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Strong Convergence Theorems under Shrinking Projection Methods for Split Common Fixed Point Problems in Two Banach Spaces

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We deal with the split common fixed point problem in two Banach spaces. Using the resolvents of maximal monotone operators, demimetric mappings, demigeneralized mappings in Banach spaces, we prove strong convergence theorems under shrinking projection methods for finding solutions of split common fixed point problems with zero points of maximal monotone operators in two Banach spaces. Using these results, we get new results which are connected with the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and Banach spaces.

Keywords: Split common fixed point problem, fixed point, metric projection, generalized projection, metric resolvent, generalized resolvent, shrinking projection method.

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1. Introduction

Let H_1 and H_2 be two Hilbert spaces. Let C and D be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [8] is to find $z \in H_1$ such that $z \in C \cap A^{-1}D$. Given set-valued mappings $B : H_1 \to 2^{H_1}$ and $G : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common null point problem* [7] is to find a point $z \in H_1$ such that $z \in B^{-1}0 \cap A^{-1}(G^{-1}0)$, where $B^{-1}0$ and $G^{-1}0$ are null point sets of B and G, respectively. Given nonlinear mappings $T : H_1 \to H_1$ and $U : H_2 \to H_2$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common fixed point problem* [9, 21] is to find a point $z \in H_1$ such that $z \in F(T) \cap A^{-1}F(U)$, where $F(T)$ and $F(U)$ are fixed point sets of T and U, respectively. Defining $T = P_C$ and $U = P_D$, where P_C and P_D are the metric projections of H_1 onto C and H_2 onto D, respectively, we have

$$z \in C \cap A^{-1}D \quad \Longleftrightarrow \quad z \in F(T) \cap A^{-1}F(U).$$

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Defining $T = J_\lambda$ and $U = Q_\mu$, where J_λ and Q_μ are the resolvents of B for $\lambda > 0$ and G for $\mu > 0$, respectively, we have

$$z \in B^{-1}0 \cap A^{-1}(G^{-1}0) \quad \Longleftrightarrow \quad z \in F(T) \cap A^{-1}F(U).$$

Thus, the split common fixed point problem generalizes the split feasibility problem and the split common null point problem. If $C \cap A^{-1}D$ is nonempty, then the condition $z \in C \cap A^{-1}D$ is equivalent to

$$z = P_C(I - \lambda A^*(I - P_D)A)z, \tag{1}$$

where $\lambda > 0$ and P_C is the metric projection of H_1 onto C. Furthermore, if the set $B^{-1}0 \cap A^{-1}(G^{-1}0)$ is nonempty, then $z \in B^{-1}0 \cap A^{-1}(G^{-1}0)$ is equivalent to

$$z = J_\lambda(I - \gamma A^*(I - Q_\mu)A)z, \tag{2}$$

where $\lambda, \mu > 0$ and $\gamma > 0$, and J_λ and Q_μ are the resolvents of B and G, respectively. There are many results for the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces; see, for instance, [3, 9, 17, 21, 39].

Takahashi [30, 31, 32] and Hojo and Takahashi [11] extended the results of (1) and (2) in Hilbert spaces to Banach spaces. Furthermore, by using the shrinking projection method, Takahashi and Takahashi [24, 26] proved strong convergence theorems for metric resolvents and generalized resolvents of maximal monotone operators in two Banach spaces; see also [25]. These theorems solved the split common null point problem in two Banach spaces. Motivated by such results for the split common null point problem, Takahashi [33] proved a strong convergence theorem under the shrinking projection method for demimetric mappings and the metric projection in two Banach spaces. This theorem solved the split common fixed point problem in two Banach spaces. Furthermore, Takahashi [35] proved such a result for demigeneralized mappings and the generalized projection in two Banach spaces.

In this paper, using the generalized resolvents of maximal monotone operators, demimetric mappings, demigeneralized mappings and the generalized projection, we first prove a strong convergence theorem under a new shrinking projection method for the split common fixed point problem in two Banach spaces. Furthermore, we prove another strong convergence theorem under the shrinking projection method for the metric resolvents of maximal monotone operators, demigeneralized mappings, demimetric mappings and the metric projection in two Banach spaces. Using these results, we get new results which are connected with the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^*\rangle$. When $\{x_n\}$ is a sequence in E, we denote the *strong* convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the *weak* convergence by $x_n \rightharpoonup x$.

The *modulus* δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x-y\| \geq \epsilon\right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be *uniformly convex* if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [10, 22].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{3}$$

exists. In the case, E is called *smooth*. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be *Fréchet differentiable* if for each $x \in U$, the limit (3) is attained uniformly for $y \in U$. The norm of E is said to be *uniformly smooth* if the limit (3) is attained uniformly for $x, y \in U$. If E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [27, 28]. We know the following result.

Lemma 2.1. ([27]) *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi_E : E \times E \to \mathbb{R}$ by

$$\phi_E(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{4}$$

In the case when E is clear, ϕ_E is simply denoted by ϕ. Observe that, in a Hilbert space H, $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{5}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x-z, Jz-Jy \rangle; \tag{6}$$

$$2\langle x-y, Jz-Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{7}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{8}$$

The following lemmas were proved by Kamimura and Takahashi [13].

Lemma 2.2. ([13]) *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Lemma 2.3. ([13]) *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and $g(\|x-y\|) \leq \phi(x, y)$ for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.*

Let C be a nonempty, closed and convex subset of a strictly convex and reflexive Banach space E. Then we know that for any $x \in E$, there exists a unique element $z \in C$ such that $\|x - z\| \leq \|x - y\|$ for all $y \in C$. Putting $z = P_C x$, we call P_C the metric projection of E onto C. We know the following result.

Lemma 2.4. ([27]) *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = P_C x$;

(2) $\langle z - y, J(x - z)\rangle \geq 0, \quad \forall y \in C$.

For any $x \in E$, we also know that there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. We know the following results.

Lemma 2.5. ([1, 2, 13]) *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x$;

(2) $\langle z - y, Jx - Jz\rangle \geq 0, \quad \forall y \in C$.

Lemma 2.6. ([1, 13]) *Let C be a nonempty, closedand convex subset of a smooth, strictly convex and reflexive Banach space. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C, \, x \in E.$$

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to [5, 23]; see also [28, Theorem 3.5.4].

