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Existence of Solutions Belonging to a Tube for Non-Convex Sweeping Processes

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We establish existence results of solutions belonging to a tube for non-convex sweeping processes. Our approach mainly combines features belonging to the class of prox-regular sets, sweeping processes associated with this type of sets and fixed point theorems techniques.

Keywords: Differential inclusion, sweeping processes, solution tube, prox-regular sets, fixed point theorems.

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1. Introduction

In this paper we study the following differential inclusion

$$\begin{cases} x'(t) \in -N_{C(t)}(x(t)) + F(t, x(t)), & a.e.\ t \in [0, T], \\ x(0) = x_0 \in C(0), \\ x(t) \in C(t),\ \forall t \in [0, T]. \end{cases} \tag{1}$$

Here $T \geq 0$ is a non-negative real number and for each $t \in [0, T]$, $C(t)$ is a nonempty closed subset of $\mathbb{R}^N$. Moreover, $N_{C(t)}(x(\cdot))$ is the normal cone to $C(t)$ at $x(t) \in C(t)$ for all $t \in [0, T]$ and $F : [0, T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is a multivalued mapping. The differential inclusion (1) is generally called a time dependent sweeping process.

The sweeping processes were introduced by Moreau in the early 1970s (see for example [35]). Since then, they have been studied under various assumptions by several authors and have found a great number of applications. Indeed, the sweeping processes are very useful in many different areas. In particular, they have been played an important role in mechanics (see for example [36, 34]). The existence and uniqueness of solutions was studied by Colombo and Monteiro Marques [13] for a system without perturbation, when the sets $C(t)$ are prox-regular and contained in the interior of a suitable subset moving continuously. Thibault [46] showed the existence of a solution of (1) when the sets $C(t)$ are non-convex moving in an absolutely con-

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tinuous way and F is a scalarly upper semicontinuous multivalued mapping. For more information on the perturbed time-dependent sweeping process, we refer to [7, 10, 17]. The state dependent sweeping process i. e. when $C(t)$ is replaced by $C(t,x)$ was studied in [32] for the convex case and in [27, 11] for the non-convex case. Moreover, in [47], Tolstonogov and Timoshin obtained the existence of a solution for an F having different semicontinuity types at different points of it's domain.

Additionally, numerical methods are developed in [9] for the study of systems composed of interacting rigid bodies. One of the methods uses the Moreau's sweeping process, which is similar to a reflected problem. Finally, let us mention that some optimal control problems governed by variational inequalities have been also studied in several works as for instance [3, 5, 26, 45, 43, 44] and the references therein.

In this paper we are interested in finding solutions belonging to a tube for the problem (1) when the sets $C(t)$ are non-convex. For this purpose we use the concept of solution-tube. This type of solutions extend the Hartman condition [28] and the notion of upper-lower solutions. The notion of solutions-tube was introduced by Frigon in [19] for a differential inclusion with periodic solutions of the form $x'(t) \in F(t, x(t))$ and $x(0) = x(1)$.

For more details, we refer the reader to [18, 20, 23, 24]. One of the motivations for studying such solutions is given by the fact that an important technique to prove existence results for differential equations is the well-known method of lower and upper solutions described for instance in [15, 37, 40].

One important feature of the solution-tube method is the fact that it allows to obtain the existence of a solution of the considered problem lying in a tube. More precisely, we have information about the existence and the location of the solutions. Another very powerful property is the fact that it allows to obtain multiplicity results, i. e. the existence of several solutions for differential inclusions located in disjoint tubes. Note that our approach does not use only viability results as in [2]. Firstly, we employ several techniques developed in [46] for sweeping processes. Secondly, we use an approach inspired by [20] where fixed point theorems are used in order to establish existence results for systems of first order differential inclusions with maximal monotone terms and periodic boundary conditions. Note that one major difficulty we encounter in our study is the fact that, in the prox-regular case, we do not have maximal monotonicity for the normal cone.

A valuable contribution of our results consists in dealing with multivalued first order differential inclusion involving a hypo-monotone term and with an initial value condition. Additionally, this term is depending on time (time-dependent sweeping process). More precisely, in this paper, we suppose that the multivalued mapping $C(\cdot) : [0,T] \rightrightarrows \mathbb{R}^N$ moves in an absolutely continuous way and has prox-regular images. As mentioned before we are able to treat two cases, i.e. the multivalued perturbation F satisfies an upper a lower semi-continuity condition. The techniques we use to treat this two cases are quite different from each other. More precisely, we obtain our main results using existence results for sweeping processes (see [46]) in the u.s.c. case and Schauder's fixed point theorem (see [16]) in the L.s.c. case. For more applications of the solution-tube method see e. g. [48] for apriori bounds for the solutions of semilinear second order differential inclusions and [21, 22] for multiplicity results for systems of first order differential inclusions.

Our study is carried out in the non-convex framework, which to our knowledge has not been considered before. The subclass of prox-regular functions was introduced by Rockafellar in [41] as the class of lower-C^2 functions in the finite-dimensional setting. Later, Poliquin and Rockafellar in [38] identified, in the finite-dimensional setting, the large class of prox-regular functions, which includes lower-C^2 and strongly amenable functions, as well as, the concept of prox-regularity for sets via the Mordukhovich (limiting) normal cone. Thereafter, Poliquin et al. studied the prox-regularity of sets in [39] in Hilbert spaces. In [4], Bernard and Thibault studied the local uniform prox-regularity of functions in Hilbert spaces. Moreover, they made a connection between the lower-C^2 property and prox-regularity of epigraphs. This class of prox-regular functions covers all lower semicontinuous, proper, convex functions, lower-C^2 functions and strongly amenable functions, hence a large core of functions of interest in variational analysis and optimization.

Our paper is organized as follows: Section 2 is reserved to some preliminaries. In the third section, we present without proofs the main existence results. In Section 4 we introduce and study several operators which will be used in our proofs. The proofs of the existence of solutions for (1) when the images of the multivalued mapping $C(\cdot)$ are uniformly prox-regular are presented in Section 5.

2. Preliminaries

This section is dedicated to some preliminaries. We start by introducing several notations. Let $C([0,T],\mathbb{R}^N)$ denote the space of continuous functions endowed with the norm $\|\cdot\|_0$[1] and $L^p([0,T],\mathbb{R}^N)$ be the space of L^p-integrable functions with the usual norm $\|\cdot\|_{L^p}$ for $p\in[1,\infty]$. Recall that

$$W^{1,p}([0,T],\mathbb{R}^N)=\left\{x(\cdot)\in C([0,T],\mathbb{R}^N)\colon \begin{array}{l}\text{such that } x(\cdot) \text{ is absolutely conti-}\\ \text{nuous and } x'(\cdot)\in L^p([0,T],\mathbb{R}^N)\end{array}\right\}$$

and its usual norm is denoted by $\|\cdot\|_{1,p}$. Finally, we consider the subset $\mathcal{A}_p(x_0)$ of $W^{1,p}([0,T],\mathbb{R}^N)$ defined by $\mathcal{A}_p(x_0)=\{x\in W^{1,p}([0,T],\mathbb{R}^N),\ \text{such that } x(0)=x_0\}$.

