Journal of Convex Analysis
Volume 28 (2021), No. 3, 959–966

Field Theory for Integrands with Low Regularity

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Received: April 6, 2020
Accepted: August 12, 2020

We develop a field theory for one-dimensional variational problems defined via integrands with low regularity and singular ellipticity. This is developed via *a priori* existence and regularity results for one-dimensional obstacle problems with such integrands.

Keywords: Integral functionals, obstacle problems, singular ellipticity, Tonelli regularity, existence and regularity in small, field theory.

2010 Mathematics Subject Classification: 49N60.

1. Main result

Consider the basic problem in the calculus of variations

$$J(u) = \int_a^b L(x, u(x), \dot{u}(x))dx \to \min \tag{1}$$

$$u(a) = A, \quad u(b) = B. \tag{2}$$

Consider a continuous Lagrangian $L = L(x, u, v)\colon \mathbb{R}^3 \to \mathbb{R}$ which is convex in v. We shall consider the following precise assumptions (H) on L: that for all compact sets $K \subseteq \mathbb{R}^3$,

(H1) there exist constants $c, \alpha > 0$ such that

$$|L(x_1, u_1, v_1) - L(x_2, u_2, v_2)| \leq c\,(|x_1 - x_2| + |u_1 - u_2| + |v_1 - v_2|)^{\alpha}$$

for all $(x_1, u_1, v_1), (x_2, u_2, v_2) \in K$; and

(H2) there exist constants $\mu > 0$ and $p > 1$ such that for all $(x, u, v_1) \in K$, there exists $l \in \partial L_v(x, u, v_1)$ such that for all $(x, u, v_2) \in K$, we have

$$L(x, u, v_2) - L(x, u, v_1) - (l, v_2 - v_1) \geq \mu |v_2 - v_1|^p.$$

Definition 1.1. A C^1-regular function $u_0 : [a, b] \to \mathbb{R}$ is called an *ϵ-weak minimizer* provided for each $u \in W^{1,\infty}[a, b]$ with the same boundary conditions (2) and with $\|u - u_0\|_{W^{1,\infty}} \leq \epsilon$ we have $J(u) \geq J(u_0)$; u_0 is called an isolated ϵ-weak minimizer if additionally $J(u) > J(u_0)$ for u different than u_0.

ISSN 0944-6532 / \$ 2.50

The main result of this paper is the following theorem.

Theorem 1.2. *Let $M_0 > 0$, $\epsilon_0 \in]0,1[$, $\alpha_0 \in]0,1[$. Let L satisfy* (H). *Assume u_i, $i = 1,\dots,N$, is a family of ϵ_0-weak isolated minimizers such that $u_{i+1} > u_i$ in $[a,b]$ and $\|u_i\|_{C^{1,\alpha_0}} \le M_0$, $i = 1,\dots,N$.*

Then for each $\eta_0 \in]0,1[$ there exists $\delta_0 > 0$ such that if $\max_i \{\|u_i - u_{i+1}\|_C\} \le \delta_0$ then for the minimization problem (1)–(2) *with $A \in [u_i(a), u_{i+1}(a)]$, $B \in [u_i(b), u_{i+1}(b)]$, $i \in \{1,\dots,N-1\}$, a solution u_0 in the class $u \in W^{1,1}[a,b]$, $u_1 \le u \le u_N$ exists; moreover all such solutions u_0 are C^1-regular functions and $u_i \le u_0 \le u_{i+1}$,* $\max\{\|u_i - u_0\|_{C^1}, \|u_{i+1} - u_0\|_{C^1}\} \le \eta_0$.

Remark 1.3. If $A = u_i(a)$, $B = u_i(b)$, where $i \in \{2,\dots,N-1\}$, then the only solution of the problem is u_i itself. This can easily been seen: due to Theorem 1.2 all solutions of the problem u_0 satisfy both the obstacles $u_i \le u_0 \le u_{i+1}$ and $u_{i-1} \le u_0 \le u_i$. Therefore $u_0 = u_i$.

The paper is organized as follows. In Section 2 we prove some auxiliary propositions. In Section 3 we prove Theorem 1.2. In Section 4 we explain why we selected the assumptions of Theorem 1.2 for the field theory in the case of integrands with low regularity.

