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Generalizing Multiplicative Convex Functions

P. Jiménez-Rodríguez
Dep. de Matemática Aplicada, Universidad de Valladolid, 42004 Soria, Spain
pablo.jimenez.rodriguez@uva.es

M. E. Martínez-Gómez, G. A. Muñoz-Fernández
Dep. de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain
mariae24@ucm.es, gustavo_fernandez@mat.ucm.es

J. B. Seoane-Sepúlveda
Inst. de Matemática Interdisciplinar (IMI), Dep. de Análisis Matemático y Matemática Aplicada, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain
jseoane@ucm.es

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This work is a natural continuation of the ongoing research started by the authors in a previous article [*Describing multiplicative convex functions*, J. Convex Analysis 27 (2020) 935–942] with respect to multiplicative convex functions, their continuity properties and their generalizations. Some results regarding algebraic genericity are also presented and open questions are posed.

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1. Introduction

The theory of convex functions keeps playing a central role in real analysis, in operator theory and in some realms of applied mathematics, such as optimization theory or management science (we refer the interested reader to [2, 5, 14] for applications of the property of convexity). Since Jensen began this research, they have been exhaustively described and many of their properties have been revealed.

Recall that a function $f : V \to \mathbb{R}$ (where V is a vector space over $\mathbb{R}$) is said to be *convex* if, whenever x, $y \in V$ and $0 \leq \lambda \leq 1$, we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y).$$

The study of convex functions has been extended considering other inequalities. In this direction, Niculescu proposed the following definition (see [15]):

Let $I \subseteq (0,\infty)$ be an interval. A function $f : I \to (0,\infty)$ is called *multiplicative convex* if, for every $x, y \in I$ and $\lambda \in [0,1]$, we have

$$f(x^{1-\lambda} y^{\lambda}) \leq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{1}$$

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Functions satisfying inequality (1) are also known as *GG-convex functions*, since they are supposed to replace the arithmetic mean in the inequality that defines the convex functions by its corresponding geometric mean.

In [11], the authors considered the definition as a way of upgrading the operations that take part in the definition of convex functions. In this direction, they see that the addition operation turns into multiplication operation, and the multiplication operation turns into power operation. But, λ and $\mu = 1-\lambda$ are related by $\lambda+\mu = 1$. So, completing the upgrade, λ and μ should be related by $\lambda \cdot \mu = 1$ and the following definition was proposed.

Definition 1.1. Let $f : (0,\infty) \to [0,\infty)$ be a function. We will say that f is *multiplicative convex* (or f is an *mc*-function, for short) if, for every $\mu > 0$ and $x, y > 0$ we have

$$f(x^\mu y^{1/\mu}) \leq f(x)^\mu f(y)^{1/\mu}. \qquad \square \tag{2}$$

The inequality in (2) implies that $f(1) \geq 1$. In [11] the following distinction was made:

Definition 1.2. A function $f : (0,\infty) \to [0,\infty)$ is said to be *1-multiplicative convex* (or f is a *mc*1-function) if, for every $\mu > 0$ and $x, y > 0$ we have

$$f(x^\mu y^{1/\mu}) \leq f(x)^\mu f(y)^{1/\mu}.$$

and the function satisfies also the condition $f(1) = 1$. We will use the notation

$$\begin{aligned} \mathcal{MC} &:= \{f : (0,\infty) \to [0,\infty) \mid f \text{ is a } mc\text{-function}\} \\ \mathcal{MC}_1 &:= \{f : (0,\infty) \to [0,\infty) \mid f \text{ is a } mc1\text{-function}\}. \qquad \square \end{aligned}$$

In [11] the property $f(1) = 1$ was of great importance to give the following characterization of *mc*1-functions:

Theorem 1.3. [11, Theorem 2.5] *Let $f : (0,\infty) \to [0,\infty)$. Then, f is a mc1-function if and only if it can be written in the form*

$$f(x) = \begin{cases} b^{\log_a(x)} & \text{if } 0 < x \leq 1, \\ b'^{\log_{a'}(x)} & \text{if } x > 1, \end{cases} \tag{3}$$

where a, b, a' and b' satisfy the following conditions:

(1) $0 < a < 1$ *and* $a' > 1$.

(2) *If $b < 1$, then $\log_b(b') \leq \log_a(a') < 0$ (which, in particular, implies $b' > 1$).*

(3) *If $b > 1$, then $\log_b(b') \geq \log_a(a')$.*

In the direction of generalizing *mc*1-functions into *mc*-functions we can find the following remark.