Let us recall some properties of multivalued mappings. Let X,Y be topological spaces and Ω_1,Ω_2 measurable spaces. Let $G: X\rightrightarrows Y$ a multivalued mapping. G is said to be upper semicontinuous (u.s.c) if, for every closed set $U\subset Y$, the set $\{x\in X: G(x)\cap U\neq\emptyset\}$ is closed in X. Similarly, we say that G is lower semicontinuous (l.s.c) if, for every open set $U\subset Y$, the set $\{x\in X: G(x)\cap U\neq\emptyset\}$ is open in X. Here, $\mathrm{D}(G)=\{x\in X|\ G(x)\neq\emptyset\}$ is the domain of definition of G and $\mathrm{Gr}(G)=\{(x,y)\in X\times Y\mid y\in G(x)\}$ is the graph of G. Moreover, consider $G:\Omega_1\to\Omega_2$ a multivalued mapping. We say that G is measurable, if for all measurable sets $U\subseteq\Omega_2$, the set $\{w\in\Omega_1: G(w)\cap U\neq\emptyset\}$ is measurable. For more informations on multivalued mappings we refer to [1, 6, 14, 30, 31] and the references therein.

For any closed set K of $\mathbb{R}^N$, we denote by $d_K(\cdot)$ the usual distance function associated with K i.e., $d_K(x)=\inf_{y\in K}\|x-y\|$. For any $x\in\mathbb{R}^N$ and $r\geq 0$, we denote by $B(x,r)$ (resp. $\overline{B}(x,r)$) the open (resp. closed) ball centered in x with radius r. For $x=0$ and $r=1$ we will use instead $\mathbb{B}$ in place of $\overline{B}(0,1)$. For any subset K of $\mathbb{R}^N$, ones denote by $\overline{co}K$ the closed convex hull of K.

[1] Recall that we have $\|f\|_0=\sup_{t\in[0,T]}|f(t)|$.

We continue with the presentation of several important definitions. Note that, proximal regularity can be defined for sets as well as for functions. We focus our attention on sets and their normal cones in this paper. For this purpose we consider in the following not necessarily convex sets $K \subset \mathbb{R}^N$.

Recall that a vector $\mu \in \mathbb{R}^N$ is a proximal normal vector of K at $x \in K$ if and only if there are constant real numbers $\sigma > 0$ and $\zeta > 0$ such that

$$\langle \mu, x' - x\rangle \le \sigma \|x' - x\|^2 \quad \text{for all } x' \in K \cap B(x, \zeta).$$

The proximal cone is given by the equality (see [12])

$$N_K^P(x) = \{\mu \in \mathbb{R}^N : \exists \rho > 0 \text{ such that } x \in \mathrm{proj}_K(x + \rho\mu)\},$$

where $$\mathrm{proj}_K(u) = \{y \in K : d_K(x) = \|u - y\|\}.$$

One also defines the *limiting normal cone* by

$$N_K^L(x) = \{\zeta \in \mathbb{R}^N : \zeta_n \to \zeta, \zeta_n \in N_K^P(x_n), x_n \xrightarrow{K} x\}.$$

Here $x_n \xrightarrow{K} x$ means that $x_n \to x$ with $x_n \in K$ for all n. Moreover, recall that the *Clarke normal cone* is given by

$$N_K(x) = \overline{co} N_K^L(x).$$

The uniformity of the positive constant ρ for the unit proximal normal vectors to K leads to the concept of uniformly ρ-prox-regular sets. In general, we only have $N_K^P(x) \subset N_K(x)$ and the inclusion can be strict. However, for a ρ-prox-regular set K, we have $N_K^P(x) = N_K(x)$, see [39]. Uniform prox-regularity defined for instance in [39] is as follows.

Definition 2.1. (cf. [39]) A closed set $K \subset \mathbb{R}^N$ is *ρ-prox-regular* (or *uniformly prox-regular* with constant $\rho^{-1} > 0$ if whenever $x \in K$ and $\mu \in N_K(x)$ with $\|\mu\| < 1$, then x is the unique nearest point of K to $x + \rho\mu$.

The condition in the previous definition means that for any $x \in K$ and any unit normal vector $\mu \in N_K(x)$ to K at x one has

$$x \in \mathrm{proj}_K(x + \rho\mu),$$

which is equivalent to $\langle \mu, x' - x\rangle \le \dfrac{1}{2\rho}\|x' - x\|^2, \quad$ for $x' \in K$.

Here, we make the convention $\frac{1}{\rho} = 0$ for $\rho = +\infty$. Note that for $\rho = \infty$ the uniform ρ-prox-regularity of the closed sets $C(t)$ is equivalent to the convexity of the sets.

We recall that in the case of prox regular sets we need to work with the truncated mapping $N_K^1 : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ defined by (see, [39])

$$N_K^1(x) = \begin{cases} N_K(x) \cap \mathbb{B} & \text{if } x \in K, \\ \emptyset & \text{if } x \notin K. \end{cases}$$

Let us mention that throughout this section f will denote a single valued function from $\mathbb{R}^N$ into $\mathbb{R} \cup \{+\infty\}$. We continue by recalling several definitions.

Definition 2.2. (proximal subdifferential [42]) A vector $x^* \in \mathbb{R}^N$ is in the *proximal subdifferential* $\partial_P f(x)$ of f at a point x if there exist constant real numbers $\sigma > 0$ and $\varepsilon > 0$ such that

$$\langle x^*, x' - x\rangle \leq f(x') - f(x) + \sigma \|x' - x\|^2 \quad \text{for all } x' \in B(x, \varepsilon).$$

Let us consider the sequential limit

$$\limsup_{x \to_f \tilde{x}} \partial_P f(x) = \{w = \lim \bar{x}_n : \bar{x}_n \in \partial_P f(x_n),\ x_n \to_f \tilde{x}\}.$$

Here $x_n \to_f \tilde{x}$ means that $\|x_n - \tilde{x}\| \to 0$ and $f(x_n) \to f(\tilde{x})$. This set is generally called *the limiting subdifferential* of f at $\tilde{x}$ and it is denoted by $\partial f(\tilde{x})$.

The indicator function I_K of K is defined by $I_K(x) = 0$ if $x \in K$ and $I_K(x) = +\infty$ otherwise. The proximal normal cone $N_K^P(x)$ of K at x is the proximal subdifferential $\partial_P I_K(x)$.

