

Structure of Sets of Strong Subdifferentiability in Dual L^1 -Spaces

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We analyse the structure of finite dimensional subspaces of the set of points of strong subdifferentiability in a dual space. In a dual $L_1(\mu)$ space, such a subspace is in the discrete part of the Yoshida-Hewitt type decomposition. In this set up, any Banach space consisting of points of strong subdifferentiability is necessarily finite dimensional. Our results also lead to streamlined and new proofs of results from the study of strong proximality for subspaces of finite co-dimension in a Banach space.

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1. Introduction

Let X be a real Banach space. We recall from [4] that a vector $x \in X$ is said to be a point of strong subdifferentiability (in short, SSD-point) if the one sided limit $\lim_{t \rightarrow 0^+} \frac{\|x+th\| - \|x\|}{t}$ exists uniformly for $h \in B_X$, where B_X denotes the closed unit ball of the Banach space X . If $x^* \in X^*$ is an SSD-point, it was shown in [4] that x^* attains its norm.

To see the connection with proximality, let $Y \subset X$ be a closed subspace of X . For $x \in X$, let $P_Y(x) = \{y \in Y : d(x, Y) = \|x - y\|\}$. An element of $P_Y(x)$ is called a best approximation to x from Y . If $P_Y(x) \neq \emptyset$ for all $x \notin Y$, then Y is said to be a proximal subspace of X . It is well known that $x^* \in X^*$ attains its norm if and only if $Y = \ker(x^*)$ is a proximal subspace of X .

In [14], L. P. Vlasov introduced the notion of an H-set which is stronger than the notion of proximality. Later in [5], G. Godefroy and V. Indumathi introduced an equivalent notion of H-set, namely strong proximality. We now give another equivalent definition of strongly proximal subspace.

Definition 1.1. A proximal subspace Y of a Banach space X is said to be *strongly proximal* in X if, for any $x \notin Y$ and for any sequence $\{y_n\}_{n \geq 1}$ in Y with $d(x, Y) = \lim_n \|x - y_n\|$ (such a sequence is called a *minimizing sequence* for x in Y), there is a subsequence $\{y_{n_k}\}$ and a sequence $\{z_k\}_{k \geq 1}$ in $P_Y(x)$ such that $\|y_{n_k} - z_k\| \rightarrow 0$. $\square$

Our main tool in the investigation of the structure of sets of SSD-points is an astonishing geometric statement from [5] that in a dual space X^* , a non-zero vector x^* in X^* is an SSD-point of X^* if and only if $\ker(x^*)$ is a strongly proximal subspace of X . Note that the definition of an SSD-point makes no reference to the predual X . Thus a lot of information about SSD-points in a dual space can be gleaned by choosing an appropriate predual of X^* . Using our analysis on SSD-points, we give one of the most efficient proofs of the proximality and strong proximality of M -ideals in a Banach space without using the Bishop-Phelps theorem (as was done in [7]) but only using simple geometry of SSD-points in a dual space.

If $L_1(\mu)$ is a dual space and F is a finite dimensional subspace of $L_1(\mu)$ consisting of SSD-points of $L_1(\mu)$, then there is a positive integer k such that $F \subset \ell_1(k)$ and $L_1(\mu) = \ell_1(k) \oplus_1 N$ for some closed subspace N , where $\ell_1(k)$ is the finite dimensional space $\mathbb{R}^k$ with ℓ_1 -norm. Moreover, we prove that any Banach space consisting of SSD-points of a dual $L_1(\mu)$ space is finite dimensional.

As an application we show that if X is a Banach space such that X^* is isometric to $L_1(\mu)$ for some positive measure μ (such a Banach space X is called an L_1 -predual space), then in the class of subspaces of finite co-dimension, being strongly proximal is transitive and is preserved under finite intersections.

We also show that the property of ‘finite intersection of strongly proximal hyperplanes being strongly proximal’ is preserved under c_0 -direct sums.

Our approach also leads to several streamlined and some new proofs from the study of strong proximality for subspaces of finite co-dimension (see [2, 5, 7, 9]).

In this article, we consider only Banach spaces over the real field $\mathbb{R}$ and all subspaces we consider are assumed to be closed. We denote the unit sphere of a Banach space X by S_X . For an arbitrary collection $\{X_i : i \in I\}$ of Banach spaces, $\bigoplus_{i \in I} X_i$ denotes

the ℓ_1 -direct sum equipped with the ℓ_1 -norm, and $\bigoplus_{i \in I}^{c_0} X_i$ and $\bigoplus_{i \in I}^{\infty} X_i$ denote the

c_0 -direct sum and ℓ_∞ -direct sum of $\{X_i : i \in I\}$ equipped with supremum norm respectively. When there is no confusion, we omit writing the index set for the direct sums.