2. Auxiliary propositions

Definition 2.1. Let F be a family of continuous functions

$$f\colon [a_f, b_f] \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \cup \{-\infty\}.$$

Then F is called a *conditionally equicontinuous family* (CEF) if for all $M > 0$, $\epsilon > 0$ there is $\delta = \delta(M,\epsilon) > 0$ such that if $f \in F$ and $|f(x_0)| \le M$ then $|f(x) - f(x_0)| \le \epsilon$ for $|x - x_0| \le \delta$. The function $\delta(\cdot,\cdot)$ is called a *modulus of conditional equicontinuity* of the family F.

This definition was first introduced in [4].

The main result of this section is the following theorem.

Theorem 2.2. *Let L satisfy the assumption* (H). *Let $f, g\colon [a,b] \to \mathbb{R}$ be obstacles ($f > g$ in $[a,b]$) such that $\|f\|_{C^{1,\alpha_0}}, \|g\|_{C^{1,\alpha_0}} \le$* const. *Then the obstacle problem has a solution u_M in the class of M-Lipschitz functions with boundary data (a_0, A_0, b_0, B_0), where $a \le a_0 < b_0 \le b$, $A_0 \in [g(a_0), f(a_0)]$, $B_0 \in [g(b_0), f(b_0)]$, provided there is at least one admissible function. Moreover all such solutions are C^1-regular and their derivatives form a* CEF *for $M \ge \max\{\|\dot f\|_C, \|\dot g\|_C\}$ (a_0, A_0, b_0, B_0, and M may vary).*

Given a function $u\colon [a,b] \to \mathbb{R}$, and a nontrivial subinterval $[x_1,x_2] \subseteq [a,b]$, we define $l_{x_1,x_2}\colon [x_1,x_2] \to \mathbb{R}$ to be the affine function such that $l_{x_1,x_2}(x_1) = u(x_1)$ and $l_{x_1,x_2}(x_2) = u(x_2)$. Thus we have

$$\dot l_{x_1,x_2}(x) = \dot l_{x_1,x_2} = \frac{u(x_2) - u(x_1)}{x_2 - x_1} \quad \text{for all} \quad x \in]x_1, x_2[.$$

We shall also adopt the notation

$$J(u;[x_1,x_2]) := \int_{x_1}^{x_2} L(x,u(x),\dot u(x))\,dx.$$

Lemma 2.3. *Let $G \subseteq \mathbb{R}^2$ be compact, and let $M > 0$. Suppose $u \in W^{1,\infty}[a,b]$ has $\|u\|_{W^{1,\infty}} \le M$ and has graph contained in G. Suppose further that $x_1 \le x_3 \le x_4 \le x_2$ are points of $[a,b]$ such that $|x_2 - x_1| \le 1$ and $|x_2 - x_1| \le 2|x_4 - x_3|$, and that the graphs of the functions l_{x_1,x_2} and l_{x_3,x_4} are contained in G.*

Let L satisfy condition (H), *with, for the compact set $K := G \times [-M,M]$, constants $c_1, \alpha > 0$ in the Hölder condition* (H1), *and constants $\mu > 0$ and $p > 1$ in the singular ellipticity condition* (H2).

Suppose further that there exist constants $c_2, \beta > 0$ such that

$$J(u;[x_1,x_2]) \le J(l_{x_1,x_2};[x_1,x_2]) + c_2|x_2 - x_1|^{1+\beta}; \tag{3}$$

$$J(u;[x_3,x_4]) \le J(l_{x_3,x_4};[x_3,x_4]) + c_2|x_4 - x_3|^{1+\beta}. \tag{4}$$

Then

$$\left|\frac{u(x_2)-u(x_1)}{x_2-x_1} - \frac{u(x_4)-u(x_3)}{x_4-x_3}\right| \le 4\left(\frac{c_2 + 2c_1(1+M)^\alpha}{\mu}\right)^{1/p} |x_2 - x_1|^{\gamma/p}, \tag{5}$$

where $\gamma := \min\{\alpha,\beta\}$.

Lemma 2.3 was proved in [2].