The proximal normal cone is related also to the distance function to K via the following equality, (see, e.g, [12])

$$\partial_P d_K(x) = N_K^P(x) \cap \mathbb{B} \text{ and } N_K^P(x) = \mathbb{R}^+ \partial_P d_K(x) \text{ for } x \in K,$$

where $\mathbb{R}^+$ denotes the set of non-negative real numbers. Moreover we have that

$$\partial_P d_K(x) = \partial d_K(x) \text{ for } x \in K,$$

whenever K is prox-regular.

Definition 2.3. A multivalued mapping A from $\mathbb{R}^N$ into $\mathbb{R}^N$ is called *monotone* if and only if

$$\forall x_1, x_2 \in \mathrm{D}(A), \forall v_i \in A(x_i), i = 1, 2,\ \langle v_1 - v_2, x_1 - x_2\rangle \geq 0.$$

A monotone multivalued mapping is maximal if there is no other monotone multivalued mapping $\tilde{A}$ whose graph contains strictly the graph of A.

An equivalent formulation is the following: A is maximal if

$$\langle v_1 - v_2, x_1 - x_2\rangle \geq 0 \quad \forall x_1 \in \mathrm{D}(A) \text{ and } \forall v_1 \in A(x_1) \Rightarrow x_2 \in \mathrm{D}(A) \text{ and } v_2 \in A(x_2).$$

Recall that for a closed, convex set K in $\mathbb{R}^N$, the multivalued map $N_K : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is maximal monotone. Moreover, $\mathrm{D}(N_K) = K$ because, by definition, we have $N_K(x) = \emptyset$ for all $x \notin K$. In the prox-regular setting we need a weaker notion of monotonicity that we present in the following.

Definition 2.4. A multivalued mapping $T : \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is r-hypo-monotone on a set $E \subset \mathbb{R}^N$ if $T + r\,\mathrm{id}$ is monotone on E. If $E = \mathbb{R}^N$ we will say that T is r-hypo-monotone.

We finish this section by recalling that prox-regularity of a set K is linked with hypo-monotonicity of $N_K^1(\cdot)$ (see for instance [39]).

Theorem 2.5. *Let K be a nonempty closed set and let $\rho > 0$. Then the following are equivalent:*

(a) *The set K is ρ-prox-regular;*

(b) *$N_K^1(\cdot) + \rho^{-1} id$ is monotone on K. More precisely, one has*

$$\langle \mu_2 - \mu_1, x_2 - x_1 \rangle \geq -\frac{1}{\rho} \|x_2 - x_1\|^2,$$

whenever $x_i \in K$ and $\mu_i \in N_K^1(x_i)$ with $i = 1, 2$.

3. Main results

The aim of this section is to present without proof existence results for the problem (1) i.e., the existence of an absolutely continuous function $x(\cdot) \in \mathcal{A}_2(x_0)$ satisfying (1) and which lies in a tube.

We adapt to our setting the notion of L^2-solution-tube of the problem (1) introduced for instance in the paper of M. Frigon [24] for the periodic case.

Definition 3.1. Let $\alpha(\cdot) \in W^{1,2}([0,T], \mathbb{R}^N)$, $\beta(\cdot) \in W^{1,2}([0,T], \mathbb{R})$ and $\rho > 0$. We say that (α, β) is an L^2*-solution-tube* of (1) if there exists $\delta \in L^2([0,T], \mathbb{R}^N)$ and an absolutely continuous function $w : [0,T] \to \mathbb{R}$ such that

1. $\delta(t) \in N_{C(t)}(\alpha(t))$ and $\|\delta(t)\| \leq w'(t)$ a.e. $t \in [0,T]$,
2. for a.e. $t \in [0,T]$, and every $x \in \mathbb{R}^N$ such that $\|x - \alpha(t)\| = \beta(t)$, there exists $\nu \in F(t,x)$ such that $\langle x - \alpha(t), \nu - \delta(t) - \alpha'(t) \rangle \leq \beta(t)\beta'(t) - \frac{w'(t)}{\rho}\beta^2(t)$,
3. $\alpha'(t) \in -\delta(t) + F(t, \alpha(t))$ a.e. on $\{t \in [0,T] : \beta(t) = 0\}$,
4. $\|x(0) - \alpha(0)\| \leq \beta(0)$.

We define $B_{\beta(\cdot)}(\alpha(\cdot)) = \{x \in C([0,T], \mathbb{R}^N) : \|x(t) - \alpha(t)\| \leq \beta(t),\ \forall t \in [0,T]$.

In order to establish our main existence results, we need to introduce several assumptions:

($\mathcal{P}$) There exists a constant $\rho > 0$ such that, the sets $C(t)$ are (uniformly) ρ-prox-regular, for all $t \in [0,T]$.

($\mathcal{H}$) There exists a non-decreasing absolutely continuous function $v : [0,T] \to \mathbb{R}$ such that, for all $s, t \in [0,T]$ and all $x \in \mathbb{R}^N$ we have

$$|d_{C(t)}(x) - d_{C(s)}(x)| \leq |v(t) - v(s)|.$$

($\mathcal{H}_u$) $F : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is a multivalued mapping with compact convex values such that

(i) $t \mapsto F(t,x)$ is measurable for all $x \in \mathbb{R}^N$,

(ii) $x \mapsto F(t,x)$ is u.s.c. a.e. $t \in [0,T]$.

($\mathcal{H}_l$) $F : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ is a multivalued mapping with compact values such that

(i) $x \mapsto F(t,x)$ is l.s.c. a.e. $t \in [0,T]$,

(ii) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}$-measurable.

Here $[0,T] \times \mathbb{R}^N$ is endowed with the σ-algebra generated by subsets $C \times D$, where $C \subset [0,T]$ and $D \subset \mathbb{R}^N$ are, respectively, Lebesgue and Borel measurable.

$(\mathcal{H}_k)$ For every $k \geq 0$, there exists $l_k \in L^2([0,T])$ such that

$$\max\{\|\mu\| : \mu \in F(t,x), \|x\| \leq k\} \leq l_k(t) \quad \text{a.e. } t \in [0,T].$$

$(\mathcal{S}-\mathcal{L}^2)$ There exists $(\alpha(\cdot),\beta(\cdot)) \in W^{1,2}([0,T],\mathbb{R}^N) \times W^{1,2}([0,T],[0,\infty))$ an L^2-solution-tube of (1).

It is easy to see that $(\mathcal{H})$ implies that $C(\cdot)$ moves in an absolutely continuous way with respect with the Hausdorff distance. Let us state the main existence results in the following. Note that the proofs are postponed in the next section.

