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Projecting onto Intersections of Halfspaces and Hyperplanes

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It is well-known that the sequence of iterations of the composition of projections onto closed affine subspaces converges linearly to the projection onto the intersection of the affine subspaces when the sum of the corresponding linear subspaces is closed. Inspired by this, we systematically study the relation between the projection onto intersection of halfspaces and hyperplanes, and the composition of projections onto halfspaces and hyperplanes. In addition, as by-products, we provide the Karush-Kuhn-Tucker conditions for characterizing the optimal solution of convex optimization with finitely many equality and inequality constraints in Hilbert spaces and construct an explicit formula for the projection onto the intersection of hyperplane and halfspace.

Keywords: Projection, halfspace, hyperplane, best approximation mapping, linear convergence, Karush-Kuhn-Tucker conditions, convex optimization.

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1. Introduction

Throughout this paper, we assume that $\mathcal{H}$ is a real Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$.

Throughout the paper, we use the convention that $\mathbb{N} := \{0, 1, 2, \ldots\}$. Let $m \in \mathbb{N} \setminus \{0\}$. For every $i \in \{1, \ldots, m\}$, let u_i be in $\mathcal{H}$ and let η_i be in $\mathbb{R}$. Set

$$W_i := \{x \in \mathcal{H} \ : \ \langle x, u_i \rangle \leq \eta_i\} \quad \text{and} \quad H_i := \{x \in \mathcal{H} \ : \ \langle x, u_i \rangle = \eta_i\}.$$

According to Deutsch's [10, Theorems 9.8 and 9.35] and an easy translation argument, the sequence of iterations of the composition of projections onto closed affine subspaces converges linearly to the projection onto the intersection of the affine subspaces when the sum of the linear subspaces which are parallel to the affine subspaces is closed. Moreover, intersections of halfspaces and hyperplanes are frequently seen in constraints of road design problems [2, Section 2], least norm problems [6, Section 5.2], constrained regression problems [7, p. 153], and least square problems [14, p. 226].

Note that although Dykstra's algorithm allows one to compute best approximations from an intersection of finitely many closed convex sets (see, e.g., [10, p. 207]), the practical manipulation of Dykstra's algorithm is very complicated. Moreover, although there exist explicit formulae for $\mathrm{P}_{W_1 \cap W_2}$ and $\mathrm{P}_{H_1 \cap W_2}$ (see, Fact 4.2, Fact 4.7,

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Theorem 5.4, and Theorem 5.6 below), given a point $x \in \mathcal{H}$, the formulae of $\mathrm{P}_{W_1 \cap W_2} x$ and $\mathrm{P}_{H_1 \cap W_2} x$ depend on the linear dependence relation of u_1 and u_2, and on the "region" where the x is located in. Hence, it is worthwhile to explore other easy ways to find the best approximation from the intersection of halfspaces and hyperplanes or feasibility point in that intersection.

Inspired by [10, Theorem 9.8], in this work, our goal is *to study the relation between the projection onto intersection of halfspaces and hyperplanes, and the composition of projections onto halfspaces and hyperplanes.*

The main results in this work are the following:

R1: Theorem 4.17 summarizes the relation between $\mathrm{P}_{W_1 \cap W_2}$ and $\mathrm{P}_{W_2} \mathrm{P}_{W_1}$. In particular, if u_1 and u_2 are linear dependent or orthogonal, then we have $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{P}_{W_1 \cap W_2}$. Otherwise, for $\gamma := \frac{|\langle u_1, u_2 \rangle|}{\|u_1\| \|u_2\|} \in [0, 1[$, if $\langle u_1, u_2 \rangle < 0$, then $(\forall x \in \mathcal{H})$ $(\forall k \in \mathbb{N})$ $\|(\mathrm{P}_{W_2} \mathrm{P}_{W_1})^k x - \mathrm{P}_{W_1 \cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{W_1 \cap W_2} x\|$; if $\langle u_1, u_2 \rangle > 0$, then $(\forall x \in \mathcal{H})$ $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x \in W_1 \cap W_2$: particularly, if we have $x \in W_1 \cup W_2$, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$; if $x \in W_1^c \cap W_2^c$ with $\mathrm{P}_{H_1} x \in W_2$, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1 \cap W_2} x$; if $x \in W_1^c \cap W_2^c$ with $\mathrm{P}_{H_1} x \notin W_2$, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x \in W_1 \cap W_2 \setminus \{\mathrm{P}_{W_1 \cap W_2} x\}$.

R2: Theorem 5.2 states the KKT conditions associated with convex optimization with finitely many equality and inequality constraints in Hilbert spaces.

R3: Theorem 5.6 shows an explicit formula for the projection onto intersection of hyperplane and halfspace.

R4: Theorem 6.5 concludes the relations of $\mathrm{P}_{H_1 \cap W_2}$ with $\mathrm{P}_{W_2} \mathrm{P}_{H_1}$ and $\mathrm{P}_{H_1} \mathrm{P}_{W_2}$. In particular, if u_1 and u_2 are linearly dependent or orthogonal, then we have $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1 \cap W_2} = \mathrm{P}_{H_1} \mathrm{P}_{W_2}$.
Otherwise, for $\gamma := \frac{|\langle u_1, u_2 \rangle|}{\|u_1\| \|u_2\|} \in [0, 1[$, we have for all $x \in \mathcal{H}$ and all $k \in \mathbb{N}$ $\|(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x - \mathrm{P}_{H_1 \cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{H_1 \cap W_2} x\|$, and if only $\mathrm{P}_{H_1} \mathrm{P}_{W_2} x \notin W_2$, then $(\forall x \in W_2)$ $(\forall k \in \mathbb{N})$ $\|\mathrm{P}_{W_2} (\mathrm{P}_{H_1} \mathrm{P}_{W_2})^k x - \mathrm{P}_{H_1 \cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{H_1 \cap W_2} x\|$, and $(\forall x \in W_2^c)$ $(\forall k \in \mathbb{N})$ $\|(\mathrm{P}_{H_1} \mathrm{P}_{W_2})^k - \mathrm{P}_{H_1 \cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{H_1 \cap W_2} x\|$.

In view of our results mentioned above, if u_1 and u_2 are linearly dependent or orthogonal, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{P}_{W_1 \cap W_2}$, and $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1} \mathrm{P}_{W_2} = \mathrm{P}_{H_1 \cap W_2}$. Let $x \in \mathcal{H}$. Moreover, the sequence $\big((\mathrm{P}_{W_2} \mathrm{P}_{W_1})^k x\big)_{k \in \mathbb{N}}$ converges linearly or in one step for finding the desired best approximation point $\mathrm{P}_{H_1 \cap W_2} x$ or converges in one step for finding the feasibility point in $W_1 \cap W_2$. In addition, the sequence $\big((\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x\big)_{k \in \mathbb{N}}$ always converges linearly to the desired best approximation point $\mathrm{P}_{H_1 \cap W_2} x$.

Although the result on KKT conditions is classical and well-known, some KKT conditions are only shown in finite-dimensional spaces (see, e.g., [7, p. 244] and [15, Theorem 28.3]), some include only inequality constraints without equality constraints (see, e.g., [1, Proposition 27.21], [5, Theorem 3.78], [13, p. 249], and [16, Pages 94]), and some act as only necessary optimality conditions (see, e.g., [5, Theorem 3.78], [13, p. 249], and [16, Pages 94 and 274]). The result on KKT conditions presented in Theorem 5.2 characterizes the optimal solution of the convex optimization with finitely many equality and inequality constraints, which is a generalization of the version presented in [7, p. 244] from finite-dimensional spaces to Hilbert spaces and from differentiable function to subdifferentiable function, and of the version shown in [1, Proposition 27.21] from inequality constrains to inequality and equality constants.

The organization of the rest of the paper is the following. We display some auxiliary results in Section 2. In Section 3, we collect the fact on the linear convergence of the composition of finitely many projections onto hyperplanes and construct an explicit formula of projection onto finitely many hyperplanes. In Section 4, we systematically study the relation between $\mathrm{P}_{W_2}\mathrm{P}_{W_1}$ and $\mathrm{P}_{W_1\cap W_2}$. In Section 5, we aim to construct explicit formulae for the projection onto the intersection of hyperplane and halfspace, which plays a critical role for us to investigate the relations of $\mathrm{P}_{H_1\cap W_2}$ with $\mathrm{P}_{W_2}\mathrm{P}_{H_1}$ and $\mathrm{P}_{H_1}\mathrm{P}_{W_2}$ in Section 6. To this end, as a by-product, in Section 5, we also provide the KKT conditions associated with convex optimization with finitely many equality and inequality constraints in Hilbert spaces.

We now turn to the notation used in this paper. Let D be a subset of $\mathcal{H}$. $D^c := \mathcal{H}\backslash D$ is the *complementary set* of D. The *interior* of D is the largest open set that is contained in D; it is denoted by $\operatorname{int} D$. The *orthogonal complement* of D is the set $D^{\perp} := \{x \in \mathcal{H} \;:\; (\forall y \in D)\ \langle x, y\rangle = 0\}$. D is an *affine subspace* of $\mathcal{H}$ if $D \neq \varnothing$ and $(\forall \rho \in \mathbb{R})\ \rho D + (1-\rho)D = D$. In addition, D is a *cone* if $D = \mathbb{R}_{++}D$. The polar cone of D is $D^{\ominus} := \{u \in \mathcal{H} \;:\; \sup\langle D, u\rangle \leq 0\}$. The *conical hull* of D is the intersection of all the cones in $\mathcal{H}$ containing D, i.e., the smallest cone in $\mathcal{H}$ containing D. It is denoted by $\operatorname{cone} D$. Let C be a nonempty convex subset of $\mathcal{H}$. Let $x \in \mathcal{H}$. The *normal cone* to C at x is

$$\mathrm{N}_C\, x := \begin{cases} \{u \in \mathcal{H} \;:\; \sup\langle C - x, u\rangle \leq 0\}, & \text{if } x \in C; \\ \varnothing, & \text{otherwise.} \end{cases}$$

The *strong relative interior* of C is $\operatorname{sri} C := \{x \in C : \operatorname{cone}(C - x) = \overline{\operatorname{span}}\,(C - x)\}$. Suppose that C is a nonempty closed convex subset of $\mathcal{H}$. The *projector* (or *projection operator*) onto C is the operator, denoted by P_C, that maps every point in $\mathcal{H}$ to its unique projection onto C.

Let $\mathcal{K}$ be a real Hilbert space. We define

$$\mathcal{B}(\mathcal{H}, \mathcal{K}) := \{T : \mathcal{H} \to \mathcal{K} \;:\; T \text{ is linear and bounded}\}.$$

Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The adjoint of T is the unique operator $T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ that satisfies $(\forall x \in \mathcal{H})\ (\forall y \in \mathcal{K})\ \langle Tx, y\rangle = \langle x, T^* y\rangle$. Let $f : \mathcal{H} \to\,]-\infty, +\infty]$ be *proper*, that is, $\operatorname{dom} f := \{x \in \mathcal{H} \;:\; f(x) < +\infty\} \neq \varnothing$. Denote the *domain of continuity* of f by $\operatorname{cont} f := \{x \in \mathcal{H} \;:\; f(x) \in \mathbb{R} \text{ and } f \text{ is continuous at } x\}$. The *subdifferential* of f is the set-valued operator

$$\partial f : \mathcal{H} \to 2^{\mathcal{H}} : \quad x \mapsto \{u \in \mathcal{H} \;:\; (\forall y \in \mathcal{H})\langle y - x, u\rangle + f(x) \leq f(y)\}.$$

The *indicator function* of a subset A of $\mathcal{H}$ is the function

$$\iota_A : \mathcal{H} \to\,]-\infty, +\infty] : \quad x \mapsto \begin{cases} 0, & \text{if } x \in A; \\ +\infty, & \text{otherwise.} \end{cases}$$

Denote by $\Gamma_0(\mathcal{H})$ the set of all proper lower semicontinuous convex functions from $\mathcal{H}$ to $]-\infty, +\infty]$. Let u be in $\mathcal{H}$. Then $\ker u := \{x \in \mathcal{H} \;:\; \langle x, u\rangle = 0\}$ is the *kernel* *of* u. The *set of fixed points of the operator* $T : \mathcal{H} \to \mathcal{H}$ is denoted by $\operatorname{Fix} T$, i.e., $\operatorname{Fix} T := \{x \in \mathcal{H} \ : \ Tx = x\}$.

For other notation not explicitly defined here, we refer the reader to [1].

2. Auxiliary results

In this section, we provide some results to be used in the sequel.

Linearly independent vectors

The following well-known results will be used frequently in our proofs later. For completeness, we attach the easy proof below as well.

Fact 2.1. *Let* u_1 *and* u_2 *be in* $\mathcal{H}$*. The following statements hold.*

(i) *Suppose* $u_1 \neq 0$*. Then* u_1, u_2 *are linearly dependent* $\Leftrightarrow u_2 = \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$ $\Leftrightarrow \|u_1\|\|u_2\| = |\langle u_1, u_2\rangle| \Leftrightarrow$ *either* $u_2 = \frac{\|u_2\|}{\|u_1\|} u_1$ *or* $u_2 = -\frac{\|u_2\|}{\|u_1\|} u_1$.

(ii) u_1, u_2 *are linearly dependent* $\Leftrightarrow \|u_1\|\|u_2\| = |\langle u_1, u_2\rangle|$.

(iii) $\|u_1\|\|u_2\| > |\langle u_1, u_2\rangle|$ *if and only if* u_1, u_2 *are linearly independent.*

Proof. (i): Because $u_1 \neq 0$, we know that u_1, u_2 are linearly dependent if and only if $u_2 = \beta u_1$ for some $\beta \in \mathbb{R}$. Take inner product with u_1 for both sides of $u_2 = \beta u_1$ to obtain that $\langle u_1, u_2\rangle = \beta \langle u_1, u_1\rangle$, which implies that $\beta = \frac{\langle u_1, u_2\rangle}{\langle u_1, u_1\rangle}$. Hence, u_1, u_2 are linearly dependent if and only if $u_2 = \frac{\langle u_1, u_2\rangle}{\langle u_1, u_1\rangle} u_1$. On the other hand,

$$0 \leq \left\langle u_2 - \frac{\langle u_1, u_2\rangle}{\langle u_1, u_1\rangle} u_1, u_2 - \frac{\langle u_1, u_2\rangle}{\langle u_1, u_1\rangle} u_1 \right\rangle = \langle u_2, u_2\rangle - \frac{\langle u_1, u_2\rangle^2}{\langle u_1, u_1\rangle} = \|u_2\|^2 - \frac{\langle u_1, u_2\rangle^2}{\|u_1\|^2}.$$

Hence, $\|u_1\|^2\|u_2\|^2 = |\langle u_1, u_2\rangle|^2$ if and only if $u_2 = \frac{\langle u_2, u_1\rangle}{\langle u_1, u_1\rangle} u_1$. Altogether, (i) is true.

(ii): If $u_1 = 0$, then $\|u_1\|\|u_2\| = 0 = |\langle u_1, u_2\rangle|$ and u_1, u_2 are linearly dependent. So, (ii) follows from (i).

(iii): By the Cauchy-Schwartz inequality, $\|u_1\|\|u_2\| \geq |\langle u_1, u_2\rangle|$. Hence, (ii) implies (iii). ☐

Fact 2.2. [12, Theorem 6.5-1] *Let* $a_1, \ldots, a_m$ *be points in* $\mathcal{H}$*. Then* $a_1, \ldots, a_m$ *are linearly independent if and only if the following Gram matrix is invertible:*

$$G(a_1, \ldots, a_m) := \begin{pmatrix} \|a_1\|^2 & \langle a_1, a_2\rangle & \cdots & \langle a_1, a_m\rangle \\ \langle a_2, a_1\rangle & \|a_2\|^2 & \cdots & \langle a_2, a_m\rangle \\ \vdots & \vdots & & \vdots \\ \langle a_m, a_1\rangle & \langle a_m, a_2\rangle & \cdots & \|a_m\|^2 \end{pmatrix}. \tag{1}$$

Lemma 2.3. *Let* $a_1, \ldots, a_m$ *be linearly independent points in* $\mathcal{H}$ *and let* $\xi_1, \ldots, \xi_m$ *be in* $\mathbb{R}$*. For every* $i \in \{1, \ldots, m\}$*, set* $M_i := \{x \in \mathcal{H} : \langle x, a_i\rangle = \xi_i\}$*. Then the set* $\cap_{i=1}^m M_i \cap \operatorname{span}\{a_1, \ldots, a_m\}$ *is a singleton. Consequently,* $\cap_{i=1}^m M_i \neq \varnothing$.

Proof. According to Fact 2.2, the Gram matrix $G(a_1,\dots,a_m)$ defined as (1) is invertible. Let $\beta_1,\dots,\beta_m \in \mathbb{R}^m$. Then,

$$\sum_{i=1}^{m}\beta_i a_i \in \cap_{i=1}^m M_i \cap \operatorname{span}\{a_1,\dots,a_m\} \Leftrightarrow \begin{cases} \langle \sum_{i=1}^m \beta_i a_i, a_1\rangle = \xi_1 \\ \quad\vdots \\ \langle \sum_{i=1}^m \beta_i a_i, a_m\rangle = \xi_m \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} \|a_1\|^2 & \langle a_1,a_2\rangle & \cdots & \langle a_1,a_m\rangle \\ \vdots & \vdots & & \vdots \\ \langle a_m,a_1\rangle & \langle a_m,a_2\rangle & \cdots & \|a_m\|^2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \end{pmatrix} = G(a_1,\dots,a_m)^{-1} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix},$$

which implies that

$$(a_1,\cdots,a_m)G(a_1,\dots,a_m)^{-1}(\xi_1,\cdots,\xi_m)^\intercal \in \cap_{i=1}^m M_i \cap \operatorname{span}\{a_1,\dots,a_m\} \neq \varnothing. \qquad \square$$

Best approximation mappings and projections

Definition 2.4. [4, Definition 3.1] Let $G : \mathcal{H} \to \mathcal{H}$, and let $\gamma \in [0,1[$. Then G is a *best approximation mapping with constant* γ (for short γ-BAM), if

(i) $\operatorname{Fix} G$ is a nonempty closed convex subset of $\mathcal{H}$,

(ii) $\mathrm{P}_{\operatorname{Fix} G} G = \mathrm{P}_{\operatorname{Fix} G}$, and

(iii) $(\forall x \in \mathcal{H})\ \|Gx - \mathrm{P}_{\operatorname{Fix} G} x\| \leq \gamma \|x - \mathrm{P}_{\operatorname{Fix} G} x\|$.

In particular, if γ is unknown or not necessary to point out, we just say that G is a BAM.

The following result plays an important role in the proofs of some of our main results later.

