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An Eaves Type Theorem for Quadratic Fractional Programming Problems and its Applications

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Using the notions of asymptotic cone and function, Lara [J. Convex Analysis 26 (2019) 15–32] proposed necessary and sufficient conditions for the existence of minimizers of quadratic fractional programming (QFP) problems. This result could be seen as the first extension of the Frank-Wolfe theorem from the quadratic to the quadratic fractional case. However, the asymptotically linear assumption is quite strong and the denominator of the objective function must be linear. In this paper, we present a new class of convex sets, which contains the previous class of asymptotically linear sets. By using the obtained results we propose necessary and sufficient conditions for the existence of solutions of QFP problems. Finally, we use the above results to investigate the boundedness or unboundedness of the global solution set and the stability for global solution map of a class of QFP problems.

Keywords: Quadratic fractional program, existence of solutions, Eaves Theorem, stability, asymptotically linear function.

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1. Introduction

Consider the following quadratic fractional programming (QFP) problem

$$(P) \qquad \text{minimize} \quad f(x) := \frac{\frac{1}{2}x^T A_1 x + a_1^T x + \alpha_1}{\frac{1}{2}x^T A_2 x + a_2^T x + \alpha_2}$$
$$\text{subject to} \quad x \in \mathbb{R}^n,\ \frac{1}{2}x^T B_i x + b_i^T x + \beta_i \le 0, \quad i = 1, \dots, m,$$

where A_1, A_2, B_i are symmetric real $(n \times n)$-matrices; B_i is positive semidefinite; $a_1, a_2, b_i \in \mathbb{R}^n$ and $\alpha_1, \alpha_2, \beta_i \in \mathbb{R}$ for every $i \in \{1, \dots, m\}$. Let

$$\mathcal{F} := \left\{ \frac{1}{2}x^T B_i x + b_i^T x + \beta_i \le 0, \quad i = 1, \dots, m \right\}.$$

Model problems of this type, which has been widely studied both from a theoretical and an algorithmic point of view, is an important issue in continuous nonlinear optimization. Its applications in economics, transportation science, and engineering have been actively studied for several decades (see [8] and the references therein).

The problem (P) reduces a quadratic programing (QP) problem under quadratic constraints as $A_2 = 0$ and $a_2 = 0$. Existence and stability for this problem have been investigated in [17, 18, 19, 20, 23].

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The existence of minimizers is one of the most important issues in optimization theory. Frank and Wolfe [7] proved that a QP problem has a solution if its objective function f is bounded below over a nonempty polyhedral convex set C (called *Frank-Wolfe Theorem*). This is an extension of the fundamental existence theorem of linear programming. There have been some other proofs for this theorem and its extended versions (see [3, 24]). A similar result [15] was established for the case, where f is quadratic and C is defined by a convex quadratic function and linear constraint functions. However, verifying whether the function is bounded from below on a given set is a rather difficult task. Eaves [5] proposed necessary and sufficient conditions for the solution existence of linearly constrained QP problems. By using the concept of recession cone in convex analysis, Kim et al. [11] established an Eaves type theorem for convex QP problems. Recently, an Eaves type theorem for nonconvex QP problems has been presented in [23]. Despite these extensive studies, the existence of the global solutions for QFP problems has received limited attention.

As $A_2 = 0$ and $\mathcal{F}$ is an asymptotically linear set (see [1, Definition 2.3.1]), due to using the asymptotic cone and function, Lara [13, Theorem 4.1] proposed necessary/sufficient conditions for the solutions existence of (P). This result could be seen as the first extension of the Frank-Wolfe theorem [7] from the quadratic to the quadratic fractional case. However, the asymptotically linear assumption is quite strong, for instance, the following convex set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 - x_2 \leq 0\}$ is not an asymptotically linear set (see Example 3.1). Therefore, studying the solution existence of (P) under weaker assumptions is very necessary. As far as we know, up to now, there is no result on this direction for the case, where $A_2 \neq 0$ and $\mathcal{F}$ is defined by many quadratic constraints.

In this paper, we present a new class of convex sets, so called *weakly asymptotically linear* set respect to a cone $K \subset \mathbb{R}^n$ (denote by K-wal set), which contains the previous class of asymptotically linear sets. We also show that $\mathcal{F}$ could be a K-wal set under some checked conditions. By using this results, we obtain the necessary/sufficient conditions for the solution existence of (P) in the general case, where the objective function is a ratio of two quadratic functions and the constraint is defined by many quadratic functions. Thanks to the above result, the boundedness/unboundedness of the global solution set and the semicontinuity for global solution map of a class of QFP problems are also fully characterized. Our approach is via properties of K-wal sets, structure of the quadratic forms, and techniques of asymptotic cones and functions (see [1, 22]).

The structure of the paper is as follows. In Section 3, we introduce the definition of K-wal sets and prove that $\mathcal{F}$ is K-wal under the assumption (H). We also show some cases, where $\mathcal{F}$ is K-wal but not asymptotically linear. In Section 4, an Eaves type theorem for solution existence of (P) is proposed. We obtain some important corollaries under weaker assumptions in comparison with previous results. A class of QFP problems with A_2 being concave is investigated in Section 5.

2. Preliminaries

The scalar product of vectors x, y and the Euclidean norm of a vector x in a finite-dimensional Euclidean space are denoted, respectively, by $x^T y$ (or $\langle x, y\rangle$) and $\|x\|$, where the superscript ^{T} denotes the transposition.

Vectors in Euclidean spaces are interpreted as columns of real numbers. The notation $x \geq y$ (resp., $x > y$) means that every component of x is greater or equal (resp., greater) the corresponding component of y.

Put $$f_1(x) := \frac{1}{2}x^T A_1 x + a_1^T x + \alpha_1, \quad f_2(x) := \frac{1}{2}x^T A_2 x + a_2^T x + \alpha_2,$$

$$g_i(x) := \frac{1}{2}x^T B_i x + b_i^T x + \beta_i, \;\; i = 1, \dots, m.$$

Denote by G the global optimal solution set of (P) and let

$$\alpha := \inf \big\{ f(x) : x \in \mathcal{F} \big\}.$$

For a symmetric real matrix $A \in \mathbb{R}^{n \times n}$, define

$$Null(A) := \{ v \in \mathbb{R}^n : v^T A v = 0 \}.$$

Let $I = \{1, ..., m\}$, $I_1 = \{ i \in I : \; B_i \neq 0 \}$, and $I_0 = \{ i \in I : \; B_i = 0 \}$.