Proof of Theorem 2.2. If u_M is a solution of such an obstacle problem we can not guarantee that given $x_1, x_2 \in [a,b]$ the graph of the associated function l_{x_1,x_2} still fits the obstacles. However we can modify l_{x_1,x_2} to fit this requirement. Indeed consider a function w such that $w = l_{x_1,x_2}$ where $l_{x_1,x_2} \le f$ and $w = f$ otherwise (the case of the obstacle g can be considered analogously). Then w is admissible for the obstacle problem and, therefore, $J(u) \le J(w)$. We now estimate the difference $|J(w) - J(l_{x_1,x_2})|$ via estimating $|L(x,w(x),\dot w(x)) - L(x,l_{x_1,x_2}(x),\dot l_{x_1,x_2}(x))|$ in each interval $[y_1,y_2]$ with $l_{x_1,x_2} = w$ at the endpoints. Since $\dot w = \dot l_{x_1,x_2}$ at a certain point of $[y_1,y_2]$ we infer

$$|\dot w(y) - \dot l_{x_1,x_2}| \le c_0|y_2 - y_1|^{\alpha_0}, \quad y \in]y_1,y_2[. \tag{6}$$

Then

$$|w(y) - l_{x_1,x_2}(y)| \le c_0|y_2 - y_1|^{1+\alpha_0}, \quad y \in]y_1,y_2[. \tag{7}$$

Since

$$\begin{aligned} &|L(x,w(x),\dot w(x)) - L(x,l_{x_1,x_2}(x),\dot l_{x_1,x_2}(x))| \\ &\le c_1\{|w(x) - l_{x_1,x_2}(x)| + |\dot w(x) - \dot l_{x_1,x_2}(x)|\}^\alpha \end{aligned} \tag{8}$$

we infer by (6) and (7) that

$$\begin{aligned} &|L(x,w(x),\dot w(x)) - L(x,l_{x_1,x_2}(x),\dot l_{x_1,x_2})| \\ &\le c_1|x_2 - x_1|^{\alpha\alpha_0} c_0^\alpha(1 + |x_2 - x_1|)^\alpha \\ &\le c_1 c_0^\alpha (1 + |b-a|)^\alpha |x_2 - x_1|^{\alpha\alpha_0}. \end{aligned} \tag{9}$$

Therefore (5) holds with $\beta = \alpha\alpha_0$ and with $c_2 = c_1 c_0^\alpha (1 + |b - a|)^\alpha$.

Now we show that each minimizer u_M is in fact $C^{1,\gamma(M)/p(M)}$-regular.

For every $x_1, x_2, x_3, x_4 \in [a_0, b_0]$ we stay under the conditions of Lemma 2.3 with $K = \mathrm{conv}G \times [-M, M]$, where $G = \{(x, u) : x \in [a, b], f \geq u \geq g\}$, $\mu = \mu(M)$, $c_1 = c_1(M)$, $\alpha = \alpha(M)$, $p = p(M)$, $c_2 > 0$, $\beta > 0$. By (5) we have

$$|\dot{l}_{x_1,x_2} - \dot{l}_{x_3,x_4}| \leq 4\left[\frac{c_2 + 2c_1(M)(1+M)^{\alpha(M)}}{\mu(M)}\right]^{1/p(M)} |x_2 - x_1|^{\gamma(M)/p(M} \tag{10}$$

provided $[x_3, x_4] \subset [x_1, x_2]$, $2|x_3 - x_4| \geq |x_1 - x_2|$. If

$$x_0 \in [x_1, x_2] \cap [x_3, x_4], \quad [x_3, x_4] \subset [x_1, x_2] \text{ and } |x_3 - x_4| \geq 2^{-n}|x_1 - x_2|$$

we obtain

$$\begin{aligned} &|\dot{l}_{x_1,x_2} - \dot{l}_{x_3,x_4}| \\ &\leq 4\left[\frac{c_2 + 2c_1(M)(1+M)^{\alpha(M)}}{\mu(M)}\right]^{1/p(M)} \sum_{k=0}^{n} 2^{-k(\gamma(M)/p(M))}|x_2 - x_1|^{\gamma(M)/p(M)} \\ &\leq 4\left[\frac{c_2 + 2c_1(M)(1+M)^{\alpha(M)}}{\mu(M)}\right]^{1/p(M)} \sum_{k=0}^{\infty} 2^{-k(\gamma(M)/p(M))}|x_2 - x_1|^{\gamma(M)/p(M)}. \end{aligned} \tag{11}$$

Therefore $\dot{u}_M(x_0)$ exists and satisfies the inequality

$$\begin{aligned} &|\dot{u}_M(x_0) - \dot{l}_{x_1,x_2}| \\ &\leq 4\left[\frac{c_2 + 2c_1(M)(1+M)^{\alpha(M)}}{\mu(M)}\right]^{1/p(M)} \sum_{k=0}^{\infty} 2^{-k(\gamma(M)/p(M))}|x_2 - x_1|^{\gamma(M)/p(M)} \\ &\leq \mathrm{const}(M)|x_2 - x_1|^{\gamma(M)/p(M)}. \end{aligned} \tag{12}$$

Then $\dot{u}_M$ is $\gamma(M)/p(M)$-Hölder continuous and bounded in this norm.

