Journal of Convex Analysis
Volume 28 (2021), No. 3, 967–982

Superlinear Weighted (p,q)-Equations with Indefinite Potential*

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Received: June 12, 2020
Accepted: October 1, 2020

We consider a nonlinear Dirichlet problem driven by a weighted (p,q)-Laplacian plus an indefinite potential term. The reaction is superlinear. We prove a three solutions theorem providing sign information for all of them (positive, negative, nodal). The nodal solution is produced using flow invariance arguments.

Keywords: Weighted (p,q)-Laplacian, nonlinear maximum principle, extremal constant sign solutions, nodal solution, indefinite potential.

2010 Mathematics Subject Classification: 35J20, 35J60.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following weighted (p,q)-equation:

$$\begin{cases} -\Delta_p^{a_1} u(z) - \Delta_q^{a_2} u(z) + \xi(z)|u(z)|^{p-2}u(z) = f(z,u(z)) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 1 < q < p < \infty. \end{cases} \tag{1}$$

Given $a \in L^\infty(\Omega)$ and $r \in (1,\infty)$ by Δ_r^a we denote the weighted r-Laplace differential operator defined by

$$\Delta_r^a u = \operatorname{div}(a(z)|Du|^{r-2}Du), \ \forall u \in W_0^{1,r}(\Omega).$$

In problem (1) we have the sum of two such operators with different weights. So,the differential operator in (1) is not homogeneous. Moreover, the fact that the weights are different, precludes the use of the existing nonlinear maximum principle (see Pucci-Serrin [17, p. 110]). For this reason, in Section 3 we prove a maximum principle which is suitable for the differential operators we deal with. In addition to the

*This work was supported by NNSF of China Grant No. 12071413, NSF of Guangxi Grant No. 2018GXNSFDA138002.

ISSN 0944-6532 / $ 2.50 © Heldermann Verlag

weighted (p, q)-Lanlacian, we also have a potential term $\xi(z)|u|^{p-2}u$ with the potential function $\xi \in L^{\infty}(\Omega)$ being indefinite (sign changing). The reaction (right hand side) of (1), is a Carathéodory function $f(z, x)$ (that is, for all $x \in \mathbb{R}, z \to f(z, x)$ is measurable and for a.a.$z \in \Omega, x \to f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits $(p-1)$-superlinear growth near $\pm\infty$, but without satisfying the usual in such cases "Ambrosetti-Robinowitz condition" (the "AR-condition" for short). Using the maximum principle that we prove here and also variational tools from the critical point theory, together with flow invariance arguments, we prove a multiplicity theorem for problem (1) producing three nontrivial solutions all with sign information (positive,negative and nodal). To the best of our knowledge this is the first three solutions theorem for nonhomogeneous superlinear problems, which provides sign information for all the solutions (that is, we obtain a nodal solution).

The first multiplicity result (three solutions theorem), for equations with a superlinear reaction, was proved by Wang [19], who considers a semilinear Dirichlet problem driven by the Laplacian, with no potential term (that is, $\xi \equiv 0$) and a reaction term of the form $f(z, x) = f(x)$ with $f \in C^1(\mathbb{R})$ satisfying the AR-condition. Under these conditions, Wang [19], proves the existence of three nontrivial solutions, but without providing sign information for the third one. The result of Wang [19] was extended to equations driven by the p-Laplacian by Sun [18], who also obtained three nontrivial solutions but no nodal one. Nodal solutions for p-Laplacian equations were produced by Bartsch-Liu [1], but under more restrictive conditions on the reaction $f(z, x)$ and using the AR-condition. Extension to (p, q)-equations with no weights was obtained by Mugnai-Panageorgiou [12] and to more general nonhomogeneous z-invariant (autonomous) differential operators, by Papageorgiou-Rădulescu [14]. More recently Gasiński-Papageorgiou [4], proved such a three solutions theorem for double phase problems with unbalanced growth, producing also a nodal solution. However, their conditions on the reaction $f(z, \cdot)$ are stronger and their method of proof is different based on the Nehari manifold.

2. Mathematical background, hypotheses

The main spaces in the analysis of problem (1), are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u|_{\partial\Omega} = 0\}$. By $\|\cdot\|$, we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By the Poincaré inequality we have

$$\|u\| = \|Du\|_p \qquad \text{for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is ordered with positive (order) cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\mathrm{int} C_+ = \{u \in C_+ : u(z) > 0 \quad \text{for all } z \in \Omega, \quad \frac{\partial u}{\partial \mathbf{n}}|_{\partial\Omega} < 0\}$$

with $\mathbf{n}(\cdot)$ being the outward unit normal on $\partial\Omega$. Given $r \in (1, \infty)$, let $r' \in (1, \infty)$ be defined by $\frac{1}{r} + \frac{1}{r'} = 1$ (the conjugate exponent of r). Then given $a \in L^{\infty}(\Omega)$ with

$\operatorname{ess\,inf}_{\Omega} a > 0$, by $A_r^a : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^*$ we denote the the nonlinear operator defined by

$$\langle A_r^a(u), h\rangle = \int_\Omega a(z)|Du|^{r-2}(Du, Dh)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

The following proposition gathers the main properties of this map (see, for example, Gasiński-Papageorgiou [3]).

Proposition 2.1. *The operator $A_r^a(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone and of type $(S)_+$, that is,*

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,r}(\Omega), \quad \text{and} \quad \limsup_{n\to\infty}\langle A_r^a(u_n), u_n - u\rangle \le 0$$
$$\text{imply that} \quad u_n \to u \quad \text{in} \quad W_0^{1,r}(\Omega), n \to \infty.$$

We will also use the spectrum of $(-\triangle_r^a, W_0^{1,r}(\Omega))$,with $a \in C^{0,\alpha}(\Omega), \alpha \in (0,1]$ as above. So, we consider the following nonlinear eigenvalue problem

$$\begin{cases} -\Delta_r^a u(z) = \hat\lambda |u(z)|^{r-2}u(z) \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \quad 1 < r < \infty. \end{cases} \tag{2}$$

We know (see Mugnai-Papageorgiou [11] and Papageorgiou-Rădulescu [14]) that this problem has a smallest eigenvalue $\hat\lambda_1(r,a)$ which has the following properties:

(a) $\hat\lambda_1(r,a) > 0$ and it is isolated (that is, if by $\hat\sigma(r,a)$, we denote the spectrum of (2), then we can find $\epsilon > 0$ such that

$$(\hat\lambda_1(r,a), \hat\lambda_1(r,a) + \epsilon) \cap \hat\sigma(r,a) = \emptyset,$$

(b) $\hat\lambda_1(r,a)$ is simple (that is, if $\hat u, \hat v \in W_0^{1,p}(\Omega)$ are eigenfunctions corresponding to $\hat\lambda_1(r,a)$, then we have $\hat u = \theta \hat v$ for some $\theta \in \mathbb{R}\backslash\{0\}$).