Theorem 2.7. ([5, 23]) *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$ we have $R(J + rB) = E^*$, where $R(J + rB)$ is the range of $J + rB$.*

Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the equation

$$0 \in J(x_r - x) + rBx_r.$$

This equation has a unique solution x_r; see [28]. We define J_r by $x_r = J_r x$. Such a J_r is denoted by $J_r = (I + rJ^{-1}B)^{-1}$ and is called the *metric resolvent* of B. For $r > 0$, the Yosida approximation $A_r : E \to E^*$ is defined by

$$A_r x = \frac{J(x - J_r x)}{r}, \quad \forall x \in E.$$

Lemma 2.8. ([[28]) *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let J_r and A_r be the metric resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\langle J_r x - u, J(x - J_r x)\rangle \geq 0, \quad \forall x \in E,\ u \in B^{-1}0;$

(2) $(J_r x, A_r x) \in B, \quad \forall x \in E;$

(3) $F(J_r) = B^{-1}0.$

For all $x \in E$ and $r > 0$, we also consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r; see [15]. We define Q_r by $x_r = Q_r x$. Such a Q_r is called the generalized resolvents of B. For $r > 0$, the *Yosida approximation* $A_r : E \to E^*$ is defined by

$$A_r x = \frac{Jx - JQ_r x}{r}, \quad \forall x \in E.$$

The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [28]. When the Banach space is a Hilbert space, we have that $J_r = Q_r$ for all $r > 0$. Such a J_r is called the *resolvent* of B simply. We also know the following result.

Lemma 2.9. ([15]) *Let E be a uniformly convex and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. Let $r > 0$ and let Q_r and A_r be the generalized resolvent and the Yosida approximation of B, respectively. Then, the following hold:*

(1) $\phi(u, Q_r x) + \phi(Q_r x, x) \leq \phi(u, x), \quad \forall x \in E,\ u \in B^{-1}0;$

(2) $(Q_r x, A_r x) \in B, \quad \forall x \in E;$

(3) $F(Q_r) = B^{-1}0.$

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$.

If C_0 satisfies
$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n, \tag{9}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [20] and we write henceforth $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [20]. The following lemma was proved by Tsukada [41].

Lemma 2.10. ([41]) *Let E be a uniformly convex Banach space. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-lim}_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{P_{C_n}x\}$ converges strongly to $P_{C_0}x$, where P_{C_n} and P_{C_0} are the mertic projections of E onto C_n and C_0, respectively.*

Let E be a smooth Banach space, let C be a nonempty subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [34] if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2,$$

where $F(U)$ is the set of fixed points of U. We also know the following dfinition. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called (η, s)-demigeneralized [37, 19] if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux) + s\phi(Ux, x). \tag{10}$$

In particular, if $s = 0$ in (10), then the mapping U is as follows:

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux)$$

for all $x \in C$ and $q \in F(U)$. Such $(\eta, 0)$-demigeneralized mappings in the class of demigeneralized mappings are important.

Examples

(1) Let H be a Hilbert space, let C be a nonempty subset of H and let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseud-contraction [6] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseud-contraction and $F(U) \neq \emptyset$, then U is k-demimetric and $(k, 0)$-demigeneralized; see [34] and [37].

(2) Let H be a Hilbert space and let C be a nonempty subset of H. A mapping $U : C \to H$ is called generalized hybrid [14] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{11}$$

for all $x, y \in C$. Such a mapping U is called (α, β)-generalized hybrid. Notice that the class of (α, β)-generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [15, 16] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Ux - Uy\|^2 \leq \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [29] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Ux - Uy\|^2 \leq \|x - y\|^2 + \|Ux - y\|^2 + \|Uy - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings may not be continuous; see [12]. If U is $(\alpha,\ \beta)$-generalized hybrid and $F(U) \neq \emptyset$, then U is 0-demimetric and $(0,0)$-demigeneralized; see [34] and [37].

(3) Let E be a smooth, strictly convex and reflexive Banach space and let D be a nonempty, closed and convex subset of E. Let P_D be the metric projection of E onto D. Then P_D is (-1)-demimetric; see [34].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the metric resolvent J_λ is (-1)-demimetric; see [34].

(5) Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let Π_C be the generalized projection of E onto C. Then Π_C is $(0,1)$-demigeneralized. In fact, since Π_C is the generalized projection of E onto C, we have that, for any $x \in E$ and $q \in C$,

$$2\langle \Pi_C x - q, Jx - J\Pi_C x\rangle \geq 0.$$

Then we get $2\langle \Pi_C x - x + x - q, Jx - J\Pi_C x\rangle \geq 0$ and hence

$$2\langle x - q, Jx - J\Pi_C x\rangle \geq 2\langle x - \Pi_C x, Jx - J\Pi_C x\rangle = \phi(x, \Pi_C x) + \phi(\Pi_C x, x).$$

This means that Π_C is $(0,1)$-demigeneralized. Furthermore, since

$$\phi(x, \Pi_C x) + \phi(\Pi_C x, x) \geq \phi(x, \Pi_C x),$$

Π_C is also $(0,0)$-demigeneralized; see [37].

(6) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the generalized resolvent Q_λ is $(0,1)$-demigeneralized. In fact, since Q_λ is the generalized resolvent of B for $\lambda > 0$, we have that, for any $x \in E$ and $q \in B^{-1}0$,

$$2\langle Q_\lambda x - q, Jx - JQ_\lambda x\rangle \geq 0.$$

Then we get $2\langle Q_\lambda x - x + x - q, Jx - JQ_\lambda x\rangle \geq 0$ and hence

$$2\langle x - q, Jx - JQ_\lambda x\rangle \geq 2\langle x - Q_\lambda x, Jx - JQ_\lambda x\rangle = \phi(x, Q_\lambda x) + \phi(Q_\lambda x, x).$$

This means that Q_λ is $(0,1)$-demigeneralized. Furthermore, since

$$\phi(x, Q_\lambda x) + \phi(Q_\lambda x, x) \geq \phi(x, Q_\lambda x),$$

Q_λ is also $(0,0)$-demigeneralized; see [37].

The following lemmas are important and crucial in the proofs of our main results.

Lemma 2.11. ([34]) *Let E be a smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let η be a real number with $\eta \in (-\infty, 1)$. Let U be an η-demimetric mapping of C into E. Then $F(U)$ is closed and convex.*

Lemma 2.12. ([37]) *Let E be a smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. Let U be an $(\eta, 0)$-demigeneralized mapping of C into E. Then $F(U)$ is closed and convex. In particular, if U is (η, s)-demigeneralized, then $F(U)$ is closed and convex.*

3. Main Results

In this section, using a new shrinking projection method, we first prove a strong convergence theorem for finding a solution of the split common fixed point problem in two Banach spaces; see also Takahashi, Takeuchi and Kubota [36]. Let E be a Banach space and let C be a nonempty, closed and convex subset of E. A mapping $U : C \to E$ is called demiclosed if for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds.

