

Poidge-Convexity

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We investigate in this paper the poidge-convexity, which is a generalization of right convexity, introduced by one of the present authors in 2014. Although not every convex body is poidge-convex, there are many families of compact sets, some of them very different from convex bodies, which are poidge-convex. We present here several such families.

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1. Introduction

At the 1974 meeting about convexity in Oberwolfach, the third author proposed the investigation of the following very general kind of convexity. Let $\mathbb{F}$ be a family of sets in a space $\mathcal{X}$. A set $M \subset \mathcal{X}$ is called $\mathbb{F}$ -convex if for any pair of distinct points $x, y \in M$ there is a set $F \in \mathbb{F}$ such that $x, y \in F$ and $F \subset M$. In this paper, $\mathcal{X}$ will be $\mathbb{R}^n$; we always assume $n \geq 2$. Usual convexity, affine linearity, arc-wise connectedness, polygonal connectedness, are just some examples of $\mathbb{F}$ -convexity (for suitably chosen families $\mathbb{F}$).

Blind, Valette and the third author [1], and also Böröczky Jr. [2], investigated the rectangular convexity, the case when $\mathbb{F}$ is the family of all non-degenerate rectangles.

Bruckner and Bruckner [7], and also Magazanik and Perles [11] investigated L_m sets, which are $\mathbb{F}$ -convex sets, $\mathbb{F}$ consisting of all polygonal paths with at most m edges in the plane. Magazanik and Perles [10] and Breen [3, 4, 5, 6] dealt with staircase connectedness, which is also a kind of $\mathbb{F}$ -convexity, $\mathbb{F}$ being the family of all staircases. The third author studied the right convexity [18] (the case with $\mathbb{F}$ consisting of all right triangles), and the last two authors, generalizing it, investigated the rt -convexity, i.e. the right triple convexity [12, 13].

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A set $M \subset \mathbb{R}^n$ is called *rt-convex* if it is $\mathbb{F}$ -convex, $\mathbb{F}$ being the family of all triples $\{x, y, z\} \subset \mathbb{R}^n$ with $\angle xyz = \pi/2$. If, for $x, y \in M$, there exists $z \in M$ such that the triangle xyz is right, we say that *the pair x, y has the rt-property*.

A set M will be called *poidge-convex* if it is $\mathbb{F}$ -convex, $\mathbb{F}$ being the family of all unions $\{x\} \cup \sigma$, called *poidges*, where x is a point, σ a line-segment, and $\text{conv}(\{x\} \cup \sigma)$ a right triangle (see Figure 1.1). If, for points $u, v \in M$, there is a poidge containing them included in M , we say that *the pair u, v has the poidge-property*.



Figure 1.1: Poidges

For convex sets, right convexity, *rt-convexity* and *poidge-convexity* are equivalent. Obviously, in general, not every *rt-convex* set is *poidge-convex*, but every rightly convex set is both *rt-convex* and *poidge-convex*. Less obvious, but true, is that not every *poidge-convex* set is *rt-convex* either.

The *poidge-convexity* also generalizes the thin rectangular convexity (for short *tr-convexity*, where $\mathbb{F}$ is the family of all boundaries of non-degenerated rectangles), and a fortiori the rectangular convexity itself.

In this paper, we start the investigation of *poidge-convexity*. Our main goal will be to identify large classes of *poidge-convex* sets. On the one hand, not all convex bodies are *poidge-convex*: take, for example, the convex hull of an ellipse different from a circle. On the other hand, sets looking quite different from being convex, like meandering topological discs or single-point-kernel starshaped sets, can be found among the *poidge-convex* sets.

2. Notation

For distinct $x, y \in \mathbb{R}^n$, let xy denote the line-segment from x to y , and $\overline{xy}$ the line through x, y .

For any compact set $M \subset \mathbb{R}^n$, let S_M be the smallest hypersphere containing M in its convex hull; also, $\overline{M}$ means the affine hull of M , $\text{int } M$ the relative interior of M (i.e., in the topology of $\overline{M}$) and $\text{bd } M$ the relative boundary of M .

By $\dim M$ we denote the Hausdorff dimension of M . Also, denote by $\mathbb{P}_A(x)$ the orthogonal projection of x onto the affine subspace A . The distance from a point x to a compact set M will be denoted by $\rho(x, M) = \min\{\|x - y\| : y \in M\}$.

Furthermore, for distinct $x, y \in \mathbb{R}^n$, we denote by H_{xy} the hyperplane through x orthogonal to $\overline{xy}$, by H_{xy}^+ the closed half-space containing y determined by H_{xy} , and by H_{xy}^- the other closed half-space determined by H_{xy} .

We denote by $[xy)$ the half-line starting at x and passing through y . Also, we set $\widehat{xky} = \text{conv}([kx) \cup [ky))$, if $k \notin xy$.

The compact ball of centre ω and radius r is denoted by $B_r(\omega)$. $\mathbf{0}$ is the origin of $\mathbb{R}^n$.

In a Baire space, we say that *most* of its elements have property **P** if those not enjoying **P** form a set of first Baire category.

3. Starshaped sets

3.1. One-sided starshaped sets

Let $\mathcal{S}$ be the Baire space of all starshaped sets in $\mathbb{R}^n$, always considered compact here. For $M \in \mathcal{S}$, a line-segment $kx \subset M$ is called a *ray* if $k \in \ker M$ and kx cannot be extended beyond x in M .

It is known (see the corollary to Theorem 1 in [17]) that most starshaped sets have a single-point kernel. Moreover, in $\mathcal{S}$, the set of those starshaped sets M with $\text{card}(M \cap S_M) = 2$ is nowhere dense. For a proof, the reader can adapt the first part of the proof of Theorem 1 in [16], which deals with convex curves instead of starshaped sets.

More generally, we call any compact set M *ordinary* if $\text{card}(M \cap S_M) \geq 3$.

Clearly, being ordinary is a necessary condition for any compact set to be poidge-convex.