2. Strong subdifferentiability and strong proximality

Since finite dimensional subspaces of X^* correspond precisely to finite co-dimensional subspaces of X , we first analyse the structure of finite co-dimensional strongly proximal subspaces of X . It is easy to see that any finite dimensional subspace is a strongly proximal subspace. If Y is a proximal subspace of X and F is a finite dimensional subspace of X , then, by [3, Corollary], we have that $F + Y$ is a proximal subspace of X (see [11] for more details). Now the following theorem from [12] gives an analogous result for strong proximality. As this is our principle tool, we reproduce the proof, for the sake of completeness.

Theorem 2.1. *Let F be a finite dimensional subspace of a Banach space X and Y be a strongly proximal subspace of X . Then $F + Y$ is a strongly proximal subspace of X .*

Proof. We already have that $F + Y$ is a proximal subspace of X . Now let $x \in X$. Let $\{f_n + y_n\}_{n \geq 1} \subset F + Y$ be a sequence such that $d(x, F + Y) = \lim_n \|x - f_n - y_n\|$, where $\{f_n\}_{n \geq 1} \subset F$ and $\{y_n\}_{n \geq 1} \subset Y$. Since any minimizing sequence is bounded, and as $F + Y$ is closed, by [13, Theorem 5.20], we may assume without loss of generality that both $\{f_n\}_{n \geq 1}$ and $\{y_n\}_{n \geq 1}$ are bounded sequences. As F is finite dimensional, by going through a subsequence, if necessary, we may and do assume that $f_n \rightarrow f_0$ for some $f_0 \in F$. We next claim that

$$d(x - f_0, Y) = d(x, F + Y) = \lim_n \|x - f_0 - y_n\|.$$

Clearly $|\|x - f_n - y_n\| - \|x - f_0 - y_n\|| \leq \|f_n - f_0\|$ so that

$$\lim_n \|x - f_n - y_n\| = d(x, F + Y) = \lim_n \|x - f_0 - y_n\| \geq d(x - f_0, Y).$$

We always have $d(x, F + Y) \leq d(x - f_0, Y)$. Hence the claim. Since Y is strongly proximal in X , we have a subsequence $\{y_{n_k}\}$ and a sequence $\{z_k\} \subset P_Y(x - f_0)$ such that $\|y_{n_k} - z_k\| \rightarrow 0$.

If $z \in P_Y(x - f_0)$, we have $\|x - f_0 - z\| = d(x - f_0, Y) = d(x, F + Y)$, so that $f_0 + z \in P_{F+Y}(x)$.

Now $\{f_0 + z_k\} \subset P_{F+Y}(x)$ and $\|f_{n_k} + y_{n_k} - (f_0 + z_k)\| \rightarrow 0$. Thus $F + Y$ is a strongly proximal subspace of X . $\square$

Corollary 2.2. *Let $f_1, \dots, f_n \in X^*$ and suppose that $Y = \bigcap_{i=1}^n \ker(f_i)$ is a strongly proximal subspace of X . Then, for any $g_1, \dots, g_k \in Y^\perp = \text{span}\{f_1, \dots, f_n\}$, $Z = \bigcap_{i=1}^k \ker(g_i)$ is a strongly proximal subspace of X .*

Proof. Since $Y \subset Z \subset X$ and Y is of finite co-dimension in X , we have that $Z = Y + F$ for some finite dimensional subspace F of X . Thus the conclusion follows from Theorem 2.1. $\square$

Even though the next result follows from [5, Theorem 2.5], we give an alternate proof of it using Theorem 2.1.

Corollary 2.3. *Let Z and Y be subspaces of a Banach space X such that $Z \subset Y \subset X$. If Z is a finite co-dimensional strongly proximal subspace of X , then Y is strongly proximal in X .*

Proof. Since Z is finite co-dimensional in Y , we have $Y = Z + F$ for some finite dimensional subspace F of X . As Z is strongly proximal in X and F is finite dimensional, by Theorem 2.1, Y is strongly proximal in X . $\square$

We now prove the following result from [5] by using Theorem 2.1.