(c) $$\hat\lambda_1(r,a) = \inf\left[\frac{\int_\Omega a(z)|Du|^r dz}{\|u\|_r^r} : u \in W_0^{1,p}(\Omega), u \ne 0\right].$$

Using these properties, we have the following lemma (see Mugnai-Papagenrgiou [11, Lemma 4.11]).

Lemma 2.2. *If $\theta \in L^\infty(\Omega), \theta(z) \le \hat\lambda_1(r,a)$ for a.a $z \in \Omega$ and $\theta \ne \hat\lambda_1(r,a)$, then there exists $\widehat{c} > 0$ such that*

$$\widehat{c}\|Du\|_r^r \le \int_\Omega a(z)|Du|^r dz - \int_\Omega \theta(z)|u(z)|^r dz \text{ for all } u \in W_0^{1,r}(\Omega).$$

Recall that if $u : \Omega \to \mathbb{R}$ is a measurable function, then we define

$$u^+ = max\{u, 0\} \quad \text{and} \quad u^- = max\{-u, 0\}.$$

We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,r}(\Omega)$, then $u^\pm \in W_0^{1,r}(\Omega)$.

Let X be a Banach space and $\varphi \in C^1(X)$. We say that $\varphi(\cdot)$ satisfies the "C-condition", if it has the following property:

Every sequence $\{u_n\} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and the sequence $(1+\|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to \infty$, admits a strongly convergent subsequence.

This is a compactness-type condition on the functional $\varphi(\cdot)$. Since the space X is not in general locally compact (in most cases of interest X is infinite dimensional), the burden of compactness is passed on the functional $\varphi(\cdot)$ (as is the case in the Leray-Schauder degree theory). Using the C-condition, one can prove a deformation theorem from which follows the minimax theory of the critical values of $\varphi(\cdot)$ (see Panageorgiou-Rădulescu-Repovš [15, Chapter 5]). Also by K_φ we denote the critical set of $\varphi(\cdot)$, that is,

$$K_\varphi = \{u \in X : \varphi'(u) = 0\}.$$

Now we are ready to introduce our hypotheses on the data of problem (1)

H$_0$: $a_1, a_2 \in C^{0,1}(\overline{\Omega}), a_1(z), a_2(z) \geq c_0 > 0$ for all $z \in \overline{\Omega}$ and $\xi \in L^\infty(\Omega)$.

H$_1$: $f\colon \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z,x)| \leq a(z)(1+|x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$ and

$$p < r < p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } N \leq p; \end{cases}$$

(ii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\lim_{x\to\pm\infty} \frac{F(z,x)}{|x|^p} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(iii) there exists $\tau \in ((r-p)max\{\frac{N}{p}, 1\}, p^*)$ such that

$$0 < \beta_0 \leq \liminf_{x\to\pm\infty} \frac{f(z,x)x - pF(z,x)}{|x|^\tau} \quad \text{uniformly for a.a. } z \in \Omega;$$

(iv) there exist a function $\theta \in L^\infty(\Omega)$ and $\widetilde{c} > 0$ such that

$$\theta(z) \leq \hat{\lambda}_1(q,a_2) \quad \text{for a.a. } z \in \Omega,\ \theta \neq \hat{\lambda}_1(q,a_2)$$
$$-\widetilde{c} \leq \liminf_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} \leq \limsup_{x\to 0} \frac{f(z,x)}{|x|^{q-2}x} \leq \theta(z) \quad \text{uniformly for a.a. } z \in \Omega;$$

(v) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a $z \in \Omega$, the function

$$x \to f(z,x) + \widehat{\xi}_\rho |x|^{p-2}x$$

is nondecreasing on $[-\rho, \rho]$.

Remark 2.3. Hypotheses H$_1$(ii),(iii) imply that

$$\lim_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = +\infty \quad \text{uniformly for a.a. } z \in \Omega.$$

So $f(z,\cdot)$ is $(p-1)$-superlinear. However, instead of the AR-condition which is common in the literature for superlinear problems (see Wang [19]), we use the less restrictive condition H$_1$(iii).

The following function satisfies hypotheses H_1 but fails to satisfy the AR-condition. For the sake of simplicity, we drop the z-dependence:

$f(x) = |x|^{p-2}x ln(1+|x|)+\theta|x|^{q-2}x$ for all $x \in \mathbb{R}$ with $1 < q < p < \infty, \theta < \widehat{\lambda}_1(q, a_2)$.

3. A nonlinear maximum principle

In this section, we prove a maximum principle for the weighted (p, q)-Laplacian. Our proof is inspired by the works of Pucci-Serrin [17] and Zhang [20].

So let $g \in L^\infty(\Omega)$ and consider the following Dirichlet problem

$$\begin{cases} -\triangle_p^{a_1} u(z) - \triangle_p^{a_2} u(z) + \xi(z)|u(z)|^{p-2}u(z) = g(z) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, 1 < q < p < \infty. \end{cases} \tag{3}$$

Proposition 3.1. *If hypotheses* H_0 *hold and* $u \in C_0^1(\overline{\Omega})$ *is a solution of* (3) *such that* $u \geq 0, u \neq 0$*, then for every* $K \subseteq \Omega$ *compact we have* $0 < c_K \leq u(z)$ *for all* $z \in K$.