Let $M \subset \mathbb{R}^2$ be a starshaped set with more than one point, and let $k \in \ker M$. The set M will be called *one-sided*, if there exist half-lines $[kx)$ and $[ky)$ such that

- (i) $M \subset \widehat{xky}$, or
- (ii) $[kx)$ and $[ky)$ are opposite, i.e. $[kx) \cup [ky)$ is a line, and M is included in one of the two half-planes with boundary $[kx) \cup [ky)$.

Now, taking $\angle xky$ to be minimal over all $x, y \in \mathbb{R}^2 \setminus \{k\}$ and $k \in \ker M$ in case (i), we call $\omega(M) = \angle xky$ the *opening* of M . In case (ii), put $\omega(M) = \pi$. Thus, $0 \leq \omega \leq \pi$. (Note that $M \neq \{k\}$ because $\text{card } M > 1$.) See Figure 3.1.

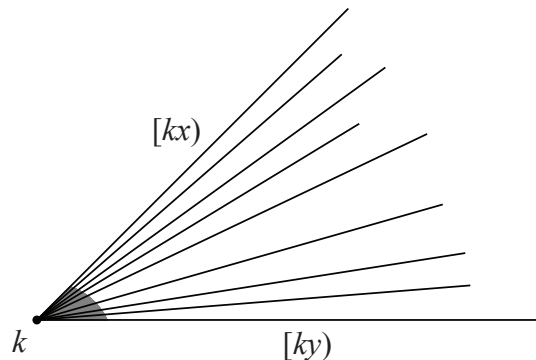


Figure 3.1: The opening of an one-sided starshaped set

We call M *acute*, if there exists no non-degenerate triangle $kxy \subset M$, where kx is a ray of M .

Theorem 3.1. *Let $M \subset \mathbb{R}^2$ be an ordinary one-sided starshaped set. If we have $0 < \omega(M) \leq \frac{\pi}{2}$, then M is poidge-convex. If M is acute and $\frac{\pi}{2} < \omega(M) < \pi$, then M is not poidge-convex.*

Proof. Take the point k from the definition of the opening to be $\{\mathbf{0}\}$. Assume $0 < \omega(M) \leq \frac{\pi}{2}$. Let $x, y \in M \setminus \{\mathbf{0}\}$, with $\|x\| \leq \|y\|$. If x, y are linearly independent, consider $z = \mathbb{p}_{\overline{\mathbf{0}y}}(x)$. Then $\{x\} \cup zy \subset M$. If x, y are linearly dependent, there exists $u \in M \setminus \overline{\mathbf{0}y}$, since $\omega(M) > 0$. Consider a point $v \in \mathbf{0}u$ close to $\mathbf{0}$ and $w = \mathbb{p}_{\overline{\mathbf{0}y}}(v)$.

Then x, y belong to $\{v\} \cup wy \subset M$. It remains to consider the pairs $(\mathbf{0}, x) \in M$. Let $\mathbf{0}y$ be the ray containing x . From $\text{card}(M \cap S_M) \geq 3$ it follows that M has a point $z' \notin \text{int conv } S_{\mathbf{0}y}$ different from $\mathbf{0}$ and y . One possibility is that $\angle z'\mathbf{0}y < \frac{\pi}{2}$.

Then $\mathbf{0}z' \cap S_{\mathbf{0}y} \neq \{\mathbf{0}, y\}$. Take $z \in \mathbf{0}z' \cap S_{\mathbf{0}y} \setminus \{\mathbf{0}, y\}$; possibly $z = z'$. Obviously, the points $\mathbf{0}, x$ belong to the poidge $\{z\} \cup \mathbf{0}y \subset M$. The other possibility is that $\angle z'\mathbf{0}y = \frac{\pi}{2}$. In this case, $\mathbf{0}, x$ belong to the poidge $\{z'\} \cup \mathbf{0}y \subset M$.

Now assume $\frac{\pi}{2} < \omega(M) < \pi$. There exist two rays $\mathbf{0}x, \mathbf{0}y$ of M such that we have $\frac{\pi}{2} < \angle x\mathbf{0}y < \pi$. We claim that x, y do not enjoy the poidge-property. Since every line-segment in M starting at x or y must be collinear with $\mathbf{0}$, otherwise M would not be acute, the only way for x, y to have the poidge-property would be to belong to a poidge with its line-segment σ along one of the rays $\mathbf{0}x, \mathbf{0}y$. But since $\angle x\mathbf{0}y > \frac{\pi}{2}$, σ would contain $\mathbf{0}$ in its relative interior. This contradicts the inequality $\omega(M) < \pi$. $\square$

It is easily seen that, for $\omega(M) = 0$, M is not poidge-convex and, for $\omega(M) = \pi$, both poidge-convexity and non-poidge-convexity are possible.

We consider the family $\mathcal{S}^\kappa$ of all one-sided $M \in \mathcal{S}$ with bounded opening $\omega(M) \leq \kappa$. This family is closed in $\mathcal{S}$, and is therefore itself complete and consequently a Baire space.

Theorem 3.2. *For $0 < \kappa \leq \pi/2$, most sets in $\mathcal{S}^\kappa$ are poidge-convex.*

For $\pi/2 < \kappa < \pi$, most sets in $\mathcal{S}^\kappa$ are not poidge-convex.

Proof. For any $\kappa \in]0, \pi[$, most sets in $\mathcal{S}^\kappa$ are ordinary. The proof parallels the one for $\mathcal{S}$ or for the space of all convex curves instead of $\mathcal{S}^\kappa$. Moreover, the sets $M \in \mathcal{S}^\kappa$ with $\omega(M) = 0$ form a nowhere dense family. Hence, by the first part of Theorem 3.1, most sets in $\mathcal{S}^\kappa$ are poidge-convex.

For any $\kappa < \pi$, most sets in $\mathcal{S}^\kappa$ are nowhere dense (the proof works like in the case of $\mathcal{S}$ instead of $\mathcal{S}^\kappa$, see the corollary to Theorem 1 in [17]), hence acute. By the second part of Theorem 3.1, most sets in $\mathcal{S}^\kappa$ are not poidge-convex. $\square$

3.2. Symmetric starshaped sets

Clearly, the space $\mathcal{S}^*$ of all (point) symmetric starshaped sets in $\mathbb{R}^n$ is closed in $\mathcal{S}$, so it is itself a Baire space.