Theorem 3.1. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $B \subset E \times E^*$ be a maximal monotone operator and let $Q_\lambda = (J + \lambda B)^{-1}J$ be the generalized resolvent of B for all $\lambda > 0$. Let τ and η be real numbers with $\tau, \eta \in (-\infty, 1)$. Let $T : E \to E$ be a τ-demimetric and demiclosed mapping with $F(T) \neq \emptyset$ and let $U : F \to F$ be an $(\eta, 0)$-demigeneralized and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that*

$$\Omega = B^{-1}0 \cap F(T) \cap A^{-1}F(U) \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F Ax_n - J_F UAx_n)\big), \\ y_n = Tz_n, \\ u_n = Q_{\lambda_n} y_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle y_n - z, J_E y_n - J_E u_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n), \\ \qquad\qquad 2\langle z_n - z, J_E(z_n - y_n)\rangle \geq (1-\tau)\|z_n - y_n\|^2 \\ \qquad\qquad \text{and} \quad 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n(1-\eta)\phi_F(Ax_n, UAx_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1 \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\} \subset (0, \infty)$ and $a, b, c \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{b}{\|A\|^2} \quad \text{and} \quad 0 < c \leq \lambda_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = \Pi_\Omega x_1$.

Proof. Since B is maximal monotone, $B^{-1}0$ is closed and convex. We also have from Lemmas 2.11 and 2.12 that $F(T)$ and $F(U)$ are closed and convex. Therefore, $\Omega = B^{-1}0 \cap F(T) \cap A^{-1}F(U)$ is closed and convex. It is obvious that C_n is closed and convex for all $n \in \mathbb{N}$. We show that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\Omega \subset C_1 = E$. Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. To show $\Omega \subset C_{k+1}$, we show

$$2\langle y_k - z, J_E y_k - J_E u_k\rangle \geq \phi_E(y_k, u_k) + \phi_E(u_k, y_k),$$

$$2\langle z_k - z, J_E(z_k - y_k)\rangle \geq (1-\tau)\|z_k - y_k\|^2$$

and $$2\langle x_k - z, J_E x_k - J_E z_k\rangle \geq r_k(1-\eta)\phi_F(Ax_k, UAx_k)$$

for all $z \in \Omega$. In fact, let $z \in \Omega$. Since Q_{λ_k} is the generalized resolvent, we have that

$$\langle Q_{\lambda_k} y_k - z, J_E y_k - J_E Q_{\lambda_k} y_k \rangle \geq 0$$

for all $z \in \Omega \subset B^{-1}0$. Thus, we get $\langle Q_{\lambda_k} y_k - y_k + y_k - z, J_E y_k - J_E Q_{\lambda_k} y_k \rangle \geq 0$ and hence

$$2\langle y_k - z, J_E y_k - J_E Q_{\lambda_k} y_k \rangle \geq 2\langle y_k - Q_{\lambda_k} y_k, J_E y_k - J_E Q_{\lambda_k} y_k \rangle.$$

We have from (7) that

$$2\langle y_k - z, J_E y_k - J_E Q_{\lambda_k} y_k \rangle \geq \phi_E(y_k, Q_{\lambda_k} y_k) + \phi_E(Q_{\lambda_k} y_k, y_k).$$

This implies that $2\langle y_k - z, J_E y_k - J_E u_k \rangle \geq \phi_E(y_k, u_k) + \phi_E(u_k, y_k)$.

We also have that, for all $z \in \Omega$,

$$\begin{aligned} &2\langle x_k - z, J_E x_k - J_E z_k \rangle = 2\langle x_k - z, r_k A^*(J_F A x_k - J_F U A x_k) \rangle \\ &= 2r_k \langle A x_k - A z, J_F A x_k - J_F U A x_k \rangle \geq r_k (1 - \eta) \phi_F(A x_k, U A x_k). \end{aligned}$$

Furthermore, we have that, for all $z \in \Omega$,

$$\begin{aligned} &2\langle z_k - z, J_E(z_k - y_k) \rangle - (1 - \tau)\|z_k - y_k\|^2 \\ &= 2\langle z_k - z, J_E(z_k - T z_k) \rangle - (1 - \tau)\|z_k - T z_k\|^2 \\ &\geq (1 - \tau)\|z_k - T z_k\|^2 - (1 - \tau)\|z_k - T z_k\|^2 = 0 \end{aligned}$$

and hence $$2\langle z_k - z, J_E(z_k - y_k) \rangle \geq (1 - \tau)\|z_k - y_k\|^2.$$

Then, we have $\Omega \subset C_{k+1}$. By mathematical induction, we have that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since Ω is a nonempty, closed and convex subset of E, there exists $z_0 \in \Omega$ such that $z_0 = \Pi_\Omega x_1$. We have from $x_n = \Pi_{C_n} x_1$ that

$$\phi_E(x_n, x_1) \leq \phi_E(y, x_1), \quad \forall y \in C_n.$$

Since $z_0 \in \Omega \subset C_n$, we have that

$$\phi_E(x_n, x_1) \leq \phi_E(z_0, x_1), \quad \forall n \in \mathbb{N}. \tag{12}$$

From $x_n = \Pi_{C_n} x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have that

$$\phi_E(x_n, x_1) \leq \phi_E(x_{n+1}, x_1).$$

Thus, $\{\phi_E(x_n, x_1)\}$ is bounded and nondecreasing. Then there exists the limit of $\{\phi_E(x_n, x_1)\}$. Put $\lim_{n\to\infty} \phi_E(x_n, x_1) = c$. For any $m, n \in \mathbb{N}$ with $m \geq n$, we have $C_m \subset C_n$. From $x_m = \Pi_{C_m} x_1 \in C_m \subset C_n$ and Lemma 2.6, we have that

$$\phi_E(\Pi_{C_n} x_1, x_1) + \phi_E(x_m, \Pi_{C_n} x_1) \leq \phi_E(x_m, x_1).$$

This implies that

$$\phi_E(x_m, x_n) \leq \phi_E(x_m, x_1) - \phi_E(x_n, x_1) \leq c - \phi_E(x_n, x_1). \tag{13}$$

Since $c-\phi_E(x_n,x_1)\to 0$ as $n\to\infty$, we have from Lemma 2.3 that $\{x_n\}$ is a Cauchy sequence. By the completeness of E, there exists a point $w_0\in E$ such that

$$x_n \to w_0. \tag{14}$$

We have from $x_{n+1}\in C_{n+1}$ that

$$2\langle x_n - x_{n+1}, J_E x_n - J_E z_n\rangle \geq r_n(1-\eta)\phi_F(Ax_n, UAx_n). \tag{15}$$

Furthermore, we claim that $\{J_E x_n - J_E z_n\}$ is bounded. We show that $\{J_E x_n - J_E z_n\}$ is bounded as follows. We have that

$$\|J_E x_n - J_E z_n\| = \|r_n A^*(J_F Ax_n - J_F UAx_n)\|.$$

Furthermore, we have that $\|J_F Ax_n\| = \|Ax_n\| \leq \|A\|\|x_n\|$.