Corollary 2.4. *Let Y be a finite co-dimensional strongly proximal subspace of a Banach space X . Then $Y^\perp \subset \{x^* \in X^* : x^* \text{ is an SSD-point of } X^*\}$.*

Proof. Let Y be a finite co-dimensional strongly proximal subspace of X . Let $f \in Y^\perp$. Then $Y \subset \ker(f) \subset X$. Since Y is finite co-dimensional in X , we have $\ker(f) = Y + G$ for a finite dimensional subspace G of X . Then, by Theorem 2.1, $\ker(f)$ is strongly proximal in X and hence, by [5], f is an SSD-point of X^* . Thus $Y^\perp \subset \{x^* \in X^* : x^* \text{ is an SSD-point of } X^*\}$. $\square$

We now recall the notion of an M -ideal in a Banach space which is known to be a stronger notion than strong proximality.

Definition 2.5. ([6]) Let X be a Banach space. A linear projection P on X is said to be an L -projection if $\|x\| = \|Px\| + \|x - Px\|$ for all $x \in X$. A subspace Y of X is said to be an L -summand in X if it is the range of an L -projection. A subspace Y of X is said to be an M -ideal in X if $Y^\perp$ is an L -summand in X^* .

In what follows, we need the following result from [6, Chapter I, Proposition 1.17]. We include a simple proof of this result for the sake of completeness.

Proposition 2.6. *Let Z and Y be subspaces of a Banach space X with $Z \subset Y \subset X$. If Z is an M -ideal in X , then Z is an M -ideal in Y .*

Proof. Let $P : X^* \rightarrow X^*$ be an L -projection such that $\ker(P) = Z^\perp$. It is easy to see that the projection $\tilde{P} : Y^* \rightarrow Y^*$ defined by $\tilde{P}(g) = P(f)|_Y$ is an L -projection with $\ker(\tilde{P}) = Z^\perp$, where f is the Hahn-Banach extension of g to X and $Z^\perp$ is the annihilator of Z in Y^* . Thus Z is an M -ideal in Y . $\square$

We next give a self contained proof of a classical result without invoking intersection properties of balls, as is usually done.

Theorem 2.7. *Let Y be an M -ideal in a Banach space X . Then Y is strongly proximal in X .*

Proof. For all $x \in X$, by Proposition 2.6, Y is an M -ideal in $\text{span}\{Y \cup \{x\}\}$. Since strong proximality of Y in $\text{span}\{Y \cup \{x\}\}$ for all $x \in X$ implies the strong proximality of Y in X , we assume without loss of generality that Y is an M -ideal of co-dimension one in X .

Let $Y = \ker(f)$ for some $f \in S_{X^*}$. Then $\text{span}(f) = Y^\perp$ is an L -summand in X^* . Since f is an SSD-point of $\text{span}(f)$ and $\text{span}(f)$ is an L -summand in X^* , by [4, Proposition 2.1], f is an SSD-point of X^* . Thus $Y = \ker(f)$ is strongly proximal in X . $\square$

The following result characterizes SSD-points of duals of L_1 -predual spaces. For a compact Hausdorff space K , we denote by $C(K)$ the space of continuous functions on K equipped with the supremum norm.

Theorem 2.8. *Let X be an L_1 -predual space and $f \in S_{X^*}$. Then f is an SSD-point of X^* if and only if $f = \sum_{i=1}^n \alpha_i f_i$ for some scalars α_i and for some distinct $f_1, \dots, f_n \in S_{X^*}$ such that $\text{span}(f_i)$ is an L -summand in X^* for all i .*

Proof. Suppose $f = \sum_{i=1}^n \alpha_i f_i$ for some distinct $f_1, \dots, f_n \in S_{X^*}$ such that $\text{span}(f_i)$ is an L -summand in X^* for all i . Then $\ker(f_i)$ is an M -ideal in X and hence, by [6, Chapter I, Proposition 1.11], $\bigcap_{i=1}^n \ker(f_i)$ is an M -ideal in X . Since we have $\bigcap_{i=1}^n \ker(f_i) \subset \ker(f) \subset X$ by Theorem 2.7 and Corollary 2.3, $\ker(f)$ is strongly proximal in X and hence f is an SSD-point of X^* .

Now suppose that f is an SSD-point of X^* . Since X is an L_1 -predual space, by [10, Section 21, Theorem 6], X^{**} is isometric to $C(K)$ for some compact Hausdorff space K and by [6, Chapter IV, Example 1.1], X^* is an L -summand in X^{***} . Thus f is an SSD-point of $X^{***} = C(K)^*$. Then, by [2, Theorem 2.1], f is a finitely supported measure on K . So we may assume without loss of generality that $f = \sum_{i=1}^n \alpha_i \phi_i$ for $\alpha_i \in [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$ and for $\phi_i \in X^{***}$ with $\text{span}(\phi_i)$ is an L -summand in X^{***} for all i .