Like most starshaped sets, most symmetric starshaped sets have a single-point kernel.

If a starshaped set is symmetric, then its kernel is also symmetric, with the same centre of symmetry, say $\mathbf{0}$. Being convex, the kernel contains $\mathbf{0}$.

Theorem 3.3. *A set $M \in \mathcal{S}^*$ is poidge-convex if and only if it is ordinary.*

Proof. Suppose $M \in \mathcal{S}^*$ is ordinary. Take $x, y \in M \setminus \{0\}$, assuming without loss of generality that $\|x\| \leq \|y\|$. Suppose first that x, y are linearly independent.

Taking $z = \mathbb{P}_{\overline{0y}}(x)$, we have $\{x, y\} \subset \{x\} \cup zy \subset M$. If x, y are linearly dependent, then $x \in y(-y)$. If $S_{y(-y)} \cap M = \{y, -y\}$, then M is not ordinary. Hence, $S_{y(-y)} \cap M$ contains some point $z \notin \{y, -y\}$, and x, y belong to the poidge $\{z\} \cup y(-y)$.

Suppose now that M is not ordinary, whence it has a single diameter $x(-x)$. There is no point $z \in S_{x(-x)} \cap M \setminus \{x, -x\}$. Moreover, the hyperplanes $H_{x(-x)}$ and $H_{(-x)x}$ meet M in $x, -x$ only. So, $x, -x$ don't belong to any poidge in M . $\square$

The next result has a corresponding one for convex bodies (with a similar proof), and uncovers a major discrepancy in comparison with the non-symmetric case.

Theorem 3.4. *Most sets belonging to $\mathcal{S}^*$ are not ordinary.*

Proof. Let $\mathcal{S}_m^*$ be the set of all $M \in \mathcal{S}^*$ having a pair of diameters at Pompeiu-Hausdorff distance at least $1/m$ from each other. We prove that $\mathcal{S}_m^*$ is nowhere dense in $\mathcal{S}^*$.

Let $\mathcal{O} \subset \mathcal{S}^*$ be open and $M \in \mathcal{O}$. Choose a diameter of M . This is also a diameter of S_M . Extend it in both directions by $\varepsilon > 0$ to a line-segment Δ_ε . Then $M_\varepsilon = M \cup \Delta_\varepsilon$ belongs to $\mathcal{S}^*$ and has the unique diameter Δ_ε . If ε is small enough, $M_\varepsilon \in \mathcal{O}$, too. There exists an open neighbourhood $\mathcal{N}$ of M_ε in $\mathcal{O}$ such that every set belonging to $\mathcal{N}$ has all its diameters at distance less than $1/2m$ from Δ_ε . This shows that they are not in $\mathcal{S}_m^*$. Thus, $\mathcal{S}_m^*$ is nowhere dense, and the set $\cup_{m=1}^\infty \mathcal{S}_m^*$ of all $M \in \mathcal{S}^*$ having more than one diameter is of first Baire category. $\square$

Corollary 3.5. *Most sets belonging to $\mathcal{S}^*$ are not poidge-convex.*

3.3. Starshaped sets close to convex

We introduce a measure of non-convexity for non-convex starshaped sets, which might be of independent interest. Line-segments xy joining $x, y \in \text{bd } M$ are called *chords* of the set $M \subset \mathbb{R}^n$. Let k belong to the kernel $\ker M$ of $M \in \mathcal{S}$. Let $\Gamma(M)$ be the set of chords xy of M not included in M . Let

$$\xi(M) = \sup_{xy \in \Gamma(M)} \inf_{k \in \ker M} \angle xky.$$

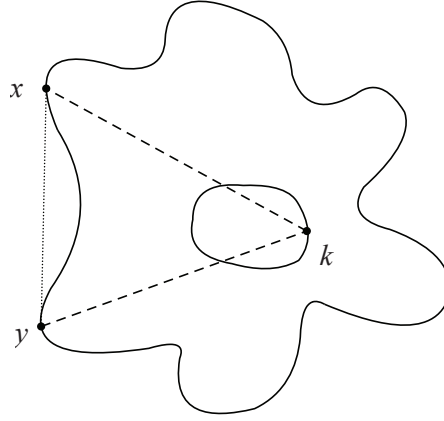
See Figure 3.2. If the sequence $\{M_m\}_{m=1}^\infty$ of starshaped sets converges, and if $\xi(M_m) \rightarrow 0$, then the limit set is convex. Thus, ξ indicates how close to being convex a starshaped set is. We shall see that starshaped sets “closer” to convex ones, i.e. with smaller value of ξ , are more likely to be poidge-convex.

Theorem 3.6. *If $M \subset \mathbb{R}^2$, $M \in \mathcal{S}$, M is ordinary, $\dim \ker M > 1$, and $\xi(M) < \frac{\pi}{2}$, then M is poidge-convex.*

Proof. Let $x, y \in M$.

Case 1. $xy \subset M$.

Being ordinary, M must have a point $z \notin (\text{int conv } S_{xy}) \cup \{x, y\}$.

Figure 3.2: The function ξ .

If $z \in S_{xy}$, then $\{z\} \cup xy$ is a suitable poidge. If $z \in H_{xy} \cup H_{yx}$, then again $\{z\} \cup xy$ is a suitable poidge. So, let z lie in one of the four components H_{xy}^- , H_{yx}^- , E_1 , E_2 of $\mathbb{R}^2 \setminus (H_{xy} \cup H_{yx} \cup \text{conv } S_{xy})$.

Take $k \in (\ker M) \setminus (\overline{xy} \cup \overline{xz} \cup \overline{yz})$. This is possible, because $\dim \ker M > 1$.

If $k \in (H_{xy} \cup H_{yx} \cup S_{xy})$, we are done.

If $k \in E_1 \cup E_2$, then $kx \cap S_{xy} \neq \emptyset$, and we are done again.

If $k \in H_{xy}^-$, then ky meets H_{xy} and the intersection point is not x . For $k \in H_{yx}^-$, we have an analogous situation. So, assume k lies in $\text{int conv } S_{xy}$.

Now, we consider again the positions that z may take.