Remark 1.4. ([11]) If f is a *mc*1-function and $C > 1$ then $g(x) = Cf(x)$ is a *mc*-function (but not *mc*1). $\square$

The problem of the characterization of the *mc*-functions which are not *mc*1 remains open, but a natural approach would be to relate both classes. We can then find the following natural question:

Problem 1.5. [11] *If f is an mc-function, is $g(x) = \frac{f(x)}{f(1)}$ an mc1-function?*

That was negatively answered with the following example.

Example 1.6. [11] The function $f(x) = x + 1$ is a *mc*-function but $g(x) = \frac{1}{2}f(x)$ is not *mc*1.

In Section 2 of this manuscript we shall focus on the study of certain features of *mc*-functions in order to see some of the algebraic properties of the set $\mathcal{MC}$. In the following sections, we shall use those properties to describe the algebraic structure of this set and we provide an example of a discontinuous *mc*-function, which leads to the conclusion that the *mc*-property does not imply continuity. We will then focus our attention to the question of whether continuous *mc*-functions may be all obtained via basic manipulations of *mc*1-functions. Some results regarding algebraic genericity are also presented and open questions will be posed in the last section. This work is a natural continuation of [11].

2. Some algebraic properties of the set $\mathcal{MC}$

Before stating the main points of this section, let us mention a well-known result that will be crucial in some of the proofs presented here. The following lemma can be found in [10] and it is very well known, however we include it here for self-containment.

Lemma 2.1. *For $1 \leq p \leq \infty$, let ℓ_p denote the space*

$$\ell_p = \left\{ x = (x_1, x_2, \dots) \in \mathbb{R}^{\mathbb{N}} : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}$$

endowed with the usual norm, which we denote by $\|\cdot\|_p$. Then, ℓ_p may be continuously embedded into ℓ_q for $1 \leq p \leq q \leq \infty$.
In particular, for every $1 \leq p \leq q \leq \infty$ and $x \in \ell_p$, $\|x\|_q \leq \|x\|_p$

A natural, and straightforward, consequence of the previous lemma is the following result.

Corollary 2.2. *If a, b are positive numbers and $\mu \geq 1$, then $a^\mu + b^\mu \leq (a + b)^\mu$.*

The first algebraic property of $\mathcal{MC}$ indicates that this set is closed under addition:

Lemma 2.3. *If f and g are mc-functions, then $h(x) = f(x)+g(x)$ in a mc-function as well.*

Proof. Let f, g be *mc*-functions, $x, y \in (0, \infty)$ and $\mu \geq 1$, then

$$\begin{aligned}
(f+g)(x^\mu y^{\frac{1}{\mu}}) &= f(x^\mu y^{\frac{1}{\mu}}) + g(x^\mu y^{\frac{1}{\mu}}) \leq f(x)^\mu f(y)^{\frac{1}{\mu}} + g(x)^\mu g(y)^{\frac{1}{\mu}} \\
&\leq f(x)^\mu (g(y) + f(y))^{\frac{1}{\mu}} + g(x)^\mu (f(y) + g(y))^{\frac{1}{\mu}} \\
&= (f(x)^\mu + g(x)^\mu)(g(y) + f(y))^{\frac{1}{\mu}} \leq \big(f(x) + g(x)\big)^\mu (g(y) + f(y))^{\frac{1}{\mu}} \\
&= [(f+g)(x)]^\mu [(f+g)(y)]^{\frac{1}{\mu}}
\end{aligned}$$

Therefore, $f + g$ is an *mc*-function. □

Corollary 2.4. *If f is a mc-function and $c \geq 1$, then $g(x) = f(x) + c$ is a mc-function as well.*

For the case $g(x) = f(x) + c$ with $c \in (0, 1)$ we will study first the case where f is a $mc1$-function.

Lemma 2.5. *If f is a mc1-function and $f(x) \geq 1$ for every $x \in (0, \infty)$, then $g(x) = f(x) + c$ is a mc-function for $0 < c < 1$.*

Proof. We know that f is decreasing in $(0, 1)$ and increasing in $(1, \infty)$, so we can prove that:

$$f(x)^\mu + c \leq (f(x) + c)^\mu. \tag{4}$$

If we define $h(x) = (f(x)+c)^\mu - f(x)^\mu - c$, then $h'(x) = \mu f'(x)[f(x)+c)^{\mu-1} - f(x)^{\mu-1}]$. Due to the monotonicity of f we have that $h(x)$ has an absolute minimum at $x = 1$ but $h(1) = (1 + c)^\mu - (1 + c) \geq 0$ for every $\mu \geq 1$, therefore $h(x) \geq 0$ for every $x \in (0, \infty)$ and we have (4).