The *recession cone* (see, for instance, [4, p. 33]) of a nonempty closed convex set S is defined by

$$S^\infty = \{ v \in \mathbb{R}^n : x + tv \in S \;\; \forall x \in S \;\; \forall t \geq 0 \}.$$

From [11, Lemma 1.1] it follows that

$$\mathcal{F}^\infty = \{ v \in \mathbb{R}^n : B_i v = 0, b_i^T v \leq 0, \; \forall i = 1, \dots, m \}. \tag{1}$$

Let S be a nonempty closed set of $\mathbb{R}^n$. Then, S is said to be an *asymptotically linear set* (see [1, Definition 2.3.1]) if for each $\rho > 0$ and for each sequence $\{x^k\}$ satisfying

$$\{x^k\} \subset S, \; \|x^k\| \to +\infty, \; x^k / \|x^k\| \to \bar{x},$$

there exists $k_0 \in \mathbb{N}$ such that $x^k - \rho \bar{x} \in S$ for all $k \geq k_0$. Clearly, every polyhedral set is asymptotically linear, but converse is not true (see after [1, Definition 2.3.2]).

The matrix A is called *copositive* on $\mathcal{F}$ if $v^T A v \geq 0$ for every $v \in \mathcal{F}^\infty$. We say that A is *strict copositive* on $\mathcal{F}$ if $v^T A v > 0$ for every $v \in \mathcal{F}^\infty \setminus \{0\}$.

We recall the notion of upper semicontinuity and lower semicontinuity of multifunctions. A multifunction $F : X \subset \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is said to be *upper semicontinuous* (usc) at $\bar{p} \in \mathbb{R}^s$ if for each open set V containing $F(\bar{p})$ there exists $\delta > 0$ such that $F(p) \subset V$ for every $p \in \mathbb{R}^s$ satisfying $\|p - \bar{p}\| < \delta$. A multifunction $F : X \subset \mathbb{R}^s \rightrightarrows \mathbb{R}^n$ is said to be *lower semicontinuous* (lsc) at $\bar{p} \in \mathbb{R}^n$ if $F(\bar{p}) \neq \emptyset$ and, for each open set V satisfying $F(\bar{p}) \cap V \neq \emptyset$, there exists $\delta > 0$ such that $F(p) \cap V \neq \emptyset$ for every $p \in \mathbb{R}^s$ satisfying $\|p - \bar{p}\| < \delta$. A multifunction F is called *continuous* if it is usc and lsc.

In this paper, we alway assume that $f_2(x) \geq \theta > 0$ for all $x \in \mathcal{F}$.

3. Weakly asymptotically linear set

To study the global solution existence for QP problems over convex sets, Flores-Bazán and Cárcamo [6] extended Frank-Wolfe Theorem [7] for the case, where the constraint set is asymptotically linear. Lara [13] also used this assumption to establish conditions for the existence of minimizers of a quadratic fractional function. However, this class of convex sets is quite narrow although it contains polyhedral sets. To see this, we consider the following well-known closed convex set.

Example 3.1. Let W:=$\{x = (x_1, x_2) \in \mathbb{R}^2 : m(x) := x_1^2 - x_2 \leq 0\}$. For some $\rho > 0$, let a sequence $\{x^k = (k, k^2)\}$ satisfy

$$\{x^k\} \subset W,\ \|x^k\| \to +\infty,\ x^k/\|x^k\| \to \bar{x} = (0, 1).$$

We have $m(x^k - \rho\bar{x}) = k^2 - (k^2 - \rho) = \rho > 0$ for all k. Thus, $x^k - \rho\bar{x} \notin W$ for all k. Therefore, W is not an asymptotically linear set. □

In this section, we present a new concept which is useful to prove the main results.

Definition 3.2. Let S be a nonempty closed set of $\mathbb{R}^n$ and let $K \subset S^\infty$ be a cone. Then, S is said to be a *K-weakly asymptotically linear set* (K-wal set) if for each sequence $\{x^k\} \subset S$ satisfying $\|x^k\| \to +\infty$ and $x^k/\|x^k\| \to \bar{h} \in K$, there exists $\delta > 0$ such that $x^k - t\bar{h} \in S$ for every $t \in (0, \delta)$ for every k large enough.

As a consequence of Definition 3.2, both the intersection of a finite number of K-wal sets and the union of a finite number of K-wal sets are K-wal sets.

Remark 3.3. Every nonempty closed convex set of $\mathbb{R}^n$ is a 0-wal set. So, K has an important role in Definition 3.2. Consider again the set W in Example 3.1. We have $W^\infty = \{(0, a) : a \in \mathbb{R}\}$. Then W is not a W^∞-wal set (Example 3.1) but W is a 0-wal set.

The first main result is established in the following theorem.

Theorem 3.4. *Let $K \subset \mathcal{F}^\infty$ be a cone. If the following assumption holds*

$$(H) \qquad b_i^T h = 0 \text{ for all } h \in K \text{ and for all } i \in I_1,$$

then $\mathcal{F}$ is a K-wal set.

Proof. Let any $\{x^k\} \subset \mathcal{F}$ satisfy $\|x^k\| \to \infty$ as $k \to \infty$, $\|x^k\| \neq 0$ for all k and $\|x^k\|^{-1}x^k \to \bar{h}$ with $\|\bar{h}\| = 1$. By [22, Theorem 8.2], we have $\bar{h} \in \mathcal{F}^\infty \setminus \{0\}$. For each $i \in I$, since $\{g_i(x^k)\}$ is bounded from above, there exists a subsequence $\{k^s\}$ of $\{k\}$ such that all the limits $\lim\limits_{s\to\infty} g_i(x^{k^s})$ exist. Without loss of generality, we can assume that $\{k^s\} \equiv \{k\}$. For each $i \in I_0$, we obtain that

$$b_i^T\bar{h} = \lim_{k\to\infty}\left(b_i^T\frac{x^k}{\|x^k\|} + \frac{\beta_i}{\|x^k\|}\right) = \lim_{k\to\infty}\frac{1}{\|x^k\|}\lim_{k\to\infty}(b_i^T x^k + \beta_i) = 0.$$