If $z \in H_{xy}^-$, then $kz \cap H_{xy} \neq \emptyset$ and the intersection point is not x .

If $z \in H_{yx}^-$, then $kz \cap H_{yx} \neq \emptyset$ and the intersection point is not y .

Finally, if $z \in E_1 \cup E_2$, then $kz \cap S_{xy} \neq \emptyset$.

Thus, in all cases a poidge in M including xy can be found.

Case 2. $xy \notin M$.

Extend xy until we obtain a chord $x'y' \in \Gamma(M)$. Since $\inf_{k \in \ker M} \angle x'ky' \leq \xi(M) < \pi/2$, we find a point $k^* \in \ker M$ satisfying $\angle x'k^*y' < \pi/2$, which implies $\angle xk^*y < \pi/2$.

Assume without loss of generality that $\|k^* - x\| \leq \|k^* - y\|$. Now, if $z = \mathbb{P}_{\overline{k^*y}}(x)$, then $\{x\} \cup zy$ is a suitable poidge. $\square$

4. Topological discs

4.1. Smooth topological discs

For a set $M \subset \mathbb{R}^2$ with differentiable boundary, a chord xy is called a *double normal* of M if it is orthogonal to both tangent lines of $\text{bd } M$ at x and y (see Figure 4.1).

Let $\mathfrak{D}_m$ be the space of all planar topological discs with a C^m -boundary.

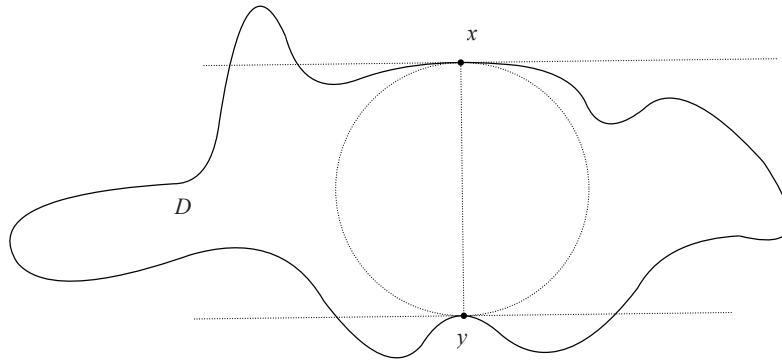


Figure 4.1: A double normal xy .

Theorem 4.1. *A topological disc $D \in \mathfrak{D}_2$ is poidge-convex if for any double normal xy of $\text{bd } D$, the curvature of $\text{bd } D$ at x or y is less than $\frac{2}{\|x-y\|}$.*

Proof. Clearly, x, y have the poidge-convex property if one of them is interior to D . So, let $x, y \in \text{bd } D$. If xy is not a double normal, then the tangent line at x or y is not orthogonal to $\overline{xy}$. Suppose the tangent line T at x is not orthogonal to $\overline{xy}$. Then there exists a small line-segment $xz \subset D$ orthogonal to $\overline{xy}$. Thus, $\{y\} \cup xz$ is a suitable poidge.

If xy is a double normal, by hypothesis, at x or y , say at x , the curvature radius satisfies $\rho(x) > \frac{\|x-y\|}{2}$. This implies that S_{xy} has a small arc starting at x , inside of D . If that arc is $\widehat{xz}$, then the poidge $\{y\} \cup xz$ is as required. (Notice that the case that D is not locally convex at x is not excluded. In that case, the curvature condition is even superfluous.) $\square$

Remark. A topological disc must have at least one double normal: take its diameter!

The condition, which works for rt -convexity, that D has at least two diameters, or that D is ordinary (i.e. $\text{card}(D \cap S_D) \geq 3$), is not sufficient for the poidge-convexity. We offer now an extension of Theorem 4.1 to topological discs with boundaries of class C^1 .

Suppose $D \subset \mathbb{R}^2$ is a topological disc, locally convex at some boundary point x . Then, if $\text{bd } D$ is differentiable at x , a lower and an upper curvature, $\gamma_i(x)$ and $\gamma_s(x)$, at x can be defined, as in [8] (page 14). The preceding theorem can be (in the same way) proven in the following more general form.

Theorem 4.2. *A topological disc $D \in \mathfrak{D}_1$ is poidge-convex if, for any double normal xy of $\text{bd } D$, at one of its endpoints, say at x , either D is locally convex and $\gamma_s(x) < \frac{2}{\|x-y\|}$, or $\mathbb{R}^2 \setminus \text{int } D$ is locally convex.*

4.2. Unions of convex sets

Here, we consider unions of compact convex sets in $\mathbb{R}^n$, possibly of distinct dimensions.

Theorem 4.3. *Every ordinary connected union of two ordinary compact convex sets is poidge-convex.*

Proof. Let $M = A \cup B$, with A, B compact convex sets, satisfying the hypothesis. Let $x, y \in M$. If both x, y lie in A , or both in B , then we have a right triangle xyz in A or in B because they are rightly convex, by Theorem 3 in [18]. So, assume without loss of generality that $x \in A \setminus B$ and $y \in B \setminus A$. If inside S_{xy} there is no point of M , then $A \cap B = \emptyset$ and M is not connected. If $M \subset \text{conv } S_{xy}$, then, M being ordinary, there exists a point $z \in S_{xy} \setminus \{x, y\}$. If $M \not\subset \text{conv } S_{xy}$, but $S_{xy} \cap M = \{x, y\}$, then either $A \subset \text{conv } S_{xy}$ and $B \cap \text{conv } S_{xy} = \{y\}$, or vice-versa. In both cases M becomes disconnected, or one of the sets A, B is not ordinary, which is false.

Hence, in any (possible) case, there exists $z \in S_{xy} \cap M \setminus \{x, y\}$. Now, if $z \in A$ then $\{y\} \cup xz$ is a suitable poidge, and if $z \in B$, a suitable poidge is $\{x\} \cup yz$. $\square$

For $m \geq 3$, an ordinary connected union of m ordinary compact convex sets may not be poidge-convex, see Figure 4.2.