Since $2\langle Ax_n - Az, J_F Ax_n - J_F UAx_n\rangle \geq (1-\eta)\phi_F(Ax_n, UAx_n)$ for $z\in A^{-1}F(U)$, we have from (7) that

$$\phi_F(Ax_n, UAx_n) + \phi_F(Az, Ax_n) - \phi_F(Az, UAx_n) \geq (1-\eta)\phi_F(Ax_n, UAx_n)$$

and hence $\eta\phi_F(Ax_n, UAx_n) + \phi_F(Az, Ax_n) \geq \phi_F(Az, UAx_n)$.

In the case of $\eta\leq 0$, we have $\phi_F(Az, Ax_n)\geq \phi_F(Az, UAx_n)$. So, we have that, for $z\in A^{-1}F(U)$,

$$\begin{aligned}(\|Az\| - \|UAx_n\|)^2 &\leq \phi_F(Az, UAx_n) \leq \phi_F(Az, Ax_n)\\ &\leq (\|Az\| + \|Ax_n\|)^2 \leq \|A\|^2(\|z\| + \|x_n\|)^2\end{aligned}$$

and hence $|\|Az\| - \|UAx_n\|| \leq \|A\|(\|z\| + \|x_n\|)$.

Using this, we get

$$\|UAx_n\| \leq \|A\|(\|z\| + \|x_n\|) + \|Az\| \leq \|A\|(\|z\| + \|x_n\|) + \|A\|\|z\|.$$

Then, we obtain $\|J_F UAx_n\| = \|UAx_n\| \leq \|A\|(2\|z\| + \|x_n\|)$.

Hence, we have that

$$\begin{aligned}\|J_E x_n - J_E z_n\| &= \|r_n A^*(J_F Ax_n - J_F UAx_n)\|\\ &\leq \frac{b}{\|A\|^2}\|A\|\ \big(\|J_F Ax_n\| + \|J_F UAx_n)\|\big)\\ &\leq \frac{b}{\|A\|^2}\|A\|\ \big(\|A\|\|x_n\| + \|A\|(2\|z\| + \|x_n\|)\big) \leq 2b(\|x_n\| + \|z\|).\end{aligned}$$

This implies that $\{J_E x_n - J_E z_n\}$ is bounded. In the case of η with $0<\eta<1$, we get

$$\eta\phi_F(Ax_n, UAx_n) + \phi_F(Az, Ax_n) \geq \phi_F(Az, UAx_n).$$

So, we have for $z\in A^{-1}F(U)$,

$$\begin{aligned}(\|Az\| - \|UAx_n\|)^2 &\leq \phi_F(Az, UAx_n) \leq \phi_F(Az, Ax_n) + \eta\phi_F(Ax_n, UAx_n)\\ &\leq (\|Az\| + \|Ax_n\|)^2 + \eta(\|Ax_n\| + \|UAx_n\|)^2\\ &\leq \big(\|Az\| + \|Ax_n\| + \sqrt{\eta}(\|Ax_n\| + \|UAx_n\|)\big)^2.\end{aligned}$$

From this, we obtain

$$|\|Az\| - \|UAx_n\|| \leq \|Az\| + \|Ax_n\| + \sqrt{\eta}(\|Ax_n\| + \|UAx_n\|$$

and hence

$$(1-\sqrt{\eta})\|UAx_n\| \leq (1+\sqrt{\eta})\|Ax_n\| + 2\|Az\| \leq (1+\sqrt{\eta})\|A\|\|x_n\| + 2\|A\|\|z\|.$$

Then, we have that

$$\|J_F UAx_n\| = \|UAx_n\| \leq \|A\|\Big(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|\Big).$$

Hence, we receive

$$\begin{aligned}
\|J_E x_n - J_E z_n\| &= \|r_n A^*(J_F Ax_n - J_F UAx_n)\| \\
&\leq \frac{b}{\|A\|^2}\|A\| \ \big(\|J_F Ax_n\| + \|J_F UAx_n)\|\big) \\
&\leq \frac{b}{\|A\|^2}\|A\| \ \Big(\|A\|\|x_n\| + \|A\|\big(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|\big)\Big) \\
&\leq b\Big(\|x_n\| + \frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|\Big).
\end{aligned}$$

This implies that $\{J_E x_n - J_E z_n\}$ is bounded. Since $r_n \geq a > 0$ for all $n \in \mathbb{N}$, we have from (15) that

$$2\langle x_n - x_{n+1}, J_E x_n - J_E z_n\rangle \geq a(1-\eta)\phi_F(Ax_n, UAx_n). \tag{16}$$

Since $\|x_n - x_{n+1}\| \to 0$ from (14) and $\{J_E x_n - J_E z_n\}$ is bounded, we get that

$$\lim_{n\to\infty} \phi_F(Ax_n, UAx_n) = 0. \tag{17}$$

Therefore, we get from Lemma 2.2 that

$$\lim_{n\to\infty} \|Ax_n - UAx_n\| = 0. \tag{18}$$

Furthermore, since F is uniformly smooth, we have from (18) that

$$\lim_{n\to\infty} \|J_F Ax_n - J_F UAx_n\| = 0. \tag{19}$$

Since $\|J_E x_n - J_E z_n\| = \|r_n A^*(J_F Ax_n - J_F UAx_n)\|$ and $\{r_n\}$ is bounded, we get from (19) that

$$\lim_{n\to\infty} \|J_E x_n - J_E z_n\| = 0. \tag{20}$$

Since E^* is uniformly smooth, we have from (20) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{21}$$

Furthermore, we have from $x_{n+1} \in C_{n+1}$ that

$$2\langle z_n - x_{n+1}, J_E(z_n - y_n)\rangle \geq (1-\tau)\|z_n - y_n\|^2$$

and hence $2\|z_n - x_{n+1}\| \geq (1-\tau)\|z_n - y_n\|$.