Now we note that for an L -summand U in a Banach space V , if $u \in S_U$ is such that $u = \frac{v_1+v_2}{2}$ for unit vectors $v_1, v_2 \in V$, then $v_1, v_2 \in U$. For, if P is an L -projection from V onto U . Then $u = P(u) = \frac{P(v_1)+P(v_2)}{2}$. Thus $\|P(v_1)\| = \|P(v_2)\| = \|v_1\| = \|v_2\| = 1$. Since P is an L -projection, we get $P(v_i) = v_i$ for all i and hence $v_1, v_2 \in U$.

Since X^* is an L -summand in X^{***} , by the above argument, we get $\phi_i \in X^*$ for all i . So if we let $f_i = \phi_i$ for all i , then $f_1, \dots, f_n$ satisfy the required properties. $\square$

The next set of corollaries are easy to prove.

Corollary 2.9. *Let X be an L_1 -predual space. Then the set of SSD-points of X^* is a subspace.*

Corollary 2.10. *Let X be an L_1 -predual space and $F \subset X^*$ be a finite dimensional subspace. Then $F \subset \{x^* \in X^* : x^* \text{ is an SSD-point of } X^*\}$ if and only if $F \subset \text{span}\{x^* \in X^* : \text{span}(x^*) \text{ is an } L\text{-summand in } X^*\}$.*

The following proposition illustrates our remarks about using different preduals. For $C(K)$ spaces with K being a dispersed compact Hausdorff space, we give a simple proof of [2, Theorem 2.1].

Proposition 2.11. *Let K be a dispersed compact Hausdorff space and let $\mu \in C(K)^*$ be a unit vector. Then μ is an SSD-point of $C(K)^*$ if and only if support of μ is a finite set.*

Proof. Suppose μ is an SSD-point of $C(K)^*$. Since K is a dispersed compact Hausdorff space, we know that $C(K)^*$ is isometric to $\ell_1(\Gamma)$ for a discrete set Γ (see Sections 5 and 18 of [10]). On the other hand if $c_0(\Gamma)$ denotes the space of functions on Γ vanishing at ∞ , then $c_0(\Gamma)^* = \ell_1(\Gamma)$. Since $\ker(\mu) \subset c_0(\Gamma)$ is now a proximal subspace, we have that μ attains its norm. It is now easy to see that μ has only finitely many non-zero coordinates. Thus μ is finitely supported as an element of $C(K)^*$.

Conversely suppose support of μ is a finite set. Then $\mu = \sum_{i=1}^n \alpha_i \delta_{k_i}$ for scalars $\alpha_1, \dots, \alpha_n$ and $k_1, \dots, k_n \in K$. Now, by [6, Chapter I, Example 1.4(a)], $\ker(\delta_{k_i}) = \{f \in C(K) : f(k_i) = 0\}$ is an M -ideal in $C(K)$. Then, by Theorem 2.7, $\ker(\delta_{k_i})$ is strongly proximal in $C(K)$ and hence δ_{k_i} is an SSD-point of $C(K)^*$ for all i . Then, by Corollary 2.9, μ is an SSD-point of $C(K)^*$ $\square$

The next result gives the structure of finite dimensional subspaces of $L_1(\mu)$ when $L_1(\mu)$ is a dual space.

Proposition 2.12. *Let $L_1(\mu)$ be a dual space and F be a finite dimensional subspace of $L_1(\mu)$. Then F consists of SSD-points of $L_1(\mu)$ if and only if there exists a positive integer k such that $F \subset \ell_1(k)$ and $L_1(\mu) = \ell_1(k) \oplus_1 N$ for a subspace N of $L_1(\mu)$.*

Proof. Suppose $F \subset \ell_1(k)$ and $L_1(\mu) = \ell_1(k) \oplus_1 N$. Since, for the canonical basis $\{e_1, \dots, e_k\}$ of $\ell_1(k)$, $\text{span}(e_i)$ is an L -summand in $L_1(\mu)$, by Theorem 2.8, we get that F consists of SSD-points of $L_1(\mu)$.

Conversely suppose that F consists of SSD-points of $L_1(\mu)$. Then, again by Theorem 2.8, there exists a positive integer k such that $F \subset \text{span}\{f_1, \dots, f_k\}$, where the f_i 's are linearly independent and $\text{span}(f_i)$ is an L -summand in $L_1(\mu)$. Clearly $\text{span}\{f_1, \dots, f_k\}$ is isometric to $\ell_1(k)$.