Fact 2.5. [4, Proposition 3.10] *Let $\gamma \in [0,1[$ and let $G : \mathcal{H} \to \mathcal{H}$. Suppose that G is a γ-BAM. Then $(\forall x \in \mathcal{H})$ $(\forall k \in \mathbb{N})$ $\|G^k x - \mathrm{P}_{\operatorname{Fix} G} x\| \leq \gamma^k \|x - \mathrm{P}_{\operatorname{Fix} G} x\|$.*

Fact 2.6. [1, Example 29.20] *Define $W := \{x \in \mathcal{H} : \langle x,u\rangle \leq \eta\}$ for $u \in \mathcal{H} \setminus \{0\}$ and $\eta \in \mathbb{R}$. Then $W \neq \varnothing$ and*

$$(\forall x \in \mathcal{H}) \quad \mathrm{P}_W x = \begin{cases} x, & \text{if } \langle x,u\rangle \leq \eta; \\ x + \dfrac{\eta - \langle x,u\rangle}{\|u\|^2} u, & \text{if } \langle x,u\rangle > \eta. \end{cases}$$

Fact 2.7. [11, Fact 1.8] *Let A and B be two nonempty closed convex subsets of $\mathcal{H}$. Let $A \subseteq B$ and $x \in \mathcal{H}$. Then $\mathrm{P}_B x \in A$ if and only if $\mathrm{P}_B x = \mathrm{P}_A x$.*

Fact 2.8. [1, Example 29.18] *Define $H := \{x \in \mathcal{H} : \langle x,u\rangle = \eta\}$ for $u \in \mathcal{H} \setminus \{0\}$ and $\eta \in \mathbb{R}$. Then*

$$(\forall x \in \mathcal{H}) \quad \mathrm{P}_H x = x + \frac{\eta - \langle x,u\rangle}{\|u\|^2} u.$$

Remark 2.9. Let $u \in \mathcal{H} \setminus \{0\}$ and $\eta \in \mathbb{R}$. Set $W := \{x \in \mathcal{H} : \langle x, u\rangle \le \eta\}$ and $H := \{x \in \mathcal{H} : \langle x, u\rangle = \eta\}$. If $\eta > 0$, then $\frac{2\eta}{\|u\|^2}u \in W^c$. Otherwise we have $u \in W^c$ and $W^c \neq \varnothing$. Moreover, combine Facts 2.6 and 2.8 to see that $(\forall x \in W^c)$ $\mathrm{P}_W x = x + \frac{\eta - \langle x,u\rangle}{\|u\|^2}u = \mathrm{P}_H x$.

Fact 2.10. [4, Lemma 2.3] *Let M and N be closed affine subspaces of $\mathcal{H}$ with $M \cap N \neq \varnothing$. Assume $M \subseteq N$ or $N \subseteq M$. Then $\mathrm{P}_M \mathrm{P}_N = \mathrm{P}_N \mathrm{P}_M = \mathrm{P}_{M\cap N}$.*

Recall that u_1 and u_2 are in $\mathcal{H}$ and η_1 and η_2 are in $\mathbb{R}$, and that

$$W_1 := \{x \in \mathcal{H} \;:\; \langle x, u_1\rangle \le \eta_1\}, \quad W_2 := \{x \in \mathcal{H} \;:\; \langle x, u_2\rangle \le \eta_2\}, \tag{2a}$$

$$H_1 := \{x \in \mathcal{H} \;:\; \langle x, u_1\rangle = \eta_1\}, \quad H_2 := \{x \in \mathcal{H} \;:\; \langle x, u_2\rangle = \eta_2\}. \tag{2b}$$

The following result will be used frequently later.

Lemma 2.11. *Suppose that $u_1 \neq 0$ and $u_2 \neq 0$. Let $x \in \mathcal{H}$. Then the following hold:*

(i) *We have the identities:*

$$\begin{aligned}\mathrm{P}_{H_2}\mathrm{P}_{H_1} x &= x + \frac{\eta_1 - \langle x,u_1\rangle}{\|u_1\|^2}u_1 + \frac{\eta_2 - \langle x,u_2\rangle}{\|u_2\|^2}u_2 - \frac{(\eta_1 - \langle x,u_1\rangle)\langle u_1,u_2\rangle}{\|u_1\|^2\|u_2\|^2}u_2\\ &= x + \frac{\eta_1 - \langle x,u_1\rangle}{\|u_1\|^2}u_1\\ &\quad + \frac{1}{\|u_1\|^2\|u_2\|^2}\big((\eta_2 - \langle x,u_2\rangle)\|u_1\|^2 - (\eta_1 - \langle x,u_1\rangle)\langle u_1,u_2\rangle\big)\, u_2.\end{aligned}$$

(ii) $\mathrm{P}_{H_1} x \notin W_2 \Leftrightarrow \|u_1\|^2(\langle x,u_2\rangle - \eta_2) > \langle u_1,u_2\rangle(\langle x,u_1\rangle - \eta_1)$. *Moreover,* $\mathrm{P}_{H_2} x \notin W_1 \Leftrightarrow \|u_2\|^2(\langle x,u_1\rangle - \eta_1) > \langle u_1,u_2\rangle(\langle x,u_2\rangle - \eta_2)$.

(iii) *Suppose that $\langle u_1,u_2\rangle = 0$. Then*

$$(\forall x \in \mathcal{H})\ \mathrm{P}_{H_2}\mathrm{P}_{H_1} x = x + \tfrac{\eta_1 - \langle x,u_1\rangle}{\|u_1\|^2}u_1 + \tfrac{\eta_2 - \langle x,u_2\rangle}{\|u_2\|^2}u_2.$$

Moreover, $\mathrm{P}_{H_2}\mathrm{P}_{H_1} = \mathrm{P}_{H_1\cap H_2}$.

(iv) *Suppose that u_1 and u_2 are linearly dependent. Then* $\mathrm{P}_{H_2}\mathrm{P}_{H_1} = \mathrm{P}_{H_2}$.

(v) *Suppose that $\langle u_1,u_2\rangle > 0$ and that $\mathrm{P}_{H_1} x \notin W_2$. Then* $\mathrm{P}_{W_2}\mathrm{P}_{H_1} x = \mathrm{P}_{H_2}\mathrm{P}_{H_1} x \in \operatorname{int} W_1 \cap H_2$.

Proof. (i): Apply Fact 2.8 two times to obtain that

$$\begin{aligned}\mathrm{P}_{H_2}\mathrm{P}_{H_1} x &= \mathrm{P}_{H_1} x + \frac{\eta_2 - \langle \mathrm{P}_{H_1} x, u_2\rangle}{\|u_2\|^2}u_2\\ &= x + \frac{\eta_1 - \langle x,u_1\rangle}{\|u_1\|^2}u_1 + \frac{1}{\|u_2\|^2}\left(\eta_2 - \left\langle x + \frac{\eta_1 - \langle x,u_1\rangle}{\|u_1\|^2}u_1, u_2\right\rangle\right)u_2\\ &= x + \frac{\eta_1 - \langle x,u_1\rangle}{\|u_1\|^2}u_1 + \frac{\eta_2 - \langle x,u_2\rangle}{\|u_2\|^2}u_2 - \frac{(\eta_1 - \langle x,u_1\rangle)\langle u_1,u_2\rangle}{\|u_1\|^2\|u_2\|^2}u_2\\ &= x + \frac{\eta_1 - \langle x,u_1\rangle}{\|u_1\|^2}u_1\\ &\quad + \frac{1}{\|u_1\|^2\|u_2\|^2}\big((\eta_2 - \langle x,u_2\rangle)\|u_1\|^2 - (\eta_1 - \langle x,u_1\rangle)\langle u_1,u_2\rangle\big)\, u_2.\end{aligned}$$

(ii): These equivalences are clear from Fact 2.8 and (2a).

(iii): Let $x \in \mathcal{H}$. Using (i) and $\langle u_1, u_2\rangle = 0$, we get

$$\mathrm{P}_{H_2} \mathrm{P}_{H_1} x = x + \tfrac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1 + \tfrac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2} u_2, \quad \langle \mathrm{P}_{H_2} \mathrm{P}_{H_1} x, u_1\rangle = \eta_1$$

and $\langle \mathrm{P}_{H_2} \mathrm{P}_{H_1} x, u_2\rangle = \eta_2$, which, by (2b), imply that $\mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in H_1 \cap H_2$. So, Fact 2.10 yields $\mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap H_2} \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap H_2} x$.

(iv): Since u_1 and u_2 are linearly dependent and $u_1 \neq 0$, Fact 2.1(i) leads to

$$\langle u_1, u_2\rangle u_2 = \pm \|u_1\|\|u_2\| \left(\pm \frac{\|u_2\|}{\|u_1\|} u_1 \right) = \|u_2\|^2 u_1,$$

which implies that

$$(\forall x \in \mathcal{H}) \quad \frac{(\eta_1 - \langle x, u_1\rangle)\langle u_1, u_2\rangle}{\|u_1\|^2\|u_2\|^2} u_2 = \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1. \tag{3}$$

Combining (3) with the first identity in (i) and using Fact 2.8, we obtain (iv).

(v): It is easy to see $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x$ from $\mathrm{P}_{H_1} x \notin W_2$ and Remark 2.9. Moreover, using $\langle u_1, u_2\rangle > 0$ and $\mathrm{P}_{H_1} x \notin W_2$, we obtain that

$$\begin{aligned} &\eta_1 - \langle \mathrm{P}_{H_2} \mathrm{P}_{H_1} x, u_1\rangle \\ &\overset{(i)}{=} -\frac{1}{\|u_1\|^2\|u_2\|^2} \left((\eta_2 - \langle x, u_2\rangle)\|u_1\|^2 - (\eta_1 - \langle x, u_1\rangle)\langle u_1, u_2\rangle \right) \langle u_2, u_1\rangle \overset{(ii)}{>} 0, \end{aligned}$$

which, by (2a), implies $\mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in \operatorname{int} W_1$.
In consequence we obtain $\mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in \operatorname{int} W_1 \cap H_2$. □

Lemma 2.12. *Suppose that $u_1 \neq 0$ and $u_2 \neq 0$ and that $\langle u_1, u_2\rangle \neq 0$. Let $x \in H_1 \cap W_2^c$. Then the following statements hold:*

(i) $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x \notin H_1$

(ii) $\mathrm{P}_{H_1} \mathrm{P}_{W_2} x = \mathrm{P}_{H_1} \mathrm{P}_{H_2} x \notin W_2$.

(iii) $(\forall k \in \mathbb{N} \setminus \{0\})$ $(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x = (\mathrm{P}_{H_2} \mathrm{P}_{H_1})^k x \notin H_1$ *and*
$\mathrm{P}_{H_1}(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x = \mathrm{P}_{H_1}(\mathrm{P}_{H_2} \mathrm{P}_{H_1})^k x \notin W_2$.

Proof. (i): As a consequence of $x \in W_2^c$ and Remark 2.9, we have $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x$. Combine the assumptions with Fact 2.8 and (2b) to see that

$$\eta_1 - \langle \mathrm{P}_{H_2} x, u_1\rangle = \eta_1 - \left\langle x + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2} u_2, u_1 \right\rangle = -\frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2}\langle u_2, u_1\rangle \neq 0,$$

which, by (2b), implies that $\mathrm{P}_{H_2} x \notin H_1$.

(ii): Clearly, $\mathrm{P}_{H_1} \mathrm{P}_{W_2} x = \mathrm{P}_{H_1} \mathrm{P}_{H_2} x$ is from (i). Apply Lemma 2.11(i) with swapping H_1 and H_2 to obtain that

$$\begin{aligned} \eta_2 - \langle \mathrm{P}_{H_1}\mathrm{P}_{H_2} x, u_2\rangle &= -\frac{1}{\|u_1\|^2\|u_2\|^2} \left((\eta_1 - \langle x, u_1\rangle)\|u_2\|^2 - (\eta_2 - \langle x, u_2\rangle)\langle u_1, u_2\rangle \right) \langle u_1, u_2\rangle \\ &= \frac{1}{\|u_1\|^2\|u_2\|^2} (\eta_2 - \langle x, u_2\rangle)\langle u_1, u_2\rangle^2 < 0, \quad (\text{by } x \in H_1 \cap W_2^c \text{ and } \langle u_1, u_2\rangle \neq 0), \end{aligned}$$

which, by (2a), implies that $\mathrm{P}_{H_1} \mathrm{P}_{H_2} x \notin W_2$.

(iii): We proceed with the proof by induction on k. For $k = 1$, bearing in mind that $x \in H_1 \cap W_2^c$, Remark 2.9 and (i), we have that $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} x \notin H_1$. Combine this with (ii) to see that $\mathrm{P}_{H_1} \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} \mathrm{P}_{H_2} x \notin W_2$. Hence, the base case is true. Suppose that for some $k \in \mathbb{N} \setminus \{0\}$,

$$(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x = (\mathrm{P}_{H_2} \mathrm{P}_{H_1})^k x \notin H_1 \quad \text{and} \quad \mathrm{P}_{H_1}(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x \notin W_2. \tag{4}$$

Denote by $y := \mathrm{P}_{H_1}(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x$. Then $y \in H_1 \cap W_2^c$ follows from the induction hypothesis and (4). Moreover,

$$\begin{aligned}(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^{k+1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} (\mathrm{P}_{W_2} \mathrm{P}_{H_1})^k x = \mathrm{P}_{W_2} y &\overset{\text{(i)}}{=} \mathrm{P}_{H_2} y \\ = (\mathrm{P}_{H_2} \mathrm{P}_{H_1})^{k+1} x, \quad \mathrm{P}_{W_2} y &\overset{\text{(i)}}{\notin} H_1, \end{aligned} \tag{5a}$$

$$\mathrm{P}_{H_1}(\mathrm{P}_{W_2} \mathrm{P}_{H_1})^{k+1} x \overset{\text{(5a)}}{=} \mathrm{P}_{H_1} \mathrm{P}_{H_2} y \overset{\text{(ii)}}{\notin} W_2. \tag{5b}$$

Hence, (5a) and (5b) yield that (4) holds for $k+1$. Therefore, (iii) holds by induction. □

3. Projection onto intersection of hyperplanes

In this section, we consider the projection onto intersection of hyperplanes and the composition of projections onto hyperplanes. Recall that $\mathrm{I} := \{1, 2, \ldots, m\}$ and that for every $i \in \mathrm{I}$, u_i is in $\mathcal{H}$, and η_i is in $\mathbb{R}$. Moreover,

$$(\forall i \in \mathrm{I}) \quad H_i := \{x \in \mathcal{H} \ : \ \langle x, u_i \rangle = \eta_i\}.$$

Remark 3.1. Suppose that $m \geq 2$ and that $u_1, \ldots, u_m$ are linearly dependent. Then without loss of generality, assume that $u_1, \ldots, u_t$ with $t \in \mathrm{I} \setminus \{m\}$ are linearly independent and that $(\forall i \in \{t+1, \ldots, m\})$ $u_1, \ldots, u_t, u_i$ are linearly dependent. Let $i \in \{t+1, \ldots, m\}$. Assume that $u_i = \sum_{j=1}^t \gamma_j u_j$ for some $(\gamma_1, \ldots, \gamma_t)^\intercal \in \mathbb{R}^t$.

$$\text{Then if } \ \eta_i = \textstyle\sum_{j=1}^t \gamma_j \eta_j, \ \text{ then } \ \cap_{j=1}^t H_j = (\cap_{j=1}^t H_j) \cap H_i.$$

Otherwise, $(\cap_{j=1}^t H_j) \cap H_i = \varnothing$. Set J as the maximally subset of I such that u_i, for all $i \in \mathrm{J}$ are linearly independent. Let $x \in \mathcal{H}$. Therefore, if only $\cap_{i \in \mathrm{I}} H_i \neq \varnothing$, to deduce $\mathrm{P}_{\cap_{i \in \mathrm{I}} H_i} x$, we only need to first find J, then $\mathrm{P}_{\cap_{i \in \mathrm{I}} H_i} x = \mathrm{P}_{\cap_{i \in \mathrm{J}} H_i} x$. Therefore, in the following Proposition 3.2, we care only the case in which $u_1, \ldots, u_m$ are linearly independent.

Proposition 3.2. *Suppose that $u_1, \ldots, u_m$ are linearly independent. Denote by*

$$(\beta_1, \ldots, \beta_m)^\intercal := G(u_1, \ldots, u_m)^{-1} \left(\langle u_1, x \rangle - \eta_1, \ldots, \langle u_m, x \rangle - \eta_m \right)^\intercal,$$

where $G(u_1, \ldots, u_m)$ is the Gram matrix defined in Fact 2.2. Then

$$(\forall x \in \mathcal{H}) \quad \mathrm{P}_{\cap_{i \in \mathrm{I}} H_i} x = x - \sum_{i \in \mathrm{I}} \beta_i u_i.$$

Proof. Define $L : \mathcal{H} \to \mathbb{R}^m$ by $(\forall x \in \mathcal{H})$ $Lx := (\langle u_1, x \rangle, \ldots, \langle u_m, x \rangle)^\intercal$.

Clearly we have $L \in \mathcal{B}(\mathcal{H}, \mathbb{R}^m)$. Then according to the definition of adjoint operator, $L^* : \mathbb{R}^m \to \mathcal{H}$ is defined by

$$(\forall \alpha := (\alpha_1, \ldots, \alpha_m)^\intercal \in \mathbb{R}^m) \quad L^* \alpha = \sum_{i \in \mathrm{I}} \alpha_i u_i.$$

It is easy to see that $LL^* : \mathbb{R}^m \to \mathbb{R}^m$ satisfies that

$$(\forall \alpha \in \mathbb{R}^m) \quad LL^*\alpha = \left(\left\langle u_1, \sum_{i\in \mathrm{I}} \alpha_i u_i \right\rangle, \ldots, \left\langle u_m, \sum_{i\in \mathrm{I}} \alpha_i u_i \right\rangle \right)^{\intercal} = G(u_1, \ldots, u_m)\alpha.$$

Combine this with Fact 2.2 and the linear independence of $u_1, \ldots, u_m$ to see that LL^* is invertible, which, connecting with [1, Example 29.17(iii)], implies that

$$(\forall x \in \mathcal{H}) \quad \mathrm{P}_{\cap_{i\in \mathrm{I}} H_i} x = x - L^*(LL^*)^{-1}(Lx - \eta) = x - \sum_{i\in \mathrm{I}} \beta_i u_i,$$

where $\eta := (\eta_1, \ldots, \eta_m)^{\intercal}$. $\square$

Remark 3.3. With Remark 3.1 and Proposition 3.2, we are able to solve the least norm problem presented in [6, Section 5.2] without the requirement that the related matrix is full rank.

Lemma 3.4. *Suppose that u_1 and u_2 are linearly independent. Define $\gamma := \frac{|\langle u_1, u_2\rangle|}{\|u_1\|\|u_2\|}$. Then $\gamma \in [0, 1[$ and $(\forall x \in \mathcal{H})(\forall k \in \mathbb{N})$*

$$\|(\mathrm{P}_{H_2} \mathrm{P}_{H_1})^k x - \mathrm{P}_{H_1 \cap H_2} x\| \leq \gamma^k \|x - \mathrm{P}_{H_1\cap H_2} x\|.$$

Proof. It is easy to see the desired results from [10, Example 9.40]. $\square$

4. Compositions of projections onto halfspaces

In this section, we study the projection onto intersection of halfspaces and the composition of projections onto halfspaces. Recall that $\mathrm{I} := \{1, 2, \ldots, m\}$, that for every $i \in \mathrm{I}$, u_i is in $\mathcal{H}$, and η_i is in $\mathbb{R}$, and that

$$(\forall i \in \mathrm{I}) \quad W_i := \{x \in \mathcal{H} \ : \ \langle x, u_i\rangle \leq \eta_i\}, \quad \text{and} \quad H_i := \{x \in \mathcal{H} \ : \ \langle x, u_i\rangle = \eta_i\}. \tag{6}$$

For every $k \in \mathbb{N}$, let $[k]$ denote "$k \mod m$". Let $x \in \mathcal{H}$. The sequence of iterations of Dykstra's algorithm is: $x_0 := x, e_{-(m-1)} = \cdots = e_{-1} = e_0 = 0$,

$$(\forall k \in \mathbb{N} \setminus \{0\}) \quad x_k := \mathrm{P}_{W_{[k]}}(x_{k-1} + e_{k-m}), \text{ and } e_k := x_{k-1} + e_{k-m} - x_k.$$

Fact 4.1. [10, Example 9.41] *Suppose that $(\forall i \in \mathrm{I})\ u_i \neq 0$ and $\cap_{i\in \mathrm{I}} W_i \neq \varnothing$. Let $x \in \mathcal{H}$. Then the sequence $(x_k)_{k\in\mathbb{N}}$ according to the Dykstra's algorithm with $x_0 = x$ converges to $\mathrm{P}_{\cap_{i\in \mathrm{I}} W_i} x$.*

Unfortunately, as we mentioned before, the practical manipulation of the Dykstra's algorithm is not easy. In the remaining of this section, we consider the case in which $m = 2$ and systematically investigate the relation between $\mathrm{P}_{W_1\cap W_2}$ and $\mathrm{P}_{W_2} \mathrm{P}_{W_1}$.

u_1 and u_2 are linearly dependent

In the whole subsection, we assume that u_1 and u_2 are linearly dependent.