Let us see that g is a mc-function. Since $1 \leq f(y)^{1/\mu}$ we have

$$\begin{aligned} g(x^\mu y^{\frac{1}{\mu}}) &= f(x^\mu y^{\frac{1}{\mu}}) + c \leq f(x)^\mu f(y)^{\frac{1}{\mu}} + c \leq f(x)^\mu f(y)^{\frac{1}{\mu}} + cf(y)^{\frac{1}{\mu}} \\ &= f(y)^{\frac{1}{\mu}}(f(x)^\mu + c) \leq (f(y) + c)^{\frac{1}{\mu}}(f(x)^\mu + c) \\ &\leq (f(x) + c)^\mu (f(y) + c)^{\frac{1}{\mu}} = g(x)^\mu g(y)^{\frac{1}{\mu}}, \end{aligned}$$

where the latter inequality is due to (4). □

Notice that, according to Theorem 1.3, the situation described in Lemma 2.5 refers to the decreasing-increasing type of functions. For the other two types, we have the following:

Lemma 2.6. *If f is a monotone mc1-function then $g(x) = f(x) + c$ is not a mc-function for $0 < c < 1$.*

Proof. We are going to suppose that f is increasing (using that if f is a decreasing and mc-function, then $f(\frac{1}{x})$ is increasing and mc as well and we can generalize the result). So that we can use the characterization given in Theorem 1.3 with $a \in (0, 1)$, $a' > 1$, $b < 1$ and $b' > 1$.

Next, taking
$$x = y = a^{\frac{\log \frac{1-c}{4}}{\log b}} \in (0, 1)$$

and $\mu = 1$ we obtain
$$g(xy) = b^{\log_a(xy)} + c,$$

$$g(x)g(y) = b^{\log_a(xy)} + c(b^{\log_a x} + b^{\log_a y} + c),$$

but
$$b^{\log_a x} + b^{\log_a y} + c = \frac{1-c}{4} + \frac{1-c}{4} + c = \frac{1+c}{2} < 1.$$

Therefore $g(xy) > g(x)g(y)$ and $g(x)$ is not an mc-function. □

Hence, being closed under addition by a positive constant smaller than 1 depends on the function: $f(x) = x + c$ is not *mc* for any $c < 1$ but $g(x) = x + 1 + c$ is *mc* for every $c > 0$.

The following result tells us that $\mathcal{MC}$ has a structure like an algebra:

Lemma 2.7. *If f and g are mc-functions, then $h(x) = f(x)g(x)$ is a mc-function.*

Proof. Let f an g be *mc*-functions and let $h(x) = f(x)g(x)$.

$$\begin{aligned} h(x^\mu y^{\frac{1}{\mu}}) &= f(x^\mu y^{\frac{1}{\mu}})g(x^\mu y^{\frac{1}{\mu}}) \leq f(x)^\mu f(y)^{\frac{1}{\mu}} g(x)^\mu g(y)^{\frac{1}{\mu}} \\ &= (f(x)g(x))^\mu (f(y)g(y))^{\frac{1}{\mu}} = h(x)^\mu h(y)^{\frac{1}{\mu}}. \end{aligned}$$ □

Corollary 2.8. *Let f be a mc-function and $\lambda \geq 1$. Then, λf is a mc-function as well.*

Lemma 2.9. *If f and g are mc-functions and f is increasing then $f(g(x))$ is a mc-function.*

Proof. $f(g(x^\mu y^{\frac{1}{\mu}})) \leq f(g(x)^\mu g(y)^{\frac{1}{\mu}}) \leq f(g(x))^\mu f(g(y))^{\frac{1}{\mu}}$. □

Corollary 2.10. *If f is a mc-function then f^n is a mc-function for every $n \in \mathbb{N}$.*

With respect to convergence of *mc*-functions, we have the following result:

Lemma 2.11. *Let $f(x) = \lim_{n\to\infty} f_n(x)$ with $f_n \in \mathcal{MC}$ (resp. $f_n \in \mathcal{MC}_1$) for every $n \in \mathbb{N}$. Then, $f(x)$ is a mc-function (resp. a mc1-function).*

Proof. Just notice that

$$f(x^\mu y^{1/\mu}) = \lim_{n\to\infty} f_n(x^\mu y^{1/\mu}) \leq \lim_{n\to\infty} f_n(x)^\mu f_n(y)^{1/\mu} = f(x)^\mu f(y)^{1/\mu}.$$

If, furthermore, $f_n(x) = 1$ for every n, $f(1) = \lim_{n\to\infty} f_n(1) = 1$. □

The previous battery of examples, specially Lemma 2.3, Lemma 2.7 and Corollary 2.8, describes a very specific structure for the $\mathcal{MC}$ functions. We will need to introduce some considerations before carrying on with the results.

Definition 2.12. [13] A *cone* is a set P endowed with two operations, addition and multiplication by positive scalars, which fulfill the usual properties (commutativity, associativity, existence of neutral element, etc.).

If this set contains a linearly independent subset of infinite cardinality, the cone is said to be *infinite*. The dimension of the cone is the maximal possible cardinality of such a linearly independent set. □

The theory of cones provides a different scope to work in settings more general than the usual Banach or vector spaces. We refer the interesting reader to [13], where the authors make an introduction to the functional analysis on cones, overviewing the basic notions and definitions and studying how some of the classical results on Banach spaces (such as the Hahn-Banach Theorem) can be extended to this theory.

In particular, cones may be useful when being closed under scalar multiplication is not fulfilled but for positive scalars. In this direction, let us briefly recall the recently introduced notion of *lineability* (see, e.g., [3, 1, 4, 7, 8, 16, 9, 6]).

Definition 2.13. Let X be a vector space, $M \subseteq X$ and let μ any (finite or infinite) cardinal number. We say that M is *μ-lineable* if $M \cup \{0\}$ contains a vector space of dimension μ. If X is a topological vector space, we shall say that M is *μ-spaceable* if $M \cup \{0\}$ contains a closed vector space of dimension μ. We say that M is *μ-coneable* if $M \cup \{0\}$ contains a cone of dimension μ. $\square$

The idea behind Definition 2.13 is the search of algebraic structures on certain sets (whose elements usually verify some pathological property) of the greatest possible dimension. Depending on the set under study, some of the structures introduced in 2.13 will be allowed and some will not. For example (see, [12]):

Theorem 2.14. *There exist two closed cones of dimension $\mathfrak{c}$, C_1, C_2, with the property that, if $f \in C_1 \setminus \{0\}$ and $g \in C_2 \setminus \{0\}$, then:*

(1) $f \in D[-1,1]$.

(2) $g \in D[-1,1]$.

(3) $f * g \notin D[-1,1]$.

Nevertheless, two analogous structures can not be found if we are requiring them to be closed vector spaces (so the spaceability analogue does not exist).

For the properties we are interested in this mansucript we can not search for cones, since being closed under multiplication by positive scalars is not ensured. Because of this, we have to consider another definition.

Definition 2.15. Let X be a vector space and $M \subseteq X$. We say that M is an infinite dimensional *truncated cone* if M fulfills the following properties:

(1) M is closed under addition.

(2) M is closed under multiplication by scalar no less than 1.

(3) M contains a set of infinite cardinality of linearly independent elements.

The maximal possible cardinality of a set like in property (3) is the *dimension* of the truncated cone.

If, furthermore, we can define a multiplication on X (satisfying the usual properties on itself and with respect to addition) and M is closed under products, then we say that M is an *algebraic truncated cone*. In that case,

(1) the *linear dimension* of the truncated cone will be the maximal possible cardinality so that there exists a subset of such cardinality and consisting on linearly independent elements.

(2) The *algebraic dimension* will be the maximal possible cardinality so that there exists a subset of such cardinality and consisting on algebraic independent elements (that is, so that the only polynomial vanishing on them is the null polynomial).