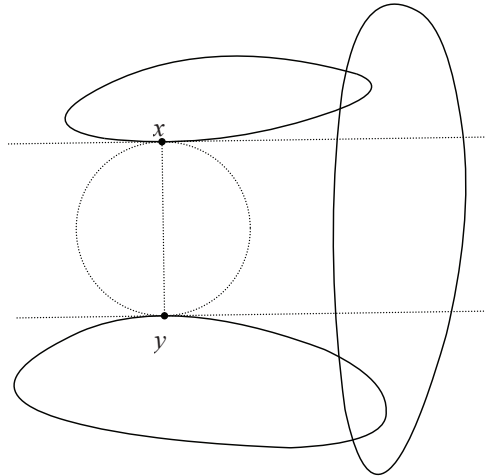


Figure 4.2: A connected union of 3 convex bodies.

5. Not simply connected sets

5.1. Convex bodies with spherical holes

It is easily seen that there are poidge-convex sets which are not simply connected. So, for example, the set $B_1(\mathbf{0}) \setminus \text{int } B_\varepsilon(\mathbf{0})$ in $\mathbb{R}^2$, for any $\varepsilon \in]0, 1[$.

Theorem 5.1. *Let $K \subset \mathbb{R}^2$ be an ordinary convex body, and assume that $\mathbf{0}$ is the centre of S_K . The set $K \setminus \text{int } B_\alpha(\mathbf{0})$ is poidge-convex, if α satisfies the following conditions. For any double normal ab of $\text{bd } K$,*

- (i) *if ab is not a chord of S_K , then $\alpha \leq \rho(\mathbf{0}, ab)$, and*
- (ii) *if ab is a chord of S_K and $c \in S_K \cap K \setminus \{a, b\}$, then $\alpha \leq \max\{\rho(\mathbf{0}, bc), \rho(\mathbf{0}, ca)\}$.*

Proof. Let $L = K \setminus \text{int } B_\alpha(\mathbf{0})$, and $x, y \in L$. We need to investigate only the case when xy is a double normal of $\text{bd } K$, which we now assume.

Case 1. xy is not a chord of S_K .

In this case, $xy \subset L$, because $xy \cap \text{int } B_\alpha(\mathbf{0}) = \emptyset$, by condition (i). Since K is ordinary, $S_{xy} \cap K \neq \{x, y\}$. Choose $w \in S_{xy} \cap K \setminus \{x, y\}$. So, $\{w\} \cup xy$ is a poidge in L .

Case 2. xy is a chord of S_K .

Since K is ordinary, $S_K \cap K \setminus \{x, y\}$ contains a point z . We have $yz \cap \text{int } B_\alpha(\mathbf{0}) = \emptyset$ or $zx \cap \text{int } B_\alpha(\mathbf{0}) = \emptyset$, by condition (ii) of the hypothesis. Assume without loss of generality that $yz \cap \text{int } B_\alpha(\mathbf{0}) = \emptyset$.

Since $\angle yzx = \pi/2$, we have the poidge $\{x\} \cup yz \subset L$. □

Is always the existence of such a number α guaranteed?

Theorem 5.2. *If $K \subset \mathbb{R}^2$ is an ordinary convex polygon, in which every double normal passing through the centre $\mathbf{0}$ of S_K is a chord of S_K , then a number α satisfying the conditions in Theorem 5.1 does exist.*

Proof. Let $\mathcal{D}_1$ be the family of those double normals of K which do not contain $\mathbf{0}$, and $\mathcal{D}_2$ the complementary family of double normals, which must be chords of S_K . Since no double normal in $\mathcal{D}_1$ contains $\mathbf{0}$, $\beta = \min\{\rho(\mathbf{0}, N) : N \in \mathcal{D}_1\} > 0$.

The set of double normals passing through $\mathbf{0}$ is finite. For each double normal $N = ab \in \mathcal{D}_2$, consider

$$\gamma_N = \max\{\max\{\rho(\mathbf{0}, ax), \rho(\mathbf{0}, bx)\} : x \in S_K \cap \text{bd } K \setminus \{a, b\}\}$$

and $\gamma = \min\{\gamma_N : N \in \mathcal{D}_2\} > 0$.

By choosing $\alpha = \min\{\beta, \gamma\}$, both conditions of Theorem 5.1 are satisfied, condition (i) for the double normals of $\mathcal{D}_1$, and condition (ii) for the double normals of $\mathcal{D}_2$. □

Corollary 5.3. *If $K \subset \mathbb{R}^2$ is an ordinary convex polygon, in which no double normal passes through the centre $\mathbf{0}$ of S_K , then a number α satisfying the conditions in Theorem 5.1 does exist.*

Theorem 5.2 presents only sufficient conditions for a convex polygon to acquire an appropriate number α . All regular polygons are poidge-convex, although they do not satisfy the conditions of Theorem 5.2.

5.2. Tetrahedral surfaces

Starting with this section, we investigate the poidge-convexity of convex surfaces in $\mathbb{R}^n$, for $n = 3$ or for larger n .

The following lemma is straightforward.

Lemma 5.4. *Every non-obtuse triangle is poidge-convex.*

Lemma 5.5. *Assume that P is an $(n-1)$ -dimensional polytope in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, and $y \in P \setminus (V(P) \cup \{x\})$, then there exists a poidge in $P \cup \{x\}$ containing both x and y .*

Proof. Indeed, $H_{yx} \cap P$ includes a non-degenerate line-segment σ with an endpoint in y . Thus, $\{x\} \cup \sigma$ is a suitable poidge. $\square$

Theorem 5.6. Suppose $abcd$ is a tetrahedron in $\mathbb{R}^3$ with a non-obtuse facet (i.e. 2-dimensional face) abc . Then $\text{bd}(abcd)$ is poidge-convex if and only if

$$d \in (H_{ba}^+ \cup H_{ca}^+) \cap (H_{ab}^+ \cup H_{cb}^+) \cap (H_{ac}^+ \cup H_{bc}^+).$$

The boundary of a tetrahedron with all facets obtuse is not poidge-convex.

Proof. Suppose abc and d are as stipulated in the first part of the theorem. We prove that $\text{bd}(abcd)$ is poidge-convex.

Let $x, y \in \text{bd}(abcd)$ be any two distinct points.

Case 1. There is a face F of $abcd$ such that $x, y \in F$.