Since $\text{span}\{f_1, \dots, f_k\} = \text{span}(f_1) + \dots + \text{span}(f_k)$ and the sum of finitely many L -summands is again an L -summand, it follows that $\text{span}\{f_1, \dots, f_k\}$ is an L -summand of $L_1(\mu)$. Thus $L_1(\mu) = \ell_1(k) \oplus_1 N$ for a subspace N of $L_1(\mu)$. $\square$

Corollary 2.13. *Let $L_1(\mu)$ be a dual space and let B be a Banach space consisting of SSD-points of $L_1(\mu)$. Then B is a finite dimensional space.*

Proof. By Theorem 2.8, $B \subset \text{span}\{f \in L_1(\mu) : \text{span}(f) \text{ is an } L\text{-summand in } L_1(\mu)\}$. Since B is a Banach space, by an application of the Baire category theorem, if B is infinite dimensional, B contains an infinite combination $\sum_1^\infty \alpha_i f_i$ where $\text{span}(f_i)$ is an L -summand for all i . But such a combination is not an SSD-point. Thus B is a finite dimensional space. $\square$

As another application of Theorem 2.8, we prove the following result from [9, Proposition 3.20].

Theorem 2.14. *Let X be an L_1 -predual space and Y be a finite co-dimensional subspace of X . Then Y is strongly proximal in X if and only if $Y^\perp \subset \{x^* \in X^* : x^* \text{ is an SSD-point of } X^*\}$.*

Proof. Suppose $Y^\perp \subset \{x^* \in X^* : x^* \text{ is an SSD-point of } X^*\}$. Then, by Theorem 2.8, $Y^\perp \subset \text{span}\{x^* \in X^* : \text{span}(x^*) \text{ is an } L\text{-summand in } X^*\}$. Since $Y^\perp$ is finite dimensional, there exists $g_1, \dots, g_k \in X^*$ such that $\text{span}(g_i)$ is an L -summand in X^* and $\bigcap_{i=1}^k \ker(g_i) \subset Y \subset X$. Since $\bigcap_{i=1}^k \ker(g_i)$ is an M -ideal in X , the strong proximality of Y in X follows from Theorem 2.7 and Corollary 2.3.

The other implication follows from Corollary 2.4. $\square$

To indicate the technique of non-unique preduals, we briefly indicate an alternate proof of the above result in the case of separable Banach spaces.

Proof. Let X be a separable Banach space and $X^* = L_1(\mu)$ for some positive measure μ .

If X^* is also separable, then it is easy to see that the set of extreme points of the unit ball of X^* is a countable discrete set (for any two independent extreme points x_1^* and x_2^* , as $\text{span}\{x_1^*\}$ and $\text{span}\{x_2^*\}$ are L -summands with $\text{span}\{x_1^*\} \cap \text{span}\{x_2^*\} = \{0\}$, $\|x_1^* \pm x_2^*\| = 2$). Thus, by [10, Section 22, p. 226, Theorem 5], $X^* = \ell_1$. If c denotes the space of convergent sequences with supremum norm, then $c^* = \ell_1$.

If X^* is not separable, then the set of extreme points of the unit ball of X^* is uncountable. Therefore, again by [10, Section 22, p. 226, Theorem 5], $X^* = C([0, 1])^*$.

In either case X^* is the dual of a $C(K)$ space for some compact Hausdorff space. Therefore since SSD-points are independent of the predual, the conclusion follows from [2, Corollary 2.3]. $\square$

Remark 2.15. It may be recalled that a long standing open problem in the isometric theory is that given an L_1 -predual space X , is there a $C(K)$ space such that $C(K)^* = X^*$.

Now using these ideas related to SSD-points, we give a simple proof of [9, Theorem 3.10] in the case of L_1 -predual spaces.

Corollary 2.16. *Let X be an L_1 -predual space and Y be a finite co-dimensional subspace of X . Then Y is strongly proximal in X if and only if $Y^{\perp\perp}$ is strongly proximal in X^{**} .*

Proof. Suppose Y is strongly proximal in X . Assume that $Y = \bigcap_{i=1}^n \ker(f_i)$ for $f_1, \dots, f_n \in X^*$. Then $Y^\perp = \text{span}\{f_i : i = 1, \dots, n\}$ is contained in set of all SSD-points of X^* . Since X^* is an L -summand in X^{***} , $Y^\perp$ is contained in set all SSD-points of X^{***} . Since $Y^{\perp\perp} = \bigcap_{i=1}^n \ker(f_i)$ in $X^{**} = C(K)$ for some compact Hausdorff space K and $Y^{\perp\perp\perp} = Y^\perp$ is contained in the set of all SSD-points of X^{***} , by Theorem 2.14, $Y^{\perp\perp}$ is strongly proximal in X^{**} .

