_3)n_3 \\
&\quad + 2(p-2)|\nabla_H dist(x,\partial\Omega^+)|^{p-4}(n_2-2x_1n_3)(n_1+2x_2n_3)n_3 \\
&= 0.
\end{aligned}$$

Plugging the above expressions into inequality (12), we arrive at

$$\int\limits_{\Omega^+}|\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}}+\gamma)\int\limits_{\Omega^+}\frac{|(n_1+2x_2n_3,n_2-2x_1n_3)|^p}{(x_1n_1+x_2n_2+x_3n_3-d)^p}|f(x)|^p dx, \quad (18)$$

which can be maximised by differentiating with respect to γ. Then we have

$$\frac{p}{p-1}|\gamma|^{\frac{1}{p-1}}+1=0, \quad \text{and hence} \quad \gamma=-\left(\frac{p-1}{p}\right)^{p-1}.$$

By inserting the value of γ into inequality (18), we get the inequality

$$\int_{\Omega^+}|\nabla_H f(x)|^p dx \geq \left(\frac{p-1}{p}\right)^p\int_{\Omega^+}\frac{|(n_1+2x_2n_3,n_2-2x_1n_3)|^p}{dist(x,\partial\Omega^+)^p}|f(x)|^p dx,$$

that proves Corollary 3.4. □

Corollary 3.6. *Let Ω^+ be a half-space on the Engel group $\mathbb{E}$. Then for every function $f\in C_0^\infty(\Omega^+)$ and for $\gamma\in\mathbb{R}$ and $p=2$, we have*

$$\begin{aligned}
\int_{\Omega^+}|\nabla_H f(x)|^2 dx \geq -\left(|\gamma|^2+\gamma\right)\int_{\Omega^+}\frac{|\nabla_H dist(x,\partial\Omega^+)|^2}{dist(x,\partial\Omega^+)^2}|f(x)|^2 dx \quad (19)\\
+\frac{\gamma}{6}\int_{\Omega^+}\frac{x_2n_4}{dist(x,\partial\Omega^+)}|f(x)|^2 dx,
\end{aligned}$$

where $dist(x,\partial\Omega^+)=\langle x,n(x)\rangle-d$ and $d\in\mathbb{R}$.

Proof of Corollary 3.6. A straightforward calculation gives

$$\begin{aligned}
dist(x,\partial\Omega^+) &= x_1n_1+x_2n_2+x_3n_3+x_4n_4-d, \\
\nabla_H dist(x,\partial\Omega^+) &= \left(n_1-\frac{x_2n_3}{2}-\frac{x_3n_4}{2}-\frac{x_1x_2n_4}{12}, n_2+\frac{x_1n_3}{2}+\frac{x_1^2n_4}{12}\right), \\
|\nabla_H dist(x,\partial\Omega^+)|^2 &= \left(n_1-\frac{x_2n_3}{2}-\frac{x_3n_4}{2}-\frac{x_1x_2n_4}{12}\right)^2+\left(n_2+\frac{x_1n_3}{2}+\frac{x_1^2n_4}{12}\right)^2,
\end{aligned}$$

and

$$
\begin{aligned}
\mathcal{L}(dist(x,\partial\Omega^+)) =& \nabla_H \cdot \nabla_H dist(x,\partial\Omega^+) \\
=& \nabla_H \cdot \left(n_1 - \frac{x_2 n_3}{2} - \frac{x_3 n_4}{2} - \frac{x_1 x_2 n_4}{12}, n_2 + \frac{x_1 n_3}{2} + \frac{x_1^2 n_4}{12} \right) \\
=& X_1 \left(n_1 - \frac{x_2 n_3}{2} - \frac{x_3 n_4}{2} - \frac{x_1 x_2 n_4}{12} \right) + X_2 \left(n_2 + \frac{x_1 n_3}{2} + \frac{x_1^2 n_4}{12} \right) \\
=& \frac{x_2 n_4}{6}.
\end{aligned}
$$