If F is a non-obtuse face, then there exists a poidge containing x, y and contained in F , by Lemma 5.4. If F is an obtuse face, then we obviously have to consider only the case that x, y are vertices of the longest edge of $\text{bd } F$. If that edge is included in $\text{bd}(abc)$, then we find a poidge containing x and y in the face abc , by Lemma 5.4. So, suppose the obtuse angle of F is not at d . Let d' be the orthogonal projection of d on the plane $\overline{abc}$.

Subcase 1.1. $d' \in H_{ba}^+ \cap H_{ca}^+ \cap H_{ab}^+ \cap H_{cb}^+ \cap H_{ac}^+ \cap H_{bc}^+$. (See Figure 5.1.)

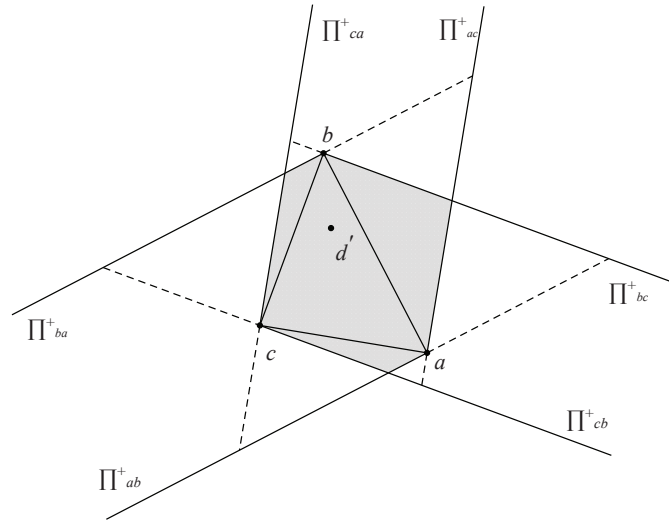


Figure 5.1: $d' \in H_{ba}^+ \cap H_{ca}^+ \cap H_{ab}^+ \cap H_{cb}^+ \cap H_{ac}^+ \cap H_{bc}^+$.

All the angles $\widehat{dac}$, $\widehat{dca}$, $\widehat{dcb}$, $\widehat{dbc}$, $\widehat{dba}$, $\widehat{dab}$ are non-obtuse. Then the obtuse face F must have an obtuse angle at d , and we obtain a contradiction.

Subcase 1.2. $d' \in (H_{ca}^+ \cap H_{ac}^+) \setminus (H_{ba}^+ \cap H_{bc}^+)$. (See Figure 5.2.)

All angles $\widehat{dac}$, $\widehat{dca}$, $\widehat{dcb}$, $\widehat{dab}$ are non-obtuse. Clearly, $F \neq cad$. If $F = bcd$, it must have its obtuse angle at b . If d'' is the orthogonal projection of d onto $\overline{ac}$, then $d'' \in ac$ and x, y belong to the poidge $cd'' \cup \{d\}$. The case $F = abd$ is analogous.

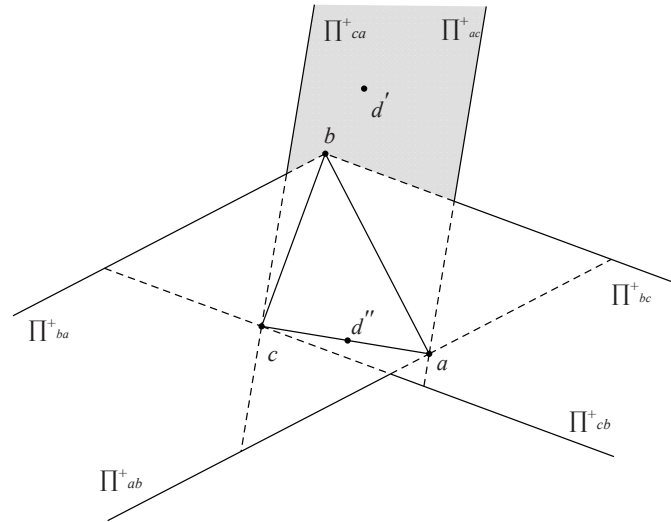


Figure 5.2: $d' \in (H_{ca}^+ \cap H_{ac}^+) \setminus (H_{ba}^+ \cap H_{bc}^+)$.

The cases $d' \in (H_{bc}^+ \cap H_{cb}^+) \setminus (H_{ac}^+ \cap H_{ab}^+)$ and $d' \in (H_{ba}^+ \cap H_{ab}^+) \setminus (H_{ca}^+ \cap H_{cb}^+)$ are analogous to subcase 1.2.

Case 2. There are two distinct faces F_1 and F_2 such that $x \in F_1$ and $y \in F_2$.

If $x \in \text{int } F_1$ or $y \in \text{int } F_2$, then they have the poidge-property, by Lemma 5.5. Otherwise, $x \in \text{bd } F_1$ and $y \in \text{bd } F_2$, which means that we are in Case 1, unless x and y are in opposite edges. Without loss of generality, we can assume that $x \in ad$ and $y \in bc$. Let w be the orthogonal projection of x onto $\overline{bc}$. Then $w \in bc$, and x, y will be contained in the poidge $\{x\} \cup wb$ or $\{x\} \cup wc$.

Now, let us show that, if abc is as required, but d not (see Figure 5.3), then $\text{bd}(abcd)$ is not poidge-convex. This will be proved by showing that d and one of the other three vertices do not enjoy the poidge-property.

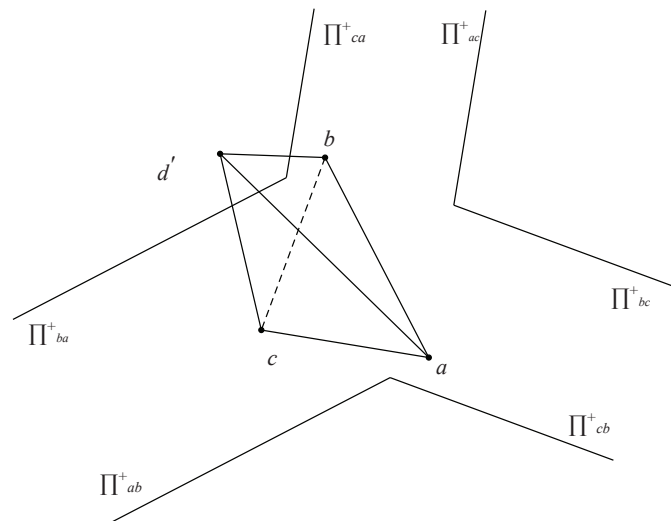


Figure 5.3: $\text{bd}(abcd)$ is not poidge-convex.