We denote by $w(\cdot)$ the absolutely continuous function given by

$$w(t) = \int_0^t (v'(s) + \hat{l}(s))ds \text{ for every } t \in [0,T]. \tag{2}$$

Here $v(\cdot)$ is the same as in $(\mathcal{P})$ and $\hat{l}(\cdot)$ is given as in Proposition 4.1 and Proposition 4.8(note that we can consider that $\hat{l}(\cdot)$ is the same for these two propositions).

For the rest of the paper we assume that $w(\cdot)$ is the function that we use in $(\mathcal{S}-\mathcal{L}^2)$ and that

$$2\int_0^t w(s)ds < \rho \text{ for every } t \in [0,T]. \tag{3}$$

Theorem 3.2. *Assume that $(\mathcal{P})$, $(\mathcal{H})$, $(\mathcal{H}_u)$, $(\mathcal{H}_k)$ and $(\mathcal{S}-\mathcal{L}^2)$ hold and $C(0)$ is bounded. Then the time depending sweeping process* (1) *has a solution in the space* $W^{1,2}([0,T],\mathbb{R}^N) \cap B_{\beta(\cdot)}(\alpha(\cdot))$.

Theorem 3.3. *Assume that $(\mathcal{P})$, $(\mathcal{H})$, $(\mathcal{H}_l)$, $(\mathcal{H}_k)$ and $(\mathcal{S}-\mathcal{L}^2)$ hold and $C(0)$ is bounded. Then the time depending sweeping process* (1) *has a solution in the space* $W^{1,2}([0,T],\mathbb{R}^N) \cap B_{\beta(\cdot)}(\alpha(\cdot))$.

4. Preliminary tools

We begin by constructing several operators which will be used in the proof of the main results. Note that they are inspired by the operators introduced in the paper of M. Frigon [20] that we adapted to the prox-regular framework.

Consider (α,β), δ and w be given as in $(S-L^2)$ and in Definition 3.1. We define

$$\overline{F} : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N \quad \text{by} \quad \overline{F} = \widehat{F} \cap \overline{G},$$

where
$$\widehat{F}(t,x) = \begin{cases} F(t,\widehat{x}_t) & \text{if } \|x-\alpha(t)\| > \beta(t), \\ F(t,x) & \text{if } \|x-\alpha(t)\| \leq \beta(t). \end{cases}$$

$$\overline{G}(t,x) = \begin{cases} \alpha'(t) + \delta(t), & \text{if } \beta(t) = 0, \\ \mathbb{R}^N, & \text{if } \|x-\alpha(t)\| \leq \beta(t) \\ & \text{and } \beta(t) > 0, \\ \{z : \langle x-\alpha(t), z-\delta(t)-\alpha'(t)\rangle & \\ \leq \beta'(t)\|x-\alpha(t)\| - \frac{w'(t)}{\rho}\|x-\alpha(t)\|^2\}, & \text{otherwise,} \end{cases}$$

and
$$\widehat{x}_t = \begin{cases} x & \text{if } \|x-\alpha(t)\| \le \beta(t), \\ \alpha(t) + \frac{\beta(t)}{\|x-\alpha(t)\|}(x-\alpha(t)) & \text{if } \|x-\alpha(t)\| > \beta(t). \end{cases} \tag{4}$$

Similarly, we define

$$\underline{F} : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N \quad \text{by} \quad \underline{F} = \widehat{F} \cap \underline{G},$$

where

$$\underline{G}(t,x) = \begin{cases} \alpha'(t) + \delta(t), & \text{if } \beta(t) = 0, \\ \mathbb{R}^N, & \text{if } \|x-\alpha(t)\| < \beta(t), \\ \{z : \langle x-\alpha(t), z-\delta(t)-\alpha'(t)\rangle \le & \\ \beta'(t)\|x-\alpha(t)\| - \frac{w'(t)}{\rho}\|x-\alpha(t)\|^2\}, & \text{otherwise.} \end{cases}$$

Proposition 4.1. *Assume that $(\mathcal{H}_u)$, $(\mathcal{H}_k)$ and $(\mathcal{S}-\mathcal{L}^2)$ hold and $C(0)$ is bounded. Then the multimap $(t,x) \to \overline{F}(t,x) - x + \widehat{x}_t$ defined on $[0,T]\times\mathbb{R}^N$ and with values in $\mathbb{R}^N$ has nonempty, closed convex values. Moreover, it is measurable with respect to t for all $x \in \mathbb{R}^N$ and it is u.s.c. with respect to x a.e. $t \in [0,T]$. Furthermore, there exists $\hat{l}(\cdot) \in L^2([0,T],[0,\infty))$ such that $\|\overline{F}(t,x) - x + \widehat{x}_t\| \le \hat{l}(t)$ a.e. $t \in [0,T]$ and for all $x \in C(t)$.*

Proof. First, notice that, under the assumptions $(\mathcal{H}_u)$, $(\mathcal{H}_k)$ and $(\mathcal{S}-\mathcal{L}^2)$, the mapping $\overline{F}$ has nonempty, closed convex values, $t \mapsto \overline{F}(t,x)$ is measurable for all $x \in \mathbb{R}^N$ and $x \mapsto \overline{F}(t,x)$ is u.s.c. a.e. $t \in [0,T]$.

Secondly, there exists $\hat{l} \in L^2([0,T],[0,\infty))$ such that

$$\|\overline{F}(t,x) - x + \widehat{x}_t\| \le \hat{l}(t) \tag{5}$$

a.e. $t \in [0,T]$ and for all $x \in C(t)$. Indeed, the previous estimation follows from $(\mathcal{H}_k)$ and the fact that $\|\widehat{x}_t - x\| \le \|\alpha(t)\| + |\beta(t)| + \|x\|$ for all $t \in [0,T]$ and for all $x \in C(t)$. We conclude by observing that $C(0)$ bounded implies that the image of $C(\cdot)$ is bounded. □

Proposition 4.2. *Assume that $(\mathcal{H}_l)$, $(\mathcal{H}_k)$, and $(\mathcal{S}-\mathcal{L}^2)$ hold and $C(0)$ is bounded. Then there exists a continuous single valued map $f : C([0,T],\mathbb{R}^N) \to L^1([0,T],\mathbb{R}^N)$, such that*

$$f(x)(t) \in \underline{F}(t,x(t)) + \widehat{x(t)}_t \quad \text{a.e. } t \in [0,T]$$

and for every $x(\cdot) \in C([0,T],\mathbb{R}^N)$. Furthermore, there exists $l(\cdot) \in L^2([0,T],[0,\infty))$ such that $\|f(x)(t)\| \le l(t)$ a.e. $t \in [0,T]$ and for every $x(\cdot) \in C([0,T],\mathbb{R}^N)$. Consequently, we have

$$f(C([0,T],\mathbb{R}^N)) \subset \{\nu \in L^2([0,T],\mathbb{R}^N) : \|\nu(t)\| \le l(t) \text{ a.e. } t \in [0,T]\}.$$

Proof. $\underline{F}$ has nonempty values because of $(\mathcal{S}-\mathcal{L}^2)$. Moreover, we will prove that $(t,x) \mapsto \underline{F}(t,x)$ is $\mathcal{L}\otimes\mathcal{B}$-measurable, and $x \mapsto \underline{F}(t,x)$ is l.s.c. a.e. $t \in [0,T]$. Indeed, let $U \subset \mathbb{R}^N$ be open, and denote by $O = \{x \in \mathbb{R}^N : \underline{F}(t,x) \cap U \ne \emptyset\}$.