Plugging the above expressions into inequality (12), we get

$$
\begin{aligned}
\int_{\Omega^+} |\nabla_H f(x)|^2 dx \geq & - (|\gamma|^2 + \gamma) \int_{\Omega^+} \frac{|\nabla_H dist(x,\partial\Omega^+)|^2}{dist(x,\partial\Omega^+)^2} |f(x)|^2 dx \\
& + \frac{\gamma}{6} \int_{\Omega^+} \frac{x_2 n_4}{dist(x,\partial\Omega^+)} |f(x)|^2 dx,
\end{aligned}
$$

that completes the proof of Corollary 3.6. □

4. Proof of the main results

The approach to prove the main results is based on the works [15] and [16] (see, also [14]-[17]). For a vector field $g \in C^\infty(\Omega)$, we compute

$$
\begin{aligned}
\int_\Omega \mathrm{div}_X g |f(x)|^p dx = & -p \int_\Omega |f(x)|^{p-1} \langle g, \nabla_X f(x) \rangle dx \\
& \leq p \left(\int_\Omega |\nabla_X f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_\Omega |g|^{\frac{p}{p-1}} |f(x)|^p dx \right)^{\frac{p-1}{p}} \\
& \leq \int_\Omega |\nabla_H f(x)|^p dx + (p-1) \int_\Omega |g|^{\frac{p}{p-1}} |f(x)|^p dx.
\end{aligned}
$$

Here we have applied the divergence theorem, the Hölder inequality and the Young inequality, respectively. By rearranging the above expression, we arrive at

$$
\int_\Omega |\nabla_X f(x)|^p dx \geq \int_\Omega (\mathrm{div}_X g - (p-1)|g|^{\frac{p}{p-1}}) |f(x)|^p dx. \tag{20}
$$

A suitable choice of the vector field g in each special case is a key argument of our proofs.

Proof of Theorem 2.2. Let us set

$$
g = \gamma \frac{|\nabla_H \langle Z(x), n(x) \rangle|^{p-2}}{|\langle Z(x), n(x) \rangle|^{p-1}} \nabla_H \langle Z(x), n(x) \rangle,
$$

then a straightforward computation gives

$$
|g|^{\frac{p}{p-1}} = |\gamma|^{\frac{p}{p-1}} \frac{|\nabla_H \langle Z(x), n(x) \rangle|^p}{|\langle Z(x), n(x) \rangle|^p}, \tag{21}
$$

and
$$\mathrm{div}_H g = \gamma \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}} - \gamma(p-1)\frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p}. \tag{22}$$

Plugging the above expressions (21) and (22) into inequality (20), we have

$$\int_\Omega |\nabla_H f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_\Omega \frac{|\nabla_H \langle Z(x), n(x)\rangle|^p}{|\langle Z(x), n(x)\rangle|^p} |f(x)|^p dx$$
$$+ \gamma \int_\Omega \frac{\mathcal{L}_p(\langle Z(x), n(x)\rangle)}{|\langle Z(x), n(x)\rangle|^{p-1}} |f(x)|^p dx,$$

which proves inequality (9). □

Proof of Theorem 3.1. Let us take

$$g = \gamma \frac{|\nabla_X dist(x, \partial\Omega^+)|^{p-2}}{dist(x, \partial\Omega^+)^{p-1}} \nabla_X dist(x, \partial\Omega^+), \tag{23}$$

so that we have
$$|g|^{\frac{p}{p-1}} = |\gamma|^{\frac{p}{p-1}} \frac{|\nabla_X dist(x, \partial\Omega^+)|^p}{dist(x, \partial\Omega^+)^p}, \tag{24}$$

and
$$\mathrm{div}_X g = \gamma \frac{\mathcal{L}_p dist(x, \partial\Omega^+)}{dist(x, \partial\Omega^+)^{p-1}} - \gamma(p-1)\frac{|\nabla_X dist(x, \partial\Omega^+)|^p}{dist(x, \partial\Omega^+)^p}. \tag{25}$$

Combining expressions (24) and (25) with inequality (20), we obtain

$$\int_\Omega |\nabla_X f(x)|^p dx \geq -(p-1)(|\gamma|^{\frac{p}{p-1}} + \gamma) \int_\Omega \frac{|\nabla_X dist(x, \partial\Omega^+)|^p}{dist(x, \partial\Omega^+)^p} |f(x)|^p dx$$
$$+ \gamma \int_\Omega \frac{\mathcal{L}_p dist(x, \partial\Omega^+)}{dist(x, \partial\Omega^+)^{p-1}} |f(x)|^p dx,$$

which proves inequality (12). □

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