If $d \notin H_{ba}^+ \cup H_{ca}^+$, then the angles $\widehat{dca}$ and $\widehat{dba}$ are obtuse. Then, obviously, a and d do not enjoy the poidge-property. The cases $d \notin H_{ba}^+ \cup H_{ca}^+$ and $d \notin H_{ab}^+ \cup H_{cb}^+$ are analogous.

The proof of the first part of the theorem is now complete.

For the second part, assume all facets of the tetrahedron $abcd$ are obtuse. Assume without loss of generality that ab is a diameter of $abcd$. The pair of points a, b has not the poidge-property. Indeed, $S \cap abcd = \{a, b\}$, $H_{ab} \cap abcd = \{a\}$ and $H_{ba} \cap abcd = \{b\}$. $\square$

5.3. Boundaries of Cartesian products and cones

As we could see in the preceding section, there exist poidge-convex convex surfaces. However, many convex surfaces are not poidge-convex: for example, no strictly convex body has a poidge-convex boundary. Therefore, most convex surfaces are not poidge-convex (see [9]). We present now some classes of convex surfaces which are poidge-convex.

We consider the Cartesian product in $\mathbb{R}^n$ of a k -dimensional compact convex set K and an $(n - k)$ -dimensional compact convex set L , hence with $\overline{K}$ orthogonal to $\overline{L}$.

Theorem 5.7. *Every Cartesian product of compact convex sets of positive dimensions has a poidge-convex boundary.*

Proof. We consider $\overline{K}$ to be spanned by the first k axes, and $\overline{L}$ by the last $n - k$ axes. Let $S = \text{bd}(K \times L)$.

Let $x, y \in S$. We have $x = u \times t_u$ and $y = v \times t_v$, where $u, v \in K$ and $t_u, t_v \in L$.

Consider the chord $t'_u t'_v = \overline{t_u t_v} \cap L$ of L .

Case 1. $x, y \in \{u\} \times L$, where $u \in \text{bd } K$.

Take $u' \in K \setminus \{u\}$ arbitrarily. Then the poidge $\{u' \times t'_u\} \cup (\{u\} \times t'_u t'_v)$ contains x, y and lies in S .

Case 2. $x \in \{u\} \times L$, $y \in \{v\} \times L$, where $u, v \in \text{bd } K$ are distinct.

If t'_u, t_u, t_v, t'_v lie in this order on $\overline{t_u t_v}$, we have $t_u t_v \subset t_u t'_v$. Then $\{x\} \cup (\{v\} \times t_u t'_v)$ is a suitable poidge in S .

Case 3. $x, y \in K \times \{t\}$, where $t \in \text{bd } L$.

For this case we have the poidge $(u \times t') \cup xy \subset S$, where $t' \in (\text{bd } L) \setminus \{t\}$.

Case 4. $x \in K \times \{t_u\}$, $y \in K \times \{t_v\}$, where $t_u, t_v \in \text{bd } L$ are distinct.

If $u \neq v$, a good poidge is $\{y\} \cup x(v \times t_u)$. Otherwise, $\{y\} \cup xz$ is a good one, z being an arbitrary point of $K \times \{t_u\}$ different from x .

Case 5. $x \in K \times \{t_u\}$, $y \in (\text{bd } K) \times L$, here $t_u \in \text{bd } L$.

Now, if we are not in Case 1 or 4, a suitable poidge is $\{x\} \cup (\{v\} \times I)$, where I is an arbitrary line-segment in L starting at t_u . $\square$

Let K be an $(n-1)$ -dimensional convex body in $\mathbb{R}^n$, and $v \in \mathbb{R}^n \setminus K$. Put $v' = \mathbb{P}_K(v)$. We call the cone $C = \text{conv}(\{v\} \cup K)$ *right*, if $v' \in K$ and, for any pair of points $a, b \in \text{bd } K$, $\angle avb \leq \pi/2$.

The boundary of a right cone will be called here a *conical surface*. See Figure 5.4.

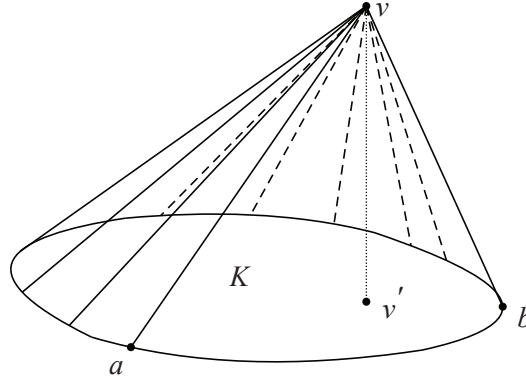


Figure 5.4: A conical surface.

Theorem 5.8. *Every conical surface is poidge-convex.*

Proof. Let $S = \text{bd } C$, where $C = \text{conv}(\{v\} \cup K)$ is a right cone. We observe the preceding notation.

Let $x, y \in S$. We assume without loss of generality that $\|v - x\| \leq \|v - y\|$.

Case 1. $x, y \in K$.

If x, y are not both boundary points of K , then they clearly have the poidge-property (in K). Suppose now $x, y \in \text{bd } K$. Since $\angle xvy \leq \pi/2$ and $\|v - x\| \leq \|v - y\|$, $z = \mathbb{P}_{\overline{vy}}(x)$ belongs to vy . Thus, $\{x\} \cup yz$ is a suitable poidge.

Case 2. The points v, x, y are collinear.

The point x is closer from v than y , possibly $x = v$.

Let $\{y'\} = \overline{xy} \cap K$.

If $y' = y$, then we consider the poidge $\{x\} \cup yz$, where $z = \mathbb{P}_K(x)$.