Proof. We show that $u(z) > 0$ for all $z \in \Omega$. Arguing by contradiction, suppose that it is not true that $u(z) > 0$ for all $z \in \Omega$. Then we can find $z_1, z_2 \in \Omega$ and $\rho > 0$ such that

$$\overline{B}_{2\rho}(z_2) \subseteq \Omega, z_1 \in \partial B_{2\rho}(z_2), u(z_1) = 0, u|_{B_{2\rho}(z_2)} > 0$$

with $B_{2\rho}(z_2) = \{z \in \mathbb{R}^N : |z - z_2| < 2\rho\}$. We can always choose $\rho > 0$ small (fix z_1 and vary z_2). We set $0 < m = \min\limits_{\partial B_\rho(z_2)} u$. Then we have

$$Du(z_1) = 0, m \to 0^+ \text{ and } \frac{m}{\rho} \to 0^+ \text{ as } \rho \to 0^+ \text{ (by L'Hospital's rule).} \tag{4}$$

We introduce the following items

$$R = \{z \in \Omega : \rho < |z - z_2| < 2\rho\}, \quad \text{and} \quad \lambda = \max\{\sup_\Omega |Da_1|, \sup_\Omega |Da_2|\} > 0.$$

On account of hypotheses H_0 and Rademacher's theorem (see Gasiński-Papageorgiou [2, p. 56]), we know that the weights a_1, a_2 are differentiable almost everywhere on $\overline{\Omega}$ and so $\lambda > 0$ is well-defined.

We set
$$\gamma = -ln\frac{m}{\rho} + \frac{N-1}{\rho} + 2\lambda$$

and introduce the following function

$$v(t) = \frac{m[e^{\frac{\gamma t}{q-1}} - 1]}{e^{\frac{\gamma\rho}{q-1}} - 1} \quad \text{for all } 0 \leq t \leq \rho.$$

We see that for $\rho > 0$ small we have

$$0 < v(t), v'(t) < 1 \quad \text{for all } t \in [0, \rho] \quad \text{(see (4)), and} \tag{5}$$

$$v''(t) = \frac{\gamma}{q-1}v'(t) \quad \text{for all } t \in [0, \rho]. \tag{6}$$

For notational simplicity, without any loss of generality, we may assume that $z_2 = 0$.

We set
$$r = |z - z_2| = |z| \quad \text{and} \quad t = 2\rho - r.$$

For $t \in [0,\rho]$ and $r \in [\rho, 2\rho]$, we define
$$y(r) = v(2\rho - r) = v(t) \quad \Rightarrow \quad y'(t) = -v'(t) \text{ and } y''(t) = v''(t).$$

We define $\widehat{y}(z) = y(r)$ for all $z \in \Omega, |z| = r$. Then $\widehat{y} \in C^2(R)$. We have

$$\begin{aligned}
&\operatorname{div}[a_1(z)|D\widehat{y}|^{p-2}D\widehat{y} + a_2(z)|D\widehat{y}|^{q-2}D\widehat{y}] - \xi(z)|\widehat{y}|^{p-2}\widehat{y} + g(z)\\
= \;& (p-1)a_1(z)v'(t)^{p-2}v''(t) - a_1(z)\frac{N-1}{r}v'(t)^{p-1} - v'(t)^{p-1}\sum_{k=0}^{N}\frac{\partial a_1}{\partial z_k}\frac{z_k}{r}\\
& +(q-1)a_2(z)v'(t)^{q-2}v''(t) - a_2(z)\frac{N-1}{r}v'(t)^{q-1} - v'(t)^{q-1}\sum_{k=0}^{N}\frac{\partial a_2}{\partial z_k}\frac{z_k}{r}\\
& -\xi(z)v(t)^{p-1} + g(z)\\
\geq \;& a_1(z)[\gamma - \frac{N-1}{r}]v'(t)^{p-1} + a_2(z)[\gamma - \frac{N-1}{r}]v'(t)^{q-1} - 2\lambda v'(t)^{q-1} - c_1\\
& \text{with } c_1 = \|\xi\|_\infty + \|g\|_\infty \geq 0 \,(\text{ use (5), (6) and the fact that } q < p)\\
\geq \;& c_0[\gamma - \frac{N-1}{r} - 2\lambda]v'(t)^{q-1} - c_1 \;(\text{ since } \gamma > \frac{N-1}{r})\\
\geq \;& c_0(-ln\frac{m}{\rho})v'(t)^{p-1} - c_1 \;(\text{ see (5) and recall } q < p).
\end{aligned} \tag{7}$$

From (4) and (7) it follows that for $\rho > 0$ small
$$-\triangle_p^{a_1}\widehat{y} - \triangle_p^{a_2}\widehat{y} + \xi(z)\widehat{y}^{p-1} \leq g(z) \quad \text{in } \Omega.$$

Note that $y|_{\partial R} \leq u|_{\partial R}$ since $y(\rho) = v(\rho) = m \leq u$ and $y(2\rho) = v(0) = 0 \leq u$. Hence by the weak comparison principle (see Pucci-serrin [17, p. 61]), we have
$$y(z) \leq u(z) \quad \text{for all } z \in \overline{R}.$$

Then we have (recall $u(z_1) = 0$)
$$\lim_{\theta \to 0^+}\frac{u(z_1 + \theta(z_2 - z_1))}{\theta} \geq \lim_{\theta \to 0^+}\frac{\widehat{y}(z_1 + \theta(z_2 - z_1)) - \widehat{y}(z_1)}{\theta} = v'(0) > 0$$
$$\Rightarrow \; Du(z_1) \neq 0, \;\; \text{a contradiction (see (4))}.$$

Therefore we have $u(z) > 0$ for all $z \in \Omega$.

This means that for every $K \subseteq \Omega$ compact subset, we can find $c_K > 0$ such that
$$0 < c_K \leq u(z) \quad \text{for all } z \in K. \qquad \square$$

Remark 3.2. In fact in the above proposition, we can drop the hypothesis that $u \in C_0^1(\overline{\Omega})$ and simply require that $u \in W_0^{1,p}(\Omega)$. Then we have
$$0 < c_K \leq u(z) \quad \text{for a.a. } z \in K.$$

In this case, we simply show that
$$\widehat{\Omega} = \{z \in \Omega : \text{there exisits } \epsilon > 0, \text{ such that } \overline{B}_\epsilon(z) \subseteq \Omega, u|_{B_\epsilon(z)} = 0\} = \varnothing.$$

We will not need this more general version.