By the assumption (H), we have $b_i^T\bar{h} = 0\ \ \forall i \in I_1$. Hence

$$b_i^T\bar{h} = 0\ \ \forall i \in I. \tag{2}$$

For each $i \in I$, for each $t > 0$, from (2) it follows that

$$\begin{aligned} g_i(x^k - t\bar{h}) &= \frac{1}{2}(x^k - t\bar{h})^T B_i(x^k - t\bar{h}) + b_i^T(x^k - t\bar{h}) + \beta_i \\ &= \frac{1}{2}(x^k)^T B_i x^k + b_i^T x^k + \beta_i = g_i(x^k) \leq 0. \end{aligned} \tag{3}$$

This means that $x^k - t\bar{h} \in \mathcal{F}$ for all k. According to Definition 3.2, we conclude that $\mathcal{F}$ is a K-wal set. The proof is complete. □

Example 3.5. Consider the following set:

$$\mathcal{F} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{1}{2}x_1^2 - 2x_3 - 3 \leq 0; x_1 + 5x_2 + 1 \leq 0; -2x_1 + 3 \leq 0\right\}.$$

We have $\mathcal{F} \neq \emptyset$ and $\mathcal{F}^\infty = \{(0, h_2, h_3) : h_2 \leq 0; h_3 \geq 0\}$.

Let $K = \{(0, h_2, 0) : h_2 \leq 0\} \subset \mathcal{F}^\infty$. We have $I_1 = \{1\}$ and

$$b_1^T h = (0, 0, -2)(0, h_2, 0)^T = 0 \ \ \forall h = (h_1, h_2, h_3) \in K.$$

This follows that the condition (H) is satisfied. According to Theorem 3.4, we obtain that $\mathcal{F}$ is a K-wal set. □

4. An Eaves type theorem

In this section, we propose necessary/sufficient conditions for the existence of minimizers of (P). This following main result is an extension of Eaves Theorem [5] from linearly constrained QP to QFP problems.

Theorem 4.1. *Let $\mathcal{F}$ be nonempty. The following statements are valid:*

(a) *For each $x \in \mathcal{F}$, if (H) is satisfied with $K := \mathcal{F}^\infty \cap Null(A_1 - f(x)A_2)$ and if*

$$h^T\big(A_1 - f(x)A_2\big)h \geq 0 \ \ \forall h \in \mathcal{F}^\infty, \tag{4}$$

$$\big(A_1x + a_1 - f(x)(A_2x + a_2)\big)^T h \geq 0 \quad \forall h \in K, \tag{5}$$

then (P) has a solution;

(b) *For each $x \in \mathcal{F}$, if (H) is satisfied with $K := \{h \in \mathcal{F}^\infty : h^T\big(A_1 - f(x)A_2\big)h \leq 0\}$ and if (5) holds then (P) has a solution;*

(c) *If G is nonempty and if there exists $x \in \mathcal{F}$ such that*

$$h^T\big(A_1 - f(x)A_2\big)h > 0 \ \forall h \in \mathcal{F}^\infty \setminus \{0\}, \tag{6}$$

then G is compact;

(d) *If $\alpha > -\infty$ then:*

$$h^T(A_1 - \alpha A_2)h \geq 0 \ \forall h \in \mathcal{F}^\infty; \tag{7}$$

$$\big(A_1x + a_1 - \alpha(A_2x + a_2)\big)^T h \geq 0 \ \ \forall x \in \mathcal{F} \ \ \forall h \in Null(A_1 - \alpha A_2) \cap \mathcal{F}^\infty. \tag{8}$$

Proof. (a) For each $k \geq 1$, let G_k be the global solution set of the following problem

$$\text{minimize} \quad f(x) \quad \text{subject to } \ x \in \mathcal{F}_k := \{x \in \mathcal{F} : \|x\| \leq k\}.$$

Then, G_k is nonempty and closed. Let x^k be the smallest norm element in S_k.

We first show that $\{x^k\}$ is bounded. Indeed, suppose that $\{x^k\}$ is unbounded. Without loss of generality, assume that $\|x^k\| \to \infty$ as $k \to \infty$, $\|x^k\| \neq 0$ for all k and $\|x^k\|^{-1}x^k \to \bar{h}$ with $\|\bar{h}\| = 1$. By [22, Theorem 8.2], we have $\bar{h} \in \mathcal{F}^\infty \setminus \{0\}$. For any $x \in \mathcal{F}$, there exists k_0 such that $x \in \mathcal{F}_k$ for all $k \geq k_0$. Then, one has $f(x^k) \leq f(x)$, i.e.,

$$\frac{\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1}{\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2} \leq f(x).$$

Since $\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2 > 0$, we have

$$\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1 \leq f(x)\left(\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2\right).$$

Multiplying the above inequality by $\|x^k\|^{-2}$ and letting $k \to \infty$, one has

$$\bar{h}^T\big(A_1 - f(x)A_2\big)\bar{h} \le 0. \tag{9}$$

Combining (9) with the assumption (4) yields

$$\bar{h} \in Null\big(A_1 - f(x)A_2\big). \tag{10}$$

By the assumption (5), we obtain

$$\big(\nabla f_1(x) - f(x)\nabla f_2(x)\big)^T\bar{h} \ge 0. \tag{11}$$

According to Theorem 3.4, $\mathcal{F}$ is a K-wal set, that is, there exist k_1 and $\delta > 0$ such that

$$x^k - t\bar{h} \in \mathcal{F} \quad \forall k \ge k_1 \quad \forall t \in (0,\delta). \tag{12}$$

On the other hand, since $\bar{h}^T\bar{h} = 1$ and $\|x^k\|^{-1}x^k \to \bar{h}$, there exists $k_2 \ge k_1$ such that $(x^k)^T\bar{h} > 0$ for all $k \ge k_2$. Consequently, there exists $\gamma > 0$ such that

$$\|x^k - t\bar{h}\|^2 \ = \|x^k\|^2 - 2t(x^k)^T\bar{h} + t^2\|\bar{h}\|^2 < \|x^k\|^2 \le k^2 \quad \forall t \in (0,\gamma). \tag{13}$$

Let $\delta := \min\{\delta, \gamma\}$. From (12) and (13), we obtain that

$$x^k - t\bar{h} \in \mathcal{F}_k \quad \forall k \ge k_1 \quad \forall t \in (0,\delta). \tag{14}$$