Fact 4.2. [1, Proposition 29.22] *Exactly one of the following cases occurs:*

(i) $u_1 = u_2 = 0$ *and* $0 \leq \min\{\eta_1, \eta_2\}$. *Then* $W_1 \cap W_2 = \mathcal{H}$ *and* $\mathrm{P}_{W_1\cap W_2} = \mathrm{Id}$.

(ii) $u_1 = u_2 = 0$ *and* $\min\{\eta_1, \eta_2\} < 0$. *Then* $W_1 \cap W_2 = \varnothing$.

(iii) $u_1 \neq 0$, $u_2 = 0$, *and* $0 \leq \eta_2$. *Then* $W_1 \cap W_2 = W_1$ *and* $\mathrm{P}_{W_1\cap W_2} = \mathrm{P}_{W_1}$.

(iv) $u_1 \neq 0$, $u_2 = 0$, *and* $\eta_2 < 0$. *Then* $W_1 \cap W_2 = \varnothing$.

(v) $u_1 = 0$, $u_2 \neq 0$, *and* $0 \leq \eta_1$. *Then* $W_1 \cap W_2 = W_2$ *and* $\mathrm{P}_{W_1 \cap W_2} = \mathrm{P}_{W_2}$.

(vi) $u_1 = 0$, $u_2 \neq 0$, *and and* $\eta_1 < 0$. *Then* $W_1 \cap W_2 = \varnothing$.

(vii) $u_1 \neq 0$, $u_2 \neq 0$, *and* $\langle u_1, u_2 \rangle > 0$. *Then* $W_1 \cap W_2 = \{x \in \mathcal{H} \; : \; \langle x, u \rangle \leq \eta\}$ *where* $u = \|u_2\| u_1$ *and* $\eta = \min\{\eta_1 \|u_2\|, \eta_2 \|u_1\|\}$, *and*

$$(\forall x \in \mathcal{H}) \quad \mathrm{P}_{W_1 \cap W_2} x = \begin{cases} x, & \text{if } \langle x, u \rangle \leq \eta; \\ x + \frac{\eta - \langle x, u \rangle}{\|u\|^2} u, & \text{if } \langle x, u \rangle > \eta. \end{cases}$$

(viii) $u_1 \neq 0$, $u_2 \neq 0$, $\langle u_1, u_2 \rangle < 0$, *and* $\eta_1 \|u_2\| + \eta_2 \|u_1\| < 0$. *Then* $W_1 \cap W_2 = \varnothing$.

(ix) $u_1 \neq 0$, $u_2 \neq 0$, $\langle u_1, u_2 \rangle < 0$, *and* $\eta_1 \|u_2\| + \eta_2 \|u_1\| \geq 0$.
Then $W_1 \cap W_2 = \{x \in \mathcal{H} : \gamma_1 \leq \langle x, u \rangle \leq \gamma_2\} \neq \varnothing$ *where* $u = \|u_2\| u_1$, $\gamma_1 = -\eta_2 \|u_1\|$, *and* $\gamma_2 = \eta_1 \|u_2\|$, *and*

$$(\forall x \in \mathcal{H}) \quad \mathrm{P}_{W_1 \cap W_2} x = \begin{cases} x - \frac{\langle x, u \rangle - \gamma_1}{\|u\|^2} u, & \text{if } \langle x, u \rangle < \gamma_1; \\ x, & \text{if } \gamma_1 \leq \langle x, u \rangle \leq \gamma_2; \\ x - \frac{\langle x, u \rangle - \gamma_2}{\|u\|^2} u, & \text{if } \langle x, u \rangle > \gamma_2. \end{cases}$$

Lemma 4.3. *Suppose that* $u_1 \neq 0$ *and* $u_2 \neq 0$.

(i) *Suppose that* $\langle u_1, u_2 \rangle > 0$.
 (a) *If* $\eta_1 \|u_2\| \leq \eta_2 \|u_1\|$, *then* $W_1 \cap W_2 = W_1$ *and* $H_1 \subseteq W_2$.
 (b) *If* $\eta_1 \|u_2\| > \eta_2 \|u_1\|$, *then* $W_1 \cap W_2 = W_2$ *and* $H_1 \subseteq W_2^c$.

(ii) *Suppose that* $\langle u_1, u_2 \rangle < 0$ *and that* $\eta_1 \|u_2\| + \eta_2 \|u_1\| \geq 0$.
Then $H_1 \subseteq W_2$ *and* $H_2 \subseteq W_1$.

Proof. (i): Set $u := \|u_2\| u_1$. Then the assumptions and Fact 2.1(i) imply that

$$u = \|u_2\| u_1 = \|u_1\| u_2. \tag{7}$$

As a consequence of (6), we see that for every $y \in \mathcal{H}$,

$$y \in W_1 \Leftrightarrow \langle y, u_1 \rangle \leq \eta_1 \Leftrightarrow \|u_2\| \langle y, u_1 \rangle \leq \|u_2\| \eta_1 \overset{(7)}{\Leftrightarrow} \langle y, u \rangle \leq \eta_1 \|u_2\|, \tag{8a}$$

$$y \in W_2 \Leftrightarrow \langle y, u_2 \rangle \leq \eta_2 \Leftrightarrow \|u_1\| \langle y, u_2 \rangle \leq \|u_1\| \eta_2 \overset{(7)}{\Leftrightarrow} \langle y, u \rangle \leq \eta_2 \|u_1\|, \tag{8b}$$

(i)(a): Suppose that $\eta_1 \|u_2\| \leq \eta_2 \|u_1\|$. Then the required results follow from (6) and (8).

(i)(b): Suppose that $\eta_1 \|u_2\| > \eta_2 \|u_1\|$. Then it is easy to see $W_2 \subseteq W_1$ from (8). Let $x \in \mathcal{H}$. Now $H_1 \subseteq W_2^c$ follows from

$$x \in H_1 \Leftrightarrow \langle x, u_1 \rangle = \eta_1 \overset{(7)}{\Leftrightarrow} \langle x, u \rangle = \eta_1 \|u_2\| > \eta_2 \|u_1\| \overset{(8b)}{\Rightarrow} x \in W_2^c.$$

(ii): Suppose that $\langle u_1, u_2 \rangle < 0$ and that $\eta_1 \|u_2\| + \eta_2 \|u_1\| \geq 0$. By assumptions and Fact 2.1(i), we know that

$$\|u_2\| u_1 = -\|u_1\| u_2. \tag{9}$$

Clearly $\eta_1 \|u_2\| + \eta_2 \|u_1\| \geq 0$ implies

$$-\frac{\|u_2\|}{\|u_1\|} \eta_1 \leq \eta_2 \quad \text{and} \quad -\frac{\|u_1\|}{\|u_2\|} \eta_2 \leq \eta_1. \tag{10}$$

Let $x \in \mathcal{H}$. Now,

$$\begin{aligned} x \in H_1 \Leftrightarrow \langle x, u_1\rangle = \eta_1 \Leftrightarrow -\frac{\|u_2\|}{\|u_1\|}\langle x, u_1\rangle &= -\frac{\|u_2\|}{\|u_1\|}\eta_1 \\ \overset{(9)}{\Leftrightarrow} \langle x, u_2\rangle &= -\frac{\|u_2\|}{\|u_1\|}\eta_1 \overset{(10)}{\leq} \eta_2 \Leftrightarrow x \in W_2, \\ x \in H_2 \Leftrightarrow \langle x, u_2\rangle = \eta_2 \Leftrightarrow -\frac{\|u_1\|}{\|u_2\|}\langle x, u_2\rangle &= -\frac{\|u_1\|}{\|u_2\|}\eta_2 \\ \overset{(9)}{\Leftrightarrow} \langle x, u_1\rangle &= -\frac{\|u_1\|}{\|u_2\|}\eta_2 \overset{(10)}{\leq} \eta_1 \Leftrightarrow x \in W_1, \end{aligned}$$

which imply that $H_1 \subseteq W_2$ and $H_2 \subseteq W_1$. □

Lemma 4.4. *Suppose that $u_1 \neq 0$, $u_2 \neq 0$, and $\langle u_1, u_2\rangle > 0$. Let $x \in \mathcal{H}$. Then exactly one of the following cases occurs:*

(i) $x \in W_1 \cap W_2$. *Then* $\mathrm{P}_{W_1\cap W_2} x = x$.

(ii) $x \in W_1 \cap W_2^c$. *Then* $\mathrm{P}_{W_1\cap W_2} x = \mathrm{P}_{H_2} x$.

(iii) $x \in W_1^c \cap W_2$. *Then* $\mathrm{P}_{W_1\cap W_2} x = \mathrm{P}_{H_1} x$.

(iv) $x \in W_1^c \cap W_2^c$. *Then*

$$\mathrm{P}_{W_1\cap W_2} x = \begin{cases} \mathrm{P}_{H_1} x, & \text{if } \eta_1\|u_2\| \leq \eta_2\|u_1\|; \\ \mathrm{P}_{H_2} x, & \text{if } \eta_1\|u_2\| > \eta_2\|u_1\|. \end{cases}$$

Proof. (i): This is trivial by the definition of projection.

(ii): The assumption shows that $W_1 \cap W_2^c \neq \varnothing$, which yields $W_1 \not\subseteq W_2$. Hence, by Lemma 4.3(i)(a), we have that $\eta_1\|u_2\| > \eta_2\|u_1\|$ and $W_1 \cap W_2 = W_2$. Therefore, by Remark 2.9, $\mathrm{P}_{W_1\cap W_2} x = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x$.

(iii): Switch W_1 and W_2 in (ii) to obtain (iii).

(iv): Follows from the assumptions, Lemma 4.3(i) and Fact 2.8. □

Lemma 4.5. *Suppose that $u_1 \neq 0$, $u_2 \neq 0$, $\langle u_1, u_2\rangle < 0$, and $\eta_1\|u_2\| + \eta_2\|u_1\| \geq 0$. Let $x \in \mathcal{H}$. Then the following statements hold:*

(i) *If $x \in W_1 \cap W_2$, then* $\mathrm{P}_{W_1\cap W_2} x = x$.

(ii) *If $x \in W_1 \cap W_2^c$, then* $\mathrm{P}_{W_1\cap W_2} x = \mathrm{P}_{H_2} x$.

(iii) *If $x \in W_1^c \cap W_2$, then* $\mathrm{P}_{W_1\cap W_2} x = \mathrm{P}_{H_1} x$.

(iv) $W_1^c \cap W_2^c = \varnothing$.

Proof. Set $\quad u := \|u_2\|u_1, \quad \gamma_1 := -\eta_2\|u_1\|, \quad$ and $\quad \gamma_2 := \eta_1\|u_2\|. \qquad (11)$

From the assumptions and Fact 2.1(i), we know that

$$u = \|u_2\|u_1 = -\|u_1\|u_2. \tag{12}$$

As a consequence of Fact 4.2(ix), we see that

$$W_1 \cap W_2 = \{x \in \mathcal{H} \ : \ \gamma_1 \leq \langle x, u\rangle \leq \gamma_2\} \neq \varnothing$$

and
$$(\forall x \in \mathcal{H}) \quad \mathrm{P}_{W_1 \cap W_2} x = \begin{cases} x - \frac{\langle x,u\rangle - \gamma_1}{\|u\|^2} u, & \text{if } \langle x,u\rangle < \gamma_1; \\ x, & \text{if } \gamma_1 \leq \langle x,u\rangle \leq \gamma_2; \\ x - \frac{\langle x,u\rangle - \gamma_2}{\|u\|^2} u, & \text{if } \langle x,u\rangle > \gamma_2. \end{cases} \tag{13}$$

Hence, according to (6), for every $y \in \mathcal{H}$,

$$y \in W_1 \Leftrightarrow \langle y, u_1\rangle \leq \eta_1 \Leftrightarrow \|u_2\|\langle y, u_1\rangle \leq \|u_2\|\eta_1$$
$$\overset{(11)}{\Leftrightarrow} \langle y, u\rangle \leq \eta_1\|u_2\| \overset{(11)}{\Leftrightarrow} \langle y, u\rangle \leq \gamma_2, \tag{14a}$$
$$y \in W_2 \Leftrightarrow \langle y, u_2\rangle \leq \eta_2 \Leftrightarrow \|u_1\|\langle y, u_2\rangle \leq \|u_1\|\eta_2 \tag{14b}$$
$$\Leftrightarrow -\eta_2\|u_1\| \leq \langle y, -\|u_1\|u_2\rangle \overset{(12)}{\Leftrightarrow} \gamma_1 \leq \langle y, u\rangle. \tag{14c}$$

(i): This is trivial.

(ii): Assume that $x \in W_1 \cap W_2^c$. Then (14) leads to $\langle x, u\rangle \leq \gamma_2$ and $\langle x, u\rangle < \gamma_1$. Hence, using (13), (11), and (12), we obtain

$$\begin{aligned} \mathrm{P}_{W_1 \cap W_2} x &= x - \frac{\langle x, u\rangle - \gamma_1}{\|u\|^2} u = x - \frac{\langle x, -\|u_1\|u_2\rangle + \eta_2\|u_1\|}{\|-\|u_1\|u_2\|^2}(-\|u_1\|u_2) \\ &= x + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2} u_2 = \mathrm{P}_{H_2} x. \end{aligned}$$

(iii): The proof is similar to the proof of (ii).

(iv): Assume to the contrary that there exists $z \in W_1^c \cap W_2^c$. Then by (6),

$$\begin{aligned} &\langle z, u_1\rangle > \eta_1 \text{ and } \langle z, u_2\rangle > \eta_2 \overset{(12)}{\Leftrightarrow} \langle z, u_1\rangle > \eta_1 \text{ and } \left\langle z, -\frac{\|u_2\|}{\|u_1\|}u_1 \right\rangle > \eta_2 \\ &\Leftrightarrow \quad -\frac{\|u_2\|}{\|u_1\|}\eta_1 > -\frac{\|u_2\|}{\|u_1\|}\langle z, u_1\rangle > \eta_2 \quad \Rightarrow \quad \|u_1\|\eta_2 + \|u_2\|\eta_1 < 0, \end{aligned}$$

which contradicts to the assumption that $\|u_1\|\eta_2 + \|u_2\|\eta_1 \geq 0$. □

Theorem 4.6. *Suppose that $W_1 \cap W_2 \neq \varnothing$. Then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{P}_{W_1 \cap W_2}$.*

Proof. Using $W_1 \cap W_2 \neq \varnothing$, Fact 4.2, and the linear dependence of u_1 and u_2, we have exactly the following cases.

Case 1: $u_1 = u_2 = 0$ and $0 \leq \min\{\eta_1, \eta_2\}$. Then, due to Fact 4.2(i), $W_1 \cap W_2 = \mathcal{H}$ and $\mathrm{P}_{W_1 \cap W_2} = \mathrm{Id}$. Moreover, $W_1 = \mathcal{H}$ and $W_2 = \mathcal{H}$. Hence, $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{Id} = \mathrm{P}_{W_1 \cap W_2}$.

Case 2: $u_1 \neq 0$, $u_2 = 0$, and $0 \leq \eta_2$. Then Fact 4.2(iii) implies that $W_1 \cap W_2 = W_1$ and $\mathrm{P}_{W_1 \cap W_2} = \mathrm{P}_{W_1}$. Moreover, $W_2 = \mathcal{H}$, and $\mathrm{P}_{W_2} = \mathrm{Id}$. Hence, $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{Id}\,\mathrm{P}_{W_1} = \mathrm{P}_{W_1} = \mathrm{P}_{W_1 \cap W_2}$.

Case 3: $u_1 = 0$, $u_2 \neq 0$, and $0 \leq \eta_1$. Then, by Fact 4.2(v), $W_1 \cap W_2 = W_2$, and $\mathrm{P}_{W_1 \cap W_2} = \mathrm{P}_{W_2}$. Moreover, $W_1 = \mathcal{H}$, and $\mathrm{P}_{W_1} = \mathrm{Id}$. Hence, $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{P}_{W_2}\,\mathrm{Id} = \mathrm{P}_{W_2} = \mathrm{P}_{W_1 \cap W_2}$.

Case 4: $u_1 \neq 0$, $u_2 \neq 0$, and $\langle u_1, u_2\rangle > 0$. Let $x \in \mathcal{H}$. Notice that

$$\mathcal{H} = (W_1 \cap W_2) \cup (W_1 \cap W_2^c) \cup (W_1^c \cap W_2) \cup (W_1^c \cap W_2^c).$$

We have exactly the following four subcases.

Case 4.1: $x \in W_1 \cap W_2$. Then, clearly, $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = x = \mathrm{P}_{W_1 \cap W_2} x$.

Case 4.2: $x \in W_1 \cap W_2^c$. Then $\mathrm{P}_{W_1} x = x$ and, by Remark 2.9, $\mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x$. Hence, Lemma 4.4(ii) implies $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

Case 4.3: $x \in W_1^c \cap W_2$. Then $W_1^c \cap W_2 \neq \varnothing$, which implies that $W_1 \cap W_2 \neq W_2$. By Lemma 4.3(i)(b), we have that $\eta_1 \|u_2\| \leq \eta_2 \|u_1\|$, $W_1 \cap W_2 = W_1$ and $H_1 \subseteq W_2$. Combine these results with $x \in W_1^c$ and Remark 2.9 to obtain that $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$.

Case 4.4: Assume that $x \in W_1^c \cap W_2^c$. Set $u := \|u_2\| u_1$ and $\eta := \min\{\eta_1 \|u_2\|, \eta_2 \|u_1\|\}$. By Lemma 4.4(iv),

$$\mathrm{P}_{W_1 \cap W_2} x = \begin{cases} \mathrm{P}_{H_1} x, & \text{if } \eta_1 \|u_2\| \leq \eta_2 \|u_1\|; \\ \mathrm{P}_{H_2} x, & \text{if } \eta_1 \|u_2\| > \eta_2 \|u_1\|. \end{cases} \tag{15}$$

Case 4.4.1: Suppose that $\eta_1 \|u_2\| \leq \eta_2 \|u_1\|$. Then $x \in W_1^c \cap W_2^c$ and Lemma 4.3(i)(a) yield

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x \overset{(15)}{=} \mathrm{P}_{W_1 \cap W_2} x.$$

Case 4.4.2: Suppose that $\eta_1 \|u_2\| > \eta_2 \|u_1\|$. Recall that $x \in W_1^c \cap W_2^c$, that u_1 and u_2 are linearly dependent with $u_1 \neq 0$ and $u_2 \neq 0$, and that we have Lemma 4.3(i)(b), Remark 2.9, and Lemma 2.11(iv). We obtain that

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} x \overset{(15)}{=} \mathrm{P}_{W_1 \cap W_2} x. \tag{16}$$

Case 5: $u_1 \neq 0$, $u_2 \neq 0$, $\langle u_1, u_2 \rangle < 0$, and $\eta_1 \|u_2\| + \eta_2 \|u_1\| \geq 0$.