4. Solutions of constant sign

Let $\widehat{\theta} > \|\xi\|_\infty$ (see hypothesis H_0) and consider the C^1-functionals $\widehat{\varphi}_\pm \colon W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned}\widehat{\varphi}_+(u) &= \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p dz + \frac{\widehat{\theta}}{p}|u^-|_p^p \\ &\quad - \int_\Omega F(z,u^+)dz, \\ \widehat{\varphi}_-(u) &= \frac{1}{p}\int_\Omega a_1(z)|Du|^p dz + \frac{1}{q}\int_\Omega a_2(z)|Du|^q dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p dz + \frac{\widehat{\theta}}{p}|u^+|_p^p \\ &\quad - \int_\Omega F(z,-u^-)dz, \qquad \text{for all } u \in W_0^{1,p}(\Omega).\end{aligned}$$

We will use these functionals to produce constant sign solutions for problem (1). More precisely working with $\widehat{\varphi}_+(\cdot)$ we will generate a positive solution, while using $\widehat{\varphi}_-(\cdot)$ we will obtain a negative. Since both functionals are C^1, we can use critical point theory.

Proposition 4.1. *If hypotheses* H_0, H_1 *hold, then the functionals* $\widehat{\varphi}_\pm(\cdot)$ *satisfy the C-condition.*

Proof. We do the proof for the functional $\widehat{\varphi}_+(\cdot)$, the proof for $\widehat{\varphi}_-(\cdot)$ being similar. Let $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ be a sequence such that

$$|\widehat{\varphi}_+(u_n)| \le c_2 \quad \text{for some } c_2 > 0, \quad \text{all } n \in \mathbb{N}, \tag{8}$$

$$(1+\|u_n\|)\widehat{\varphi}'_+(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \quad \text{as } n \to \infty. \tag{9}$$

From (9) we have

$$\begin{aligned}&|\langle A_p^{a_1}(u_n),h\rangle + \langle A_q^{a_2}(u_n),h\rangle + \int_\Omega \xi(z)(u_n^+)^{p-1}h dz - \int_\Omega [\xi(z)+\widehat{\vartheta}](u_n^-)^{p-1}h dz \\ &- \int_\Omega f(z,u_n^+)h dz| \le \frac{\varepsilon_n\|h\|}{1+\|u_n\|}, \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+.\end{aligned} \tag{10}$$

We test (10) with $h = -u_n^- \in W_0^{1,p}(\Omega)$ and obtain

$$\begin{aligned}&c_0\|Du_n^-\|_p^p \le \epsilon_n, \quad \text{for all } n \in \mathbb{N} \text{ (recall that } \widehat{\theta} > \|\xi\|_\infty,) \\ &\qquad \Rightarrow \; u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to \infty.\end{aligned} \tag{11}$$

From (8) and (11) it follows that

$$\begin{aligned}&\int_\Omega a_1(z)|Du_n^+|^p dz + \frac{p}{q}\int_\Omega a_2(z)|Du_n^+|^q dz + \int_\Omega \xi(z)(u_n^+)^p dz \\ &\quad - \int_\Omega pF(z,u_n^+)dz \le c_3 \quad \text{for some } c_3 > 0, \text{ all } n \in \mathbb{N}.\end{aligned} \tag{12}$$

In (10) we choose the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$. We obtain

$$-\int_\Omega a_1(z)|Du_n^+|^p dz - \int_\Omega a_2(z)|Du_n^+|^q dz - \int_\Omega \xi(z)(u_n^+)^p dz$$
$$+ \int_\Omega f(z, u_n^+)u_n^+ dz \leq \epsilon_n \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

We add (4) and(13). Since $q < p$, we obtain,

$$\int_\Omega [f(z, u_n^+)u_n^+ - pF(z, u_n^+)]dz \leq c_4 \text{ for some } c_4 > 0, \text{ all } n \in \mathbb{N}. \tag{14}$$

Hypotheses H_1(i),(iii) imply that given $\beta_1 \in (0, \beta_0)$, we can find $c_5 = c_5(\beta_1) > 0$ such that

$$\beta_1|x|^\tau - c_5 \leq f(z, x)x - pF(z, x) \quad \text{for a.a } z \in \Omega, \text{all } x \in \mathbb{R}. \tag{15}$$

We return to (14) and use (15). We obtain that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq L^\tau(\Omega) \text{ is bounded.} \tag{16}$$

Form H_1(iii) it is clear that we may assume that $\tau < r < p^*$. We choose $t \in (0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{p^*}. \tag{17}$$

On account of the interpolation inequality, we have

$$\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t}\|u_n^+\|_{p^*}^t \;\Rightarrow\; \|u_n^+\|_r^r \leq c_6\|u_n^+\|^{tr} \text{ for some } c_6 > 0, \text{ all } n \in \mathbb{N} \tag{18}$$

(see (16) and use the Sobolev embedding theorem). In (10) we use the test function $h = u_n^+ \in W_0^{1,p}(\Omega)$ and obtain

$$c_0\|Du_n^+\|_p^p + \int_\Omega [\xi(z) + \widehat{\theta}](u_n^+)^p dz \leq \epsilon_n + \widehat{\theta}\|u_n^+\|_p^p + \int_\Omega f(z, u_n^+)u_n^+ dz. \tag{19}$$

On account of hypotheses H_1(i) we can always take $p < r < p^*$ big (that is, close to p^*) so that $\tau \geq p$. Then from (19) ,(16) and (18) it follows that

$$\|u_n^+\|^p \leq c_7[1 + \|u_n^+\|^{tr}] \quad \text{for some } c_7 > 0, \text{ all } n \in \mathbb{N}. \tag{20}$$

First assume that $p \neq N$.

If $p > N$, the $p^* = +\infty$ and so from (17) we have $tr = r - \tau < \rho$ (see hypothesis H_1(iii)), which implies that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (20)).} \tag{21}$$

If $p < N$, then $p^* = \frac{Np}{N-p}$, and from (17) we obtain

$$tr = \frac{p^*(r - \tau)}{p^* - \tau} < p \quad (\text{ see hypothesis } \mathrm{H}_1(\text{iii})).$$

Then again from (20) we infer that (21) is true.