By (10), (11), and (14), we have

$$\begin{aligned}
&\big(\nabla f_1(x^k) - f(x^k)\nabla f_2(x^k)\big)^T\bar{h} \ge 0\\
\Leftrightarrow\ & f_1(x^k)\nabla f_2(x^k)^T\bar{h} \le f_2(x^k)\nabla f_1(x^k)^T\bar{h}\\
\Leftrightarrow\ & tf_1(x^k)\nabla f_2(x^k)^T\bar{h} \le tf_2(x^k)\nabla f_1(x^k)^T\bar{h} - \tfrac{1}{2}t^2\bar{h}^T\big(A_1 - f(x^k)A_2\big)\bar{h}\\
\Leftrightarrow\ & f_2(x^k)[\tfrac{1}{2}t^2\bar{h}^TA_1\bar{h} - t(A_1x^k + a_1)\bar{h} + f_1(x^k)]\\
&\qquad\qquad \le f_1(x^k)[\tfrac{1}{2}t^2\bar{h}^TA_2\bar{h} - t(A_2x^k + a_2)\bar{h} + f_2(x^k)]\\
\Leftrightarrow\ & f_2(x^k)f_1(x^k - t\bar{h}) \le f_1(x^k)f_2(x^k - t\bar{h})\\
\Leftrightarrow\ & \frac{f_1(x^k - t\bar{h})}{f_2(x^k - t\bar{h})} \le \frac{f_1(x^k)}{f_2(x^k)} \Leftrightarrow f(x^k - t\bar{h}) \le f(x^k).
\end{aligned} \tag{15}$$

Combining (14) with (15) yields

$$x^k - t\bar{h} \in G_k \quad \forall k \ge k_1, \forall t \in (0,\delta). \tag{16}$$

By (13) and (16), we have $x^k - t\bar{h} \in G_k$ and $\|x^k - t\bar{h}\| < \|x^k\|$ for all $k \ge k_2$, $t \in (0,\delta)$. This contradicts the fact that x^k is the smallest norm element in G_k. Therefore, we conclude that $\{\|x^k\|\}$ must be bounded.

Since $\{\|x^k\|\}$ is bounded, it has a convergent subsequence. Without loss of generality, we can assume that $x^k \to \bar{x}$ as $k \to \infty$. By the fact that $\mathcal{F}$ is closed and $x^k \in \mathcal{F}$, we have $\bar{x} \in \mathcal{F}$. Since $\{x^k\}$ is a minimizing sequence and f is lower semicontinuous, we obtain that

$$f(\bar{x}) \le \liminf_{k\to+\infty} f(x^k) = \inf_{x\in\mathcal{F}} f(x).$$

This leads that $\bar{x}$ is a solution of (P).

(b) Repeating the previous argument and using (5), we obtain (9) and (11). From the assumption (H) with $K = \{h \in \mathcal{F}^\infty : h^T\big(A_1 - f(x)A_2\big)h \leq 0\}$, we have (14). By a similar argument as in part (b), we get the desired conclusion.

(c) Suppose on the contrary that G is unbounded. There then exists a sequence $\{x^k\} \subset G$ such that $\|x^k\| \to \infty$ as $k \to \infty$. Without loss of generality, we may assume that $x^k/\|x^k\| \to \bar{h} \in \mathbb{R}^n \setminus \{0\}$. Since $x^k \in \mathcal{F}$, by [22, Theorem 8.2], we obtain $\bar{h} \in \mathcal{F}^\infty$. For any $x \in \mathcal{F}$, one has $f(x^k) \leq f(x)$, i.e.,

$$\frac{\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1}{\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2} \leq f(x).$$

Since $\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2 > 0$, we have

$$\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1 \leq f(x)\bigg(\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2\bigg).$$

Multiplying the above inequality by $\|x^k\|^{-2}$ and letting $k \to \infty$ yields

$$\bar{h}^T\big(A_1 - f(x)A_2\big)\bar{h} \leq 0.$$

This contradicts the assumption (6). Therefore, G is a nonempty compact set.

(d) Let any $h \in \mathcal{F}^\infty$. Since $x + th \in \mathcal{F}$ for every $t \geq 0$ and for every $x \in \mathcal{F}$, we have $f(x+th) \geq \alpha$ for every $t \geq 0$, that is,

$$f(x+th) = \frac{f_1(x+th)}{f_2(x+th)} \geq \alpha. \tag{17}$$

From the fact that $x + th \in \mathcal{F}$ it follows $f_2(x+th) \geq \theta$. By this and (17), we obtain that $f_1(x+th) \geq \alpha f_2(x+th)$. This leads to

$$\frac{1}{2}t^2 h^T(A_1 - \alpha A_2)h + t(\nabla f_1(x) - \alpha \nabla f_2(x)) + f_1(x) - \alpha f_2(x) \geq 0 \tag{18}$$

for every $t \geq 0$. If $h^T(A_1 - \alpha A_2)h < 0$ then the left side of the inequality (18) tends to $-\infty$ as $t \to +\infty$, a contradiction. Hence, $h^T(A_1 - \alpha A_2)h \geq 0$.

For any $h \in Null(A_1 - f(x)A_2) \cap \mathcal{F}^\infty$, if $\big(\nabla f_1(x) - \alpha \nabla f_2(x)\big)^T h < 0$ then the left side of the inequality (18) tends to $-\infty$ as $t \to +\infty$, a contradiction. Thus,

$$\big(\nabla f_1(x) - \alpha \nabla f_2(x)\big)^T h \geq 0,$$

and the proof is complete. □

To illustrate for Theorem 4.1, we consider the following example.

Example 4.2. Consider the following problem

$$\min\left\{f(x) = \frac{\frac{1}{2}x_3^2 - 3x_1 - 5h_2 - 4x_3 - 7}{-\frac{1}{2}x_1^2 + 2x_1 + 2x_3 + 2} : x \in \mathcal{F}\right\}, \quad \text{where}$$

$$\mathcal{F} = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{1}{2}x_1^2 - 2x_3 - 3 \leq 0; x_1 + 5x_2 + 1 \leq 0; -2x_1 + 3 \leq 0\right\}.$$

Then, $f_2(x) = -(\frac{1}{2}x_1^2 - 2x_3 - 3) - (-2x_1 + 3) + 2 \geq 2 > 0$. For each $x \in \mathcal{F}$, we have

$$\mathcal{F}^\infty = \{(0, h_2, h_3) : h_2 \leq 0; h_3 \geq 0\}$$

and $$Null(A_1 - f(x)A_2) = \{(h_1, h_2, h_3) : f(x)h_1^2 + h_3^2 = 0\}.$$

Hence, $$K = \mathcal{F}^\infty \cap Null(A_1 - f(x)A_2) = \{(0, h_2, 0) : h_2 \leq 0\}.$$

From Example 3.5 it follows that (H) is satisfied. For each $x \in \mathcal{F}$, we obtain that

$$h^T\big(A_1 - f(x)A_2\big)h = h_3^2 \geq 0 \quad \forall h \in \mathcal{F}^\infty,$$

$$\big(\nabla f_1(x) - f(x)\nabla f_2(x)\big)^T h = -5h_2 \geq 0 \quad \forall h \in K.$$

Both (4) and (5) are satisfied. According to Theorem 4.1, this problem has a solution. □

An important corollary of Theorem 4.1 is established as follows.