Trivially we obtain that the algebraic dimension is not greater than the linear dimension. $\square$

The relationship between truncated cones and cones, appart from the usual geometric one, can be represented in the following:

Proposition 2.16. *Let A be an infinite dimensional truncated cone of linear dimension μ. Then $A - A = \{a_1 - a_2 : a_i \in A\}$ is an infinite dimensional cone of the same dimension.*

Proof. Let $a_1, a_2, b_1, b_2 \in A$, $a \in A - A$ and $\lambda \geq 0$. Then

$$\lambda a = \lambda(a_1 - a_2) = [(\lambda+1)a_1 + a_2] - [(\lambda+1)a_2 + a_1] \in A - A,$$
$$(a_1 - a_2) + (b_1 - b_2) = (a_1 + b_1) - (a_2 + b_2) \in A - A.$$

Let next $\{f_i : i \in \Gamma\} \subseteq A$ be a linearly independent set of cardinality μ, $i^0, i^{(1)} \in \Gamma$ be different elements and $h : (\Gamma \setminus \{i^{(0)}\}) \times (\Gamma \setminus \{i^{(1)}\}) \to \Gamma$ be a bijection. If we define the set

$$B = \left\{ f_{h(i,i^{(0)})} - f_{h(i^{(1)},i)} : i \in \Gamma \setminus \{i^{(0)}, i^{(1)}\} \right\}$$

then $\mathrm{card}(B) = \mu$ and we claim that $B \subseteq A - A$ is a linearly independent set: Indeed, assume $\lambda_k \in \mathbb{R}$, $i_k \in \Gamma \setminus \{i^{(0)}, i^{(1)}\}$ for $1 \leq k \leq n$. Then, we notice that the elements $h(i_k, i^{(0)})$, $h(i^{(1)}, i_k) \in \Gamma$ are all different and therefore

$$0 = \sum_{k=1}^{n} \lambda_k \left(f_{h(i_k,i^{(0)})} - f_{h(i^{(1)},i_k)} \right) = \sum_{k=1}^{n} \lambda_k f_{h(i_k,i^{(0)})} - \sum_{k=1}^{n} \lambda_k f_{h(i^{(1)},i_k)}$$

leads to the conclusion $\lambda_k = 0$ for every $1 \leq k \leq n$. □

Let us come back to our object of study and focus our attention on continuous *mc*-functions. We can summarize the main results of this section so far in the following theorem:

Theorem 2.17. *Let $A = C(0,\infty) \cap \mathcal{MC}$. Then,*

(1) *A is closed for the compact-open topology.*

(2) *A is an algebraic truncated cone.*

The question that we may ask ourselves is if the truncated cone from Theorem 2.17 is of infinite dimension and, in that case, what is the maximal dimension we can consider.

Theorem 2.18. *The set A defined in Theorem 2.17 is a truncated cone of algebraic dimension $\mathfrak{c}$ (the continuum), that is, of the largest possible dimension.*

Proof. Let us define, for $b \in (0,1)$,

$$f_b(x) = \begin{cases} b^{\log_{1/2}(x)} & \text{if } 0 < x \leq 1, \\ \left(\frac{1}{b}\right)^{\log_2(x)} & \text{if } x > 1. \end{cases}$$

Then, by Theorem 1.3, f_b is a *mc*1-function for every $b \in (0,1)$. Let us show, to finish with, that this set is algebraically independent. Indeed, let $\lambda_i \in \mathbb{R}$, $b_j \in (0,1)$ and $k_{i,j} \in \mathbb{N}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. We define

$$g(x) = \sum_{i=1}^{n} \prod_{j=1}^{m} f_{b_j}(x)^{k_{i,j}}.$$

Assume then that $g(x) = 0$. In particular, evaluating in $g(\frac{1}{2^s})$ for $0 \leq s \leq n-1$ we obtain

$$\begin{aligned} 0 = g(1) &= \sum_{i=1}^{n} \lambda_i, \\ 0 = g\left(\frac{1}{2}\right) &= \sum_{i=1}^{n} \lambda_i b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}}, \\ 0 = g\left(\frac{1}{2^2}\right) &= \sum_{i=1}^{n} \lambda_i b_1^{2k_{i,1}} b_2^{2k_{i,2}} \cdot \ldots \cdot b_m^{2k_{i,m}} = \sum_{i=1}^{n} \lambda_i \left(b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}}\right)^2, \end{aligned} \tag{5}$$

and so on, also having that:

$$\begin{aligned} 0 = g\left(\frac{1}{2^{n-1}}\right) &= \sum_{i=1}^{n} \lambda_i b_1^{(n-1)k_{i,1}} b_2^{(n-1)k_{i,2}} \cdot \ldots \cdot b_m^{(n-1)k_{i,m}} \\ &= \sum_{i=1}^{n} \lambda_i \left(b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}}\right)^{n-1}. \end{aligned} \tag{6}$$