Note that $\mathcal{H} = (W_1 \cap W_2) \cup (W_1 \cap W_2^c) \cup (W_1^c \cap W_2) \cup (W_1^c \cap W_2^c)$. Then combine Lemma 4.5(iv) with $W_1^c \cap W_2^c = \varnothing$ to know that we have exactly the following three subcases.

Case 5.1: $x \in W_1 \cap W_2$. This is trivial.

Case 5.2: $x \in W_1 \cap W_2^c$. Then by Remark 2.9 and Lemma 4.5(ii), $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

Case 5.3: $x \in W_1^c \cap W_2$. Then the identities $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1 \cap W_2} x$ follow easily from Lemma 4.3(ii) and Lemma 4.5(iii).

Altogether, $(\forall x \in \mathcal{H})$ $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$, which means that the proof is complete. □

u_1 and u_2 are linearly independent

In the whole subsection, we assume that u_1 and u_2 are linearly independent. Define

$$C_1 := \left\{ x \in \mathcal{H} : \langle x, u_1 \rangle > \eta_1 \text{ and } \|u_1\|^2 (\langle x, u_2 \rangle - \eta_2) \leq \langle u_1, u_2 \rangle (\langle x, u_1 \rangle - \eta_1) \right\}, \tag{17a}$$

$$C_2 := \left\{ x \in \mathcal{H} : \langle x, u_2 \rangle > \eta_2 \text{ and } \|u_2\|^2 (\langle x, u_1 \rangle - \eta_1) \leq \langle u_1, u_2 \rangle (\langle x, u_2 \rangle - \eta_2) \right\}, \tag{17b}$$

$$C_3 := \left\{ x \in \mathcal{H} : \begin{array}{l} \|u_1\|^2 (\langle x, u_2 \rangle - \eta_2) > \langle u_1, u_2 \rangle (\langle x, u_1 \rangle - \eta_1), \\ \|u_2\|^2 (\langle x, u_1 \rangle - \eta_1) > \langle u_1, u_2 \rangle (\langle x, u_2 \rangle - \eta_2) \end{array} \right\}. \tag{17c}$$

Fact 4.7. [1, Proposition 29.23] *Let $x \in \mathcal{H}$. Then $W_1 \cap W_2 \neq \varnothing$ and*

$$\mathrm{P}_{W_1 \cap W_2} x = x - \gamma_1 u_1 - \gamma_2 u_2,$$

where exactly one of the following holds:

(i) *If $x \in W_1 \cap W_2$, then $\gamma_1 = \gamma_2 = 0$.*

(ii) *If $x \in C_1$, then $\gamma_1 = \frac{\langle x,u_1\rangle - \eta_1}{\|u_1\|^2} > 0$ and $\gamma_2 = 0$.*

(iii) *If $x \in C_2$, then $\gamma_1 = 0$ and $\gamma_2 = \frac{\langle x,u_2\rangle - \eta_2}{\|u_2\|^2} > 0$.*

(iv) *If $x \in C_3$, then*
$\gamma_1 = \frac{\|u_2\|^2(\langle x,u_1\rangle - \eta_1) - \langle u_1,u_2\rangle(\langle x,u_2\rangle - \eta_2)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1,u_2\rangle|^2} > 0$, $\gamma_2 = \frac{\|u_1\|^2(\langle x,u_2\rangle - \eta_2) - \langle u_1,u_2\rangle(\langle x,u_1\rangle - \eta_1)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1,u_2\rangle|^2} > 0.$

Lemma 4.8. *Let $x \in C_3$. Then $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_1 \cap H_2} x$.*

Proof. As a consequence of Fact 4.7(iv) we have

$$\mathrm{P}_{W_1 \cap W_2} x = x - \gamma_1 u_1 - \gamma_2 u_2, \quad \text{where} \tag{18}$$

$\gamma_1 = \frac{\|u_2\|^2(\langle x,u_1\rangle - \eta_1) - \langle u_1,u_2\rangle(\langle x,u_2\rangle - \eta_2)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1,u_2\rangle|^2} > 0$ and $\gamma_2 = \frac{\|u_1\|^2(\langle x,u_2\rangle - \eta_2) - \langle u_1,u_2\rangle(\langle x,u_1\rangle - \eta_1)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1,u_2\rangle|^2} > 0.$

According to equations (29.36) and (29.37) in the proof of [1, Proposition 29.23] which is the Fact 4.7, we obtain

$$0 = \gamma_1 \left(\langle x - \gamma_1 u_1 - \gamma_2 u_2, u_1\rangle - \eta_1\right) \overset{(18)}{=} \gamma_1 \left(\langle \mathrm{P}_{W_1 \cap W_2} x, u_1\rangle - \eta_1\right), \tag{19a}$$

$$0 = \gamma_2 \left(\langle x - \gamma_1 u_1 - \gamma_2 u_2, u_2\rangle - \eta_2\right) \overset{(18)}{=} \gamma_2 \left(\langle \mathrm{P}_{W_1 \cap W_2} x, u_2\rangle - \eta_2\right). \tag{19b}$$

Bearing in mind that $\gamma_1 > 0$ and $\gamma_2 > 0$, and that we have (6), we know that (19a) and (19b) imply $\mathrm{P}_{W_1 \cap W_2} x \in H_1 \cap H_2$. Hence, apply Fact 2.7 with $A = H_1 \cap H_2$ and $B = W_1 \cap W_2$ to obtain $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_1 \cap H_2} x$. □

Remark 4.9. (i) In view of (6) and Lemma 2.11(ii), we know that $C_1 = \{x \in \mathcal{H} : x \notin W_1 \text{ and } \mathrm{P}_{H_1} x \in W_2\}$, $C_2 = \{x \in \mathcal{H} : x \notin W_2 \text{ and } \mathrm{P}_{H_2} x \in W_1\}$, and $C_3 = \{x \in \mathcal{H} : \mathrm{P}_{H_1} x \notin W_2 \text{ and } \mathrm{P}_{H_2} x \notin W_1\}$.

(ii) By Fact 4.7, Fact 2.8, and Lemma 4.8, ($x \in C_1$) $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_1} x$, ($x \in C_2$) $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_2} x$, and ($x \in C_3$) $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_1 \cap H_2} x$.

Lemma 4.10. (i) *Suppose that $\langle u_1, u_2\rangle < 0$. Then the following statements hold:*
(a) *$C_1 \subseteq W_1^c \cap \operatorname{int}(W_2)$.*
(b) *$C_2 \subseteq W_2^c \cap \operatorname{int}(W_1)$.*
(c) *$(W_1^c \cap H_2) \cup (W_2^c \cap H_1) \cup (W_1^c \cap W_2^c) \subseteq C_3$.*

(ii) *Suppose that $\langle u_1, u_2\rangle \geq 0$. Then the following statements hold:*
(a) *$W_1^c \cap W_2 \subseteq C_1$.*
(b) *$W_1 \cap W_2^c \subseteq C_2$.*
(c) *$C_3 \subseteq W_1^c \cap W_2^c$. In particular, if $\langle u_1, u_2\rangle = 0$, then $C_3 = W_1^c \cap W_2^c$.*

Proof. Note that

$$\begin{aligned}\mathcal{H} &= (W_1 \cap W_2) \cup (W_1 \cap W_2^c) \cup (W_1^c \cap W_2) \cup (W_1^c \cap W_2^c) &\text{(20a)}\\ &= (W_1 \cap W_2) \cup (\operatorname{int} W_1 \cap W_2^c) \cup (H_1 \cap W_2^c) \cup (W_1^c \cap \operatorname{int} W_2)\\ &\qquad \cup (W_1^c \cap H_2) \cup (W_1^c \cap W_2^c). &\text{(20b)}\end{aligned}$$

On the other hand, Fact 4.7 implies that

$$\mathcal{H} = (W_1 \cap W_2) \cup C_1 \cup C_2 \cup C_3, \tag{21}$$

and the sets $W_1 \cap W_2$, C_1, C_2 and C_3 are pairwise disjoint.

(i)(a): Let $x \in C_1$. Clearly, $\langle x, u_1\rangle > \eta_1$ is equivalent to $x \in W_1^c$. Moreover, $\langle u_1, u_2\rangle < 0$ yields $\langle u_1, u_2\rangle(\langle x, u_1\rangle - \eta_1) < 0$. In consequence we obtain that $\|u_1\|^2(\langle x, u_2\rangle - \eta_2) \leq \langle u_1, u_2\rangle(\langle x, u_1\rangle - \eta_1)$ implies $\langle x, u_2\rangle - \eta_2 < 0$, i.e., $x \in \operatorname{int}(W_2)$. Hence, $C_1 \subseteq W_1^c \cap \operatorname{int}(W_2)$.

(i)(b): The proof is similar to the proof of (i)(a).

(i)(c): This follows from (i)(a), (i)(b), (20b) and (21).

(ii)(a): Let $x \in W_1^c \cap W_2$. Then $\langle x, u_1\rangle > \eta_1$ and $\langle x, u_2\rangle \leq \eta_2$ are from (6). Moreover, $\langle u_1, u_2\rangle \geq 0$ leads to $\|u_1\|^2(\langle x, u_2\rangle - \eta_2) \leq 0 \leq \langle u_1, u_2\rangle(\langle x, u_1\rangle - \eta_1)$, which, due to (17a), implies $x \in C_1$. Hence, $W_1^c \cap W_2 \subseteq C_1$.

(ii)(b): By an analogous proof to (ii)(a) we know that (ii)(b) holds.

(ii)(c): Combine (ii)(a), (ii)(b), (20a) and (21) to obtain that $C_3 \subseteq W_1^c \cap W_2^c$. Suppose that $\langle u_1, u_2\rangle = 0$. Let $x \in W_1^c \cap W_2^c$. Then by (6), $\|u_1\|^2(\langle x, u_2\rangle - \eta_2) > 0 = \langle u_1, u_2\rangle(\langle x, u_1\rangle - \eta_1)$ and $\|u_2\|^2(\langle x, u_1\rangle - \eta_1) > 0 = \langle u_1, u_2\rangle(\langle x, u_2\rangle - \eta_2)$, which, by (17c), implies that $x \in C_3$. Therefore, the last assertion in (ii)(c) holds. □

Lemma 4.11. *Suppose that $\langle u_1, u_2\rangle < 0$. Let $x \in C_3$. The following hold:*

(i) *Assume that $x \in W_1$. Then $\|\mathrm{P}_{H_2} x - \mathrm{P}_{H_1 \cap H_2} x\| \leq \|\mathrm{P}_{H_2} \mathrm{P}_{H_1} x - \mathrm{P}_{H_1 \cap H_2} x\|$.*

(ii) *Assume that $x \in W_1^c$. Then*

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in C_3 \quad \textit{and}$$
$$\mathrm{P}_{W_1 \cap W_2} \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap H_2} \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap H_2} x.$$

Proof. (i): Using Fact 2.8, we know that

$$\begin{aligned} \mathrm{P}_{H_1} x &= x + \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1, \quad \mathrm{P}_{H_2} x = x + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2} u_2, \text{ and} \\ \mathrm{P}_{H_2} \mathrm{P}_{H_1} x &= \mathrm{P}_{H_1} x + \frac{\eta_2 - \langle \mathrm{P}_{H_1} x, u_2\rangle}{\|u_2\|^2} u_2, \end{aligned} \tag{22}$$

which implies that

$$\|x - \mathrm{P}_{H_1} x\|^2 = \frac{(\eta_1 - \langle x, u_1\rangle)^2}{\|u_1\|^2}, \quad \text{and} \quad \|x - \mathrm{P}_{H_2} x\|^2 = \frac{(\eta_2 - \langle x, u_2\rangle)^2}{\|u_2\|^2}. \tag{23}$$

Apply the third and the first identities in (22) to the following first and second equation, respectively, to obtain

$$\begin{aligned} \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2} \mathrm{P}_{H_1} x\|^2 &= \frac{(\eta_2 - \langle \mathrm{P}_{H_1} x, u_2\rangle)^2}{\|u_2\|^2} \\ &= \frac{1}{\|u_1\|^4 \|u_2\|^2} \left(\|u_1\|^2(\eta_2 - \langle x, u_2\rangle) - \langle u_1, u_2\rangle(\eta_1 - \langle x, u_1\rangle)\right)^2. \end{aligned} \tag{24}$$

On the one hand,

$$\begin{aligned}
&\|\mathrm{P}_{H_2} x - \mathrm{P}_{H_1\cap H_2} x\| \leq \|\mathrm{P}_{H_2}\mathrm{P}_{H_1} x - \mathrm{P}_{H_1\cap H_2} x\| &(25a)\\
\Leftrightarrow\ & \|x-\mathrm{P}_{H_1\cap H_2} x\|^2 - \|x-\mathrm{P}_{H_2} x\|^2 \leq \|\mathrm{P}_{H_1} x-\mathrm{P}_{H_1\cap H_2} x\|^2 - \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2}\mathrm{P}_{H_1} x\|^2\\
\Leftrightarrow\ & \|x-\mathrm{P}_{H_1\cap H_2} x\|^2 - \|x-\mathrm{P}_{H_2} x\|^2 \leq \|x-\mathrm{P}_{H_1\cap H_2} x\|^2 - \|x-\mathrm{P}_{H_1} x\|^2\\
&\qquad - \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2}\mathrm{P}_{H_1} x\|^2\\
\Leftrightarrow\ & \|x-\mathrm{P}_{H_1} x\|^2 + \|\mathrm{P}_{H_1} x - \mathrm{P}_{H_2}\mathrm{P}_{H_1} x\|^2 \leq \|x-\mathrm{P}_{H_2} x\|^2\\
\Leftrightarrow\ & \frac{(\eta_1 - \langle x,u_1\rangle)^2}{\|u_1\|^2} + \frac{1}{\|u_1\|^4\|u_2\|^2}\big(\|u_1\|^2(\eta_2-\langle x,u_2\rangle) - \langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\big)^2\\
&\qquad \leq \frac{(\eta_2-\langle x,u_2\rangle)^2}{\|u_2\|^2}\\
\Leftrightarrow\ & \|u_1\|^2\|u_2\|^2(\eta_1-\langle x,u_1\rangle)^2 + \big(\|u_1\|^2(\eta_2-\langle x,u_2\rangle) - \langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\big)^2\\
&\qquad \leq \|u_1\|^4(\eta_2-\langle x,u_2\rangle)^2\\
\Leftrightarrow\ & \|u_1\|^2\|u_2\|^2(\eta_1-\langle x,u_1\rangle)^2 + \langle u_1,u_2\rangle^2(\eta_1-\langle x,u_1\rangle)^2\\
&\qquad \leq 2\|u_1\|^2(\eta_2-\langle x,u_2\rangle)\langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\\
\Leftrightarrow\ & \|u_1\|^2(\eta_1-\langle x,u_1\rangle)\big(\|u_2\|^2(\eta_1-\langle x,u_1\rangle) - \langle u_1,u_2\rangle(\eta_2-\langle x,u_2\rangle)\big)\\
&+ \langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\big(\langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle) - \|u_1\|^2(\eta_2-\langle x,u_2\rangle)\big) \leq 0, &(25b)
\end{aligned}$$

where the first two equivalences are from [3, Proposition 2.10], the fourth equivalence is from (23) and (24), and $\big(\|u_1\|^2(\eta_2-\langle x,u_2\rangle) - \langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\big)^2 = \|u_1\|^4(\eta_2-\langle x,u_2\rangle)^2 - 2\|u_1\|^2(\eta_2-\langle x,u_2\rangle)\langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle) + \langle u_1,u_2\rangle^2(\eta_1-\langle x,u_1\rangle)^2$ yields the sixth equivalence. Hence, it is equivalent to show (25b).

Recall that $x \in C_3$. Then, (17c) yields $\|u_1\|^2(\langle x,u_2\rangle-\eta_2) > \langle u_1,u_2\rangle(\langle x,u_1\rangle-\eta_1)$ and $\|u_2\|^2(\langle x,u_1\rangle-\eta_1) > \langle u_1,u_2\rangle(\langle x,u_2\rangle-\eta_2)$. By (6), $x \in W_1$ means $\eta_1 - \langle x,u_1\rangle \geq 0$. Hence, by assumptions, $x \in C_3$, $x \in W_1$ and $\langle u_1,u_2\rangle < 0$,

$$\|u_1\|^2(\eta_1-\langle x,u_1\rangle)\big(\|u_2\|^2(\eta_1-\langle x,u_1\rangle) - \langle u_1,u_2\rangle(\eta_2-\langle x,u_2\rangle)\big) \leq 0,$$

$$\langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\big(\langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle) - \|u_1\|^2(\eta_2-\langle x,u_2\rangle)\big) \leq 0,$$

which imply that (25b) holds. Accordingly, (i) holds.

(ii): Recall again that $x \in C_3$ means

$$\|u_1\|^2(\langle x,u_2\rangle-\eta_2) > \langle u_1,u_2\rangle(\langle x,u_1\rangle-\eta_1) \tag{26}$$

and $\|u_2\|^2(\langle x,u_1\rangle-\eta_1) > \langle u_1,u_2\rangle(\langle x,u_2\rangle-\eta_2)$. Combine (26) with the assumption $\langle u_1,u_2\rangle < 0$ to obtain

$$\langle u_1,u_2\rangle\big(\|u_1\|^2(\eta_2-\langle x,u_2\rangle) - \langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\big) > 0. \tag{27}$$

Clearly, $x \in C_3$ and Remark 4.9(i) imply that $\mathrm{P}_{H_1} x \notin W_2$. Combine this with the assumption, $x \in W_1^c$ and Remark 2.9, to obtain that $\mathrm{P}_{W_2}\mathrm{P}_{W_1} x = \mathrm{P}_{W_2}\mathrm{P}_{H_1} x = \mathrm{P}_{H_2}\mathrm{P}_{H_1} x$. Moreover, by Lemma 2.11(i), we get

$$\begin{aligned}
\mathrm{P}_{H_2}\mathrm{P}_{H_1} x = x &+ \frac{\eta_1-\langle x,u_1\rangle}{\|u_1\|^2}u_1\\
&+ \frac{1}{\|u_2\|^2\|u_1\|^2}\big(\|u_1\|^2(\eta_2-\langle x,u_2\rangle) - \langle u_1,u_2\rangle(\eta_1-\langle x,u_1\rangle)\big)u_2,
\end{aligned}$$

so $\eta_1 - \langle \mathrm{P}_{H_2} \mathrm{P}_{H_1} x, u_1\rangle$

$$
\begin{aligned}
&= \eta_1 - \langle x, u_1\rangle - (\eta_1 - \langle x, u_1\rangle) \\
&\qquad - \frac{1}{\|u_2\|^2\|u_1\|^2}\left(\|u_1\|^2 (\eta_2 - \langle x, u_2\rangle) - \langle u_1, u_2\rangle (\eta_1 - \langle x, u_1\rangle)\right)\langle u_2, u_1\rangle \\
&= -\frac{1}{\|u_1\|^2\|u_2\|^2}\langle u_2, u_1\rangle \left(\|u_1\|^2 (\eta_2 - \langle x, u_2\rangle) - \langle u_1, u_2\rangle (\eta_1 - \langle x, u_1\rangle)\right) \overset{(27)}{<} 0,
\end{aligned}
$$

which shows that $\mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in W_1^c$. Hence, by Lemma 4.10(i)(c) we obtain that $\mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in W_1^c \cap H_2 \subseteq C_3$. Therefore, by Lemma 4.8 and Fact 2.10, we finally get $\mathrm{P}_{W_1\cap W_2} \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1\cap H_2} \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1\cap H_2} x$. $\square$

Theorem 4.12. *Assume that $\langle u_1, u_2\rangle < 0$. Define $\gamma := \frac{|\langle u_1,u_2\rangle|}{\|u_1\|\|u_2\|}$. Then $\gamma \in [0, 1[$ and $\mathrm{P}_{W_2} \mathrm{P}_{W_1}$ is a γ-BAM. Consequently,*

$$(\forall x \in \mathcal{H})\ (\forall k \in \mathbb{N})\ \|(\mathrm{P}_{W_2} \mathrm{P}_{W_1})^k x - \mathrm{P}_{W_1\cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{W_1\cap W_2} x\|.$$

Proof. Recall that u_1 and u_2 are linearly independent. Then $\gamma \in [0, 1[$ follows from Lemma 3.4. As a consequence of [8, Corollary 4.5.2], $\operatorname{Fix} \mathrm{P}_{W_2} \mathrm{P}_{W_1} = W_1 \cap W_2$ is a nonempty closed convex subset of $\mathcal{H}$. Hence, by Theorem 2.4 and Fact 2.5, it remains to show that for every $x \in \mathcal{H}$,

(i) $\mathrm{P}_{W_1\cap W_2} \mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_1\cap W_2} x$, and

(ii) $\|\mathrm{P}_{W_2} \mathrm{P}_{W_1} x - \mathrm{P}_{W_1\cap W_2} x\| \leq \gamma \|x - \mathrm{P}_{W_1\cap W_2} x\|$.