Corollary 4.3. *Let A_2 be negative semidefinite, that is, $v^T A_2 v \leq 0$ for every $v \in \mathbb{R}^n$, and let $\mathcal{F}$ be nonempty. The following assertions hold:*

(i) *(P) has a solution if (H) is satisfied with $K = \mathcal{F}^\infty \cap Null(A_1)$ and if A_1 is copositive on $\mathcal{F}$ and*

$$\big(A_1x + a_1 - f(x)a_2\big)^T h \geq 0 \ \ \forall x \in \mathcal{F} \ \ \forall h \in K; \tag{19}$$

(ii) *If (H) is satisfied with $K = \{h \in \mathcal{F}^\infty : h^T A_1 h \leq 0\}$ and if (19) holds then (P) has a solution;*

(iii) *If A_1 is strict copositive on $\mathcal{F}$ then G is nonempty compact;*

(iv) *If $\alpha > -\infty$ then A_1 is copositive on $\mathcal{F}$ and*

$$\big(A_1x + a_1 - \alpha a_2\big)^T h \geq 0 \ \ \forall x \in \mathcal{F} \ \ \forall h \in \mathcal{F}^\infty \cap Null(A_1).$$

Proof. Let $\{x^k\} \subset \mathcal{F}$ satisfy $\|x^k\| \to \infty$ as $k \to \infty$, $\|x^k\| \neq 0$ for all k and $\|x^k\|^{-1}x^k \to \bar{h}$ with $\|\bar{h}\| = 1$. Then,

$$\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2 > 0. \tag{20}$$

Multiplying both sides of the inequality (20) by $\|x^k\|^{-2}$ and letting $k \to \infty$, one has $\bar{h}^T A_2 \bar{h} \geq 0$. Since A_2 is negative semidefinite, we have $\bar{h}^T A_2 \bar{h} = 0$. Therefore, $A_2\bar{h} = 0$. According to Theorem 4.1, we get (i), (ii), and (iv).

We shall show the coercivity of f on $\mathcal{F}$. Indeed, suppose that f is not coercive on $\mathcal{F}$. There then exists a sequence $\{x^k\} \subset \mathcal{F}$ such that $\|x^k\| \to \infty$ and $f(x^k) \leq M$ for some $M \in \mathbb{R}$. Without loss of generality, we can assume that $x^k \neq 0$ for all k and $x^k/\|x^k\| \to h$. According to [22, Theorem 8.2], we obtain $h \in \mathcal{F}^\infty$. By the fact that $f(x^k) \leq M$, one has

$$\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1 \leq M\bigg(\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2\bigg).$$

Multiplying the above inequality by $\|x^k\|^{-2}$ and letting $k \to \infty$, one has $\bar{h}^T A_1 \bar{h} \le 0$. This contradicts the assumption. Hence, the coercivity of f is proved. Therefore, $G \neq \emptyset$.

Finally, suppose that there exists a sequence $\{z^k\} \subset G$ such that $\|z^k\| \to \infty$ and $z^k/\|z^k\| \to v \in \mathcal{F}^\infty$. For some $x \in \mathcal{F}$, one gets $f(z^k) \le f(x)$. Dividing the last inequality by $\|z^k\|^2$ and passing $k \to \infty$ yields $v^T A_1 v \le 0$. This contradicts the assumption. Thus, conclusion (iii) follows. □

Remark 4.4. Theorem 4.1 in [13] is a consequence of Corollary 4.3 ((i) and (iv)) for the case where $A_2 = 0$ and $K = \mathcal{F}^\infty$. The following example shows that a QFP problem has a solution by using Corollary 4.3 but Theorem 4.1 in [13] can not be applied.

Example 4.5. Consider the following problem

$$\min\left\{ f(x) = \frac{\frac{1}{2}x_3^2 - 3x_1 - 5x_2 - 4x_3 - 7}{2x_1 - 1} : x \in \mathcal{F} \right\}, \quad \text{where}$$

$$\mathcal{F} = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \frac{1}{2}x_1^2 - 2x_3 \le 0; x_1 + 5x_2 + 1 \le 0; -2x_1 + 3 \le 0 \right\}.$$

This problem has a solution by a similar argument as in Example 4.2.

For some $\rho > 0$, let a sequence $\{x^k = (2k+2, -\frac{2}{5}k - \frac{3}{5}, k^2 + 2k - \frac{1}{2})\}$ satisfy

$$\{x^k\} \subset \mathcal{F},\ \|x^k\| \to +\infty,\ x^k/\|x^k\| \to \bar{x} = (0,0,1).$$

We have $g_1(x^k - \rho\bar{x}) = 2\rho > 0$ for all k. Hence, $x^k - \rho\bar{x} \notin \mathcal{F}$ for all k. This means that $\mathcal{F}$ is not a asymptotically linear set. Therefore, Theorem 4.1 in [13] can not be applied to this problem. □

In the case where $f_2(x) = \alpha_2 > 0$, by Corollary 4.3, we get the following corollary.

Corollary 4.6. *Let $A_2 = 0$, $a_2 = 0$, and $K = \mathcal{F}^\infty \cap Null(A_1)$. Let $\mathcal{F}$ be a K-wal set. The following assertions are equivalent:*

(i) *f_1 is bounded below over $\mathcal{F}$;*

(ii) *A_1 is copositive on $\mathcal{F}$ and $\left(A_1 x + a_1\right)^T h \ge 0 \ \forall x \in \mathcal{F} \ \forall h \in K$;*

(iii) *(P) has a solution.*

Remark 4.7. Corollary 4.6 is an extension of [6, Theorem 4.1] from the asymptotically linear set to the K-wal set.