Suppose $Y^{\perp\perp}$ is strongly proximal in X^{**} . Again as above, let $Y = \bigcap_{i=1}^n \ker(f_i)$ for $f_1, \dots, f_n \in X^*$. Then $Y^\perp = \text{span}\{f_i : i = 1, \dots, n\}$ in X^* . Since $Y^{\perp\perp} = \bigcap_{i=1}^n \ker(f_i)$ is strongly proximal in $X^{**} = C(K)$ for some compact Hausdorff space K , $Y^{\perp\perp\perp}$ is contained in set all of all SSD-points of X^{***} . Now as $Y^\perp$ is finite dimensional, $Y^\perp = Y^{\perp\perp\perp}$ and hence $Y^\perp$ is contained in set all SSD-points of X^{***} . Since $Y^\perp \subset X^*$, it follows that $Y^\perp$ is contained in set all SSD-points of X^* . Then, by Theorem 2.14, Y is strongly proximal in X . $\square$

Our next lemma is motivated by the proof of [2, Proposition 3.8].

Lemma 2.17. *Let X be an L_1 -predual space and Y be a finite co-dimensional strongly proximal subspace of X . If $\phi \in X^*$ is such that $\phi|_Y$ is a non-zero SSD-point of Y^* , then ϕ is an SSD-point of X^* .*

Proof. Let $F = \phi|_Y$ and F be an SSD-point of Y^* . Suppose $\{\varphi_1, \dots, \varphi_n\}$ is a basis of $Y^\perp$. Then, by Theorem 2.8, φ_i is an SSD-point of X^* for all i . Then for every i , there exist $f_{i1}, \dots, f_{in_i} \in X^*$ and scalars $\alpha_{i1}, \dots, \alpha_{in_i}$ such that $\varphi_i = \sum_{j=1}^{n_i} \alpha_{ij} f_{ij}$ and $\text{span}\{f_{ij}\}$ is an L -summand in X^* for all j . Thus $\ker(f_{ij})$ is an M -ideal in X for all i and j . Therefore $J = \bigcap_{i,j} \ker(f_{ij})$ is an M -ideal in X . Thus $X^* = J^\perp \oplus_1 J^\#$, where $J^\# = \{x^* \in X^* : \|x^*\| = \|x^*|_J\|\}$ and $J^\#$ is isometric to J^* (see [6, Chapter I, Proof of Proposition 1.12]). Since $J \subset Y \subset X$, by Proposition 2.6, J is also an M -ideal in Y . Therefore $Y^* = J_1^\perp \oplus_1 J_1^\#$, where $J_1^\perp = \{y^* \in Y^* : y^*|_J = 0\}$ and $J_1^\# = \{y^* \in Y^* : \|y^*\| = \|y^*|_J\|\}$. Observe that $J_1^\#$ is isometric to J^* as J is an M -ideal in Y .

Let $F = F_1 + F_2$ for $F_1 \in J_1^\perp$ and $F_2 \in J_1^\#$. Then, by [2, Lemma 3.7], F_1 and F_2 are SSD-points of $J_1^\perp$ and $J_1^\#$ respectively. Since F_2 is an SSD-point of $J_1^\#$, $F_2|_J$ attains its norm on J .

Let $\phi = \phi_1 + \phi_2$ for $\phi_1 \in J^\perp$ and $\phi_2 \in J^\#$. Since $\phi_1 \in \text{span}_{i,j}\{f_{ij}\}$, by Theorem 2.8, ϕ_1 is an SSD-point of $J^\perp$. It remains to show that ϕ_2 is an SSD-point of $J_1^\#$.

If ϕ_2 is not an SSD-point of $J_1^\#$, then $\phi_2|_J$ is not an SSD-point of J^* (see [6, Chapter I, Proof of Proposition 1.12]). Then, by [5, Lemma 1.1] there exists an $\varepsilon > 0$ and a sequence (h_n) of unit vectors from J such that $\phi_2(h_n) \rightarrow \|\phi_2\|$ and

$$d(h_n, \{x \in J : \|x\| = 1 \text{ and } \phi_2(x) = \|\phi_2\|\}) \geq \varepsilon.$$