Denote $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $h_i = \left(b_1^{k_{i,1}} b_2^{k_{i,2}} \cdot \ldots \cdot b_m^{k_{i,m}}\right)$ and

$$A = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ h_1 & h_2 & h_3 & \cdots & h_n \\ h_1^2 & h_2^2 & h_3^2 & \cdots & h_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1^{n-1} & h_2^{n-1} & h_3^{n-1} & \cdots & h_n^{n-1} \end{bmatrix}.$$

Then, the system given by (5) and (6) may be summarized as $A\Lambda^T = 0$, where A is a Van der Monde type matrix and therefore it is invertible, leading us to conclude $\Lambda = (0, \ldots, 0)$. □

3. Discontinuous mc-functions

In this section we focus on the question of whether every mc-function is continuous or not. The following example shows that there are mc-functions that are not continuous.

Example 3.1. $f(x) = \begin{cases} 2 & if \quad x < 1 \\ 4 & if \quad x \geq 1 \end{cases}$ is a mc-function which is not continuous.

Proof. $f(x^\mu y^{\frac{1}{\mu}}) \leq 4 \leq 2^{(\mu + \frac{1}{\mu})} \leq f(x)^\mu f(y)^{\frac{1}{\mu}}$. □

Following Example 3.1, we generalize it in our next result:

Theorem 3.2. *Let $\alpha > 1$, $\alpha < \beta \leq \alpha^2$ and f be a $mc1$-function. Define the function*

$$g(x) = \begin{cases} \alpha f(x) & \text{if } 0 < x \leq 1, \\ \beta f(x) & \text{if } x > 1. \end{cases}$$

Then g is a discontinuous mc-function.

Proof. It is trivial to prove that g is discontinuous, since

$$\lim_{x\to 1^-} g(x) = \alpha < \beta = \lim_{x\to 1^+} g(x).$$

Let $x, y > 0$ and $\mu > 0$.

Assume first that $x^\mu y^{1/\mu} \le 1$. Then

$$\begin{aligned} g(x^\mu y^{1/\mu}) &= \alpha f(x^\mu y^{1/\mu}) \le \alpha f(x)^\mu f(y)^{1/\mu} \\ &\le \alpha^{\mu+1/\mu} f(x)^\mu f(y)^{1/\mu} = [\alpha f(x)]^\mu [\alpha f(y)]^{1/\mu} \\ &\le g(x)^\mu g(y)^{1/\mu}. \end{aligned}$$

If now $x^\mu y^{1/\mu} > 1$ and $x, y > 1$, then $g(x^\mu y^{1/\mu}) \le g(x)^\mu g(y)^{1/\mu}$, analogously as in the previous case.

If $x \le 1$ (without loss of generality), then

$$\begin{aligned} g(x^\mu y^{1/\mu}) &= \beta f(x^\mu y^{1/\mu}) \le \alpha^2 f(x^\mu y^{1/\mu}) \\ &\le \alpha^{\mu+1/\mu} f(x)^\mu f(y)^{1/\mu} = [\alpha f(x)]^\mu [\alpha f(y)]^{1/\mu} \\ &\le g(x)^\mu g(y)^{1/\mu}. \end{aligned}$$ □

The functions defined in Theorem 3.2 are all discontinuous at $x = 1$. Concerning other points of discontinuity, we may prove the following result:

Proposition 3.3. *Let $x_0 > 0$. Then, there exists a mc-function that is discontinuous at x_0.*

Proof. Assume first $x_0 < 1$ and let f be an increasing $mc1$-function and g be a function from Theorem 3.2. Then,

$$h_{x_0}(x) = g\Big(\frac{1}{f(x_0)} f(x)\Big)$$

is an mc-function which is discontinuous at x_0.

If now $x_0 > 1$, then we may just define

$$h^{(x_0)}(x) = h_{\frac{1}{x_0}}\Big(\frac{1}{x}\Big).$$ □

Even though we can not ensure that the set of discontinuous mc-functions is a truncated cone (since being closed under multiplication is not guaranteed), we can take the idea from Definition 2.13 and search for certain algebraic structures:

Proposition 3.4. *The set*

$$A = \{f : (0,\infty) \to [0,\infty) \ : \ f \text{ is a discontinuous mc-function}\}$$

contains an algebraic truncated cone of algebraic dimension $\mathfrak{c}$.