Remark 4.8. According to Martínez-Legaz et al. [16], a convex set $\mathcal{F}$ is said to be a *Frank-Wolfe set (FW-set)* if every quadratic function f which is bounded below over $\mathcal{F}$ attains its infimum. The authors proved that the Minkowski sum of a convex compact set and a polyhedral cone is a FW-set. This was shown firstly by Kummer [12]. By the argument in Section 3, every FW-set is a 0-wal set. In comparison with Corollary 4.6, $\mathcal{F}$ is a FW-set if it is a $\mathcal{F}^\infty$-wal set. It is easy to check that the Minkowski sum of a convex set S and a polyhedral cone K is a $(S^\infty + K)$-wal set.

5. Application to a class of QFP problems

In this section, we consider QFP problems in the case where A_2 is negative semidefinite. By using the above obtained results, the unboundedness of the global solution set and the continuity of the global solution map are characterized.

5.1. Unboundedness of the global solution set

A sufficient condition for compactness of G has been shown in Corollary 4.3(iii). The following theorem presents a necessary and sufficient conditions for unboundedness of G.

Theorem 5.1. *Assume that $G \neq \emptyset$ and (H) holds. Then, the following three statements are equivalent:*

(i) *G is unbounded;*

(ii) *there exist $\bar{x} \in G$ and $0 \neq \bar{h} \in Null(A_1) \cap \mathcal{F}^\infty$ satisfying $A_2\bar{h} = 0$ and $a_2\bar{h} \geq 0$ such that*
$$\big(A_1\bar{x} + a_1 - \alpha a_2\big)^T \bar{h} = 0; \tag{21}$$

(iii) *G has a solution ray, that is, there exist $\bar{x} \in G$ and $\bar{h} \in \mathbb{R}^n \setminus \{0\}$ such that $\{\bar{x} + t\bar{h} : t \geq 0\} \subset G$.*

Proof. (i)⇒(ii): Suppose that G is unbounded. There then exists a sequence $\{x^k\} \subset G$ such that $\|x^k\| \to \infty$. We can assume, without loss of generality, that $x^k/\|x^k\| \to \bar{h}$ for some $\bar{h} \in \mathbb{R}^n$ and $\|\bar{h}\| = 1$. According to [22, Theorem 8.2], we obtain $\bar{h} \in \mathcal{F}^\infty$. By a similar argument as in Corollary 4.3, we have $a_2^T\bar{h} \geq 0$ and $A_2\bar{h} = 0$. From the fact that $x^k \in G$ for all k it follows that $f(x^k) = \alpha$, i.e.,
$$\frac{\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1}{\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2} = \alpha.$$

Since $\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2 > 0$, we have
$$\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1 = \alpha\left(\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2\right).$$

Multiplying the above inequality by $\|x^k\|^{-2}$ and letting $k \to \infty$, one has
$$\bar{h} \in Null\big(A_1\big). \tag{22}$$

From the assumption (H) and Theorem 3.4, we obtain that for some k_1 there exists $\sigma_1 > 0$ such that $x^{k_1} - t\bar{h} \in \mathcal{F}$ for all $t \in (0, \sigma_1)$. Since $f_2(x^{k_1}) > 0$, there exists $\sigma_2 > 0$ such that $f_2(x^{k_1} - t\bar{h}) > 0$ for every $t \in (0, \sigma_2)$. Let $\sigma = \min\{\sigma_1, \sigma_2\}$. For some $t \in (0, \sigma)$, by (22) and the fact that $x^{k_1} \in G$, we have
$$\begin{aligned}
0 \leq f(x^{k_1} - t\bar{h}) - f(x^{k_1}) &= \frac{f_1(x^{k_1} - t\bar{h})}{f_2(x^{k_1} - t\bar{h})} - \frac{f_1(x^{k_1})}{f_2(x^{k_1})} \\
&= \big[f_2(x^{k_1})f_2(x^{k_1} - t\bar{h})\big]^{-1}\big[f_2(x^{k_1})f_1(x^{k_1} - t\bar{h}) - f_1(x^{k_1})f_2(x^{k_1} - t\bar{h})\big] \\
&= \big[f_2(x^{k_1})f_2(x^{k_1} - t\bar{h})\big]^{-1}\big[f_2(x^{k_1})[\tfrac{1}{2}t^2\bar{h}^T A_1\bar{h} - t(A_1x^{k_1} + a_1)\bar{h} + f_1(x^{k_1}) \\
&\qquad - f_1(x^{k_1})[\tfrac{1}{2}t^2\bar{h}^T A_2\bar{h} - t(A_2x^{k_1} + a_2)\bar{h} + f_2(x^{k_1})]\big]
\end{aligned}$$

$$= \big[f_2(x^{k_1})f_2(x^{k_1} - t\bar{h})\big]^{-1}\big[tf_1(x^{k_1})\nabla f_2(x^{k_1})^T\bar{h} - tf_2(x^{k_1})\nabla f_1(x^{k_1})^T\bar{h}$$
$$+ \tfrac{1}{2}f_2(x^{k_1})t^2\bar{h}^T\big(A_1 - \alpha A_2\big)\bar{h}\big]$$
$$= \big[f_2(x^{k_1} - t\bar{h})\big]^{-1}t\big[\alpha\nabla f_2(x^{k_1})^T\bar{h} - \nabla f_1(x^{k_1})^T\bar{h}\big]. \tag{23}$$

Hence, $\big(\nabla f_1(x^{k_1}) - \alpha\nabla f_2(x^{k_1})\big)^T\bar{h} \le 0.$

On the other hand, applying Corollary 4.3 for $x^{k_1} \in G$ and $\bar{h} \in Null\big(A_1\big)$, we obtain $\big(\nabla f_1(x^{k_1}) - \alpha\nabla f_2(x^{k_1})\big)^T\bar{h} \ge 0$. Thus, $\big(\nabla f_1(x^{k_1}) - \alpha\nabla f_2(x^{k_1})\big)^T\bar{h} = 0$. Letting $\bar{x} = x^{k_1}$, (21) follows.