Clearly $\phi_2(h) = \phi(h) = F_2(h) = F(h)$ for all $h \in J$. Hence

$$\lim_n F_2(h_n) = \lim_n \phi_2(h_n) = \|\phi_2\| = \|\phi_2|_J\| = \|F_2|_J\| = \|F_2\|$$

and $d(h_n, \{x \in J : \|x\| = 1 \text{ and } F_2(x) = \|F_2\|\}) \geq \varepsilon$. Thus, by [5, Lemma 1.1], $F_2|_J$ is not an SSD-point of J^* . This contradicts the fact that F_2 is an SSD-point of $J_1^\#$. Thus ϕ_2 is an SSD-point of $J^\#$ and, by [2, Lemma 3.7], ϕ is an SSD-point of X^* . $\square$

From Theorem 2.14, it follows that if X is an L_1 -predual space and $F \subset X^*$ is a finite dimensional subspace consisting of SSD-points of X^* , then the pre-annihilator $Z = \bigcap_{f \in F} \ker(f)$ is a strongly proximal subspace of X . We extend this result to any finite co-dimensional strongly proximal subspaces of an L_1 -predual space.

Theorem 2.18. *Let X be an L_1 -predual space and Y be a finite co-dimensional strongly proximal subspace of X . Suppose $F \subset Y^*$ is a finite dimensional subspace consisting of SSD-points of Y^* . Then $Z = \bigcap_{f \in F} \ker(f)$ is a strongly proximal subspace of X and hence is a strongly proximal subspace of Y .*

Proof. Let $Z = \bigcap_{f \in F} \ker(f)$. Then $Z^\perp = F$. Since Y is of finite co-dimension in X , without loss of generality we may assume that $Z = \bigcap_{1 \leq i \leq n} \ker f_i$ for some $f_i \in X^*$ and $f_i \notin Y^\perp$. Thus $f_i|_Y \in Z^\perp = F$ and hence $f_i|_Y$ is an SSD-point of Y^* .

Now by Lemma 2.17, we get that f_i is an SSD-point of X^* as well. Thus, by Corollary 2.9, annihilator $Z^\perp$ of Z in X^* consists of SSD-points of X^* and hence, by Theorem 2.14, we get that Z is strongly proximal in X and hence a strongly proximal subspace of Y . $\square$

The following result is an immediate consequence of Theorem 2.18.

Corollary 2.19. *Let X be an L_1 -predual space. Then being strongly proximal is a transitive property for finite co-dimensional closed subspaces of X . Moreover, intersection of finitely many strongly proximal subspaces of finite co-dimension is again a strongly proximal subspace.*

An interesting question that arises from Theorem 2.18 is that in an L_1 -predual space X , if one can characterize strongly proximal subspaces of finite co-dimension that are again L_1 -predual spaces. Our next theorem answers this question. Since being strongly proximal is a relative property of the subspace Y in X and being an L_1 -predual space is a geometric property of the individual space Y , our surprising answer is based on the choice of a specific type of basis for $Y^\perp$.

Theorem 2.20. *Let X be an L_1 -predual space and Y be a strongly proximal subspace of finite co-dimension n in X . Let $\{\Lambda_1, \dots, \Lambda_n\} \subset S_{X^*}$ be a basis of $Y^\perp \subset \ell_1(k)$ for some $k > 0$ and $X^* = \ell_1(k) \oplus_1 N$ for some closed subspace N of X^* . Suppose $\Lambda_i \in \text{conv}\{\pm e_j : \text{both } e_j \text{ and } -e_j \text{ do not occur simultaneously in the convex combination, } 1 \leq j \leq k\}$ for $1 \leq i \leq n$, where conv denotes the convex hull. Assume that in the convex combination corresponding to each Λ_i , there is an e_j or $-e_j$, which is different for each i and the coefficients of such e_j or $-e_j$ in the convex combination for each Λ_i are also distinct and are in $(\frac{1}{2}, 1]$. Then Y is an L_1 -predual space.*

Conversely if Y is a strongly proximal subspace of finite co-dimension in X and is also an L_1 -predual space, then $Y^\perp$ admits a basis of the above form.

Proof. To fix ideas, we will first work with $n = 1$ and then indicate how the general result follows.

Let $\Lambda = \sum_1^k \lambda_i e_i \in \ell_1(k)$, for some $\lambda_i \in [0, 1]$ with $\sum_1^k \lambda_i = 1 = \|\Lambda\|$. Assume that for some $i_0 > 0$, $\lambda_{i_0} > \frac{1}{2}$.