Let $x \in \mathcal{H}$. Note that if $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_1\cap W_2} x$, then clearly (i) and (ii) hold.

Taking Fact 4.7 into account, we have exactly the following four cases.

Case 1: $x \in W_1 \cap W_2$. This is trivial.

Case 2: $x \in C_1$. Then Fact 4.7(ii) and Fact 2.6 imply that

$$\mathrm{P}_{W_1\cap W_2} x = x + \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1 = \mathrm{P}_{H_1} x \in H_1 \cap W_2. \tag{28}$$

According to Lemma 4.10(i)(a), $x \in C_1 \subseteq W_1^c \cap \operatorname{int}(W_2)$, which, by Remark 2.9 and (28), implies

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1\cap W_2} x.$$

Case 3: $x \in C_2$. Then enforcing Fact 4.7(iii) and Fact 2.6, we know that

$$\mathrm{P}_{W_1\cap W_2} x = x + \frac{\eta_2 - \langle x, u_2\rangle}{\|u_2\|^2} u_2 = \mathrm{P}_{H_2} x. \tag{29}$$

From Lemma 4.10(i)(b) we get $x \in C_2 \subseteq W_2^c \cap \operatorname{int}(W_1)$, and by Remark 2.9 and (29),

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x = \mathrm{P}_{W_1\cap W_2} x. \tag{30}$$

Case 4: $x \in C_3$. Then Lemma 4.8 leads to

$$\mathrm{P}_{W_1\cap W_2} x = \mathrm{P}_{H_1\cap H_2} x. \tag{31}$$

Note that $C_3 \subseteq \mathcal{H} = (W_1 \cap W_2) \cup (W_1 \cap W_2^c) \cup W_1^c$, Fact 4.7, and $C_3 \cap (W_1 \cap W_2) = \varnothing$. We have exactly the following two subcases.

Case 4.1: $x \in W_1 \cap W_2^c$. Combine $x \in C_3$ with Remark 2.9, Remark 4.9(i), and Lemma 4.10(i)(c) to see that

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x \in H_2 \cap W_1^c \subseteq C_3. \tag{32}$$

Using Lemma 4.8 and Fact 2.10, we obtain

$$\begin{aligned} \mathrm{P}_{W_1 \cap W_2} \mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_1 \cap H_2} \mathrm{P}_{W_2} \mathrm{P}_{W_1} x &\overset{(32)}{=} \mathrm{P}_{H_1 \cap H_2} \mathrm{P}_{H_2} x = \mathrm{P}_{H_1 \cap H_2} x \\ &\overset{(31)}{=} \mathrm{P}_{W_1 \cap W_2} x. \end{aligned} \tag{33}$$

Moreover, $\|\mathrm{P}_{W_2} \mathrm{P}_{W_1} x - \mathrm{P}_{W_1 \cap W_2} x\| \overset{(32)}{=} \|\mathrm{P}_{H_2} x - \mathrm{P}_{W_1 \cap W_2} x\| \overset{(31)}{=} \|\mathrm{P}_{H_2} x - \mathrm{P}_{H_1 \cap H_2} x\| \leq \|\mathrm{P}_{H_2} \mathrm{P}_{H_1} x - \mathrm{P}_{H_1 \cap H_2} x\| \leq \gamma \|x - \mathrm{P}_{H_1 \cap H_2} x\| \overset{(31)}{=} \gamma \|x - \mathrm{P}_{W_1 \cap W_2} x\|$, where the last two inequalities come from Lemma 4.11(i) and Lemma 3.4. Hence, (i) and (ii) hold.

Case 4.2: $x \in W_1^c$. Because $x \in C_3$, by Remark 4.9(i), $\mathrm{P}_{H_1} x \notin W_2$. Hence, by Remark 2.9 and Lemma 4.11(ii),

$$\begin{aligned} &\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x \quad \text{and} \\ &\mathrm{P}_{W_1 \cap W_2} \mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_1 \cap H_2} x \overset{(31)}{=} \mathrm{P}_{W_1 \cap W_2} x. \end{aligned} \tag{34}$$

Moreover, we obtain directly $\|\mathrm{P}_{W_2} \mathrm{P}_{W_1} x - \mathrm{P}_{W_1 \cap W_2} x\| \overset{(34)}{=} \|\mathrm{P}_{H_2} \mathrm{P}_{H_1} x - \mathrm{P}_{W_1 \cap W_2} x\| \overset{(31)}{=} \|\mathrm{P}_{H_2} \mathrm{P}_{H_1} x - \mathrm{P}_{H_1 \cap H_2} x\| \leq \gamma \|x - \mathrm{P}_{H_1 \cap H_2} x\| \overset{(31)}{=} \gamma \|x - \mathrm{P}_{W_1 \cap W_2} x\|$, where the inequality comes from Lemma 3.4. Therefore, (i) and (ii) are true. $\square$

Proposition 4.13. *Suppose that* $\langle u_1, u_2 \rangle \geq 0$. *Let* $x \in \mathcal{H}$. *Then the following statements hold:*

(i) *Suppose that* $x \in W_1 \cap W_2$. *Then* $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = x = \mathrm{P}_{W_1 \cap W_2} x$.

(ii) *Suppose that* $x \in W_1^c \cap W_2$. *Then* $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1 \cap W_2} x$.

(iii) *Suppose that* $x \in W_1 \cap W_2^c$. *Then* $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_2} x = \mathrm{P}_{W_1 \cap W_2} x$.

Proof. (i): This is trivial.

(ii): According to Lemma 4.10(ii)(a), $x \in W_1^c \cap W_2 \subseteq C_1$. Then Fact 4.7(ii) and Fact 2.8 imply

$$\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_1} x. \tag{35}$$

In view of (6), $x \in W_1^c \cap W_2$ means $\langle x, u_1 \rangle > \eta_1$ and $\langle x, u_2 \rangle \leq \eta_2$, which combining with $\langle u_1, u_2 \rangle \geq 0$, imply that $\eta_2 - \langle x, u_2 \rangle \geq 0$ and $-\langle u_1, u_2 \rangle (\eta_1 - \langle x, u_1 \rangle) \geq 0$. Hence,

$$\|u_1\|^2 (\eta_2 - \langle x, u_2 \rangle) - \langle u_1, u_2 \rangle (\eta_1 - \langle x, u_1 \rangle) \geq 0, \tag{36}$$

which, connecting with Lemma 2.11(ii), yields that $\mathrm{P}_{H_1} x \in W_2$. Hence,

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x \overset{(35)}{=} \mathrm{P}_{W_1 \cap W_2} x.$$

(iii): Due to Lemma 4.10(ii)(b) we have $x \in W_1 \cap W_2^c \subseteq C_2$. Then by Fact 4.7(iii) and Fact 2.8 we obtain

$$\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_2} x. \tag{37}$$

Hence, $x \in W_1 \cap W_2^c$ and Remark 2.9 imply that

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} x = \mathrm{P}_{H_2} x \overset{(37)}{=} \mathrm{P}_{W_1 \cap W_2} x. \qquad \square$$

Proposition 4.14. *Suppose that $\langle u_1, u_2 \rangle > 0$. Then $(\forall x \in \mathcal{H})$ $\mathrm{P}_{W_2}\mathrm{P}_{W_1} x \in W_1 \cap W_2$. In particular, if $x \in W_1 \cup W_2$, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$; if $x \in W_1^c \cap W_2^c$ with $\mathrm{P}_{H_1} x \in W_2$, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1 \cap W_2} x$; if $x \in W_1^c \cap W_2^c$ with $\mathrm{P}_{H_1} x \notin W_2$, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x \in W_1 \cap W_2 \setminus \{\mathrm{P}_{W_1 \cap W_2} x\}$.*

Proof. Let $x \in \mathcal{H}$. If $x \in W_1 \cup W_2$, then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x \in W_1 \cap W_2$ follows from Proposition 4.13.

Suppose that $x \in W_1^c \cap W_2^c$. Then Remark 2.9 leads to $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x$. If $\mathrm{P}_{H_1} x \in W_2$, then by Remark 4.9(i), $x \in C_1$. Hence, by Remark 4.9(ii), $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1 \cap W_2} x$. Assume that $\mathrm{P}_{H_1} x \notin W_2$. Then employing Lemma 2.11(v), we have that

$$\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in \operatorname{int} W_1 \cap H_2 \subseteq W_1 \cap W_2. \tag{38}$$

On the other hand, combine Remark 4.9(i)(ii) with $x \in W_1^c \cap W_2^c$ and $\mathrm{P}_{H_1} x \notin W_2$ to see that $x \in C_2 \cup C_3$ and that either $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_1 \cap H_2} x$ or $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_2} x$. Clearly, $\mathrm{P}_{H_1 \cap H_2} x \notin \operatorname{int} W_1$, so by (38), $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x \neq \mathrm{P}_{H_1 \cap H_2} x$. Moreover, by Fact 2.8 and by the first identity in Lemma 2.11(i), if $\mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} x$, then

$$0 = \frac{\eta_1 - \langle x, u_1 \rangle}{\|u_1\|^2} u_1 - \frac{(\eta_1 - \langle x, u_1 \rangle)\langle u_1, u_2 \rangle}{\|u_1\|^2 \|u_2\|^2} u_2,$$

which contradicts with the assumption that u_1 and u_2 are linearly independent. Therefore, by (38), $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x \neq \mathrm{P}_{H_2} x$. $\square$

Lemma 4.15. *Suppose $\langle u_1, u_2 \rangle = 0$. Let $x \in \mathcal{H}$. Let $i \in \{1, 2\}$ and $j \in \{1, 2\} \setminus \{i\}$. Then $x \in W_i$ if and only if $P_{H_j} x \in W_i$.*

Proof. Combine (6) with $\langle u_1, u_2 \rangle = 0$ and Fact 2.8 to obtain that $x \in W_i \Leftrightarrow \langle x, u_i \rangle \leq \eta_i \Leftrightarrow \left\langle x + \frac{\eta_j - \langle x, u_j \rangle}{\|u_j\|^2} u_j, u_i \right\rangle \leq \eta_i \Leftrightarrow \langle \mathrm{P}_{H_j} x, u_i \rangle \leq \eta_i \Leftrightarrow \mathrm{P}_{H_j} x \in W_i$. $\square$

Theorem 4.16. *Suppose $\langle u_1, u_2 \rangle = 0$. Then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{P}_{W_2 \cap W_1}$.*

Proof. According to Proposition 4.13, it suffices to show that $(\forall x \in W_1^c \cap W_2^c)$ $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_1 \cap W_2} x$. Let $x \in W_1^c \cap W_2^c$. Since $\langle u_1, u_2 \rangle = 0$, thus, by Lemma 4.10(ii)(c), $W_1^c \cap W_2^c = C_3$. Hence, Lemma 4.8 leads to $\mathrm{P}_{W_1 \cap W_2} x = \mathrm{P}_{H_1 \cap H_2} x$. Moreover, by Lemma 4.15, $x \notin W_2$ implies that $\mathrm{P}_{H_1} x \notin W_2$. So, Lemma 2.11(iii) and $x \in W_1^c \cap W_2^c$ deduce that $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap H_2} x$. Altogether, $\mathrm{P}_{W_2} \mathrm{P}_{W_1} x = \mathrm{P}_{H_1 \cap H_2} x = \mathrm{P}_{W_1 \cap W_2} x$. $\square$

We conclude the main results obtained in this section below.

Theorem 4.17. *Recall that u_1 and u_2 are in $\mathcal{H}$, that η_1 and η_2 are in $\mathbb{R}$, that $W_1 := \{x \in \mathcal{H} : \langle x, u_1 \rangle \leq \eta_1\}$, and $W_2 := \{x \in \mathcal{H} : \langle x, u_2 \rangle \leq \eta_2\}$. Then the following statements hold:*

(i) *Suppose that u_1, u_2 are linearly dependent and that $W_1 \cap W_2 \neq \varnothing$. Then $\mathrm{P}_{W_2} \mathrm{P}_{W_1} = \mathrm{P}_{W_1 \cap W_2}$ (see Theorem 4.6).*

(ii) *Suppose that u_1, u_2 are linearly independent.*

(a) *If $\langle u_1, u_2\rangle = 0$, then $\mathrm{P}_{W_2}\mathrm{P}_{W_1} = \mathrm{P}_{W_1\cap W_2}$ (see Theorem 4.16).*

(b) *If $\langle u_1, u_2\rangle < 0$, then $\gamma := \frac{|\langle u_1,u_2\rangle|}{\|u_1\|\|u_2\|} \in [0,1[$, and $(\forall x \in \mathcal{H})$ $\|(\mathrm{P}_{W_2}\mathrm{P}_{W_1})^k x - \mathrm{P}_{W_1\cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{W_1\cap W_2} x\|$ (see Theorem 4.12).*

(c) *If $\langle u_1, u_2\rangle > 0$, then $(\forall x \in \mathcal{H})$ $\mathrm{P}_{W_2}\mathrm{P}_{W_1} x \in W_1 \cap W_2$. In particular, if $x \in W_1 \cup W_2$, then $\mathrm{P}_{W_2}\mathrm{P}_{W_1} x = \mathrm{P}_{W_1\cap W_2} x$; if $x \in W_1^c \cap W_2^c$ with $\mathrm{P}_{H_1} x \in W_2$, then $\mathrm{P}_{W_2}\mathrm{P}_{W_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{W_1\cap W_2} x$; if $x \in W_1^c \cap W_2^c$ with $\mathrm{P}_{H_1} x \notin W_2$, then $\mathrm{P}_{W_2}\mathrm{P}_{W_1} x \in W_1 \cap W_2 \setminus \{\mathrm{P}_{W_1\cap W_2} x\}$ (see Proposition 4.14).*

5. Projection onto intersection of hyperplane and halfspace

Karush-Kuhn-Tucker conditions

In order to deduce the formula of projection onto intersection of hyperplane and halfspace, we need the following Karush-Kuhn-Tucker conditions to characterize the optimal solution of convex optimization with finitely many equality and inequality constraints in Hilbert spaces.

Before showing the main result, Theorem 5.2, in this subsection, we show first the following result.

Lemma 5.1. *Let $F : \mathcal{H} \to \mathbb{R}$ be affine and continuous. Then there exists a unique vector $u \in \mathcal{H}$ such that $(\forall x \in \mathcal{H})$ $F(x) = \langle x, u\rangle + F(0)$.*

Proof. Define $(\forall x \in \mathcal{H})$ $T(x) := F(x) - F(0)$. Let $x \in \mathcal{H}$, $y \in \mathcal{H}$, $\alpha \in \mathbb{R}$, and $\beta \in \mathbb{R}$. Because F is affine, we have that $T(\alpha x + \beta y) = F(\alpha x + \beta y) - F(0) = \frac{1}{2}F(2\alpha x) + \frac{1}{2}F(2\beta y) - F(0) = \frac{1}{2}(F(2\alpha x) + F(0)) + \frac{1}{2}(F(2\beta x) + F(0)) - 2F(0) = F(\alpha x + (1-\alpha)0) + F(\beta y + (1-\beta)0) - 2F(0) = \alpha(F(x) - F(0)) + \beta(F(y) - F(0)) = \alpha T(x) + \beta T(y)$, which implies that T is linear.

Moreover, because F is continuous implies that T is continuous, we know that $T \in \mathcal{B}(\mathcal{H}, \mathbb{R})$. Now, apply [1, Fact 2.24] with $f = T$ to obtain that there exists a unique vector $u \in \mathcal{H}$ such that $(\forall x \in \mathcal{H})$ $T(x) = \langle x, u\rangle$, which, by the definition of T, yields that $(\forall x \in \mathcal{H})$ $F(x) = \langle x, u\rangle + F(0)$. □

Because, differentiable function must be continuous, by Lemma 5.1, the condition "h_j is affine" in [7, p. 244] is equivalent to "$h_j(x) = \langle u_j, x\rangle - \eta_j$ where $u_j \in \mathcal{H}$ and $\eta_j \in \mathbb{R}$". Therefore, the following KKT conditions are generalization of the version shown in [7, p. 244] from finite-dimensional spaces to Hilbert spaces and from differentiable functions to subdifferentiable functions. Actually, writing the affine function h_j as the form $h_j(x) = \langle u_j, x\rangle - \eta_j$ plays a critical role in the proof of Theorem 5.2.

The main idea of the proof of Theorem 5.2 is from [1, Proposition 27.21] that characterizes the optimal solution of convex optimization with inequality constraints in Hilbert spaces, but Theorem 5.2 is not a direct result from [1, Proposition 27.21], because if there was equality constraint $g(x) = 0$ in [1, Proposition 27.21], then $(\mathrm{lev}_{<0}\, g) \cap (\mathrm{lev}_{<0} -g) = \varnothing$ implies that the Slater condition (27.50) in [1, Proposition 27.21] fails.