Therefore $$|\dot{u}_M(x) - \dot{u}_M(y)| \leq 2\mathrm{const}(M)|x - y|^{\gamma(M)/p(M)}. \tag{13}$$

This shows that $\{\dot{u}_N\}$ forms a CEF. If $|\dot{u}_N(x)| \leq M$ then, given $\epsilon > 0$, for a certain $\delta > 0$ we have $|\dot{u}_N(x) - \dot{u}_N(y)| \leq \epsilon$ provided $|x - y| \leq \delta$. It suffices to select $\delta > 0$ such that $2\mathrm{const}(M+1)\delta^{\gamma(M+1)/p(M+1)} \leq \epsilon$.

Theorem 2.2 is proved. □

3. Theorem 1.2

Proof of Theorem 1.2. The first assumption on δ_0 is that $\delta_0 > 0$ is so small that

$$\max_i \|u_{i+1} - u_i\|_{C^1} \leq \min\{\epsilon_0, \eta_0\}/8. \tag{14}$$

Take $M \geq M_0 + 1$ and for $A \in [u_i(a), u_{i+1}(a)], B \in [u_i(b), u_{i+1}(b)]$ consider the minimization problem in M-Lipschitz functions u such that $u_1 \leq u \leq u_N$. Let u_M

be a solution. Then $u_M \in C^1$ and $\dot{u}_M$ belongs to a certain CEF, see Section 2. Let $\delta(\cdot,\cdot)$ be a modulus of conditional equicontinuity of the family $\dot{u}_M$, for $M \geq M_0+1$.

We will prove that $u_i \leq u_M \leq u_{i+1}$. Assume the contrary. Then there exists $j \geq i+1$ such that $u_M > u_j$ at certain points and $u_M \leq u_{j+1}$ everywhere (the case $u_M < u_j, j \leq i$ at certain points and $u_M \geq u_{j-1}$ everywhere can be considered analogously). Then there exist $a_1 \geq a$, $b_1 \leq b$ such that $u_M = u_j$ at a_1, b_1 and $u_M > u_j$ in $]a_1, b_1[$. We will show that

$$\|u_M - u_j\|_{C^1[a_1,b_1]} \leq \epsilon_0. \tag{15}$$

This will be a contradiction with the assumption that u_j is an ϵ_0-weak isolated minimizer. Assume that (15) fails to hold. There exists $x_1 \in]a_1, b_1[$ such that $\dot{u}_M(x_1) = \dot{u}_j(x_1)$. Then there is $x_2 \in]a_1, b_1[$ such that

$$\dot{u}_M(x_2) = \dot{u}_j(x_2) + 7\epsilon_0/8. \tag{16}$$

The case $\dot{u}_M(x_2) = \dot{u}_j(x_2) - 7\epsilon_0/8$ can be considered analogously.

Let $x_3 = x_2 + \nu$, where
$$M_0 \nu^{\alpha_0} \leq \epsilon_0/8, \tag{17}$$
i.e.
$$\nu \leq (\epsilon_0/8M_0)^{1/\alpha_0}. \tag{18}$$

There is an interval $I := [x_2, x_4] = [x_2, x_2+\nu] \cap [x_2, x_2 + \delta(M_0+1, \epsilon_0/8)]$ on which we have

$$\dot{u}_M(x) - \dot{u}_j(x) \geq 5\epsilon_0/8. \tag{19}$$

Then
$$u_M(x_4) - u_j(x_4) \geq 5\epsilon_0|I|/8 = 5\epsilon_0 \min\{\nu, \delta(M_0+1, \epsilon/8)\}/8. \tag{20}$$

Then if
$$\delta_0 < 4\epsilon_0|I|/8 = 4\epsilon_0/8 \min\{(\epsilon_0/8M_0)^{1/\alpha_0}, \delta(M_0+1, \epsilon_0/8)\} \tag{21}$$

we obtain that due to (14), at a certain point $x_0 \in [x_2, x_4]$ the minimizer u_M intersects the graph of u_{j+1}. At this point due to (14) we have $\dot{u}_{j+1}(x_0) < \dot{u}_M(x_0 - 0)$ (see (19))—a contradiction since then $\dot{u}_M(x_0+0) \neq \dot{u}_M(x_0-0)$. Thus we have showed (15), which means that $u_i \leq u_M \leq u_{i+1}$.