We have several cases. If $\beta(t) = 0$, $\underline{F}(t,x) = \alpha'(t) + \delta(t)$ for all $x \in \mathbb{R}^N$, and therefore O is open. Let $\beta(t) > 0$ and consider $x \in O$. Whenever $\|x - \alpha(t)\| < \beta(t)$, there exists $\zeta > 0$ such that $\|u - \alpha(t)\| < \beta(t)$ for all $u \in B(x, \zeta)$. Note that $\underline{F}(t, u) = F(t, u)$ for all $u \in B(x, \zeta)$. Consequently, the lower semicontinuity of F with respect to its second variable implies that there exists a neighborhood of x in O and therefore O is open.

Moreover, if $\|x - \alpha(t)\| \geq \beta(t)$, there exists

$$z_0 \in U \cap F(t, \widehat{x}_t) \cap \{z : \langle x - \alpha(t), z - \delta(t) - \alpha'(t)\rangle \leq \beta'(t)\|x - \alpha(t)\| - \frac{w'(t)}{\rho}\|x - \alpha(t)\|^2\}.$$

Set $\varepsilon > 0$ such that $B(z_0, \varepsilon) \subset U$ and fix $\mu = -\varepsilon(x - \alpha(t))/(2\|x - \alpha(t)\|)$. For y such that $\|y\| \leq 1$ and $\lambda \geq 0$ we have

$$\begin{aligned}
&\langle x + \lambda y - \alpha(t), z_0 + \mu - \delta(t) - \alpha'(t)\rangle \\
&= \langle x - \alpha(t), z_0 - \delta(t) - \alpha'(t)\rangle + \lambda\langle y, z_0 + \mu - \delta(t) - \alpha'(t)\rangle + \langle x - \alpha(t), \mu\rangle \\
&\leq \|x - \alpha(t)\|\beta'(t) - \frac{w'(t)}{\rho}\|x - \alpha(t)\|^2 + \lambda\|z_0 + \mu - \delta(t) - \alpha'(t)\| \\
&\qquad\qquad - (\varepsilon/2)\|x - \alpha(t)\|.
\end{aligned}$$

Additionally, we obtain that

$$\|x + \lambda y - \alpha(t)\|^2 \leq \|x - \alpha(t)\|^2 + \lambda(1 + 2\|x - \alpha(t)\|).$$

Therefore we have

$$\begin{aligned}
&\langle x + \lambda y - \alpha(t), z_0 + \mu - \delta(t) - \alpha'(t)\rangle \\
&\leq \|x + \lambda y - \alpha(t)\|\beta'(t) - \rho\|x + \lambda y - \alpha(t)\|^2 + (\frac{w'(t)}{\rho}\lambda)(1 + 2\|x - \alpha(t)\|) \\
&\quad + |\beta'(t)| + \|z_0 + \mu - \delta(t) - \alpha'(t)\|) - (\varepsilon/2)\|x - \alpha(t)\|.
\end{aligned}$$

For

$$0 < \lambda_0 < \varepsilon\|x - \alpha(t)\|/2(\frac{w'(t)}{\rho}(1 + 2\|x - \alpha(t)\|) + |\beta'(t)| + \|z_0 + \mu - \delta(t) - \alpha'(t)\|),$$

we have, for all $u \in B(x, \lambda_0)$, that

$$\begin{aligned}
z_0 + \mu \in \{z : \langle u - \alpha(t), z - \delta(t) - \alpha'(t)\rangle \\
\leq \beta'(t)\|u - \alpha(t)\| - \frac{w'(t)}{\rho}\|u - \alpha(t)\|^2\} \cap U,
\end{aligned}$$

Furthermore, the lower semicontinuity of F with respect to its second variable ensures that there exists $\lambda_1 > 0$ such that $B(x, \lambda_1) \subset \{u : F(t, \widehat{u}_t) \cap U \neq \emptyset\}$. Hence,

$$\begin{aligned}
B(x, \lambda_2) \subset \{u \in \mathbb{R}^N : U \cap F(t, \widehat{u}_t) \cap \{z : \langle u - \alpha(t), z - \delta(t) - \alpha'(t)\rangle \\
\leq \beta'(t)\|u - \alpha(t)\| - \frac{w'(t)}{\rho}\|u - \alpha(t)\|^2\} \neq \emptyset\} \subset O,
\end{aligned}$$

where $\lambda_2 = min\{\lambda_0, \lambda_1\}$. Consequently, O is an open set.

Moreover, following the same arguments as in [20] we can prove that assumptions $(\mathcal{H}_k)$ and $(\mathcal{S}-\mathcal{L}^2)$ guarantee that, there exists $l \in L^2([0,T],\mathbb{R}^N)$ such that $\underline{F}(t,x(t)) + \widehat{x(t)}_t \subset B(0,l(t))$ a.e. $t \in [0,T]$ and for every $x(\cdot) \in C([0,T],\mathbb{R}^N)$, because $\|\widehat{x(t)}_t\| \leq \|\alpha(t)\| + |\beta(t)|$ for all $t \in [0,T]$.

Furthermore, we conclude that, the mapping $t \mapsto \underline{F}(t,x(t))$ is measurable with closed nonempty values for all $x \in C([0,T],\mathbb{R}^N)$ (see [29]). By the Kuratowski, Ryll-Nardzewski selection theorem [33], this mapping has a measurable selection. Consequently the mapping $\underline{\mathfrak{F}} : C([0,T],\mathbb{R}^N) \rightrightarrows L^2([0,T],\mathbb{R}^N)$ defined by

$$\underline{\mathfrak{F}}(x) = \{\nu \in L^2([0,T],\mathbb{R}^N) : \nu(t) \in \underline{F}(t,x(t)) + \widehat{x(t)}_t \text{ a.e. } t \in [0,T]\}$$

has bounded nonempty values. Furthermore, $\underline{\mathfrak{F}}$ is l.s.c. with closed, decomposable values (i.e. for every $\nu, z \in \underline{\mathfrak{F}}(x)$ and every measurable set $\Omega \subset [0,T]$ we have $\nu\chi_\Omega + z\chi_{\Omega^c} \in \underline{\mathfrak{F}}(x)$).