Theorem 5.2. *Let s and t be in $\mathbb{N} \setminus \{0\}$, set $\mathrm{I} := \{1, \dots, s\}$ and $\mathrm{J} := \{1, \dots, t\}$, and let $\bar{x} \in \mathcal{H}$. Suppose that f and $(g_i)_{i\in\mathrm{I}}$ are functions in $\Gamma_0(\mathcal{H})$. Set $(\forall j \in \mathrm{J})$ $(\forall x \in \mathcal{H})$ $h_j(x) := \langle u_j, x\rangle - \eta_j$ where $u_j \in \mathcal{H}$ and $\eta_j \in \mathbb{R}$. Assume that*

$$(\forall i \in \mathrm{I}) \quad \operatorname{lev}_{\leq 0} g_i \subseteq \operatorname{int}\operatorname{dom} g_i, \tag{39a}$$
$$(\cap_{i\in\mathrm{I}} \operatorname{lev}_{<0} g_i) \cap (\cap_{j\in\mathrm{J}} \ker h_j) \neq \varnothing, \quad \textit{and} \tag{39b}$$
$$0 \in \operatorname{sri}\left((\cap_{i\in\mathrm{I}} \operatorname{lev}_{\leq 0} g_i) \cap (\cap_{j\in\mathrm{J}} \ker h_j) - \operatorname{dom} f\right). \tag{39c}$$

Consider the problem

$$\begin{aligned} &\text{minimize} \quad f(x),\\ &\text{subject to} \quad g_i(x) \leq 0,\ i \in \mathrm{I},\ h_j(x) = 0,\ j \in \mathrm{J}. \end{aligned} \tag{40}$$

Then $\bar{x}$ is a solution to (40) if and only if

$$\begin{matrix} (\exists(\bar{\lambda}_i)_{i\in\mathrm{I}} \in \mathbb{R}^s_+) \\ (\exists(\bar{v}_i)_{i\in\mathrm{I}} \in \times_{i\in\mathrm{I}}\partial g_i(\bar{x})) \\ (\exists(\bar{\beta}_j)_{j\in\mathrm{J}} \in \mathbb{R}^t) \end{matrix} \quad \begin{cases} -\sum_{i\in\mathrm{I}} \bar{\lambda}_i \bar{v}_i - \sum_{j\in\mathrm{J}} \bar{\beta}_j u_j \in \partial f(\bar{x}), \\ (\forall i \in \mathrm{I}) \quad g_i(\bar{x}) \leq 0, \bar{\lambda}_i g_i(\bar{x}) = 0, \\ (\forall j \in \mathrm{J}) \quad h_j(\bar{x}) = 0, \end{cases} \tag{41}$$

in which case $(\bar{\lambda}_i)_{i\in\mathrm{I}} \times (\bar{\beta}_j)_{j\in\mathrm{J}}$ are Lagrange multipliers associated with $\bar{x}$, and $\bar{x}$ solves the problem

$$\min_{x\in\mathcal{H}} f(x) + \sum_{i\in\mathrm{I}} \bar{\lambda}_i g_i + \sum_{j\in\mathrm{J}} \bar{\beta}_j h_j. \tag{42}$$

Moreover, if f and $(g_i)_{i\in\mathrm{I}}$ are Gâteaux differentiable at $\bar{x}$, then (41) becomes

$$\begin{matrix} \left(\exists(\bar{\lambda}_i)_{i\in\mathrm{I}} \in \mathbb{R}^s_+\right) \\ \left(\exists(\bar{\beta}_j)_{j\in\mathrm{J}} \in \mathbb{R}^t\right) \end{matrix} \quad \begin{cases} \nabla f(\bar{x}) + \sum_{i\in\mathrm{I}} \bar{\lambda}_i \nabla g_i(\bar{x}) + \sum_{j\in\mathrm{J}} \bar{\beta}_j u_j = 0 \\ (\forall i \in \mathrm{I}) \quad g_i(\bar{x}) \leq 0, \bar{\lambda}_i g_i(\bar{x}) = 0, \ \textit{and} \\ (\forall j \in \mathrm{J}) \quad h_j(\bar{x}) = 0. \end{cases} \tag{43}$$

Proof. We split the proof into the following five steps.

Step 1: By (39b), $(\cap_{j\in\mathrm{J}} \ker h_j) \neq \varnothing$. Take $\bar{z} \in \cap_{j\in J} \ker h_j$. Then for every $j \in \mathrm{J}$, by definition of h_j,

$$h_j(x + \bar{z}) = \langle u_j, x\rangle. \tag{44}$$

Define

$$(\forall x \in \mathcal{H}) \quad T(x) := x + \bar{z}, \tag{45a}$$
$$\tilde{f} := f \circ T, \quad (\forall i \in \mathrm{I})\ \tilde{g}_i := g_i \circ T \quad \text{and} \quad (\forall j \in \mathrm{J})\ \tilde{h}_j := h_j \circ T = \langle u_j, \cdot\rangle. \tag{45b}$$

Then we have that $\tilde{f}$ and $(\tilde{g}_i)_{i\in\mathrm{I}}$ are functions in $\Gamma_0(\mathcal{H})$ with $\operatorname{dom}\tilde{f} = \operatorname{dom} f - \bar{z}$, and that

$$\begin{aligned} \partial f(\bar{x}) = \partial\tilde{f}(\bar{x} - \bar{z}), \quad (\forall i \in \mathrm{I})\ \partial g_i(\bar{x}) = \partial\tilde{g}_i(\bar{x} - \bar{z}), \quad \text{and}\\ (\forall j \in \mathrm{J})\ \nabla h_j(\bar{x}) = \nabla\tilde{h}_j(\bar{x} - \bar{z}) = u_j. \end{aligned} \tag{46}$$

Because $(\forall i \in \mathrm{I})$ $\operatorname{lev}_{\leq 0} \tilde{g}_i = \operatorname{lev}_{\leq 0} g_i - \bar{z}$, $\operatorname{int}\operatorname{dom}\tilde{g}_i = \operatorname{int}\operatorname{dom} g_i - \bar{z}$, $\cap_{i\in\mathrm{I}} \operatorname{lev}_{<0} \tilde{g}_i = \cap_{i\in\mathrm{I}} \operatorname{lev}_{<0} g_i - \bar{z}$ and $\cap_{j\in\mathrm{J}} \ker\tilde{h}_j = \cap_{j\in\mathrm{J}} \ker h_j - \bar{z}$, by (39), we have that

$$(\forall i \in \mathrm{I}) \quad \operatorname{lev}_{\leq 0} \tilde{g}_i \subseteq \operatorname{int}\operatorname{dom} \tilde{g}_i, \tag{47a}$$

$$(\cap_{i\in \mathrm{I}} \operatorname{lev}_{<0} \tilde{g}_i) \cap \left(\cap_{j\in \mathrm{J}} \ker \tilde{h}_j\right) \neq \varnothing, \text{ and} \tag{47b}$$

$$0 \in \operatorname{sri}\left((\cap_{i\in \mathrm{I}} \operatorname{lev}_{\leq 0} \tilde{g}_i) \cap \left(\cap_{j\in \mathrm{J}} \ker \tilde{h}_j\right) - \operatorname{dom} \tilde{f}\right). \tag{47c}$$

Let $y \in \mathcal{H}$. Substitute x in (40) by $y + \bar{z}$ to obtain that

$$\begin{aligned} &\text{minimize } \ \tilde{f}(y) \\ &\text{subject to } \ \tilde{g}_i(y) \leq 0, \ i \in \mathrm{I}, \ \tilde{h}_j(y) = 0, \ j \in \mathrm{J}\,. \end{aligned} \tag{48}$$

Hence, $\qquad \bar{x}$ is a solution to (40) $\Leftrightarrow$ $\bar{x} - \bar{z}$ is a solution to (48). $\qquad$ (49)

Step 2: In this part, we characterize the solutions to (48). Set

$$C := (\cap_{i\in \mathrm{I}} W_i) \cap (\cap_{j\in \mathrm{J}} L_j), \quad \text{where} \tag{50}$$

$(\forall i \in \mathrm{I})$ $W_i := \operatorname{lev}_{\leq 0} \tilde{g}_i = \{y \in \mathcal{H} : \tilde{g}_i(y) \leq 0\}$ and $(\forall j \in \mathrm{J})$ $L_j := \ker \tilde{h}_j \overset{(45b)}{=} \ker u_j$.
Recall that C is a closed and convex subset of $\mathcal{H}$ and $\tilde{f} \in \Gamma_0(\mathcal{H})$. Then (47c) and [1, Proposition 27.8] imply

$$\bar{x} - \bar{z} \text{ is a solution to (48)} \Leftrightarrow (\exists w \in \mathrm{N}_C(\bar{x} - \bar{z}))\,, -w \in \partial \tilde{f}(\bar{x} - \bar{z}). \tag{51}$$

Step 3: Denote by $\bar{y} := \bar{x} - \bar{z}$. We characterize the set $\mathrm{N}_C(\bar{y})$ in this part. As a consequence of [1, Definition 6.38], if $\bar{y} \notin C$, then $\mathrm{N}_C(\bar{y}) = \varnothing$. Suppose that $\bar{y} \in C$.

Set $\qquad \mathrm{I}_+ := \{i \in \mathrm{I} \ : \ \tilde{g}_i(\bar{y}) = 0\}$ and $\mathrm{I}_- := \{i \in \mathrm{I} \ : \ \tilde{g}_i(\bar{y}) < 0\}$.

Because $\bar{y} \in C \subseteq \cap_{i\in \mathrm{I}} W_i$, we know that

$$\mathrm{I} = \mathrm{I}_+ \cup \mathrm{I}_-\,. \tag{52}$$

Let $i \in \mathrm{I}$. Then employing (47a) in the last inclusion below, we know that

$$\bar{y} \in C \subseteq W_i = \operatorname{lev}_{\leq 0} \tilde{g}_i \subseteq \operatorname{int}\operatorname{dom} \tilde{g}_i. \tag{53}$$

Note that $\tilde{g}_i$ is convex implies that $\operatorname{dom} \tilde{g}_i$ is convex. Then, according to [1, Propostion 6.45], (53) implies that

$$(\forall i \in \mathrm{I}) \quad \mathrm{N}_{\operatorname{dom} \tilde{g}_i} \bar{y} = \{0\}. \tag{54}$$

Hence, using [1, Proposition 6.2(i)] in the second identity below, we have that

$$\begin{aligned} \mathrm{N}_{\operatorname{dom} \tilde{g}_i} \bar{y} \cup (\operatorname{cone} \partial \tilde{g}_i(\bar{y})) &\overset{(54)}{=} \{0\} \cup (\operatorname{cone} \partial \tilde{g}_i(\bar{y})) = \{0\} \cup (\mathbb{R}_{++} \partial \tilde{g}_i(\bar{y})) \\ &= \mathbb{R}_+ \partial \tilde{g}_i(\bar{y}). \end{aligned} \tag{55}$$

Moreover, since (47b) implies that $(\forall i \in \mathrm{I})$ $\operatorname{lev}_{<0} \tilde{g}_i \neq \varnothing$, thus combining [1, Lemma 27.20] with (53), (54) and (55), we obtain that

$$(\forall i \in \mathrm{I}) \quad \mathrm{N}_{W_i}(\bar{y}) = \begin{cases} \mathbb{R}_+ \partial \tilde{g}_i(\bar{y}), & \text{if } i \in \mathrm{I}_+; \\ \{0\}, & \text{if } i \in \mathrm{I}_-\,. \end{cases} \tag{56}$$

Because $(\forall j \in \mathrm{J})$ $L_j = \ker u_j$ is a linear subspace, $L_j - L_j = \ker u_j$, $\bar{y} \in C \subseteq \cap_{j\in\mathrm{J}} L_j$, using [1, Example 6.43], we have that

$$(\forall j \in \mathrm{J}) \quad \mathrm{N}_{L_j}(\bar{y}) = (L_j - L_j)^{\perp} = (\ker u_j)^{\perp} = \operatorname{span}\{u_j\}. \tag{57}$$

Recall that $\{\tilde{g}_i\}_{i\in\mathrm{I}} \subseteq \Gamma_0(\mathcal{H})$. As a consequence of [1, Corollary 8.39(ii)], $(\forall i \in \mathrm{I})$ $\operatorname{cont}\tilde{g}_i = \operatorname{int}\operatorname{dom}\tilde{g}_i$. Combine this with (47a) to know that $(\forall i \in \mathrm{I})$ $\tilde{g}_i$ is continuous on $\operatorname{lev}_{<0}\tilde{g}_i \subseteq \operatorname{lev}_{\leq 0}\tilde{g}_i \subseteq \operatorname{int}\operatorname{dom}\tilde{g}_i$. Moreover, because (47b) implies that $(\forall i \in \mathrm{I})$ $\operatorname{lev}_{<0}\tilde{g}_i \neq \varnothing$, by [1, Corollary 8.47(i)], $(\forall i \in \mathrm{I})$ $\operatorname{int} W_i = \operatorname{lev}_{<0}\tilde{g}_i$, which yields that

$$(\cap_{j\in\mathrm{J}} L_j) \cap (\cap_{i\in\mathrm{I}} \operatorname{int} W_i) = (\cap_{j\in\mathrm{J}} L_j) \cap (\cap_{i\in\mathrm{I}} \operatorname{lev}_{<0}\tilde{g}_i) \overset{(47\mathrm{b})}{\neq} \varnothing. \tag{58}$$

Apply [1, Example 16.50(iv)] with $m = s+1$, $(\forall i \in \mathrm{I})$ $f_i = \iota_{W_i}$ and $f_{s+1} = \iota_{\cap_{j=1}^{t} L_j}$ to obtain that

$$\partial\left(\sum_{i\in\mathrm{I}} \iota_{W_i} + \iota_{\cap_{j\in\mathrm{J}} L_j}\right)(\bar{y}) = \left(\Big(\sum_{i\in\mathrm{I}} \partial\iota_{W_i}\Big) + \partial\iota_{\cap_{j\in\mathrm{J}} L_j}\right)(\bar{y}). \tag{59}$$

In addition, note that $(\forall j \in \mathrm{J})$ $\sum_{k=1}^{j} L_k^{\perp} = \sum_{k=1}^{j} (\ker u_k)^{\perp} = \sum_{k=1}^{j} \operatorname{span}\{u_k\}$ is a closed finite-dimensional linear subspace of $\mathcal{H}$. Taking [4, Lemma 4.3(ii)] into account, we see that

$$(\forall j \in \mathrm{J}\setminus\{1\}) \quad L_j + \cap_{k=1}^{j-1} L_k \text{ is closed.} \tag{60}$$

Notice that $(\forall j \in \mathrm{J})$ $\operatorname{dom}\iota_{L_j} = L_j = \ker u_j$ is a linear subspace. Use (60), and apply [1, Example 16.50(iii)] with $m = t$, $(\forall k \in \{1,\ldots,m\})$ $f_k = \iota_{L_k}$ to obtain

$$\partial\Big(\sum_{j\in\mathrm{J}} \iota_{L_j}\Big)(\bar{y}) = \sum_{j\in\mathrm{J}} \partial\iota_{L_j}(\bar{y}). \tag{61}$$

Now, using intensively the definition of indicator function, we obtain

$$\begin{aligned}
\mathrm{N}_C(\bar{y}) &= \partial\iota_C(\bar{y}) \quad \text{(by [1, Example 16.13])}\\
&\overset{(50)}{=} \partial\iota_{(\cap_{i\in\mathrm{I}} W_i)\cap(\cap_{j\in\mathrm{J}} L_j)}(\bar{y}) = \partial\left(\Big(\sum_{i\in\mathrm{I}} \iota_{W_i}\Big) + \iota_{\cap_{j\in\mathrm{J}} L_j}\right)(\bar{y})\\
&\overset{(59)}{=} \left(\Big(\sum_{i\in\mathrm{I}} \partial\iota_{W_i}\Big) + \partial\iota_{\cap_{j\in\mathrm{J}} L_j}\right)(\bar{y})\\
&\overset{(52)}{=} \sum_{i\in\mathrm{I}_+} \partial\iota_{W_i}(\bar{y}) + \sum_{i\in\mathrm{I}_-} \partial\iota_{W_i}(\bar{y}) + \partial\iota_{\cap_{j\in\mathrm{J}} L_j}(\bar{y})\\
&= \sum_{i\in\mathrm{I}_+} \partial\iota_{W_i}(\bar{y}) + \sum_{i\in\mathrm{I}_-} \partial\iota_{W_i}(\bar{y}) + \partial\Big(\sum_{j\in\mathrm{J}} \iota_{L_j}\Big)(\bar{y})\\
&\overset{(61)}{=} \sum_{i\in\mathrm{I}_+} \partial\iota_{W_i}(\bar{y}) + \sum_{i\in\mathrm{I}_-} \partial\iota_{W_i}(\bar{y}) + \sum_{j\in\mathrm{J}} \partial\iota_{L_j}(\bar{y})\\
&= \sum_{i\in\mathrm{I}_+} \mathrm{N}_{W_i}(\bar{y}) + \sum_{i\in\mathrm{I}_-} \mathrm{N}_{W_i}(\bar{y}) + \sum_{j\in\mathrm{J}} \mathrm{N}_{L_j}(\bar{y}) \quad \text{(by [1, Example 16.13])}\\
&= \sum_{i\in\mathrm{I}_+} \mathbb{R}_+\partial\tilde{g}_i(\bar{y}) + \sum_{j\in\mathrm{J}} \operatorname{span}\{u_j\},
\end{aligned} \tag{62}$$

where the last identity follows from (56) and (57).

Combine (62) with (51) to obtain that $\bar{y}$ is a solution to (48) if and only if

$$\begin{array}{l}(\exists(\bar{\lambda}_i)_{i\in \mathrm{I}} \in \mathbb{R}^s_+)\\ (\exists(\bar{v}_i)_{i\in \mathrm{I}} \in \times_{i\in \mathrm{I}}\partial\tilde{g}_i(\bar{y}))\\ (\exists(\bar{\beta}_j)_{j\in \mathrm{J}} \in \mathbb{R}^t)\end{array} \quad \begin{cases} -\sum_{i\in \mathrm{I}}\bar{\lambda}_i\bar{v}_i - \sum_{j\in \mathrm{J}}\bar{\beta}_j u_j \in \partial\tilde{f}(\bar{y}) \\ (\forall i \in \mathrm{I}) \quad \tilde{g}_i(\bar{y}) \leq 0, \bar{\lambda}_i\tilde{g}_i(\bar{y}) = 0, \\ (\forall j \in \mathrm{J}) \quad \tilde{h}_j(\bar{y}) = 0, \end{cases} \tag{63}$$

Note that we have only I_+ in (62), but I in (63). In fact, for every $i \in \mathrm{I}$, if $i \in \mathrm{I}_+$, then by definition of I_+, $\tilde{g}_i(\bar{y}) = 0$, so $\bar{\lambda}_i\tilde{g}_i(\bar{y}) = 0$. Otherwise, if $i \in \mathrm{I}_-$, then to satisfy (62) and $-\sum_{i\in \mathrm{I}}\lambda_i v_i - \sum_{j\in \mathrm{J}}\beta_j u_j \in \partial\tilde{f}(\bar{y})$, we set $\lambda_i = 0$, which implies $\lambda_i\tilde{g}_i(\bar{y}) = 0$ as well.

Using (45) and (46), we know that (63) is equivalent to the desired statement (41).

Therefore, by (49), $\bar{x}$ is a solution to (40) if and only if (41) holds.

Step 4: Suppose that $\bar{x}$ is a solution of (40). Then by Step 3 above, (41) holds. Now, for the $(\bar{\lambda}_i)_{i\in \mathrm{I}} \in \mathbb{R}^s_+$, $(\bar{v}_i)_{i\in \mathrm{I}} \in \times_{i\in \mathrm{I}}\partial g_i(\bar{x})$, and $(\bar{\beta}_j)_{j\in \mathrm{J}} \in \mathbb{R}^t$ in (41), combine (41) and the definition of subdifferential to obtain that

$$\begin{aligned} 0 \in \partial f(\bar{x}) + \sum_{i\in \mathrm{I}}\bar{\lambda}_i\bar{v}_i + \sum_{j\in \mathrm{J}}\bar{\beta}_j u_j &\subseteq \partial f(\bar{x}) + \sum_{i\in \mathrm{I}}\bar{\lambda}_i\partial g_i(\bar{x}) + \sum_{j\in \mathrm{J}}\bar{\beta}_j\partial h_j(\bar{x}) \\ &\subseteq \partial\left(f + \sum_{i\in \mathrm{I}}\bar{\lambda}_i g_i + \sum_{j\in \mathrm{J}}\bar{\beta}_j h_j\right)(\bar{x}), \end{aligned}$$

which, by [1, Theorem 16.3], implies that $\bar{x}$ solves (42).