Finally if $p = N$, then by the Sobolev embedding theorem we know that

$$W_0^{1,p}(\Omega) \hookrightarrow L^s(\Omega) \quad \text{for all } 1 \le s < \infty, \text{ continuously.}$$

Then for the previous argument to work, we need to replace $p^* = +\infty$ by $\eta > r > \tau$. As before, we choose $t \in (0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{\eta} \ \Rightarrow\ tr = \frac{\eta(r-\tau)}{\eta-\tau}.$$

We see that
$$\frac{\eta(r-\tau)}{\eta-\tau} \to r-\tau \quad \text{as } \eta \to +\infty.$$

Since $r - \tau < p$ (see hypothesis H_1(iii)), we choose $\eta > r$ big so that

$$tr = \frac{\eta(r-\tau)}{\eta-\tau} < p.$$

Then again we have (21). From (11) and (21) it follows that

$$\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded .}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \ \text{ and } \ u_n \to u \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{22}$$

In (10) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (22). Then we have

$$\begin{aligned}
&\lim_{n\to\infty} \left[\langle A_p^{a_1}(u_n), u_n - u\rangle + \langle A_q^{a_2}(u_n), u_n - u\rangle\right] = 0,\\
\Rightarrow\ &\limsup_{n\to\infty} \left[\langle A_p^{a_1}(u_n), u_n - u\rangle + \langle A_q^{a_2}(u), u_n - u\rangle\right] \le 0 \text{ (since } A_q^{a_2}(\cdot) \text{ is monotone)},\\
\Rightarrow\ &\limsup_{n\to\infty} \langle A_p^{a_1}(u_n), u_n - u\rangle \le 0 \quad \text{(see (22))},\\
\Rightarrow\ &u_n \to u \quad \text{in } W_0^{1,p}(\Omega) \quad \text{as } n \to \infty \text{ (see Proposition 2.1)}.
\end{aligned}$$

This proves that $\widehat{\varphi}_+(\cdot)$ satisfies the C-condition.
Similarly we show that $\widehat{\varphi}_-(\cdot)$ satisfies the C-condition. □

Proposition 4.2. *If hypotheses* H_0, H_1 *hold, then* $u = 0$ *is a local minimizer for both functionals* $\widehat{\varphi}_+(\cdot)$ *and* $\widehat{\varphi}_-(\cdot)$.

Proof. Again we do the proof for $\widehat{\varphi}_+(\cdot)$, the proof for $\widehat{\varphi}_-(\cdot)$ being similar.

On account of hypotheses H_1(i),(ii), given $\epsilon > 0$, we can find $c_8 = c_8(\epsilon) > 0$ such that

$$F(z,x) \le \frac{1}{q}[\vartheta(z) + \epsilon]|x|^q + c_8|x|^r \quad \text{for a.a } z \in \Omega, \text{all } x \in \mathbb{R}. \tag{23}$$

$$\begin{aligned}
\widehat{\varphi}_+(u) &\geq \frac{1}{p}\int_\Omega a_1|Du|^p dz + \frac{1}{q}\int_\Omega a_2|Du|^q dz + \frac{1}{p}\int_\Omega \xi(z)|u|^p dz + \frac{\widehat{\vartheta}}{p}\|u^-\|_p^p \\
&\quad -\frac{1}{q}\int_\Omega [\vartheta(z)+\epsilon]|u|^q dz - c_8\|u\|_r^r \quad \text{(see (23))} \\
&\geq \frac{c_0}{p}\|u\|^p + \frac{1}{q}[\int_\Omega a_2|Du|^q dz - \int_\Omega \vartheta(z)|u|^q dz - \frac{\epsilon}{\widehat{\lambda}_1(q,a_2)}\|u\|^q] \\
&\quad -c_9\|u\|^r \quad \text{for some } c_9>0 \text{ (recall that } \widehat{\theta} > \|\xi\|_\infty \text{ and } p<r) \\
&\geq \frac{c_0}{p}\|u\|^p + \frac{1}{q}[\widehat{c} - \frac{\epsilon}{\widehat{\lambda}_1(q,a_2)}]\|u\|^q - c_9\|u\|^r \text{ (see Lemma 2.1).}
\end{aligned}$$

Choosing $\epsilon \in (0, \hat{\lambda}_1(q,a_2)\widehat{c})$, we obtain $\widehat{\varphi}_+(u) \geq \frac{c_0}{p}\|u\|^p - c_9\|u\|^r$.

Since $p<r$, we see that if $\rho \in (0,1)$ is small, we have

$$\widehat{\varphi}_+(u) > \widehat{\varphi}_+(0) = 0 \quad \text{if } 0 < \|u\| \leq \rho,$$

$$\Rightarrow \; u=0 \text{ is a (strict) local minimizer of } \widehat{\varphi}_+(\cdot).$$

Similarly we show this result for the functional $\widehat{\varphi}_-(\cdot)$. □

Proposition 4.3. *If Hypotheses* H_0, H_1 *hold and* $u \in intC_+$, *then* $\hat{\varphi}_\pm(tu) \to -\infty$ *as* $t \to \pm\infty$.

Proof. We show the proof for $\hat{\varphi}_+(\cdot)$.

H_1(i),(ii) imply that given $\mu>0$, we can find $c_\mu>0$ such that

$$F(z.x) \geq \frac{\mu}{p}|x|^p - c_\mu \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}$$

Let $|\cdot|_N$ denote the Lebesgue measure on $\mathbb{R}^N$. We have

$$\begin{aligned}
\widehat{\varphi}_+(tu) &\leq \frac{t^p}{p}\|u\|^p + \frac{t^q}{q}\|Du\|_q^q + \frac{t^p}{p}\int_\Omega \xi(z)u^p dz - \frac{t^p}{p}\mu\|u\|_p^p + c_\mu|\Omega|_N \\
&\leq \frac{t^p}{p}[c_{10}\|u\|^p - \mu\|u\|_p^p] + \frac{t^q}{q}\|Du\|_q^q + c_\mu|\Omega|_N \text{ for some } c_{10}>0;
\end{aligned}$$

(recall $\xi \in L^\infty(\Omega)$). Choosing $\mu > c_{10}\frac{\|u\|^p}{\|u\|_p^p}$, we obtain

$$\widehat{\varphi}_+(tu) \leq c_{11}t^q - c_{12}t^p + c_\mu|\Omega|_N \text{ for some } c_{11}>0, c_{12}>0.$$

Since $q<p$, we infer that $\widehat{\varphi}_+(tu) \to -\infty$ as $t \to +\infty$.