(ii)⇒(iii): Suppose that there exist $\bar{x} \in G$ and $\bar{h} \in Null(A_1) \cap \mathcal{F}^\infty$ satisfying (21). Since $f_2(\bar{x}) > 0$, for any $t > 0$, we obtain that

$$f_2(\bar{x} + t\bar{h}) = f_2(\bar{x}) + \tfrac{1}{2}t^2\bar{h}^TA_2\bar{h} + t(A_2\bar{x} + a_2)^T\bar{h} > 0.$$

Then, for every $t \ge 0$, we have

$$\begin{aligned} f(\bar{x} + t\bar{h}) - f(\bar{x}) &= \big[f_2(\bar{x} + t\bar{h})f_2(\bar{x})\big]^{-1}\big[f_1(\bar{x} + t\bar{h})f_2(\bar{x}) - f_2(\bar{x} + t\bar{h})f_1(\bar{x})\big] \\ &= \big[f_2(\bar{x} + t\bar{h})f_2(\bar{x})\big]^{-1}\big[tf_2(\bar{x})\nabla f_1(\bar{x})^T\bar{h} - tf_1(\bar{x})\nabla f_2(\bar{x})^T\bar{h} \\ &\qquad + \tfrac{1}{2}f_2(\bar{x})t^2\bar{h}^T\big(A_1 - \alpha A_2\big)\bar{h}\big] \\ &= \big[f_2(\bar{x} + t\bar{h})f_2(\bar{x})\big]^{-1}tf_2(\bar{x})\big[\nabla f_1(\bar{x})^T\bar{h} - \alpha\nabla f_2(\bar{x})^T\bar{h}\big] = 0. \end{aligned}$$

Therefore, $\{\bar{x} + t\bar{h} : t \ge 0\} \subset G$.

(iii)⇒(i): The conclusion is obvious. The proof is complete. □

5.2. Some stability results

Let $\mathbb{R}^{n\times n}_S$ be the space of real symmetric $(n \times n)$–matrices equipped with the matrix norm induced by the vector norm in $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}_{S+}$ be the set of positive semidefinite real symmetric $(n \times n)$–matrices. Let

$$\Gamma := (\mathbb{R}^{n\times n}_S \times \mathbb{R}^n \times \mathbb{R}) \times (\mathbb{R}^{n\times n}_{S+} \times \mathbb{R}^n \times \mathbb{R})^{m+1} \subset \mathbb{R}^s$$

with $s = (m+2)(n^2+n+1)$. Let $u_1 := (A_1, a_1, \alpha_1)$, $u_2 := (A_2, a_2, \alpha_2)$, $u := (u_1, u_2)$, $\omega_i = (B_i, b_i, \beta_i)$ and let $\gamma := (u_1, -u_2, \omega_1, \dots, \omega_m) \in \Gamma$. Consider the following parametric QFP problem

$$(P(\gamma)) \qquad \text{minimize} \quad f(x,u) := \frac{\frac{1}{2}x^TA_1x + a_1^Tx + \alpha_1}{\frac{1}{2}x^TA_2x + a_2^Tx + \alpha_2}$$

$$\text{subject to} \;\; g_i(x, \omega_i) := \tfrac{1}{2}x^TB_ix + b_i^Tx + \beta_i \le 0, \;\; i = 1, \dots, m,$$
$$f_2(x,u) := \tfrac{1}{2}x^TA_2x + a_2^Tx + \alpha_2 \ge \theta > 0,$$

depending on the parametric γ. Put $f_1(x,u) := \frac{1}{2}x^TA_1x + a_1^Tx + \alpha_1$.

The Mangasarian-Fromovitz Constraint Qualification (MFCQ) holds at $\bar{x} \in \mathcal{F}(\gamma)$ if there exists $v^0 \in \mathbb{R}^n$ such that

$$(B_i\bar{x} + b_i)^Tv^0 < 0 \;\; \forall i \in I(\bar{x}, \gamma),$$

where $I(\bar{x}, \gamma) := \{i \in \{1, \dots m\} : g_i(\bar{x}, \omega_i) = 0\}$ is the *active constraint index* set.

We say that the system $g_i(x,\omega_i) \le 0$, $i = 1,\dots,m$, satisfies *Slater's condition* if there exists $x^0 \in \mathbb{R}^n$ such that $g_i(x^0,\omega_i) < 0$ for all $i = 1,\dots,m$. Since B_i, $i = 1,\dots,m$, are positive semidefinite, Slater's condition is equivalent to $(MFCQ)$ (see, for instance, [14, p.47-48]).

The following lemma is well-known (see, for example,[23], [21, Theorem 1] and [2, Theorem 3.1.5]).

Lemma 5.2. (i) *The system $g_i(x,\omega_i) \le 0,\ i = 1,\dots,m$, satisfies (MFCQ) if and only if the set-valued map $\mathcal{F} : \Gamma \rightrightarrows \mathbb{R}^n$, defined by*

$$\mathcal{F}(\tilde{\gamma}) = \{x \in \mathbb{R}^n :\ g_i(x,\tilde{\omega}_i) \le 0,\ i = 1,\dots,m\},$$

is lsc at γ.

(ii) *If the system $g_i(x,\omega_i) \le 0,\ i = 1,\dots,m$, does not satisfy (MFCQ), then there exists $\gamma^k \to \gamma$ such that, for each k, the system $g_i(x,\omega_i^k) \le 0,\ i = 1,\dots,m$, has no solution.*

Lemma 5.3. *The set $\mathcal{Q} := \{\gamma \in \Gamma : h^T A_1 h > 0\ \forall h \in \mathcal{F}^\infty \setminus \{0\}\}$ is open in Γ.*

Proof. If $\mathcal{Q}$ is not open in Γ, there exists $\{\gamma^s\} \subset \Gamma \setminus \mathcal{Q}$ tending to $\gamma \in \mathcal{Q}$ and $h^s \in \mathbb{R}^n$ such that

$$\|h^s\| = 1,\ (h^s)^T A_1 h^s \le 0,\ B_i h^s = 0,\ b_i^T h^s \le 0,\ i = 1,\dots,m.$$

We may assume, without loss of generality, that $\{h^s\}$ itself converges to some $\bar{h} \in \mathbb{R}^n$. Passing limits in the above as $s \to \infty$ yields

$$\|\bar{h}\| = 1,\ \bar{h}^T A_1 \bar{h} \le 0,\ B_i \bar{h} = 0,\ b_i^T \bar{h} \le 0,\ i = 1,\dots,m,$$

contrary to the fact that $\gamma \in \mathcal{Q}$. Therefore, $\mathcal{Q}$ is open. □

The continuity of $G(\cdot)$ is characterized as follows.