Since $\|z_n - x_n\| \to 0$ and $\|x_n - x_{n+1}\| \to 0$, we have that $\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0$ and hence $\lim_{n\to\infty} \|z_n - y_n\| = 0$. Using $y_n = Tz_n$, we have that

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{22}$$

We also have from $x_{n+1} \in C_{n+1}$ that

$$2\langle y_n - x_{n+1}, Jy_n - Ju_n\rangle \geq \phi_E(y_n, u_n) + \phi_E(u_n, y_n).$$

From $y_n - z_n \to 0$, $z_n - x_n \to 0$ and $x_n - x_{n+1} \to 0$, we have $\|y_n - x_{n+1}\| \to 0$.

Then we get $\lim_{n\to\infty} \phi_E(y_n, u_n) = 0$ and hence

$$\lim_{n\to\infty} \|y_n - Q_{\lambda_n} y_n\| = 0. \tag{23}$$

Since $x_n \to w_0$ and A is bounded and linear, we have that $\{Ax_n\}$ converges strongly to Aw_0 and hence $\{Ax_n\}$ converges weakly to Aw_0. Since U is demiclosed, we have from (18) that $Aw_0 = UAw_0$. From $x_n \to w_0$ and (21), we also have that $\{z_n\}$ converges weakly to w_0. Since T is demiclosed, we have from (22) that $w_0 = Tw_0$. We show $w_0 \in B^{-1}0$. Since E is uniformly smooth, from $u_n = Q_{\lambda_n} y_n$ and (23) we have that

$$\lim_{n\to\infty} \|Jy_n - Ju_n\| = 0.$$

From $\lambda_n \geq c$, we have $\lim_{n\to\infty} \frac{1}{\lambda_n}\|Jy_n - Ju_n\| = 0$. Therefore, we have

$$\lim_{n\to\infty} \|A_{\lambda_n} y_n\| = \lim_{n\to\infty} \frac{1}{\lambda_n}\|Jy_n - Ju_n\| = 0.$$

For $(p, p^*) \in B$, from the monotonicity of B and Lemma 2.9, we have

$$\langle p - u_n, p^* - A_{\lambda_n} y_n\rangle \geq 0$$

for all $n \in \mathbb{N}$. From $u_n \to w_0$ and $A_{\lambda_n} y_n \to 0$, we get $\langle p - w_0, p^*\rangle \geq 0$. From the maximality of B, we have $w_0 \in B^{-1}0$. Therefore, we have $w_0 \in \Omega$.

From $z_0 = \Pi_\Omega x_1$, $w_0 \in \Omega$ and (26), we have that

$$\phi_E(z_0, x_1) \leq \phi_E(w_0, x_1) = \lim_{n\to\infty} \phi_E(x_n, x_1) \leq \phi_E(z_0, x_1).$$

Then we get $z_0 = w_0$. Therefore, we have $x_n \to z_0$. This completes the proof. $\square$

Next, using the shrinking projection method, we prove another strong convergence theorem for finding a solution of the split common fixed point problem in two Banach spaces.

Theorem 3.2. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let $B \subset E \times E^*$ be a maximal monotone operator and let $J_\lambda = (I + \lambda J^{-1}B)^{-1}$ be the metric resolvent of B for all $\lambda > 0$. Let τ and η be real numbers with $\tau, \eta \in (-\infty, 1)$. Let $T : E \to E$ be a $(\tau, 0)$-demigeneralized and demiclosed mapping with $F(T) \neq \emptyset$ and let $U : F \to F$ be an η-demimetric and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that*

$$\Omega = B^{-1}0 \cap F(T) \cap A^{-1}F(U) \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n J_E^{-1} A^* J_F(Ax_n - UAx_n), \\ y_n = Tz_n, \\ u_n = J_{\lambda_n} y_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle y_n - z, J(y_n - u_n) \rangle \geq \|y_n - u_n\|^2, \\ \qquad\qquad 2\langle z_n - z, J_E z_n - J_E y_n \rangle \geq (1-\tau)\phi_E(z_n, y_n) \\ \qquad\qquad\qquad\qquad \text{and} \quad \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\} \subset (0, \infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{1-\eta}{\|A\|^2} \quad \text{and} \quad 0 < b \leq \lambda_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $w_1 \in \Omega$, where $w_1 = P_\Omega x_1$.

Proof. As in the proof of Theorem 3.1, it follows that C_n is closed and convex for all $n \in \mathbb{N}$. We also have that $\Omega = B^{-1}0 \cap F(T) \cap A^{-1}F(U)$ is nonempty, closed and convex. We show that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\Omega \subset C_1 = E$. Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. To show $\Omega \subset C_{k+1}$, let us show that

$$\langle z_k - z, J_E(x_k - z_k) \rangle \geq 0, \quad 2\langle z_k - z, J_E z_k - J_E y_k \rangle \geq (1-\tau)\phi_E(z_k, y_k),$$

and
$$\langle y_k - z, J(y_k - u_k) \rangle \geq \|y_k - u_k\|^2 \ \text{ for all } \ z \in \Omega.$$

In fact, we have for all $z \in \Omega$,

$$\begin{aligned} &\langle z_k - z, J_E(x_k - z_k) \rangle = \langle z_k - x_k + x_k - z, J_E(x_k - z_k) \rangle \\ &= \langle -r_k J_E^{-1} A^* J_F(Ax_k - UAx_k) + x_k - z, J_E(r_k J_E^{-1} A^* J_F(Ax_k - UAx_k)) \rangle \\ &= \langle -r_k J_E^{-1} A^* J_F(Ax_k - UAx_k) + x_k - z, r_k A^* J_F(Ax_k - UAx_k) \rangle \\ &\geq -r_k^2 \|A\|^2 \|Ax_k - UAx_k\|^2 + \frac{r_k(1-\eta)}{2} \|Ax_k - UAx_k\|^2 \\ &= r_k\Big(\frac{1-\eta}{2} - r_k\|A\|^2\Big)\|Ax_k - UAx_k\|^2 \geq 0 \end{aligned} \tag{24}$$

and
$$\begin{aligned} & 2\langle z_k - z, J_E z_k - J_E y_k\rangle - (1-\tau)\phi_E(z_k, y_k) \\ &= 2\langle z_k - z, J_E z_k - J_E T z_k)\rangle - (1-\tau)\phi_E(z_k, T z_k) \\ &\geq (1-\tau)\phi_E(z_k, T z_k) - (1-\tau))\phi_E(z_k, T z_k) = 0. \end{aligned} \tag{25}$$