Similarly we show that $\widehat{\varphi}_-(tu) \to -\infty$ as $t \to -\infty$. □

Now we can produce two nontrivial constant sign solutions for problem (1).

Proposition 4.4. *If hypotheses* H_0, H_1 *hold, then problem* (1) *has two constant sign solutions* $u_0 \in intC_+$ *and* $v_0 \in -intC_+$.

Proof. Proposition 4.2 and Theorem 5.7.6 of Papageorgiou-Rădulescu-Repovš [16, p. 449], imply that we can find $\rho \in (0,1)$ small such that

$$\widehat{\varphi}_+(0) = 0 < inf[\widehat{\varphi}_+(u) : \|u\| = \rho] = \widehat{m}_+. \tag{24}$$

Then (24) together with Propositions 4.1 and 4.3, permit the use of the mountain pass theorem. So we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$u_0 \in K_{\widehat{\varphi}_+} \quad \text{and} \quad \widehat{\varphi}_+(0) = 0 < \widehat{m}_+ \le \widehat{\varphi}_+(u_0) \quad (\text{ see (24) }) \tag{25}$$

From (25) we have $u_0 \neq 0$ and

$$\begin{aligned}
&\varphi'_+(u_0) = 0 \;\Rightarrow \\
&\langle A_p^{\alpha_1}(u_0), h\rangle + \langle A_q^{\alpha_2}(u_0), h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h dz - \widehat{\theta}\int_\Omega (u_0^-)^{p-1} h dz \\
&= \int_\Omega f(z, u_0^+) h dz \quad \text{ for all } h \in W_0^{1,p}(\Omega).
\end{aligned} \tag{26}$$

In (26) we choose $h = -v_0^- \in W_0^{1,p}(\Omega)$. Then since $\widehat{\theta} > \|\xi\|_\infty$, we obtain

$$c_0\|Du_0^-\|_p^p \le 0 \Rightarrow u_0 \ge 0, u_0 \neq 0.$$

From (26) it follows that

$$\begin{cases} -\Delta_p^{a_1} u_0(z) - \Delta_q^{a_2} u_0(z) + \xi(z)u_0(z)^{p-1} = f(z, u_0(z)) & \text{in } \Omega, \\ u_0|_{\partial\Omega} = 0. \end{cases} \tag{27}$$

From (27) and Theorem 7.1 of Ladyzhenskaya-Uraltseva [9, p. 286] we obtain that $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [10] implies that $u_0 \in C_+ \setminus \{0\}$.

Let $\widehat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. Proposition 3.1 and Lemma 2.3 of Guo-Webb [6] imply that we can find $c_{13} > 0, c_{14} > 0$ such that

$$c_{13}\widehat{d} \le u \le c_{14}\widehat{d}. \tag{28}$$

Lemma 14.16 of Gilbarg-Trudinger [5, p. 335], says that there exists $\delta > 0$ small such that

$$\widehat{d} \in C^2(\Omega_\delta) \quad \text{with} \quad \Omega_\delta = \{z \in \widehat{\Omega} : \widehat{d}(z) < \delta\} \Rightarrow \widehat{d} \in intC_+.$$

Then from (28) it follows that $u_0 \in intC_+$.

Similarly, working this time with the functional $\widehat{\varphi}_-(\cdot)$ we produce a negative solution $v_0 \in -intC_+$. □

In fact we can show that the problem has extremal constant sign solutions, that is, there is a smallest positive solution and a biggest negative solution for problem (1).

In what follows by S_+ we denote the set of positive solutions of (1) and by S_- the set of negative solutions of (1). We already know that

$$\emptyset \neq S_+ \subseteq intC_+ \quad \text{and} \quad \emptyset \neq S_- \subseteq -intC_+.$$

Proposition 4.5. *If hypotheses* H_0, H_1 *hold, then problem* (1) *has a smallest positive solution* $u^* \in intC_+$ *and a biggest negative solution* $v^* \in -intC_+$.

Proof. From Papageorgiou-Rădulescu-Repovš [15] (see the proof of Proposition 4.4), we have that S_+ is downward directed (that is, if $u_1, u_2 \in S_+$, then we can find $u \in S_+$ such that $u \leq u_1, u \leq u_2$). Then invoking Lemma 3.10 of Hu-Papageorgiou [8, p. 178], we can find $\{u_n\}_{n \in N} \subseteq S_+$ decreasing such that

$$\inf_{n \geq 1} u_n = \inf S_+.$$

We have for all $h \in W_0^{1,\rho}(\Omega)$ and all $n \in \mathbb{N}$,

$$\langle A_p^{a_1}(u_n), h \rangle + \langle A_q^{a_2}(u_n), h \rangle + \int_\Omega \xi(z) u_n^{p-1} h dz = \int_\Omega f(z, u_n) h dz, \tag{29}$$

$$\text{and} \quad 0 \leq u_n \leq u_1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{30}$$

In (29) we choose $h = u_n \in W_0^{1,p}(\Omega)$. From (30) and hypothesis H_1(i) it follows that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u^* \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{31}$$

We test (29) with $h = u_n - u^* \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to \infty$ and use (31), then as in the proof of Proposition 4.1, using the $(S)_+$-property of $A_\rho^{a_1}(\cdot)$ (see Proposition 2.1), we obtain that

$$u_n \to u^* \text{ in } W_0^{1,p}(\Omega). \tag{32}$$

Suppose that $u^* = 0$. We set $y_n = \frac{u_n}{\|u_n\|}, n \in \mathbb{N}$. We have $\|y_n\| = 1, y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \text{ and } y_n \to y \text{ in } L^r(\Omega). \tag{33}$$

From (29) we have

$$\begin{aligned} &\|u_n\|^{p-q} \langle A_p^{a_1}(y_n), h \rangle + \langle A_q^{a_2}(y_n), h \rangle + \|u_n\|^{p-q} \int_\Omega \xi(z) y_n^{p-1} h dz \\ &= \int_\Omega \frac{f(z, u_n)}{\|u_n\|^{q-1}} h dz \ \text{ for all } h \in W_0^{1,p}(\Omega), \ \text{all } n \in \mathbb{N}. \end{aligned} \tag{34}$$

In consequence of hypotheses H_1(i),(iv) we have

$$\begin{aligned} &|f(z, x)| \leq c_{15}(x^{q-1} + x^{r-1}) \text{ for a.a. } z \in \Omega, \ \text{all } x \geq 0, \ \text{some } c_{15} > 0, \\ &\Rightarrow \ \left\{ \frac{f(\cdot, u_n(\cdot))}{\|u_n\|^q} \right\}_{n \in \mathbb{N}} \subseteq L^\infty(\Omega) \ \text{ is bounded (see (30)).} \end{aligned} \tag{35}$$

Also we have

$$\|u_n\| \to 0 \ \text{ as } \ n \to \infty \tag{36}$$

(see (32) and recall that we assumed that $u^* = 0$).