Proof. Similarly as in Theorem 2.18, we define, for $b \in (0,1)$,

$$f_b(x) = \begin{cases} 2b^{\log_{1/2}(x)} & \text{if } 0 < x \leq 1, \\ 4\left(\frac{1}{b}\right)^{\log_2(x)} & \text{if } x > 1. \end{cases}$$

Each f_b is a discontinuous mc-function, by means of Theorem 3.2, and they form a linearly independent set (the proof follows similarly as the proof in Theorem 2.18). We can just define the following truncated cone

$$T = \left\{ \lambda \prod_{i=1}^{n} f_{b_i}(x) \,:\, \lambda \geq 1,\, n \in \mathbb{N}\, b_1, \ldots, b_n \in (0,1) \right\},$$

finishing the proof. □

4. Other continuous mc-functions

Multiplicative convexity is closed under the majority of the algebraic operations, similarly as continuity. In this section we will examine whether the set of $mc1$-functions can serve as a system of generators of the set

$$A = C(0,\infty) \cap \mathcal{MC},$$

that is, if we can obtain every element of the set A by means of algebraic operations over elements from $\mathcal{MC}_1$.

Lemma 4.1. *The function $h(x) = f(x) + c$ with $f \in \mathcal{MC}_1$, $f(x) \geq 1$ for every $x \in (0,\infty)$ and $c \in (0,1)$ verifies that $h \in A$ but can not be set as $\sum_{n=1}^{m} \lambda_n f_n(x)$ with $f_n \in \mathcal{MC}_1$ and $\lambda_n \in [1,\infty)$.*

Proof. We have shown in Lemma 2.5 that if f is a $mc1$-function that satisfies $f(x) \geq 1$ for every $x \in (0,\infty)$ and $c \in (0,1)$, then $h(x) = f(x) + c$ is also a mc-function. Since f is continuous we have that $h \in A$.

Notice that $h(1) = 1 + c < 2$. If $m > 1$, $\lambda_1, \ldots, \lambda_m \geq 1$ and $f_1, \ldots, \lambda_m$ are $mc1$-functions, then $\sum_{n=1}^{m} \lambda_n f_n(x) > 2$, so that if $h(x) = \sum_{n=1}^{m} \lambda_n f_n(x)$ then $m = 1$ and the function h will be a product of an $mc1$-function and a constant bigger than 1. Let us see that the function $g(x) = \frac{f(x)+c}{1+c}$ is not $mc1$. Taking $\mu = 1$, $x = y = a^{log_b 2} \in (0,1)$ we obtain

$$g(x)g(y) = \left(\frac{f(x)+c}{1+c}\right)^2 = \left(\frac{b^{log_a a^{log_b 2}} + c}{1+c}\right)^2 = \left(\frac{2+c}{1+c}\right)^2 = \frac{c^2+4c+4}{c^2+2c+1},$$

$$g(xy) = \frac{f(xy)+c}{1+c} = \frac{b^{log_a (a^{log_b 2})} + c}{1+c} = \frac{b^{2log_b 2} + c}{1+c} = \frac{4+c}{1+c} = \frac{c^2+5c+4}{c^2+2c+1}.$$

Since $g(xy) > g(y)g(x)$ we conclude that $g \notin \mathcal{MC}_1$. □

5. Open questions

We finish this work by posing some open directions of study in order to continue this ongoing research theory of multiplicative convex functions.

(1) The cardinality of the set $\{f : \mathbb{R} \to \mathbb{R}\}$ is $2^{\mathfrak{c}}$. What is the cardinality of $\mathcal{MC}$?

(2) In Proposition 3.4 we proved the existence of an algebraic truncated cone of algebraic dimension $\mathfrak{c}$ contained in $\mathcal{MC} \setminus C(0,\infty)$. What is the maximum possible algebraic dimension of such a truncated cone (if it is greater than $\mathfrak{c}$)?

(3) Is $\mathcal{MC} \setminus C(0,\infty)$ a truncated cone itself? That is, is $\mathcal{MC} \setminus C(0,\infty)$ closed under addition and multiplication?

(4) What can be said about the general behavior of not continuous *mc*-functions? Can they be of the three types described in [11] for *mc*1-functions (monotone or decreasing-increasing type)?

(5) What can be said about the set of points of discontinuity of a multiplicative convex function, in general?

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