Since J_{λ_k} is the metric resolvent, we have that, for $z \in \Omega \subset B^{-1}0$,

$$\langle J_{\lambda_k} y_k - z, J_E(y_k - J_{\lambda_k} y_k)\rangle \geq 0.$$

From this, we get that $\langle J_{\lambda_k} y_k - y_k + y_k - z, J_E(y_k - J_{\lambda_k} y_k)\rangle \geq 0$ and hence

$$\langle y_k - z, J_E(y_k - J_{\lambda_k} y_k)\rangle \geq \|y_k - J_{\lambda_k} y_k\|^2.$$

This implies that $\langle y_k - z, J_E(y_k - u_k)\rangle \geq \|y_k - u_k\|^2$.
Then $\Omega \subset C_{k+1}$. We have by mathematical induction that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since Ω is nonempty, closed and convex, there exists $w_1 \in \Omega$ such that $w_1 = P_\Omega x_1$. From $x_n = P_{C_n} x_1$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - y\|$$

for all $y \in C_n$. Since $w_1 \in \Omega \subset C_n$, we have that

$$\|x_1 - x_n\| \leq \|x_1 - w_1\|. \tag{26}$$

Let $C_0 = \bigcap_{n=1}^\infty C_n$. Since $\emptyset \neq \Omega \subset C_0$, we have that C_0 is nonempty. Since $C_0 =$ M-$\lim_{n\to\infty} C_n$ and $x_n = P_{C_n} x_1$ for all $n \in \mathbb{N}$, by Lemma 2.10 we have that

$$x_n \to z_0 = P_{C_0} x_1. \tag{27}$$

We have from $x_{n+1} \in C_{n+1}$ that $\langle z_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq 0$ and hence

$$\langle z_n - x_n + x_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq 0.$$

This implies that $\langle x_n - x_{n+1}, J_E(x_n - z_n)\rangle \geq \|z_n - x_n\|^2$ and hence

$$\|x_n - x_{n+1}\| \geq \|z_n - x_n\|.$$

Since $\|x_n - x_{n+1}\| \to 0$ from (27), we get that $x_n - z_n \to 0$. On the other hand, we have that

$$\begin{aligned} \|x_n - z_n\| &= \|J_E(x_n - z_n)\| = \|r_n A^* J_F(Ax_n - UAx_n)\| \\ &= r_n\|A^* J_F(Ax_n - UAx_n)\| \geq a\|A^* J_F(Ax_n - UAx_n)\|. \end{aligned}$$

Since $\lim_{n\to\infty}\|x_n - z_n\| = 0$, we have that

$$\lim_{n\to\infty}\|A^* J_F(Ax_n - UAx_n)\| = 0.$$

Since U is η-demimetric, we have that, for $Az \in F(U)$,

$$\begin{aligned} 2\langle x_n - z, A^* J_F(Ax_n - UAx_n)\rangle &= 2\langle Ax_n - Az, J_F(Ax_n - UAx_n)\rangle \\ &\geq (1-\eta)\|Ax_n - UAx_n\|^2. \end{aligned}$$

Then we get that
$$\lim_{n\to\infty}\|Ax_n - UAx_n\| = 0.$$

Furthermore, we have from $x_{n+1} \in C_{n+1}$ that

$$2\langle z_n - x_{n+1}, J_E z_n - J_E y_n\rangle \geq (1-\tau)\phi_E(z_n, y_n)$$

and hence

$$2\langle z_n - x_n + x_n - x_{n+1}, J_E z_n - J_E y_n\rangle \geq (1-\tau)\phi_E(z_n, y_n). \tag{28}$$

Let us show that $\{J_E z_n - J_E y_n\}$ is bounded. Since T is τ-demigeneralized, we have that, for $z \in F(T)$,

$$2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq (1-\tau)\phi_E(z_n, y_n)$$

and hence $\phi_E(z_n, y_n) + \phi_E(z, z_n) - \phi_E(z, y_n) \geq (1-\tau)\phi_E(z_n, y_n)$.

This implies that $\tau\phi_E(z_n, y_n) + \phi_E(z, z_n) \geq \phi_E(z, y_n)$.

In the case of $\tau \leq 0$, we have $\phi_E(z, z_n) \geq \phi_E(z, y_n)$. Thus we have that, for $z \in F(T)$,

$$(\|z\| - \|y_n\|)^2 \leq \phi_E(z, y_n) \leq \phi_E(z, z_n) \leq (\|z\| + \|z_n\|)^2.$$

Using this, we obtain $|\|z\| - \|y_n\|| \leq \|z\| + \|z_n\|$ and hence $\|y_n\| \leq 2\|z\| + \|z_n\|$.

Hence, we have that

$$\|J_E z_n - J_E y_n\| \leq \|J_E z_n\| + \|J_E y_n\| = \|z_n\| + \|y_n\| \leq 2\|z\| + 2\|z_n\|.$$

This implies that $\{J_E z_n - J_E y_n\}$ is bounded.

In the case of η with $0 < \tau < 1$, we have

$$\tau\phi_E(z_n, y_n) + \phi_E(z, z_n) \geq \phi_E(z, y_n).$$

Thus, we have that, for $z \in F(T)$,

$$\begin{aligned}(\|z\| - \|y_n\|)^2 &\leq \phi_E(z, y_n) \leq \phi_E(z, z_n) + \tau\phi_E(z_n, y_n) \\ &\leq (\|z\| + \|z_n\|)^2 + \tau(\|z_n\| + \|y_n\|)^2 \leq \big(\|z\| + \|z_n\| + \sqrt{\tau}(\|z_n\| + \|y_n\|)\big)^2.\end{aligned}$$

From this, we have that

$$|\|z\| - \|y_n\|| \leq \|z\| + \|z_n\| + \sqrt{\tau}(\|z_n\| + \|y_n\|$$

and hence $(1 - \sqrt{\tau})\|y_n\| \leq (1 + \sqrt{\tau})\|z_n\| + 2\|z\|$.

Then, we have that $\|y_n\| \leq \Big(\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}\|z_n\| + \frac{2}{1-\sqrt{\tau}}\|z\|\Big)$.