We test (34) with $h = y_n - y \in W_0^{1,p}(\Omega)$), pass to the limit as $n \to \infty$ and use (35), and (36) (recall $q < p$). We obtain

$$\lim_{n\to\infty} \langle A_q^{a_2}(y_n), y_n - y\rangle = 0 \ \Rightarrow\ y_n \to y \text{ in } W_0^{1,q}(\Omega) \ \text{ (see Proposition 2.1).}$$

Then we may assume that $Dy_n(z) \to Dy(z)$ in $\mathbb{R}^N$ for a.a. $z \in \Omega$. Also from (33) we see that $\{|Dy_n|^p\}_{n\in\mathbb{N}} \subseteq L^1(\Omega)$ is uniformly integrable. Hence by Vitali's theorem we have

$$\|Dy_n\|_p \to \|Dy\|_p \Rightarrow y_n \to y \ \text{ in } W_0^{1,p}(\Omega) \tag{37}$$

(see (33) and recall that $W_0^{1,p}(\Omega)$ has the Kadec-Klee property),

$$\Rightarrow \|y\| = 1,\ y \geq 0. \tag{38}$$

From (35) and hypothesis H_1(iv) we have at least for a subsequence that

$$\frac{f(\cdot, u_n(\cdot))}{\|u_n\|^{q-1}} \xrightarrow{w} \vartheta_0(\cdot)y(\cdot)^{q-1} \ \text{ in } L^{q'}(\Omega) \text{ as } n \to \infty \tag{39}$$

with $\vartheta_0 \in L^\infty(\Omega)$, $\vartheta_0(z) \leq \vartheta(z)$ for a.a. $z \in \Omega$.

If in (34) we pass to the limit as $n \to \infty$ and use (36),(37),(39), we obtain

$$\langle A_q^{a_2}(y), h\rangle = \int_\Omega \vartheta_0(z)y^{q-1}hdz \text{ for all } h \in W_0^{1,p}(\Omega),$$

$$\Rightarrow \ \int_\Omega a_2(z)|Dy|^q dz - \int_\Omega \vartheta_0(z)y^q dz = 0 \text{ (testing with } h = y \in W_0^{1,p}(\Omega)),$$

$$\Rightarrow \ c_{16}\|Dy\|^q dz \leq 0 \quad \text{for some } c_{16} > 0 \text{ (see Lemma 2.1),}$$

$$\Rightarrow \ y = 0, \text{ which contradicts (38) .}$$

Therefore $u^* \neq 0$ and we have for all $h \in W_0^{1,\rho}(\Omega)$ (see (32))

$$\langle A_p^{a_1}(u^*), h\rangle + \langle A_q^{a_2}(u^*), h\rangle + \int_\Omega \xi(z)(u^*)^{p-1}hdz = \int_\Omega f(z, u^*)hdz,$$

$$\Rightarrow \ u^* \in S_+,\ u^* = inf\, S_+.$$

Similarly we produce a biggest negative solution $v^* \in -intC_+$. Note that S_- is upward directed (that is, if $v_1, v_2 \in S_-$), then we can find $v \in S_-$ such that $v_1 \leq v, v_2 \leq v$. □

5. Nodal solution

In this section we produce a third nontrivial solution which is nodal. The proof uses the extremal constant sign solutions from Proposition 4.5 and flow invariance arguments. (see also Papageorgiou-Papalini [13] and He-Lei-Zhang-Sun [7])

So, let $u^* \in intC_+$ and $v_* \in -intC_+$ be the two extremal constant sign solutions of problem (1) produced in Proposition 4.5. We define

$$[v_*, u_*] = \{y \in W_0^{1,\rho}(\Omega) : v^*(z) \leq y(z) \leq u^*(z) \text{ for a.a. } z \in \Omega\}.$$

Proposition 5.1. *If hypotheses* H_0, H_1 *hold, then problem* (1) *has a nodal solution* $y_0 \in [v^*, u^*] \cap C_0^1(\bar{\Omega})$.

Proof. Let $\rho = max\{\|v^*\|_\infty, \|v^*\|_\infty\}$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis H_1(v). We can always take $\widehat{\xi}_\rho > \|\xi\|_\infty$. We introduce the Carathéodory function $\widehat{f}(z,x)$ defined by

$$\widehat{f}(z,x) = \begin{cases} f(z, v^*(z)) + \widehat{\xi}_\rho |v^*(z)|^{p-2} v^*(z) & \text{if } x < v^*(z), \\ f(z,x) + \widehat{\xi}_\rho |x|^{p-2} x & \text{if } v^*(z) \le x \le u^*(z), \\ f(z, u^*(z)) + \widehat{\xi}_\rho u^*(x)^{p-1} & \text{if } u^*(z) < x. \end{cases} \tag{40}$$

In what follows by $N_{\widehat{f}}$ we denote the Nemytski (superposition) map corresponding to $\widehat{f}(z,x)$, that is, for every $u : \Omega \to \mathbb{R}$ measurable, we have $N_{\widehat{f}}(u)(\cdot) = \widehat{f}(\cdot, u(\cdot))$. Let $K : W_0^{1,p}(\Omega) \to W_0^{-1,p'}(\Omega) = W_0^{1,p}(\Omega)^*$ $(\frac{1}{p} + \frac{1}{p'} = 1)$ be defined by

$$\langle K(u), h \rangle = \langle A_p^{a_1}(u), h \rangle + \langle A_q^{a_2}(u), h \rangle + \int_\Omega [\xi(z) + \widehat{\xi}_\rho] |u|^{p-2} u h dz$$

for all $h \in W_0^{1,p}(\Omega)$. The map is continuous, strictly monotone (hence maximal monotone too) and coercive. Therefore $K(\cdot)$ is surjective (see Papageorgiou-Rădulescu-Repovš [16, p. 135]) and being strictly monotone it is also injective. So we can define $K^{-1} : W^{-1,p'}(\Omega) \to W_0^{1,p}(\Omega)$. Then let $L(u) = (K^{-1} \circ N_{\widehat{f}})(u)$. The map $L(\cdot)$ is locally Lipschitz and by the Sobolev embedding theorem we have that $L(\cdot)$ is compact. Moreover, the nonlinear regularity theory (see [9, 10]) implies that

$$L(W_0^{1,p}) \subseteq C_0^1(\bar{\Omega}), \; L(C_+) \subseteq C_+ \text{ (see hypothesis } \mathrm{H}_1(\mathrm{v})).$$