In addition, because it is clear that $\left(\mathbb{R}^s_- \times \{0\}^t\right)^{\ominus} = \mathbb{R}^s_+ \times \mathbb{R}^t$, using statement (41) and [1, Remark 19.26] and applying [1, Proposition 19.25(v)] with $\mathcal{K} = \mathbb{R}^{s+t}$, $K = \mathbb{R}^s_- \times \{0\}^t$, and $R : x \mapsto (g_i(x))_{i\in \mathrm{I}} \times (h_j(x))_{j\in \mathrm{J}}$, we obtain that $(\bar{\lambda}_i)_{i\in \mathrm{I}} \times (\bar{\beta}_j)_{j\in \mathrm{J}}$ are Lagrange multipliers associated with $\bar{x}$.

Step 5: Suppose that f and $(g_i)_{i\in \mathrm{I}}$ are Gâteaux differentiable at $\bar{x}$. Then, using the assumptions that f and $(g_i)_{i\in \mathrm{I}}$ are functions in $\Gamma_0(\mathcal{H})$, and [1, Proposition 17.31(i)], we obtain that

$$\partial f(\bar{x}) = \{\nabla f(\bar{x})\} \text{ and } (\forall i \in \mathrm{I})\ \partial g_i(\bar{x}) = \{\nabla g_i(\bar{x})\}. \tag{64}$$

Therefore, (41) is equivalent to (43) under the Gâteaux differentiable assumptions. Altogether, the proof is complete. □

Remark 5.3. (i) For special cases satisfying the condition (39c) in Theorem 5.2, we refer the interested readers to [1, Proposition 6.19]. In particular, if $\operatorname{dom} f = \mathcal{H}$, then applying Proposition 6.19(vii) of the book [1] with $L = \mathrm{Id}$, $D = (\cap_{i\in \mathrm{I}} \operatorname{lev}_{\leq 0} g_i) \cap (\cap_{j\in \mathrm{J}} \ker h_j)$, and $C = \operatorname{dom} f = \mathcal{H}$, we know that (39b) implies (39c). Hence, if $\operatorname{dom} f = \mathcal{H}$, (39c) is unnecessary.

(ii) If $J = \varnothing$ in Theorem 5.2, then as it is shown in the proof of [1, Proposition 27.21], in view of [1, Propositions 17.50 and 6.19(vii)], the conditions (27.50) in [1, Proposition 27.21] imply our conditions (39). Hence, we know that if $J = \varnothing$, then Theorem 5.2 reduces to [1, Proposition 27.21].

Explicit projection formula onto intersection of hyperplane and halfspace

Recall that $u_1, u_2 \in \mathcal{H}$, $\eta_1, \eta_2 \in \mathbb{R}$, and that $H_1 := \{x \in \mathcal{H} : \langle x, u_1\rangle = \eta_1\}$, and $W_2 := \{x \in \mathcal{H} : \langle x, u_2\rangle \leq \eta_2\}$, $H_2 := \{x \in \mathcal{H}\ \langle x, u_2\rangle = \eta_2\}$.

In this subsection, we shall provide an explicit formula for $\mathrm{P}_{H_1\cap W_2}$.

We can also prove the following result by using [9, Theorem 3.1].

Theorem 5.4. *Suppose that u_1 and u_2 are linearly dependent and that $H_1 \cap W_2 \neq \varnothing$. Then the following hold:*

(i) *Assume that $u_1 = 0$. Then $H_1 = \mathcal{H}$, $H_1 \cap W_2 = W_2$ and $\mathrm{P}_{H_1\cap W_2} = \mathrm{P}_{W_2}$.*

(ii) *Assume that $u_1 \neq 0$. Then $H_1 \cap W_2 = H_1$ and $\mathrm{P}_{H_1\cap W_2} = \mathrm{P}_{H_1}$.*

Proof. (i): Clearly, $u_1 = 0$ and $H_1 \cap W_2 \neq \varnothing$ imply that $H_1 \neq \varnothing$, $\eta_1 = 0$ and $H_1 = \mathcal{H}$. Hence, (i) holds.

(ii): Using Fact 2.1(i), $u_1 \neq 0$, and the linear dependence of u_1 and u_2, we have that

$$\text{either } u_2 = \frac{\|u_2\|}{\|u_1\|}u_1 \quad \text{or} \quad u_2 = -\frac{\|u_2\|}{\|u_1\|}u_1. \tag{65}$$

Recall that $H_1 \cap W_2 \neq \varnothing$. Take $\bar{z} \in H_1 \cap W_2$, that is,

$$\langle \bar{z}, u_1\rangle = \eta_1, \quad \text{and} \quad \langle \bar{z}, u_2\rangle \leq \eta_2. \tag{66}$$

Then $(\forall x \in H_1)$ $\langle x - \bar{z}, u_1\rangle = \eta_1 - \eta_1 = 0$. Moreover, for every $x \in H_1$,

$$\begin{aligned}\langle x, u_2\rangle - \eta_2 = \langle x - \bar{z}, u_2\rangle + \langle \bar{z}, u_2\rangle - \eta_2 &\overset{(65)}{=} \pm\frac{\|u_2\|}{\|u_1\|}\langle x - \bar{z}, u_1\rangle + \langle \bar{z}, u_2\rangle - \eta_2\\ = \langle \bar{z}, u_2\rangle - \eta_2 &\overset{(66)}{\leq} 0,\end{aligned}$$

which implies that $x \in W_2$. So we obtain $H_1 \subseteq W_2$. Hence, $H_1 \cap W_2 = H_1$ and $\mathrm{P}_{H_1\cap W_2} = \mathrm{P}_{H_1}$. □

The following easy result is necessary to prove the Theorem 5.6 below.

Lemma 5.5. *Suppose that u_1 and u_2 are linearly independent. Then $H_1 \cap H_2 \neq \varnothing$ and $H_1 \cap \operatorname{int} W_2 \neq \varnothing$.*

Proof. $H_1 \cap H_2 \neq \varnothing$ follows directly from Lemma 2.3. Because we have $(\forall i \in \{1, 2\})$ $(\operatorname{span}\{u_i\})^\perp = \ker u_i$, and u_1 and u_2 are linearly independent, we conclude that $\operatorname{span}\{u_2\} \not\subseteq \operatorname{span}\{u_1\}$, and so $\ker u_1 = (\operatorname{span}\{u_1\})^\perp \not\subseteq (\operatorname{span}\{u_2\})^\perp = \ker u_2$. Take $\bar{z} \in H_1 \cap H_2$, and $\bar{y} \in \ker u_1 \setminus \ker u_2$. Now,

$$\begin{aligned}\Big\langle \bar{z} - \langle \bar{y}, u_2\rangle\bar{y}, u_1\Big\rangle &= \langle \bar{z}, u_1\rangle - \langle \bar{y}, u_2\rangle\langle \bar{y}, u_1\rangle = \eta_1,\\ \Big\langle \bar{z} - \langle \bar{y}, u_2\rangle\bar{y}, u_2\Big\rangle &= \langle \bar{z}, u_2\rangle - \langle \bar{y}, u_2\rangle^2 = \eta_1 - \langle \bar{y}, u_2\rangle^2 < \eta_2,\end{aligned}$$

which imply that $\bar{z} - \langle \bar{y}, u_2\rangle\bar{y} \in H_1 \cap \operatorname{int} W_2 \neq \varnothing$. □

The main idea of the following result is from [1, Proposition 29.23], but this time we use the KKT conditions proved in Theorem 5.2.

Theorem 5.6. *Suppose that u_1 and u_2 are linearly independent. Let $x \in \mathcal{H}$. Then $H_1 \cap W_2 \neq \varnothing$ and $\mathrm{P}_{H_1\cap W_2} x = x - \xi_1 u_1 - \xi_2 u_2$, where exactly one of the following statements holds:*

(i) $(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle > 0$. *Then*
$\xi_1 = \frac{(\langle x,u_1\rangle-\eta_1)\|u_2\|^2-(\langle x,u_2\rangle-\eta_2)\langle u_1,u_2\rangle}{\|u_1\|^2\|u_2\|^2-|\langle u_1,u_2\rangle|^2}$, $\xi_2 = \frac{(\langle x,u_2\rangle-\eta_2)\|u_1\|^2-(\langle x,u_1\rangle-\eta_1)\langle u_1,u_2\rangle}{\|u_1\|^2\|u_2\|^2-|\langle u_1,u_2\rangle|^2} > 0.$
Moreover, $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1\cap H_2} x$.

(ii) $(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle \leq 0$. *Then* $\xi_1 = \frac{\langle x,u_1\rangle-\eta_1}{\|u_1\|^2}$ *and* $\xi_2 = 0$. *Moreover,* $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1} x$.

Proof. According to the definition of projection, $\mathrm{P}_{H_1\cap W_2} x$ is the unique solution of the problem

$$\begin{aligned}&\text{minimize } \ f(y) = \frac{1}{2}\|y-x\|^2\\ &\text{subject to } \ g(y) = \langle y, u_2\rangle - \eta_2 \leq 0, \quad h(y) = \langle y, u_1\rangle - \eta_1 = 0.\end{aligned}$$

Now $\operatorname{dom} f = \mathcal{H}$, $\operatorname{dom} g = \mathcal{H}$, f and g are differentiable functions in $\Gamma_0(\mathcal{H})$, and $\operatorname{lev}_{\leq 0} g = W_2 \subseteq \mathcal{H} = \operatorname{int}\operatorname{dom} g$. By Lemma 5.5, $\operatorname{lev}_{<0} g \cap \ker h = \operatorname{int} W_2 \cap H_1 \neq \varnothing$. Moreover, because $\operatorname{dom} f = \mathcal{H}$, by Remark 5.3(i), $0 \in \operatorname{sri}((\operatorname{lev}_{\leq 0} g \cap \ker h) - \operatorname{dom} f)$. Note that $(\forall y \in \mathcal{H})\ \nabla f(y) = y - x$, and $\nabla g(y) = u_2$. Hence, apply Theorem 5.2 with $s = 1$, $t = 1$, $f(y) = \frac{1}{2}\|y-x\|^2$, $g_1 = g$ and $h_1 = h$ to obtain that there exist $\xi_2 \in \mathbb{R}_+$ and $\xi_1 \in \mathbb{R}$ such that

$$\mathrm{P}_{H_1\cap W_2} x - x + \xi_2 u_2 + \xi_1 u_1 = 0, \tag{67a}$$
$$\langle u_2, \mathrm{P}_{H_1\cap W_2} x\rangle - \eta_2 \leq 0, \quad \xi_2(\langle u_2, \mathrm{P}_{H_1\cap W_2} x\rangle - \eta_2) = 0, \tag{67b}$$
$$\langle u_1, \mathrm{P}_{H_1\cap W_2} x\rangle - \eta_1 = 0. \tag{67c}$$

Hence, we have that

$$\mathrm{P}_{H_1\cap W_2} x = x - \xi_1 u_1 - \xi_2 u_2, \quad \text{(by (67a))} \tag{68a}$$
$$\langle u_2, x - \xi_1 u_1 - \xi_2 u_2\rangle - \eta_2 \leq 0, \quad \text{(by (68a) and (67b))} \tag{68b}$$
$$\langle u_1, x - \xi_1 u_1 - \xi_2 u_2\rangle - \eta_1 = 0, \quad \text{(by (68a) and (67c))} \tag{68c}$$
$$\xi_2(\langle u_2, x - \xi_1 u_1 - \xi_2 u_2\rangle - \eta_2) = 0. \quad \text{(by (68a) and (67b))} \tag{68d}$$

Now, we have exactly the following two cases.

Case 1: $\xi_2 > 0$. Then (68d) implies

$$\langle u_2, x - \xi_1 u_1 - \xi_2 u_2\rangle - \eta_2 = 0. \tag{69}$$

Combine (69) with (68c) to obtain that

$$\begin{pmatrix}\langle u_1, u_1\rangle & \langle u_1, u_2\rangle\\ \langle u_2, u_1\rangle & \langle u_2, u_2\rangle\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}\langle u_1, x\rangle - \eta_1\\ \langle u_2, x\rangle - \eta_2\end{pmatrix}. \tag{70}$$

Fact 2.2 and the linear independence of u_1 and u_2 imply that the Gram matrix $G(u_1, u_2)$ defined as (1) is invertible. Solve system (70) of linear equations to obtain

$$\xi_1 = \frac{(\langle x, u_1\rangle - \eta_1)\|u_2\|^2 - (\langle x, u_2\rangle - \eta_2)\langle u_1, u_2\rangle}{\|u_1\|^2\|u_2\|^2 - |\langle u_1, u_2\rangle|^2}$$

and

$$\xi_2 = \frac{(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle}{\|u_1\|^2\|u_2\|^2 - |\langle u_1, u_2\rangle|^2}. \tag{71}$$

Use u_1 and u_2 are linearly independent again and apply Fact 2.1(iii) to obtain

$$\xi_2 > 0 \Leftrightarrow (\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle > 0.$$

Hence, (i) is exactly the Case 1. Combine (68a) with (71) to know that the first part of (i) is true. In addition,

$$\begin{aligned}
\langle \mathrm{P}_{H_1\cap W_2} x, u_2\rangle - \eta_2 &\overset{(68a)}{=} \langle x - \xi_1 u_1 - \xi_2 u_2, u_2\rangle - \eta_2\\
&= \langle x, u_2\rangle - \eta_2 - \xi_1\langle u_1, u_2\rangle - \xi_2\|u_2\|^2\\
&\overset{(71)}{=} (\langle x, u_2\rangle - \eta_2) - \frac{(\langle x, u_1\rangle - \eta_1)\|u_2\|^2 - (\langle x, u_2\rangle - \eta_2)\langle u_1, u_2\rangle}{\|u_1\|^2\|u_2\|^2 - |\langle u_1, u_2\rangle|^2}\langle u_1, u_2\rangle\\
&\quad - \frac{(\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle}{\|u_1\|^2\|u_2\|^2 - |\langle u_1, u_2\rangle|^2}\|u_2\|^2\\
&= (\langle x, u_2\rangle - \eta_2) - \frac{(\langle x, u_1\rangle - \eta_1)(\|u_2\|^2\langle u_1, u_2\rangle - \langle u_1, u_2\rangle\|u_2\|^2)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1, u_2\rangle|^2}\\
&\quad - \frac{(\langle x, u_2\rangle - \eta_2)(\|u_1\|^2\|u_2\|^2 - \langle u_1, u_2\rangle^2)}{\|u_1\|^2\|u_2\|^2 - |\langle u_1, u_2\rangle|^2}\\
&= (\langle x, u_2\rangle - \eta_2) - (\langle x, u_2\rangle - \eta_2) = 0,
\end{aligned}$$

which implies that $\mathrm{P}_{H_1\cap W_2} x \in H_2$. So, $\mathrm{P}_{H_1\cap W_2} x \in H_1 \cap H_2$. Hence, apply Fact 2.7 with $A = H_1 \cap H_2$ and $B = H_1 \cap W_2$ to obtain $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1\cap H_2} x$.

Case 2: $\xi_2 = 0$. Then by (68c), we know that

$$\xi_1 = \frac{\langle u_1, x\rangle - \eta_1}{\|u_1\|^2}. \tag{72}$$

Taking (68a) and Fact 2.8 into account, we know that

$$\xi_2 = 0 \Leftrightarrow \mathrm{P}_{H_1\cap W_2} x = x - \xi_1 u_1 = x + \frac{\eta_1 - \langle u_1, x\rangle}{\|u_1\|^2}u_1 = \mathrm{P}_{H_1} x. \tag{73}$$

Apply Fact 2.7 with $A = H_1 \cap W_2$ and $B = H_1$ and use Lemma 2.11(ii) to obtain

$$\begin{aligned}
\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1} x \;&\Leftrightarrow\; \mathrm{P}_{H_1} x \in W_2\\
&\Leftrightarrow (\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle \leq 0.
\end{aligned} \tag{74}$$

Combine (73) with (74) to obtain

$$\xi_2 = 0 \Leftrightarrow (\langle x, u_2\rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1\rangle - \eta_1)\langle u_1, u_2\rangle \leq 0. \tag{75}$$

Hence, (ii) is exactly the Case 2. Moreover, (72) and (73) imply (ii). Altogether, the proof is complete. □

Lemma 5.7. *Suppose that u_1 and u_2 are linearly independent. Let $x \in \mathcal{H}$. Then the following hold:*

(i) *If $\mathrm{P}_{H_1} x \notin W_2$, then $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1\cap H_2} x$; otherwise, $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1} x$.*

(ii) *Assume that $\mathrm{P}_{H_1}\mathrm{P}_{W_2} x \notin W_2$. Then $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1\cap H_2} x$.*

Proof. (i): This is clear from Theorem 5.6 and Lemma 2.11(ii).

(ii): By (i), it suffices to show $\mathrm{P}_{H_1} x \notin W_2$. If $x \in W_2$, then $\mathrm{P}_{H_1} x = \mathrm{P}_{H_1} \mathrm{P}_{W_2} x \notin W_2$. Suppose $x \notin W_2$. Assume to the contrary that $\mathrm{P}_{H_1} x \in W_2$. Then, by Fact 2.8,

$$\begin{aligned} 0 \leq \eta_2 - \langle \mathrm{P}_{H_1} x, u_2\rangle \;\Leftrightarrow\; & 0 \leq \eta_2 - \left\langle x + \frac{\eta_1 - \langle x, u_1\rangle}{\|u_1\|^2} u_1, u_2 \right\rangle \\ \Leftrightarrow\; & (\eta_1 - \langle x, u_1\rangle)\langle u_1, u_2\rangle \leq \|u_1\|^2(\eta_2 - \langle x, u_2\rangle), \end{aligned}$$

which, by Lemma 2.11(i) with swapping H_1 and H_2, and by $x \notin W_2$ and the Cauchy-Schwarz inequality, implies that

$$\begin{aligned} & \eta_2 - \langle \mathrm{P}_{H_1} \mathrm{P}_{H_2} x, u_2\rangle x \\ &= -\frac{1}{\|u_1\|^2\|u_2\|^2} \big((\eta_1 - \langle x, u_1\rangle)\|u_2\|^2 - (\eta_2 - \langle x, u_2\rangle)\langle u_1, u_2\rangle\big) \langle u_1, u_2\rangle \\ &\geq -\frac{1}{\|u_1\|^2\|u_2\|^2} \big(\|u_1\|^2\|u_2\|^2(\eta_2 - \langle x, u_2\rangle) - (\eta_2 - \langle x, u_2\rangle)\langle u_1, u_2\rangle^2\big) \\ &= (\eta_2 - \langle x, u_2\rangle) \left(\frac{\langle u_1, u_2\rangle^2}{\|u_1\|^2\|u_2\|^2} - 1\right) \geq 0, \end{aligned}$$

which implies that $\mathrm{P}_{H_1} \mathrm{P}_{H_2} x \in W_2$ and contradicts the assumption. $\square$

6. Compositions of projections onto hyperplane and halfspace

Similarly to Fact 4.1, given finitely many hyperplanes and halfspaces, by [10, Theorem 9.24], the Boyle-Dykstra Theorem, we are able to use only the projections onto these individual hyperplanes and halfspaces to generate the sequence according to the Dykstra's algorithm for finding the projection onto the intersection of these hyperplanes and halfspaces.