Hence, we have that

$$\|J_E z_n - J_E y_n\| \leq \|z_n\| + \|y_n\| \leq \|z_n\| + \Big(\frac{1+\sqrt{\tau}}{1-\sqrt{\tau}}\|z_n\| + \frac{2}{1-\sqrt{\tau}}\|z\|\Big).$$

This implies that $\{J_E z_n - J_E y_n\}$ is bounded. From (28), $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - z_n\| \to 0$, we have that $\lim_{n\to\infty} \phi_E(z_n, y_n) = 0$. Then we get from Lemma 2.2 that $\|z_n - y_n\| \to 0$ and hence

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{29}$$

Since $x_{n+1} \in C_{n+1}$, we have that

$$\langle y_n - x_{n+1}, J(y_n - u_n)\rangle \geq \|y_n - u_n\|^2$$

and hence $\|y_n - x_{n+1}\| \geq \|y_n - u_n\|$. From $y_n - z_n \to 0$, $z_n - x_n \to 0$ and $x_n - x_{n+1} \to 0$, we have $\|y_n - x_{n+1}\| \to 0$. Then we get that

$$\lim_{n\to\infty} \|y_n - u_n\| = 0 \tag{30}$$

and hence

$$\lim_{n\to\infty} \|y_n - J_{\lambda_n} y_n\| = 0. \tag{31}$$

Since $x_n \to z_0$ and T is demiclosed, we have from (29) that $z_0 \in F(T)$. Furthermore, since A is bounded and linear, we have that $\{Ax_n\}$ converges strongly to Az_0. Since U is demiclosed, we have from (3) that $Az_0 = UAz_0$. Next, we show $z_0 \in B^{-1}0$.

From $\lambda_n \geq b$ and (31), we have $\lim_{n\to\infty} \frac{1}{\lambda_n}\|J_E(y_n - J_{\lambda_n} y_n)\| = 0.$

Therefore, we have $\lim_{n\to\infty} \|A_{\lambda_n} y_n\| = \lim_{n\to\infty} \frac{1}{\lambda_n}\|J_E(y_n - J_{\lambda_n} y_n)\| = 0.$

For $(p, p^*) \in B$, from the monotonicity of B, we have $\langle p - J_{\lambda_n} y_n, p^* - A_{\lambda_n} y_n\rangle \geq 0$ for all $n \in \mathbb{N}$. Letting $n \to \infty$, we get from $J_{\lambda_n} y_n \to z_0$ that $\langle p - z_0, p^*\rangle \geq 0$. By the maximality of B, we have $z_0 \in B^{-1}0$. Therefore, we have $z_0 \in \Omega$.

From $w_1 = P_\Omega x_1$, $z_0 \in \Omega$ and (26), we have that

$$\|x_1 - w_1\| \leq \|x_1 - z_0\| = \lim_{n\to\infty} \|x_1 - x_n\| \leq \|x_1 - w_1\|.$$

Then we get that $\|x_1 - w_1\| = \|x_1 - z_0\|$ and hence $z_0 = w_1$. Therefore, we have $x_n \to z_0 = w_1$. This completes the proof. □

4. Applications

In this section, using Theorems 3.1 and 3.2, we get new strong convergence theorems which are connected with the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and Banach spaces. We know the following result obtained by Marino and Xu [18]; see also [38].

Lemma 4.1. ([18, 38]) *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [14]; see also [40].

Lemma 4.2. ([14, 40]) *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Using Lemmas 4.1 and 4.2 and Theorem 3.1, we prove strong convergence theorems for finding solutions of split common fixed point problems with zero points of maximal monotone operators in Hilbert spaces.

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces. Let $B \subset H_1 \times H_1$ be a maximal monotone operator and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for all $\lambda > 0$. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that*

$$\Omega = B^{-1}0 \cap F(T) \cap A^{-1}F(U) \neq \emptyset.$$

For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - UAx_n), \quad y_n = Tz_n, \quad u_n = J_{\lambda_n} y_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle y_n - z, y_n - u_n \rangle \geq \|y_n, -u_n\|^2, \\ \qquad\qquad\qquad 2\langle z_n - z, z_n - y_n \rangle \geq \|z_n - y_n\|^2 \\ \qquad\qquad \text{and} \quad 2\langle x_n - z, x_n - z_n \rangle \geq r_n(1-k)\|Ax_n - UAx_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\} \subset (0, \infty)$ and $a, b, c \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{b}{\|A\|^2} \quad \text{and} \quad c \leq \lambda_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = P_\Omega x_1$.

Proof. Since U is a k-strict pseudo-contraction of H_2 into itself such that $F(U) \neq \emptyset$, from (1) in Examples, U is $(k, 0)$-demigeneralized. Furthermore, from Lemma 4.1, U is demiclosed. We also have that the nonexpansive maping T is 0-demimetric and demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Theorem 4.4. *Let H_1 and H_2 be Hilbert spaces. Let $B \subset H_1 \times H_1$ be a maximal monotone operator and let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for all $\lambda > 0$. Let $T : H_1 \to H_1$ be a nonspreading mapping with $F(T) \neq \emptyset$ and let $U : H_2 \to H_2$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that*

$$\Omega = B^{-1}0 \cap F(T) \cap A^{-1}F(U) \neq \emptyset.$$

For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - UAx_n), \quad y_n = Tz_n, \quad u_n = J_{\lambda_n} y_n, \\ C_{n+1} = \Big\{ z \in C_n : \langle y_n - z, y_n - u_n \rangle \geq \|y_n, -u_n\|^2, \\ \qquad\qquad\qquad 2\langle z_n - z, z_n - y_n \rangle \geq \|z_n - y_n\|^2 \\ \qquad\qquad \text{and} \quad 2\langle x_n - z, x_n - z_n \rangle \geq r_n\|Ax_n - UAx_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\}, \{\lambda_n\} \subset (0, \infty)$ and $a, b, c \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{b}{\|A\|^2} \quad \text{and} \quad c \leq \lambda_n, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in \Omega$, where $z_0 = P_\Omega x_1$.

Proof. Since U is a generalized hybrid mapping of H_2 into itself such that $F(U) \neq \emptyset$, from (2) in Examples, U is $(0,0)$-demigeneralized. Furthermore, from Lemma 4.2, U is demiclosed. We also have that the nonspreading maping T is 0-demimetric and demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Using Theorem 3.1, we have a new strong convergence theorem which is connected with the split feasibility problem in two Banach spaces.

Theorem 4.5. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and F, respectively. Let P_C be the metric projection of E onto C and let Π_C be the generalized projections of F onto D. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F \Pi_D A x_n)\big), \quad y_n = P_C z_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Ax_n, \Pi_D A x_n) \\ \qquad\qquad\qquad\qquad \text{and} \quad \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2 \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0,\infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{b}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = \Pi_{C\cap A^{-1}D} x_1$.