We introduce the C^1-functional $\widehat{\psi} : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\begin{aligned} \widehat{\psi}(u) = & \frac{1}{p} \int_\Omega a_1(z) |Du|^p dz + \frac{1}{q} \int_\Omega a_2(z) |Du|^q dz + \frac{1}{p} \int_\Omega [\xi(z) + \widehat{\xi}_\rho] |u|^p dz \\ & - \int_\Omega \widehat{F}(z,u) dz \text{ for all } u \in W_0^{1,p}(\Omega). \end{aligned}$$

Here $\widehat{F}(z,x) = \int_0^x \widehat{f}(z,s) ds$. On account of (40) we have

$$K_{\widehat{\psi}} \subseteq [v^*, u^*] \bigcap C_0^1(\bar{\Omega})$$

and so $K_{\widehat{\psi}}$ are solutions of (1) and in fact due to the extremality of u^* and v^*, the nontrivial elements of $K_{\widehat{\psi}}$ distinct from u^*, v^* are nodal solutions of (1). Also $K_{\widehat{\psi}}$ coincides with the fixed point set of $L(\cdot)$.

We consider the following abstract Cauchy problem

$$\frac{d\sigma(t,u)}{dt} + \sigma(t,u) = L(\sigma(t,u)) \text{ for } t \ge 0, \sigma(0,u) = u. \tag{41}$$

According to Theorem 5.1.22 of Gasiński-Papageorgiou [2, p. 618], problem (41) admits a unique flow $\sigma(\cdot, u)$ defined on a maximal time interval $[0, \tau(u))$ given by

$$\sigma(t,u) = e^{-t} u + \int_0^t e^{-(t-s)} L(\sigma(s,u)) \, ds \;\; \text{for all } 0 \le t < \tau(u).$$

The sets $[v^*, u^*]$ and $int_{C_0^1(\bar{\Omega})}[v^*, u^*]$(= the interior in $C_0^1(\bar{\Omega})$ of $[v^*, u^*] \bigcap C_0^1(\bar{\Omega})$) are σ invariant (see [13]).

We introduce the sets

$$D_1 = \left\{ u \in C_0^1(\bar{\Omega}) : \begin{array}{l} \text{there exists } t^* \in [0, \tau(u)) \text{ such that} \\ \sigma(t, u) \in int_{C_0^1(\bar{\Omega})}[v^*, u^*] \text{ for all } t^* \leq t < \tau(u) \end{array} \right\},$$

$$D_2 = \left\{ u \in C_0^1(\bar{\Omega}) : \begin{array}{l} \text{there exists } t^* \in [0, \tau(u)) \text{ such that} \\ \sigma(t, u) \in int\, C_+ \ \text{ for all } t^* \leq t < \tau(u) \end{array} \right\}.$$

Reasoning as in Papageoriou-Papalini [13] (proof of Theorem 2, p. 793; see also He-Lei-Zhang-Sun [7, Section 4]), we establish the following properties for the sets D_1, D_2:

(a) D_1 is open, $0 \in D_1, D_1$ and ∂D_1 are σ-invariant, $\inf\limits_{u \in \partial D_1} \widehat{\psi} > -\infty$.

(b) D_2 is open, $0 \in \partial D_2$ and ∂D_2 is σ-invariant.

Let $c_* = \inf[\widehat{\psi}(u) : u \in \partial D_1 \bigcap \partial D_2] > -\infty$ and we have that c_* is a critical value of $\widehat{\psi}$. Indeed, if this is not the case and since $K_{\widehat{\psi}}$ is closed, we can find $\varepsilon > 0$ such that

$$(c_* - \varepsilon, c_* + \varepsilon) \bigcap \widehat{\psi}(K_{\widehat{\psi}}) = \emptyset. \tag{42}$$

Let $\widehat{u} \in \partial D_1 \bigcap \partial D_2$ be such that $\widehat{\psi}(\widehat{u}) \leq c_* + \varepsilon$.

Then as in the proof of the deformation theorem (see Papageorgiou-Rădulescu-Repovš [16, p. 372]), because of (42), we have

$$\widehat{\psi}(\sigma(\widehat{t}, u)) \leq c_* - \varepsilon \ \text{ for some } \ \widehat{t} < \tau(u).$$

But the sets $\partial D_1, \partial D_2$ are σ-invariant. Therefore $\widehat{\psi}(\sigma(\widehat{t}, u)) \geq c_*$, a contradiction. So, c_* is a critical value of $\widehat{\psi}$. Then let $y_0 \in \partial D_1 \bigcap \partial D_2$ such that

$$\widehat{\psi}'(y_0) = 0, \ \widehat{\psi}(y_0) = c_*.$$

Since $y_0 \in \partial D_1$, we infer that $y_0 \neq 0$. Moreover $y_0 \in [v^*, u^*] \cap C_0^1(\bar{\Omega})$ and since $y_0 \in \partial D_2$, we see that $y_0 \neq v^*, y_0 \neq u^*$ (recall $v^* \in -intC_+$ and $u^* \in intC_+$). Therefore $y_0 \in C_0^1(\bar{\Omega})$ is the nodal solution of (1). □

We can state the following multiplicity theorem for problem (1). It generalizes all earlier multiplicity theorems for superlinear problems.

Theorem 5.2. *If the hypotheses* H_0, H_1 *hold, then problem* (1) *has at least three nontrivial solutions*

$$u_0 \in intC_+, \ v_0 \in -intC_+$$

and $y_0 \in [v_0, u_0] \bigcap C_0^1(\bar{\Omega})$ *nodal.*

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