Finally, Fryskowski, Bressan-Colombo selection theorem [8, 25] allows us to complete the proof. □

We continue by recalling the following results from [46]. Note that in [46] ρ may depend on t and the results are also stated in the infinite dimensional setting. We consider the following equations:

$$x'(t) \in -N_{C(t)}(x(t)) \text{ a.e. } t \in [0,T],\ x(t) \in C(t) \text{ for all } t \in [0,T] \text{ and } x(0) = x_0, \quad (6)$$

$$x'(t) \in -v'(t)\partial d_{C(t)}(x(t)) \text{ a.e. } t \in [0,T], \quad x(0) = x_0. \quad (7)$$

Here $v(\cdot)$ is the function given in $(\mathcal{H})$. We assume that $(\mathcal{P})$ holds and that

$$2\int_0^t v'(s)ds < \rho, \text{ for all } t \in [0,T]. \quad (8)$$

Proposition 4.3. *Assume that $(\mathcal{P})$, $(\mathcal{H})$ and (8) hold. Then, the differential inclusions (6) and (7) have the same unique solution $x(\cdot)$, and this solution satisfies $\|x'(t)\| \leq v'(t)$ for almost all $t \in [0,T]$.*

Consider a multivalued mapping $\varGamma : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ In addition, assume that there exists an integrable function $m(\cdot)$ over $[0,T]$ such that

$$\sup\{\|z\| : z \in \varGamma(t,x)\} \leq m(t) \text{ for all } x \in \mathbb{R}^N. \quad (9)$$

Consider that $(\mathcal{P})$ holds and denote by $n(\cdot)$ the absolutely continuous function defined by

$$n(t) = \int_0^t (v'(s) + m(s))ds, \text{ for all } t \in [0,T]. \quad (10)$$

Assume that we have

$$2\int_0^t n(s)ds < \rho, \text{ for all } t \in [0,T]. \quad (11)$$

We consider the following constrained differential inclusion

$$\begin{cases} x'(t) \in -N_{C(t)}(x(t)) + \varGamma(t, x(t)), \text{ a.e. } t \in [0, T], \\ x(t) \in C(t) \text{ for all } t \in [0, T] \text{ and } x(0) = x_0 \end{cases} \tag{12}$$

and the unconstrained differential inclusion

$$x'(t) \in -n'(t)\partial d_{C(t)}(x(t)) + \varGamma(t, x(t)) \text{ a.e. } t \in [0, T] \text{ and } x(0) = x_0. \tag{13}$$

Theorem 4.4. *Consider that* $(\mathcal{H})$, $(\mathcal{P})$ *and* (11) *hold. Let* $\varGamma : [0, T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ *be a multivalued mapping with nonempty compact convex values that is measurable with respect to* (t, x) *and u.s.c with respect to* x. *Then, the differential inclusions* (12) *and* (13) *have the same solution set and this set is nonempty. Furthermore, any solution* $x(\cdot)$ *satisfies*

$$\|x'(t)\| \leq v'(t) + 2m(t) \quad \textit{for almost all } t \in [0, T],$$

whenever, instead of (9) *we assume that for all* $t \in [0, T]$ *and all* $x \in C(t)$

$$|\varGamma(t, x)| \leq m(t).$$

The following results from [46] do not require prox-regularity.

Proposition 4.5. *Assume that* $(\mathcal{H})$ *holds. Then, there exists an absolutely continuous solution* $x(\cdot)$ *of the differential inclusion* (7) *satisfying the constraints*

$$x(t) \in C(t) \quad \textit{for all } t \in [0, T].$$

Moreover $x(\cdot)$ *satisfies* (6) *as well as the following estimation*

$$\|x'(t)\| \leq v'(t) \quad \textit{for almost all } t \in [0, T].$$

Theorem 4.6. *Assume that* $(\mathcal{H})$ *holds. Let* $\varGamma : [0, T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ *be a multivalued mapping with nonempty compact convex values that is measurable with respect to* (t, x) *and u.s.c with respect to* x. *Assume that there exists an integrable function* m *over* $[0, T]$ *such that for all* $t \in [0, T]$ *and all* $x \in C(t)$

$$\min\{\|z\| : z \in \varGamma(t, x)\} \leq m(t). \tag{14}$$

Then the perturbed sweeping process (12) *admits at least one absolutely continuous* $x(\cdot)$. *Moreover, this solution also satisfies* (13). *Furthermore, we have the following estimation*

$$\|x'(t)\| \leq v'(t) + m(t) \quad \textit{for almost all } t \in [0, T],$$

whenever, instead of (14) *we assume that for all* $t \in [0, T]$ *and all* $x \in C(t)$

$$|\varGamma(t, x)| \leq m(t).$$

Another important result for completing our proofs is the following theorem from [1]:

Theorem 4.7. *Consider a sequence of absolutely continuous functions* $\{x_k(t)\}_k$ *from an interval* I *of* $\mathbb{R}$ *to a Banach space* X *satisfying:*

(a) *the set* $\{x_k(t)\}_k$ *is relatively compact subset of* X*, for all* $t \in I$*,*

(b) *there exists a positive function* $c(\cdot) \in L^1(I)$ *such that* $\|x_k'(t)\| \leq c(t)$ *for almost all* $t \in I$*.*

Then there exists a subsequence also denoted $\{x_k(t)\}_k$ *converging to an absolutely continuous function* $x(\cdot)$ *from* I *to* X *in the sense that:*

(a) $\{x_k(t)\}_k$ *converges uniformly to* $x(\cdot)$ *over compact subsets of* I*,*

(b) $\{x_k'(t)\}_k$ *converges weakly to* $x'(\cdot)$ *in* $L^1(I, X)$*.*

The previous results enable us to construct an important tool that we need for completing our proof for the l.s.c. case. More precisely, we define the following operator $P : L^2([0,T], \mathbb{R}^N) \to W^{1,2}([0,T], \mathbb{R}^N)$ given by $P(y(\cdot)) = x(\cdot)$ where $x(\cdot) \in W^{1,2}([0,T], \mathbb{R}^N)$ is the solution of

$$x'(t) \in y(t) - x(t) - N_{C(t)}(x(t)) \text{ a. e. } t \in [0,T], \quad x(0) = x_0 \in C(0), \tag{15}$$

provided that $(\mathcal{P})$ holds. Let $l \in L^2([0,T], [0,\infty))$ and

$$V = \{\nu \in L^2([0,T], \mathbb{R}^N) : \|\nu(t)\| \leq l(t) \text{ a.e. } t \in [0,T]\}$$

supplied with the topology of $L^1([0,T], \mathbb{R}^N)$. Consider $y(\cdot) \in V$ and define the mapping $\Gamma : [0,T] \times \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ by

$$|\Gamma(t,x)| = y(t) + x.$$

Observe that there exists $\bar{l}(\cdot) \in L^2([0,T], [0,\infty))$ such that $|\Gamma(t,x)| \leq \bar{l}(t)$ for for all $t \in [0,T]$ and all $x \in C(t)$, whenever $C(0)$ is supposed bounded. Consequently $P(\cdot)$ is well defined due to Theorem 4.4. We can show that $P : V \to C([0,T], \mathbb{R}^N)$ is continuous and compact. Indeed, consider the absolutely continuous function $z(\cdot)$ given by

$$z(t) = \int_0^t (v'(s) + \bar{l}(s))ds \text{ for every } t \in [0,T]. \tag{16}$$