Proof. Putting $B = 0$ in Theorem 3.1, we have $J_{\lambda_n} = I$ for all $n \in \mathbb{N}$. Furthermore, since P_C is the metric projection of E onto C, P_C is (-1)-demimetric from (3) in Examples. Furthermore, since Π_D is the generalized projection of F onto D, from (5) in Examples, Π_D is $(0,0)$-demigeneralized. We also have that if $\{v_n\}$ is a sequence in E such that $v_n \rightharpoonup q$ and $v_n - P_C v_n \to 0$, then $q = P_C q$. In fact, assume that $v_n \rightharpoonup q$ and $v_n - P_C v_n \to 0$. It is clear that $P_C v_n \rightharpoonup q$. Furthermore, we have that $\|J_E(v_n - P_C v_n)\| \to 0$. Since P_C is the metric projection of E onto C, we have that

$$\langle P_C v_n - P_C q, J_E(v_n - P_C v_n) - J_E(q - P_C q)\rangle \geq 0.$$

Therefore, $\langle q - P_C q, -J_E(q - P_C q)\rangle \geq 0$. This implies that $\|q - P_C q\|^2 \leq 0$ and hence $q = P_C q$. Therefore, P_C is demiclosed. Similarly, we can show that Π_D is demiclosed. In fact, assume that $u_n \rightharpoonup p$ and $u_n - \Pi_D u_n \to 0$. It is clear that $\Pi_D x_n \rightharpoonup p$. Furthermore, since F is uniformly smooth, we have that $\|J_F u_n - J_F \Pi_D u_n\| \to 0$. Since Π_D is the generalized projection of F onto D, we have that

$$\langle \Pi_D u_n - \Pi_D p, J_F u_n - J_F \Pi_D u_n - (J_F p - J_F \Pi_D p)\rangle \geq 0.$$

Therefore, $\langle p - \Pi_D p, -(J_F p - J_F \Pi_D p)\rangle \geq 0$. This implies that

$$\phi_F(p, \Pi_D p) + \phi_F(\Pi_D p, p) \leq 0$$

and hence $p = \Pi_D p$. Therefore, Π_D is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

Using Theorem 3.1, we have a new strong convergence theorem which is connected with the split common null point problem in two Banach spaces.

Theorem 4.6. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let G_1 and G_2 be maximal monotone operators of E into E^* and F into F^*, respectively. Let J_λ be the metric resolvent of G_1 for $\lambda > 0$ and let Q_μ be the generalized resolvent of G_2 for $\mu > 0$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G_1^{-1}0 \cap A^{-1}(G_2^{-1}0) \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F Q_\mu A x_n)\big), \quad y_n = J_\lambda z_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Ax_n, Q_\mu A x_n) \\ \qquad\qquad\qquad\qquad \text{and} \quad \langle z_n - z, J_E(z_n - y_n)\rangle \geq \|z_n - y_n\|^2 \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0,\infty)$ and $a, b \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{b}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G_1^{-1}0 \cap A^{-1}(G_2^{-1}0)$, where $z_0 = \Pi_{G_1^{-1}0 \cap A^{-1}(G_2^{-1}0)} x_1$.

Proof. Putting $B = 0$ in Theorem 3.1, we have $J_{\lambda_n} = I$ for all $n \in \mathbb{N}$. Furthermore, since J_λ is the metric resolvent of G_1 on E, J_λ is (-1)-demimetric from (4) in Examples. Furthemore, since Q_μ is the generalized resolvent of G_2 on F, from (6) in Examples, Q_μ is $(0,0)$-demigeneralized. We also have that if $\{v_n\}$ is a sequence in E such that $v_n \rightharpoonup q$ and $v_n - J_\lambda v_n \to 0$, then $q = J_\lambda q$. In fact, assume that $v_n \rightharpoonup q$ and $v_n - J_\lambda v_n \to 0$. It is clear that $J_\lambda v_n \rightharpoonup q$. Furthermore, we have that $\|J_E(v_n - J_\lambda v_n)\| \to 0$. Since J_λ is the metric resolvent of G_1, we have from [4] that

$$\langle J_\lambda v_n - J_\lambda q, J_E(v_n - J_\lambda v_n) - J_E(q - J_\lambda q)\rangle \geq 0.$$

Therefore, $\langle q - J_\lambda q, -J_E(q - J_\lambda q)\rangle \geq 0$. This implies that $\|q - J_\lambda q\|^2 \leq 0$ and hence $q = J_\lambda q$. Therefore, J_λ is demiclosed. Similarly, we can show that Q_μ is demiclosed. In fact, assume that $u_n \rightharpoonup p$ and $u_n - Q_\mu u_n \to 0$. It is clear that $Q_\mu u_n \rightharpoonup p$. Since F is uniformly smooth, we have that $\|J_F u_n - J_F Q_\mu u_n\| \to 0$. Since Q_μ is the generalized resolvent of G_2, we have from [4] that

$$\langle Q_\mu u_n - Q_\mu p, J_F u_n - J_F Q_\mu u_n - (J_F p - J_F Q_\mu p)\rangle \geq 0.$$

Therefore, $\langle p - Q_\mu p, -(J_F p - J_F Q_\mu p)\rangle \geq 0$. This implies that

$$\phi_F(p, Q_\mu p) + \phi_F(Q_\mu p, p) \leq 0$$

and hence $p = Q_\mu p$. Therefore, Q_μ is demiclosed. Therefore, we have the desired result from Theorem 3.1. $\square$

Similarly, using Theorem 3.2 and the proofs in Theorems 4.5 and 4.6, we have the following strong convergence theorems under the shrinking projection method which are connected with the split feasibility problem and the split common null point problem in two Banach spaces.

Theorem 4.7. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and F, respectively. Let Π_C be the generalized projection of E onto C and let P_D be the metric projection of F onto D. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n J_E^{-1} A^*(J_F A x_n - P_D A x_n), \quad y_n = \Pi_C z_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle z_n - z, J_E z_n - J_E y_n \rangle \geq \phi_E(z_n, y_n) \\ \qquad\qquad\qquad\qquad \text{and} \quad \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{2}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = P_{C \cap A^{-1}D} x_1$.

Theorem 4.8. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let G_1 and G_2 be maximal monotone operators of E into E^* and F into F^*, respectively. Let Q_μ be the generalized resolvent of G_1 for $\mu > 0$ and let J_λ be the metric resolvent of G_2 for $\lambda > 0$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G_1^{-1}0 \cap A^{-1}(G_2^{-1}0) \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n J_E^{-1} A^* J_F(A x_n - J_\lambda A x_n), \quad y_n = Q_\mu z_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle z_n - z, J_E z_n - J_E y_n \rangle \geq \phi_E(z_n, y_n) \\ \qquad\qquad\qquad\qquad \text{and} \quad \langle z_n - z, J_E(x_n - z_n) \rangle \geq 0 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities

$$0 < a \leq r_n \leq \frac{2}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G_1^{-1}0 \cap A^{-1}(G_2^{-1}0)$, where $z_0 = P_{G_1^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

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