The inequality
$$\max\{\|u_i - u_M\|_{C^1}, \|u_{i+1} - u_M\|_{C^1}\} \leq \eta_0 \tag{22}$$

can be proved by arguments similar to those used to prove (15) (for δ_0 sufficiently small of course). Then (22) holds for all u_M, $M \geq M_0+1$ and, therefore, u_{M_0+1} is the minimizer of the problem in the class of Lipschitz functions.

Now we want to prove that u_{M_0+1} is also the minimizer in the class of Sobolev functions $u \in W^{1,1}$ such that $u_1 \leq u \leq u_N$. Assume the contrary. Assume first that there is $w \in W^{1,1}$ such that $J(w) < J(u_{M_0+1})$. We can find nonnegative $\theta(v)\colon \mathbb{R} \to \mathbb{R}$ such that $\theta \in C^3$, $\theta(v) = 0$ for $v \in [0, M_0+2]$, $\theta_{vv} \geq 0$, $\theta(v)/|v| \to \infty$ as $|v| \to \infty$ and

$$\int_a^b \theta(\dot{w}(x))dx < \infty.$$

For the integrand $L_\mu = L + \mu\theta$ with sufficiently small $\mu > 0$ we have

$$J_\mu(w) < J_\mu(u_{M_0+1}) = J(u_{M_0+1}). \tag{23}$$

There is a solution $\tilde{w}$ of the obstacle problem $u_1 \leq w \leq u_N$ for J_μ. We intend to show that

$$u_i \leq \tilde{w} \leq u_{i+1}.$$

As proved in [2], $\dot{\tilde{w}}$ belongs to a certain CEF and its modulus of conditional equicontinuity $\tilde{\delta}(M,\epsilon)$ coincides with $\delta(M,\epsilon)$ for $M \leq M_0 + 1$, $\epsilon \leq 1$. Then we have (15) by the same arguments as for u_M. Therefore we have

$$u_i \leq \tilde{w} \leq u_{i+1}. \tag{24}$$

Since $|\dot{\tilde{w}}| = \infty$ at certain points, for δ_0 sufficiently small the graph of $\tilde{w}$ intersects that of either u_i or u_{i+1} at a certain point y_0 with $|\dot{\tilde{w}}| \geq M_0 + 1$. This contradiction shows that u_{M_0+1} is the solution of the obstacle problem $u_1 \leq u \leq u_N$ in the Sobolev class.

The remaining case when there is $w \in W^{1,1} \setminus W^{1,\infty}$ such that $J(u_{M_0+1}) = J(w)$ can be treated analogously.

Theorem 1.2 is proved. □

4. Appendix

In this section we explain why we use assumptions of Theorem 1.2.

Consider the one-dimensional variational problem

$$J(u) = \int_a^b L(x, u(x), \dot{u}(x))dx \to \min, \tag{25}$$

$$u(a) = A, u(b) = B. \tag{26}$$

Here and throughout this section we assume $L(x,u,v) \in C^3(\mathbb{R}^3)$, $L_{vv} > 0$. These conditions will be referred as *basic* ones.

If a C^1-regular function $u\colon [a,b] \to \mathbb{R}$ gives a local minimum to the problem (25)–(26), i.e. $J(u) \leq J(u+\phi)$ for $\phi \in C_0^1[a,b]$ with sufficiently small $\|\phi\|_{C^1[a,b]}$ then the Euler-Lagrange equation is valid:

$$\frac{d}{dx}L_v(x,u(x),\dot{u}(x)) = L_u(x,u(x),\dot{u}(x)). \tag{27}$$

In this case due to Hilbert's theorem we can assert that $u \in C^2[a,b]$ and the equation (27) can be resolved with respect to the second derivative of the function u, i.e. the Euler equation is valid:

$$\ddot{u} = \frac{L_u - L_{vx} - L_{vu}\dot{u}}{L_{vv}} = F(x,u,\dot{u}). \tag{28}$$

Solutions of the Euler equation (28) are called *extremals*.

A family of continuous functions $w\colon [a,b] \to \mathbb{R}$ is called a *field* if it covers without intersections a set $G = \{(x,u) \in \mathbb{R}^2 : x \in [a,b], g(x) \le u \le f(x)\}$, where automatically $g, f \in C[a,b]$ as the elements of the family and $f > g$ everywhere in $[a,b]$. The elements of the field can be parametrized by the value at the point a and, then, $w(\cdot, w(a))$ depends continuously on the parameter $w(a)$ in C-norm.