Here $v(\cdot)$ is the same as in $(\mathcal{P})$. We assume that

$$2\int_0^t z(s)ds < \rho \text{ for every } t \in [0,T]. \tag{17}$$

Proposition 4.8. *Assume that* $(\mathcal{P})$*,* $(\mathcal{H})$*,* (17) *hold and* $C(0)$ *is bounded. Then, the operator* $P : V \in L^2([0,T], \mathbb{R}^N) \to W^{1,2}([0,T], \mathbb{R}^N)$*, is continuous and compact, where* V *is supplied with the topology of* $L^1([0,T], \mathbb{R}^N)$ *and* $W^{1,2}([0,T], \mathbb{R}^N)$ *is supplied with the topology of* $C([0,T], \mathbb{R}^N)$*.*

Proof. Note that we have to show that, if $y_n \to y$ in $L^1([0,T], \mathbb{R}^N)$, then we have that $x_n = P(y_n) \to x = P(y)$ in $C([0,T], \mathbb{R}^N)$ when $n \to \infty$.

Indeed, for almost all $t \in [0,T]$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|x_n(t)-x(t)\|^2 = &\ \langle x_n'(t)-x'(t), x_n(t)-x(t)\rangle \\ \leq &\ \langle y_n - y(t), x_n(t)-x(t)\rangle + \left(\frac{z'(t)}{\rho}-1\right)\|x_n(t)-x(t)\|^2 \\ \leq &\ \|y_n(t)-y(t)\|\cdot 2M + \left(\frac{z'(t)}{\rho}-1\right)\|x_n(t)-x(t)\|^2\end{aligned}$$

where $M = \sup_{t\in[0,T]}\{\|c(t)\|, c(t)\in C(t)\}$. Then, for all $t\in[0,T]$ we have

$$\frac{1}{2}\|x_n(t)-x(t)\|^2 \leq 2M\int_0^t \|y_n(s)-y(s)\|ds + \int_0^t \left(\frac{z'(s)}{\rho}-1\right)\|x_n(s)-x(s)\|^2 ds.$$

Using Gronwall's Lemma, we obtain that

$$\begin{aligned}\frac{1}{2}\|x_n(t)-x(t)\|^2 \leq 2M\int_0^t \|y_n(s)-y(s)\|ds \\ + \int_0^t 4M\left(\frac{z'(s)}{\rho}-1\right)\left(\int_0^s \|y_n(\tau)-y(\tau)\|d\tau\right)e^{\int_s^t\left(\frac{z'(u)}{\rho}-1\right)du}ds.\end{aligned}$$

Since $y_n \to y$ in $L^1([0,T],\mathbb{R}^N)$ it follows that $x_n = P(y_n) \to x = P(y)$ uniformly on $[0,T]$. We deduce that P is continuous.

Moreover, let $\{y_n(\cdot)\}_n$ be an arbitrary sequence in V. By Proposition 4.3 we have

$$x_n'(t) \in y_n(t) - x_n(t) - N_{C(t)}(x_n(t))$$

is equivalent with $x_n'(t) \in y_n(t)-x_n(t)-z'(t)\partial d_{C(t)}(x_n(t))$ for almost every $t\in[0,T]$. We also have that $\{x_n(\cdot)\}_n \subseteq W^{1,2}([0,T],\mathbb{R}^N)$ and for every $t\in[0,T]$ the set $\{x_n(t)\}_n$ is relatively compact because the image of $C(\cdot)$ is bounded. Moreover,

$$\|x_n'(t)\| \leq 2\bar{l}(t) + v'(t)$$

for every $t\in[0,T]$. Observe that $2\bar{l}(\cdot)+v'(\cdot)\in L^1([0,T],\mathbb{R})$. Therefore, we can apply Theorem 4.7 we obtain that there exists a subsequence $\{x_{n_k}(\cdot)\}_k$ of $\{x_n(\cdot)\}_n$ that converges uniformly. Consequently, P is compact and the proof is finished. □

5. Proofs of the main results

In Section 3, we denoted by $w(\cdot)$ the absolutely continuous function given by

$$w(t) = \int_0^t (v'(s)+\hat{l}(s))ds \text{ for every } t\in[0,T]. \tag{18}$$

Recall that, here $v(\cdot)$ is the same as in $(\mathcal{P})$ and $\hat{l}(\cdot)$ is given as in Proposition 4.1 and Proposition 4.8(note that we can consider that $\hat{l}(\cdot)$ is the same for these two propositions).

Moreover we have assumed that $w(\cdot)$ is the function that we use in $(\mathcal{S}-\mathcal{L}^2)$ and

$$2\int_0^t w(s)ds < \rho \text{ for every } t \in [0,T]. \tag{19}$$

We consider the following two problems

$$\begin{cases} x'(t)+x(t) \in -N_{C(t)}(x(t)) + \overline{F}(t,x(t)) + \widehat{x(t)}_t, & a.e.\ t \in [0,T], \\ x(0) = x_0 \in C(0), \\ x(t) \in C(t),\ \forall t \in [0,T]; \end{cases} \tag{20}$$

and

$$\begin{cases} x'(t)+x(t) \in -N_{C(t)} + (x(t)) + \underline{F}(t,x(t)) + \widehat{x(t)}_t, & a.e.\ t \in [0,T], \\ x(0) = x_0 \in C(0), \\ x(t) \in C(t),\ \forall t \in [0,T]. \end{cases} \tag{21}$$

We continue by stating the following result.