Theorem 5.4. *Let (MFCQ) be satisfied at γ and A_1 be strict copositive on $\mathcal{F}$. Then, the following assertions hold:*

(i) *For each sequence $\{\gamma^k\} \subset \Gamma$ converging to γ and $x^k \in G(\gamma^k)$, there exists a sequence $\{x^{k_s}\} \subset \{x^k\}$ such that $x^{k_s} \to \bar{x}$ for some $\bar{x} \in G(\gamma)$;*

(ii) *$G(\cdot)$ is usc at γ;*

(iii) *$G(\cdot)$ is continuous at γ if $|G(\gamma)| = 1$.*

Proof. (i) Let any sequence $\{\gamma^k\} \subset \Gamma$ tend to γ and let $x^k \in G(\gamma^k)$. For any $x \in \mathcal{F}(\gamma)$, by Lemma 5.2, there exists $y^k \in \mathcal{F}(\gamma^k)$ such that $y^k \to x$. If $\{x^k\}$ is unbounded, we then may assume that $\|x^k\| \to \infty$ and $x^k/\|x^k\| \to \bar{h}$ as $k \to \infty$ for some $\bar{h} \in \mathbb{R}^n$. According to [22, Theorem 8.2], we obtain $\bar{h} \in \mathcal{F}(\gamma)^\infty$. Since $x^k \in G(\gamma^k)$, one has $f(x^k) \le f(y^k)$. This follows that

$$\frac{1}{2}(x^k)^T A_1 x^k + a_1^T x^k + \alpha_1 \le f(y^k)\Big(\frac{1}{2}(x^k)^T A_2 x^k + a_2^T x^k + \alpha_2\Big).$$

Dividing both sides of the above inequality by $\|x^k\|^2$ and letting $k \to \infty$, we get $\bar{h}^T A_1 \bar{h} \le 0$, contrary to the assumption. Therefore there exists a subsequence

$\{x^{k_j}\} \subset \{x^k\}$ such that $x^{k_j} \to \bar{x} \in \mathcal{F}$ and $f(x^{k_j}) \leq f(y^{k_j})$. Take limits in the above inequalities as $j \to \infty$, we get $f(\bar{x}) \leq f(x)$, that is, $\bar{x} \in G(\gamma)$.

(ii): Suppose on the contrary that there exists an open set U containing $G(\gamma) \neq \emptyset$, a sequence $\{\gamma^k\}$ tending to γ, and a sequence $\{x^k\}$ satisfying $x^k \in G(\gamma^k) \setminus U$ for every $k \in N$. If $\{x^k\}$ is bounded, we can assume that $x^k \to \hat{x}$ for some $\hat{x} \in \mathcal{F}(\gamma)$. By Lemma 5.2, for any $x \in \mathcal{F}(\gamma)$, there exists $z^k \in \mathcal{F}(\gamma^k)$ satisfying $\lim_{k\to\infty} z^k = x$. Since $x^k \in G(\gamma^k)$, we have $f(x^k) \leq f(z^k)$. Letting $k \to \infty$, we get $f(\hat{x}) \leq f(x)$. This follows $\hat{x} \in G(\gamma) \subset U$. Since $x^k \notin U$ for all k and U is open, we have $\hat{x} \notin U$, a contradiction. Therefore, $\{x^k\}$ is unbounded.

Without loss of the generality, assume that $\|x^k\|^{-1}x^k \to \bar{h} \in \mathcal{F}(\gamma)^\infty$. By Lemma 5.2, for any $x \in \mathcal{F}(\gamma)$, there exists $z^k \in \mathcal{F}(\gamma^k)$ satisfying $\lim_{k\to\infty} z^k = x$. Since $x^k \in G(\gamma^k)$, we have $f(x^k) \leq f(z^k)$. This implies $f_1(x^k) \leq f_2(z^k)f(z^k)$. Dividing both sides of the above inequality by $\|x^k\|^2$ and letting $k \to \infty$, we get $\bar{h}^T A_1 \bar{h} \leq 0$. This contradicts the assumption.

(iii): Suppose that $|G(\gamma)| = \bar{x}$ and U is an open set containing $\bar{x}$. According to Lemma 5.2, there exists $\rho_1 > 0$ such that we have $\mathcal{F}(\tilde{\gamma}) \neq \emptyset$ for every $\tilde{\gamma}$ satisfying $\|\tilde{\gamma} - \gamma\| < \rho_1$. By the assumption that A_1 is strict copositive on $\mathcal{F}$ and by Lemma 5.3, there exists $\rho_2 > 0$ such that $h^T \widetilde{\mathcal{A}} h > 0\ \forall h \in \mathcal{F}(\tilde{\gamma})^\infty \setminus \{0\}$ for every $\tilde{\gamma}$ satisfying $\|\tilde{\gamma} - \gamma\| \leq \rho_2$. Let $\rho := \min\{\rho_1, \rho_2\}$. From Corollary 4.3 it follows that $G(\tilde{\gamma}) \neq \emptyset$ for all $\tilde{\gamma}$ satisfying $\|\tilde{\gamma} - \gamma\| < \rho$.

By the assumption and part (ii), we obtain that $G(\cdot)$ is usc at γ, that is, $G(\tilde{\gamma}) \subset U$ for every $\tilde{\gamma}$ satisfying $\|\tilde{\gamma} - \gamma\| < \rho$ with $\rho > 0$ being small enough. This implies that $G(\tilde{\gamma}) \cap U \neq \emptyset$ for every $\tilde{\gamma}$ satisfying $\|\tilde{\gamma} - \gamma\| < \rho$. Hence, $G(\cdot)$ is lsc at γ. Therefore, the conclusion follows. The proof is complete. □

6. Conclusions

In this paper, we have given a new concept, K-wal set, and showed that, under a checked condition (H), the constraint set $\mathcal{F}$ is a K-wal set (Theorem 3.4). Due to this result, an Eaves type theorem for QFP problems has been proposed (Theorem 4.1). To our knowledge, this is the first time to establish an Eaves type theorem for generalized quadratically constrained quadratic ratio optimization problems. Our results can be used in some cases where the existing results cannot be used. By using obtained results, we characterized the unboundedness of the global solution set and semicontinuity of the global solution map for a class of QFP problems (Theorems 5.1 and 5.4).

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