In this section, for simplicity, we consider only one hyperplane and one halfspace. We shall systematically investigate the relations of $\mathrm{P}_{H_1 \cap W_2}$ and $\mathrm{P}_{W_2} \mathrm{P}_{H_1}$, and of $\mathrm{P}_{H_1 \cap W_2}$ and $\mathrm{P}_{H_1} \mathrm{P}_{W_2}$.

Recall that $u_1, u_2 \in \mathcal{H}$, $\eta_1, \eta_2 \in \mathbb{R}$, and that $H_1 := \{x \in \mathcal{H} : \langle x, u_1\rangle = \eta_1\}$, $W_2 := \{x \in \mathcal{H} : \langle x, u_2\rangle \leq \eta_2\}$, and $H_2 := \{x \in \mathcal{H} : \langle x, u_2\rangle = \eta_2\}$.

u_1 and u_2 are linearly dependent

Theorem 6.1. *Suppose that u_1 and u_2 are linearly dependent and that $H_1 \cap W_2 \neq \varnothing$. Then $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1 \cap W_2} = \mathrm{P}_{H_1} \mathrm{P}_{W_2}$. In particular, the following statements hold:*

(i) *Assume that $u_1 = 0$. Then $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{W_2} = \mathrm{P}_{H_1} \mathrm{P}_{W_2}$.*
(ii) *Assume that $u_1 \neq 0$. Then $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1} = \mathrm{P}_{H_1} \mathrm{P}_{W_2}$.*

Proof. (i): According to Theorem 5.4(i), $H_1 = \mathcal{H}$ and $H_1 \cap W_2 = W_2$. Then clearly (i) holds.

(ii): In view of Theorem 5.4(ii) we directly have $H_1 \cap W_2 = H_1$. Then clearly $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1} = \mathrm{P}_{H_1 \cap W_2}$.

On the other hand, if $u_2 = 0$, then because $H_1 \cap W_2 \neq \varnothing$ implies that $W_2 \neq \varnothing$, we have that $\eta_2 \geq 0$ and $W_2 = \mathcal{H}$. So, $\mathrm{P}_{H_1} \mathrm{P}_{W_2} = \mathrm{P}_{H_1} \mathrm{P}_{\mathcal{H}} = \mathrm{P}_{H_1}$. Suppose that $u_2 \neq 0$. Let $x \in \mathcal{H}$. If $x \in W_2$, then $\mathrm{P}_{H_1} \mathrm{P}_{W_2} x = \mathrm{P}_{H_1} x$. Assume that $x \notin W_2$. Then use Remark 2.9 and apply Lemma 2.11(iv) with swapping H_1 and H_2 to yield $\mathrm{P}_{H_1} \mathrm{P}_{W_2} x = \mathrm{P}_{H_1} \mathrm{P}_{H_2} x = \mathrm{P}_{H_1} x$. Hence, $\mathrm{P}_{H_1} \mathrm{P}_{W_2} = \mathrm{P}_{H_1}$.

Altogether, we have that $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1 \cap W_2} = \mathrm{P}_{H_1} = \mathrm{P}_{H_1} \mathrm{P}_{W_2}$. □

u_1 and u_2 are linearly independent

In the whole subsection, we set

$$C := \left\{ x \in \mathcal{H} \ : \ (\langle x, u_2 \rangle - \eta_2)\|u_1\|^2 - (\langle x, u_1 \rangle - \eta_1)\langle u_1, u_2 \rangle > 0 \right\}. \tag{76}$$

Proposition 6.2. *Suppose that u_1 and u_2 are linearly independent. Then the following hold:*

(i) *Let $x \in \mathcal{H}$. Then $x \in C$ if and only if $\mathrm{P}_{H_1} x \notin W_2$.*

(ii) *Suppose that $\langle u_1, u_2 \rangle = 0$. Then $C = W_2^c$ and $\mathrm{P}_{W_2} \mathrm{P}_{H_1} = \mathrm{P}_{H_1 \cap W_2} = \mathrm{P}_{H_1} \mathrm{P}_{W_2}$.*

(iii) *Let $x \in C^c$. Then $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap W_2} x$.*

(iv) *Suppose that $\langle u_1, u_2 \rangle \neq 0$. Let $x \in C$. Then $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x \in C$. Moreover, $\mathrm{P}_{H_1 \cap W_2} \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap H_2} x = \mathrm{P}_{H_1 \cap W_2} x$.*

Proof. (i): This is clear from Lemma 2.11(ii) and (76).

(ii): It is easy to see that $C = W_2^c$ follows from $\langle u_1, u_2 \rangle = 0$, (76) and the definition of W_2. Let $x \in \mathcal{H}$. Then we have exactly the following two cases:

Case 1: $x \in C$. Then (i) and Remark 2.9 imply $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x$. Hence, by $\langle u_1, u_2 \rangle = 0$, Theorem 5.6(i) and Lemma 2.11(iii),

$$\mathrm{P}_{H_1 \cap W_2} x = x + \frac{\eta_1 - \langle u_1, x \rangle}{\|u_1\|^2} u_1 + \frac{\eta_2 - \langle u_2, x \rangle}{\|u_2\|^2} u_2 = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x.$$

Recall that $x \in C = W_2^c$. Use Remark 2.9 and Theorem 5.6(i), and apply Lemma 2.11(iii) with swapping H_1 and H_2 to obtain that

$$\mathrm{P}_{H_1} \mathrm{P}_{W_2} x = \mathrm{P}_{H_1} \mathrm{P}_{H_2} x = x + \frac{\eta_1 - \langle u_1, x \rangle}{\|u_1\|^2} u_1 + \frac{\eta_2 - \langle u_2, x \rangle}{\|u_2\|^2} u_2 = \mathrm{P}_{H_1 \cap W_2} x.$$

Case 2: $x \in C^c$. Then by Theorem 5.6(ii) and Lemma 5.7(i), $\mathrm{P}_{H_1 \cap W_2} x = \mathrm{P}_{H_1} x$ and $\mathrm{P}_{H_1} x \in W_2$. Hence, $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap W_2} x$. Note that $x \in C^c = (W_2^c)^c = W_2$. Hence, $\mathrm{P}_{H_1} \mathrm{P}_{W_2} x = \mathrm{P}_{H_1} x = \mathrm{P}_{H_1 \cap W_2} x$.

(iii): Theorem 5.6(ii) and Lemma 5.7(i) yield $\mathrm{P}_{H_1 \cap W_2} x = \mathrm{P}_{H_1} x$ and $\mathrm{P}_{H_1} x \in W_2$. Hence, (iii) holds.

(iv): Clearly, (i) and Remark 2.9 lead to $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_2} \mathrm{P}_{H_1} x$. Because we have $\mathrm{P}_{H_1} x \in H_1 \cap W_2^c$, apply Lemma 2.12(ii) with $x = \mathrm{P}_{H_1} x$ to see that

$$\mathrm{P}_{H_1} \mathrm{P}_{W_2} \mathrm{P}_{H_1} x = \mathrm{P}_{H_1} \mathrm{P}_{H_2} \mathrm{P}_{H_1} x \notin W_2.$$

Hence, apply (i) with $x = \mathrm{P}_{W_2} \mathrm{P}_{H_1} x$ to know that $\mathrm{P}_{W_2} \mathrm{P}_{H_1} x \in C$. Combine this with $x \in C$ and Theorem 5.6 to obtain $\mathrm{P}_{H_1 \cap W_2} x = \mathrm{P}_{H_1 \cap H_2} x$ and, finally,

$\mathrm{P}_{H_1\cap W_2}\mathrm{P}_{W_2}\mathrm{P}_{H_1}x = \mathrm{P}_{H_1\cap H_2}\mathrm{P}_{W_2}\mathrm{P}_{H_1}x$. Recall that $\mathrm{P}_{W_2}\mathrm{P}_{H_1}x = \mathrm{P}_{H_2}\mathrm{P}_{H_1}x$. Combine the last three identities with Fact 2.10 used in the third equation below to obtain

$$\begin{aligned}\mathrm{P}_{H_1\cap W_2}\mathrm{P}_{W_2}\mathrm{P}_{H_1}x &= \mathrm{P}_{H_1\cap H_2}\mathrm{P}_{W_2}\mathrm{P}_{H_1}x = \mathrm{P}_{H_1\cap H_2}\mathrm{P}_{H_2}\mathrm{P}_{H_1}x = \mathrm{P}_{H_1\cap H_2}x\\ &= \mathrm{P}_{H_1\cap W_2}x. \qquad\square\end{aligned}$$

Theorem 6.3. *Suppose that u_1 and u_2 are linearly independent. Set $\gamma := \frac{|\langle u_1,u_2\rangle|}{\|u_1\|\|u_2\|}$. Then $\gamma \in [0,1[$ and $\mathrm{P}_{W_2}\mathrm{P}_{H_1}$ is a γ-BAM. Consequently,*

$$(\forall x\in\mathcal{H})(\forall k\in\mathbb{N}) \quad \|(\mathrm{P}_{W_2}\mathrm{P}_{H_1})^k x - \mathrm{P}_{H_1\cap W_2}x\| \le \gamma^k\|x - \mathrm{P}_{H_1\cap W_2}x\|.$$

Proof. According to Lemma 3.4, $\gamma\in[0,1[$. Let $x\in\mathcal{H}$. By [8, Corollary 4.5.2], $\mathrm{Fix}\,\mathrm{P}_{W_2}\mathrm{P}_{H_1} = H_1\cap W_2$ is a nonempty closed convex subset of $\mathcal{H}$. Hence, using Definition 2.4 and Fact 2.5, we only need to prove the following two statements:

(i) $\mathrm{P}_{H_1\cap W_2}\mathrm{P}_{W_2}\mathrm{P}_{H_1}x = \mathrm{P}_{H_1\cap W_2}x$.

(ii) $\|\mathrm{P}_{W_2}\mathrm{P}_{H_1}x - \mathrm{P}_{H_1\cap W_2}x\| \le \gamma\|x - \mathrm{P}_{H_1\cap W_2}x\|$.

If $x\in C^c$ or if $\langle u_1,u_2\rangle = 0$, then by Proposition 6.2(ii)&(iii), $\mathrm{P}_{W_2}\mathrm{P}_{H_1}x = \mathrm{P}_{H_1\cap W_2}x$, which implies (i) and (ii).

Suppose that $x\in C$ and that $\langle u_1,u_2\rangle\neq 0$. Then Proposition 6.2(iv) implies that (i) holds and that $\mathrm{P}_{W_2}\mathrm{P}_{H_1}x = \mathrm{P}_{H_2}\mathrm{P}_{H_1}x$ and $\mathrm{P}_{H_1\cap W_2}x = \mathrm{P}_{H_1\cap H_2}x$. Combine these identities with Lemma 3.4 to obtain

$$\begin{aligned}\|\mathrm{P}_{W_2}\mathrm{P}_{H_1}x - \mathrm{P}_{H_1\cap W_2}x\| &= \|\mathrm{P}_{H_2}\mathrm{P}_{H_1}x - \mathrm{P}_{H_1\cap H_2}x\| \le \gamma\|x - \mathrm{P}_{H_1\cap H_2}x\|\\ &= \gamma\|x - \mathrm{P}_{H_1\cap W_2}x\|,\end{aligned}$$

which yields (ii). Altogether, the proof is complete. $\square$

Proposition 6.4. *Suppose u_1 and u_2 are linearly independent. Define $\gamma := \frac{|\langle u_1,u_2\rangle|}{\|u_1\|\|u_2\|}$. Then exactly one of the following statements holds:*

(i) $\mathrm{P}_{H_1}\mathrm{P}_{W_2}x\in W_2$. *Then* $\mathrm{P}_{H_1}\mathrm{P}_{W_2}x\in H_1\cap W_2$.

(ii) $\mathrm{P}_{H_1}\mathrm{P}_{W_2}x\notin W_2$. *Then* $(\forall k\in\mathbb{N})$ $\mathrm{P}_{W_2}(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^k x\notin H_1$ *and* $(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^{k+1}x\notin W_2$. *Moreover, for every $k\in\mathbb{N}$,*

$$(\forall x\in W_2) \quad \|\mathrm{P}_{W_2}(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^k x - \mathrm{P}_{H_1\cap W_2}x\| \le \gamma^k\|x - \mathrm{P}_{H_1\cap W_2}x\|, \tag{77a}$$

$$(\forall x\in W_2^c) \quad \|(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^k - \mathrm{P}_{H_1\cap W_2}x\| \le \gamma^k\|x - \mathrm{P}_{H_1\cap W_2}x\|. \tag{77b}$$

Proof. (i): This is trivial.

(ii): Apply Lemma 3.4 with swapping H_1 and H_2 to obtain that $\gamma\in[0,1[$ and for every $x\in\mathcal{H}$ and $k\in\mathbb{N}$,

$$\|(\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k x - \mathrm{P}_{H_1\cap H_2}x\| \le \gamma^k\|x - \mathrm{P}_{H_1\cap H_2}x\|, \tag{78a}$$

$$\|(\mathrm{P}_{H_1}\mathrm{P}_{H_2})^k x - \mathrm{P}_{H_1\cap H_2}x\| \le \gamma^k\|x - \mathrm{P}_{H_1\cap H_2}x\|. \tag{78b}$$

According to Lemma 5.7(ii), $\mathrm{P}_{H_1\cap W_2}x = \mathrm{P}_{H_1\cap H_2}x$. Set $y := \mathrm{P}_{H_1}\mathrm{P}_{W_2}x\in H_1\cap W_2^c$.

Apply Lemma 2.12(iii) to the point y to obtain that for every $k \in \mathbb{N}$, ($k = 0$ is trivial)

$$\begin{aligned} &(\mathrm{P}_{W_2}\mathrm{P}_{H_1})^k y = (\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k y \notin H_1, \text{ i.e.,} \\ &\qquad \mathrm{P}_{W_2}(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^k x = (\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k \mathrm{P}_{W_2} x \notin H_1 \text{ and} \end{aligned} \tag{79}$$

$$\begin{aligned} &\mathrm{P}_{H_1}(\mathrm{P}_{W_2}\mathrm{P}_{H_1})^k y = \mathrm{P}_{H_1}(\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k y \notin W_2, \text{ i.e.,} \\ &\qquad (\mathrm{P}_{H_1}\mathrm{P}_{W_2})^{k+1} x = \mathrm{P}_{H_1}(\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k \mathrm{P}_{W_2} x \notin W_2. \end{aligned} \tag{80}$$

Moreover, if $x \in W_2$, then by (79), $\mathrm{P}_{W_2}(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^k x = (\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k x$. Hence, (78a) and $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1\cap H_2} x$ yield (77a).

If $x \notin W_2$, then by (80), $(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^{k+1} x = \mathrm{P}_{H_1}(\mathrm{P}_{H_2}\mathrm{P}_{H_1})^k \mathrm{P}_{H_2} x = (\mathrm{P}_{H_1}\mathrm{P}_{H_2})^{k+1} x$. Hence, (78b) and $\mathrm{P}_{H_1\cap W_2} x = \mathrm{P}_{H_1\cap H_2} x$ imply that (77b) is true.

Altogether, the proof is complete. □

To conclude this section, we summarize the results obtained in this section in the following theorem.

Theorem 6.5. *Recall that* $u_1, u_2 \in \mathcal{H}$, $\eta_1, \eta_2 \in \mathbb{R}$, $H_1 := \{x \in \mathcal{H} : \langle x, u_1\rangle = \eta_1\}$, $W_2 := \{x \in \mathcal{H} : \langle x, u_2\rangle \leq \eta_2\}$, $H_2 := \{x \in \mathcal{H} : \langle x, u_2\rangle = \eta_2\}$. *Then exactly one of the following statements hold.*

(i) u_1 *and* u_2 *are linearly dependent. Then* $\mathrm{P}_{W_2}\mathrm{P}_{H_1} = \mathrm{P}_{H_1\cap W_2} = \mathrm{P}_{H_1}\mathrm{P}_{W_2}$ (*see Theorem* 6.1).

(ii) u_1 *and* u_2 *are linearly independent. Then the following hold:*

- (a) *Suppose that* $\langle u_1, u_2\rangle = 0$. *Then* $\mathrm{P}_{W_2}\mathrm{P}_{H_1} = \mathrm{P}_{H_1\cap W_2} = \mathrm{P}_{H_1}\mathrm{P}_{W_2}$ (*see Proposition* 6.2(ii)).
- (b) *Suppose that* $\langle u_1, u_2\rangle \neq 0$. *Define* $\gamma := \frac{|\langle u_1, u_2\rangle|}{\|u_1\|\|u_2\|}$. *Then we have* $\gamma \in [0, 1[$. *Moreover,*
  - (1) $(\forall x \in \mathcal{H})$ $(\forall k \in \mathbb{N})$ $\|(\mathrm{P}_{W_2}\mathrm{P}_{H_1})^k x - \mathrm{P}_{H_1\cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{H_1\cap W_2} x\|$ (*see Theorem* 6.3).
  - (2) *Let* $x \in \mathcal{H}$ *and* $k \in \mathbb{N}$. *If* $\mathrm{P}_{H_1}\mathrm{P}_{W_2} x \notin H_1 \cap W_2$, *then* $(\forall x \in W_2)$ $\|\mathrm{P}_{W_2}(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^k x - \mathrm{P}_{H_1\cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{H_1\cap W_2} x\|$, *and* $(\forall x \in W_2^c)$ $\|(\mathrm{P}_{H_1}\mathrm{P}_{W_2})^k - \mathrm{P}_{H_1\cap W_2} x\| \leq \gamma^k \|x - \mathrm{P}_{H_1\cap W_2} x\|$ (*see Proposition* 6.4).

7. Conclusion and future work

We provided an explicit formula of the projection onto the intersection of finitely many hyperplanes. We also showed KKT conditions for characterizing the optimal solution of convex optimization with finitely many inequality and equality constraints in Hilbert spaces. Moreover, we constructed formulae of projections onto the intersection of hyperplane and halfspace. In addition, we systematically investigated the relations between: $\mathrm{P}_{W_2}\mathrm{P}_{W_1}$ and $\mathrm{P}_{W_1\cap W_2}$, $\mathrm{P}_{H_1\cap W_2}$ and $\mathrm{P}_{W_2}\mathrm{P}_{H_1}$, and $\mathrm{P}_{H_1\cap W_2}$ and $\mathrm{P}_{H_1}\mathrm{P}_{W_2}$.

According to the explicit formulae for $\mathrm{P}_{W_1\cap W_2}$ and $\mathrm{P}_{H_1\cap W_2}$ (see, Fact 4.2, Fact 4.7, Theorem 5.4, and Theorem 5.6), given a point $x \in \mathcal{H}$, the formulae of $\mathrm{P}_{W_1\cap W_2} x$ and $\mathrm{P}_{H_1\cap W_2} x$ depend on the linear dependence relation of u_1 and u_2, and on the "region" where the x is located in. Hence, in many proofs of this work, we mainly argued by cases and considered our questions on two halfspaces, or on one halfspace and one hyperplane. It is easy to see that if we wanted to use the current logic of proofs to extend our results from two sets to finitely many sets, then the number of cases to argue would increase exponentially. In the future, we shall try to find some techniques or tricks to make the extension work easy and the statements of the results simple.

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