We recall that a Banach space is an L_1 -predual space if and only if it's bidual is isometric to a space of continuous functions on a compact set. We will first show

that $\ker(\Lambda)$ is the range of a projection of norm one in $X^{**} = \ell_\infty(k) \oplus_\infty N^*$. This is clearly equivalent to showing $\ker(\lambda_1, \dots, \lambda_k) \subset \ell_\infty(k)$ is the range of a projection of norm one in $\ell_\infty(k)$. Since $\lambda_{i_0} > \frac{1}{2}$, it follows from the Theorem of [1] that $\ker(\lambda_1, \dots, \lambda_k)$ is the range of a projection of norm one in $\ell_\infty(k)$. Since X is an L_1 -predual space and $\ker(\Lambda) \subset X^{**}$ is the range of a projection of norm one in X^{**} , it follows from [10, Theorem 6 of Section 1 and Theorem 2 of Section 11] that $\ker(\Lambda) \subset X$ is an L_1 -predual space.

The converse part of the theorem is also proved in a similar way, by first noting that if Y is an L_1 -predual space, then $Y^{\perp\perp}$, being a dual space and a $C(K)$ -space for some compact Hausdorff space K , is the range of a projection of norm one in $X^{**} = \ell_\infty(k) \oplus_\infty N^*$ (see [10, Section 11 and Section 21]). By the structure of $Y^{\perp\perp}$, this again implies that $Y^{\perp\perp} \cap \ell_\infty(k)$ is the range of a projection of norm one in $\ell_\infty(k)$. Thus the choice of a basis for $Y^{\perp\perp}$ follows from the Theorem of [1].

Now if Y is of co-dimension n , from our earlier analysis we have reduced the problem to conditions under which finite dimensional subspaces of $\ell_\infty(k)$ are ranges of projections of norm one. By again appealing to the Theorem of [1], we see that the conditions in the statement of the theorem are equivalent to the requirement for ranges of projections of norm one in $\ell_\infty(k)$. $\square$

The next corollary follows from the arguments given above, in conjugation with our earlier results.

Corollary 2.21. *Let X be an L_1 -predual space. Then intersection of finitely many subspaces of finite co-dimension which are both L_1 -predual spaces and strongly proximal subspaces of X , is again an L_1 -predual space and a strongly proximal subspace of X .*

In the next theorem, we show that the property of ‘finite intersection of strongly proximal hyperplanes being strongly proximal’ is preserved under c_0 -direct sums. We note that this property is equivalent to saying that ‘intersection of kernels of finitely many SSD-points is a strongly proximal subspace’.

Theorem 2.22. *Let $\{X_i\}_{i \in I}$ be a family of Banach spaces such that for every X_i and for any collection of finitely many SSD-points of X_i^* , the intersection of their kernels is a strongly proximal subspace of X_i . Then the same conclusion holds for $X = \bigoplus_{c_0} X_i$.*

Proof. We first prove the case when $I = \{1, 2\}$. Let $F_1, F_2 \in X^*$ be SSD-points of $X^* = X_1^* \oplus_1 X_2^*$. Then $F_1 = (F_1(1), F_1(2))$ and $F_2 = (F_2(1), F_2(2))$, where $F_1(i)$ and $F_2(i)$ are either SSD-points of X_i^* or 0 for $i = 1, 2$.

Now consider the case where all the functionals are non-zero. We can easily see that

$$(\ker(F_1(1)) \cap \ker(F_2(1))) \bigoplus_\infty (\ker(F_1(2)) \cap \ker(F_2(2))) \subset \ker(F_1) \cap \ker(F_2) \subset X.$$

Since the ℓ_∞ -sum of finitely many strongly proximal subspaces is strongly proximal, we see, using that $(\ker(F_1(1)) \cap \ker(F_2(1))) \bigoplus_\infty (\ker(F_1(2)) \cap \ker(F_2(2)))$ is a finite co-dimensional strongly proximal subspace of X and hence, by Corollary 2.3, $\ker(F_1) \cap \ker(F_2)$ is strongly proximal in X .

Similar arguments can be used to prove other cases and also to prove the case of finitely many SSD-points of X^* .

Now let $F_1, \dots, F_k \in X^* = \bigoplus_1 X_i^*$ be SSD-points of X^* . Then, by [8, Theorem 3.10], there exists a finite subset A of I such that $F_i(j) = 0$ for all $i = 1, \dots, k$ and $j \notin A$. Then we may assume that $F_1, \dots, F_k \in \bigoplus_{i \in A} X_i^*$.

Then by arguments similar to the ones given during the proof of the first case, we get a finite co-dimensional strongly proximal subspace Z of $\bigoplus_{i \in A} X_i$ such that

$$\left(Z \oplus_{\infty} \left(\bigoplus_{i \notin A} X_i \right) \right) \subset \bigcap_{i=1}^k \ker(F_i) \subset X.$$

Hence, by Corollary 2.3, $\bigcap_{i=1}^k \ker(F_i)$ is strongly proximal in X . $\square$

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