In [6, 3] we proved the following.

Theorem 4.1. *Let L satisfy the basic assumptions. Suppose there is a field of extremals of* (28)*. Then each element of the field gives a minimum in the class of Sobolev functions $u \in W^{1,1}[a,b]$ with the same boundary data* (26) *and with graph from the set covered by the field.*

Previously this result was established in context of Weierstrass-Hilbert theory under additional assumptions on the field, see [1].

Lemma 4.2. *Let L satisfy the basic assumptions and let there be a field of extremals. Then these extremals are bounded uniformly in C^2-norm.*

Proof. In [6] we proved that extremals of the field are bounded in C^1-norm. Precisely, $p(x,u)$ is a continuous function, where $p(x,u)$ is equal to the derivative of the extremal passing via point (x,u) at the point x.

Then, due to the equation (28), all extremals are uniformly bounded in C^2-norm. $\square$

At the same time it is known that under condition (H) ϵ-weak minimizers have C^{1,α_0}-regularity, see [5]. Therefore we use this regularity in the statement of Theorem 1.2.

The following theorem is valid.

Theorem 4.3. *Let L satisfy the basic assumptions. Assume f and g are extremals such that $f > g$ in $[a,b]$, $f(a) > A > g(a)$, $f(b) > B > g(b)$ and $\|f - g\|_C \le 1/8M$, where*

$$M \ge \max\Big\{|F(x,u,v)| : x \in [a,b], g(x) \le u \le f(x), |v| \le \max\{\|\dot{f}\|_C, \|\dot{g}\|_C\} + 1\Big\}$$

with F being the right-hand side of (28) *and with $1/M < |b - a|$.*

Then the obstacle problem (25)–(26) *in the class of functions $u \in W^{1,\infty}[a,b]$ such that $g < u < f$ has solution u_0 and $\|\dot{u}_0\|_C \le \max\{\|\dot{f}\|_C, \|\dot{g}\|_C\} + 1$; furthermore u_0 is an extremal of* (28)*.*

Proof. Consider the minimization problem (25)–(26) with the obstacles f and g in the class of N-Lipschitz functions, where $N > l + 1$, $l = \max\Big\{\|\dot{f}\|_C, \|\dot{g}\|_C\Big\}$. Such a solution u_N exists and is a C^1-regular function, as proved in Section 2.

Assume $\dot{u}_N(x_0) \ge l+1$ at some point x_0 (the case $\dot{u}_N(x_0) \le -l-1$ can be considered analogously). Then, due to equation (28) we have

$$\dot{u}_N(x) \ge l + 1/2 \quad \text{in the interval} \quad I := [x_0 - 1/2M, x_0 + 1/2M].$$

Assume $[x_0, x_0 + 1/2M] \subset [a,b]$ (the case $[x_0 - 1/2M, x_0] \subset [a,b]$ can be considered analogously). Then the oscillation of u_N on $[x_0, x_0 + 1/2M]$ is larger than

$$(l + 1/2)/2M = l/2M + 1/4M.$$

Since $\|f - g\|_C \leq 1/8M$ we obtain that the graph of u_N intersects the graph of f at a certain point $y_0 \in]x_0, x_0 + 1/2M[$. Then $\dot{u}_N(y_0 - 0) \geq l + 1/2$ and $\dot{u}_N(y_0 + 0) \leq l$. This contradiction with the fact $u_N \in C^1$ shows that $\|\dot{u}_N\|_C < l + 1$. Then u_N is an extremal and a solution of the obstacle problem (25)–(26) with the obstacles $g \leq u \leq f$.

Theorem 4.3 is proved. □

As we see if we are given a field of extremals then for each extremal u_0 we have that it gives a minimum in the class $g \leq u \leq f$ where g and f are also extremals and $\|g - f\|_C \leq 1/8M$. Indeed, by Theorem 4.3 there is a solution u_N of the obstacle problem. But u_0 is the only extremal for the boundary value problem which satisfies the obstacles. Indeed, if there is another extremal u_N then $u_N(x_0) = w(x_0)$, $\dot{u}_N(x_0) = \dot{w}(x_0)$ for an extremal w of the field. But then $u_N = w$ everywhere by the uniqueness of solution of the Cauchy problem for the Euler equation (28). By this contradiction we obtain that $u_N = u_0$.

This gives us motivation for the other assumptions in Theorem 1.2.

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