Proposition 5.1. *Let $(S-L^2)$ and $(\mathcal{P})$ and $(\mathcal{H})$ be satisfied. Then every solution $x(\cdot) \in \mathcal{A}_2(x_0)$ of* (20) *or* (21) *belongs to $B_{\beta(\cdot)}(\alpha(\cdot))$.*

Proof. We follow some arguments presented in [24]. We assume that $x(\cdot)$ is a solution of (20). Consequently we know that there exists $\nu(\cdot), \delta_x(\cdot), z(\cdot) \in L^2([0,T],\mathbb{R}^N)$ such that $\delta_x(t) \in N_{C(t)}(x(t))$, $z(t) \in \overline{F}(t,x(t))$ with $\|\delta_x(t)\| \le w'(t)$ and

$$x'(t)+x(t) = -\delta_x(t) + z(t) + \widehat{x(t)}_t \text{ a.e. } t \in [0,T].$$

It remains to show that $\|x(t)-\alpha(t)\| \le \beta(t)$ for all $t \in [0,T]$.

Suppose that there exists $t_1 \in (0,T]$ with $\|x(t_1)-\alpha(t_1)\| > \beta(t_1)$. The condition (4) in Definition 3.1 implies that there exists $t_0 \in [0,t_1)$ such that $\|x(t_0)-\alpha(t_0)\| = \beta(t_0)$ and $\|x(t)-\alpha(t)\| > \beta(t) > 0$ for $t \in (t_0,t_1)$.

Recall that
$$\widehat{x(t)}_t = \alpha(t) + \frac{\beta(t)}{\|x-\alpha(t)\|}(x-\alpha(t))$$

for all $t \in [0,T]$ and the operator $N^1_{C(\cdot)}$ is ρ-hypomonotone. Additionally, recall that $\delta(t) \in N_{C(t)}(\alpha(t))$ with $\|\delta(t)\| \le w'(t)$ and $z(t) \in \overline{G}(t,x(t)), t \in [0,T]$. Consequently, from condition (2) in Definition 3.1 we deduce that we have the following estimation a.e. on (t_0,t_1) :

$$\begin{aligned} \frac{\langle x(t)-\alpha(t), x'(t)-\alpha'(t)\rangle}{\|x(t)-\alpha(t)\|} &= \frac{\langle \widehat{x(t)}_t - \alpha(t), x'(t)-\alpha'(t)\rangle}{\beta(t)} \\ = \frac{\langle \widehat{x(t)}_t - \alpha(t), -x(t)-\delta_x(t)+z(t)+\widehat{x(t)}_t - \alpha'(t)\rangle}{\beta(t)} \\ = \frac{\langle \widehat{x(t)}_t - \alpha(t), -x(t)+\widehat{x(t)}_t\rangle}{\beta(t)} &+ \frac{\langle \widehat{x(t)}_t - \alpha(t), -\delta_x(t)+z(t)+\alpha'(t)\rangle}{\beta(t)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\langle \widehat{x(t)}_t - \alpha(t), -x(t) + \widehat{x(t)}_t\rangle}{\beta(t)} + \frac{\langle \widehat{x(t)}_t - \alpha(t), -\delta(t) + z(t) + \alpha'(t)\rangle}{\beta(t)} \\
&\qquad - \frac{\langle \widehat{x(t)}_t - \alpha(t), \delta_x(t) - \delta(t)\rangle}{\beta(t)} \\
&= \beta(t) - \|x(t) - \alpha(t)\| + \frac{\langle \widehat{x(t)}_t - \alpha(t), -\delta(t) + z(t) + \alpha'(t)\rangle}{\beta(t)} \\
&\qquad - \frac{\langle x(t) - \alpha(t), \delta_x(t) - \delta(t)\rangle}{\|x(t) - \alpha(t)\|} \\
&< \beta'(t) - \frac{w'(t)}{\rho}\|x(t) - \alpha(t)\| + \frac{w'(t)}{\rho}\|x(t) - \alpha(t)\| \\
&< \beta'(t).
\end{aligned}
$$

Finally, for $t \in (t_0, t_1)$ such that $\|x(t) - \alpha(t)\| > \beta(t) = 0$ we deduce that

$$
z(t) = \alpha'(t) + \delta(t), \;\; x'(t) = \alpha'(t) + \delta(t) + \widehat{x(t)}_t - x(t) - \delta_x(t)
$$

and

$$
\begin{aligned}
&\frac{\langle x(t) - \alpha(t), x'(t) - \alpha'(t)\rangle}{\|x(t) - \alpha(t)\|} \\
&= \frac{\langle x(t) - \alpha(t), \widehat{x(t)}_t - x(t)\rangle}{\|x(t) - \alpha(t)\|} - \frac{\langle x(t) - \alpha(t), \delta_x(t) - \delta(t)\rangle}{\|x(t) - \alpha(t)\|} \\
&\leq \beta(t) - \|x(t) - \alpha(t)\| - \frac{w'(t)}{\rho}\|x(t) - \alpha(t)\| < 0 = \beta'(t).
\end{aligned}
$$

Consequently, the function $\|x(\cdot) - \alpha(\cdot)\| - \beta(\cdot)$ is decreasing on (t_0, t_1), which is a contradiction. the proof for (21) is similar and we omit it. □

We have all the tools for completing the proofs of the main existence theorems.

Proof of Theorem 3.2. Note that Proposition 4.1 implies that the mapping $\overline{F}$ has nonempty, closed convex values, $t \mapsto \overline{F}(t, x)$ is measurable for all $x \in \mathbb{R}^N$ and $x \mapsto \overline{F}(t, x)$ is u.s.c. a.e. $t \in [0, T]$.

Secondly, there exists $\hat{l} \in L^2([0, T], [0, \infty))$ such that

$$
\|\overline{F}(t, x) - x + \widehat{x}_t\| \leq \hat{l}(t) \tag{22}
$$

a.e. $t \in [0, T]$ and for all $x \in C(t)$. Therefore, we can apply Theorem 4.4 and we obtain the existence of a solution $x(\cdot)$ of (21). Finally, this solution belongs to $B_{\beta(\cdot)}(\alpha(\cdot))$ by Proposition 5.1. Finally, Proposition 5.1 ensures that this solution belongs to $B_{\beta(\cdot)}(\alpha(\cdot))$ and therefore satisfies (1) as well. □

Proof of Theorem 3.3. Note that Proposition 4.2 implies that the single valued mapping $f^\rho : C([0, T], \mathbb{R}^N) \to L^1([0, T], \mathbb{R}^N)$ is continuous and it's image is included in V. Recall that $P_+ : V \to C([0, T], \mathbb{R}^N)$. Consequently, Proposition 4.8 allows us to conclude that $P_+ \circ f^\rho : C([0, T], \mathbb{R}^N) \to C([0, T], \mathbb{R}^N)$ is a single valued continuous compact operator. Schauder's fixed point theorem implies that there exists a fixed point $x(\cdot)$ to the operator $P_+ \circ f^\rho$, and hence a solution to problem (21). Finally, Proposition 5.1 ensures that this solution belongs to $B_{\beta(\cdot)}(\alpha(\cdot))$ and therefore satisfies (1) as well. $\square$

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