If $y' \neq y$, then take $u \in (\text{bd } K) \setminus \{y'\}$ so close to y' that $\|v - u\| > \|v - y\|$. Then $\angle vuy < \pi/2$, and the projection $z = \mathbb{P}_{\overline{vu}}(x)$ belongs to vu . So, we obtain the poidge $\{z\} \cup xy \subset S$.

Case 3. $x \in vx'$, $y \in vy'$, with $x', y' \in \text{bd } K$.

We may suppose $x \neq v$, $y \neq v$, and $x' \neq y'$, otherwise we are in Case 2. Since $\angle xvy \leq \pi/2$, we have $z \in vy \setminus \{y\}$, where $z = \mathbb{P}_{\overline{vy}}(x)$. Thus, $\{x\} \cup yz$ is a suitable poidge.

Case 4. $x \in K$, $y \notin K$.

Let $z = \mathbb{P}_K(y)$. If $z \neq x$, then $\{y\} \cup xz$ is a good poidge. If $z = x$, take any point $z' \in K$ different from z . The poidge $\{y\} \cup zz'$ will do it. $\square$

For results on right convexity in cylinders and cones, see [18].

5.4. Polyhedral surfaces

We have already seen that some tetrahedral surfaces are poidge-convex, some are not. The same is true in the larger frame of all polyhedral surfaces.

The *star* S_x at a vertex x of a polytope in $\mathbb{R}^n$ is the union of all facets (i.e. $(n-1)$ -dimensional faces) having x as a vertex. Two vertices of a polytope in $\mathbb{R}^n$ will be called *opposite* if H_{xy} and H_{yx} are supporting hyperplanes of the polytope.

Recall that a compact set $M \subset \mathbb{R}^n$ is called ordinary, if $\text{card}(S_M \cap M) \geq 3$. This is equivalent to the property that, for any pair of points $x, y \in M$, not all points of M different from x, y lie inside S_{xy} , i.e. $M \setminus \text{int conv } S_{xy} \neq \{x, y\}$.

Now, we say that a polytope $P \in \mathbb{R}^n$ is *extraordinary*, if, for any pair of opposite vertices $x, y \in P$, not all points of $S_x \cup S_y$ different from x, y lie inside S_{xy} , i.e. $(S_x \cup S_y) \setminus \text{int conv } S_{xy} \neq \{x, y\}$.

Of course, every extraordinary polytope is ordinary, but not vice-versa.

Among the extraordinary polytopes we find all those admitting a circumscribed sphere, i.e. a sphere containing all vertices. The Platonic and the Archimedean Solids are well-known examples.

Theorem 5.9. *Every extraordinary polytope in $\mathbb{R}^n$ has a poidge-convex boundary.*

Proof. By Lemma 5.5, we only need to verify the poidge-property for pairs of vertices x, y of the given extraordinary polytope P .

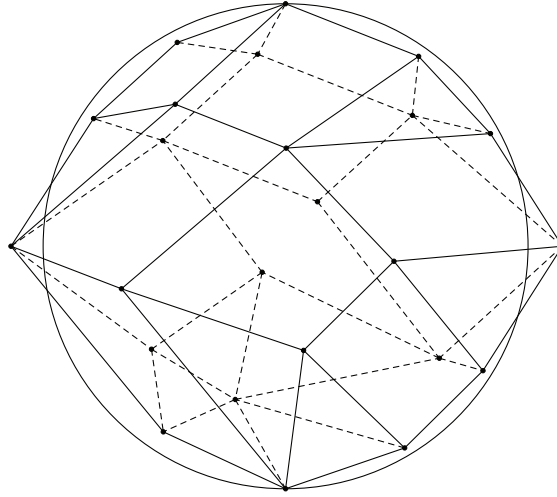


Figure 5.5: An ordinary, not extraordinary polytope.

Take $x, y \in V(P)$. Let H_x be a supporting hyperplane of P at x . Consider the hyperplane $H_y \ni y$ parallel to H_x .

If H_y is supporting P , then our hypothesis implies the existence of some point $z \in (S_x \cup S_y) \setminus \text{int conv } S_{xy}$ different from x and y . Assume without loss of generality that $z \in S_x$. Then $zx \cap S_{xy} \neq \{x\}$. If $z' \in zx \cap S_{xy} \setminus \{x\}$, then $xz' \subset S_x$, as $\{x\}$ is the kernel of the starshaped set S_x . Moreover, $\angle xz'y = \pi/2$. Thus, $\{y\} \cup xz'$ is a suitable poidge.

If H_y is not supporting P , then both H_y and S_{xy} are locally cutting $\text{bd } P$ at y . Locally, the intersection $S_{xy} \cap S_y$ is a union of pieces of spheres of dimension $n - 2$. Take a point z in $S_{xy} \cap S_y$ different from y . Then $\{x\} \cup yz$ is a suitable poidge. $\square$

Figure 5.5 shows an ordinary polytope which is not extraordinary.

Sometimes even the 1-skeleta of polytopes are poidge-convex.

In $\mathbb{R}^3$, let $\mathbf{T}_1$, $\mathbf{C}_1$, $\mathbf{O}_1$, $\mathbf{D}_1$, $\mathbf{I}_1$ be the boundary 1-complexes of the regular tetrahedron, cube, regular octahedron, regular dodecahedron, and regular icosahedron, respectively.

Theorem 5.10. $\mathbf{T}_1$, $\mathbf{C}_1$, $\mathbf{O}_1$ are poidge-convex, while $\mathbf{D}_1$, $\mathbf{I}_1$ are not.

Proof. We leave to the reader the first part of the statement. Consider $\mathbf{D}_1$. Let $abcde$ be a face of $\mathbf{D}_1$, and aa' an edge different from ab and ea . Consider the points $x \in aa'$ and $y \in bc$. Obviously, $H_{xy} \cap aa' = \{x\}$, and $H_{yx} \cap bc = \{y\}$. Moreover, $xy \not\subset \mathbf{D}_1$. Hence, x, y have not the poidge-property.

For $\mathbf{I}_1$, the proof is similar. $\square$

6. Problems

We end the paper with two problems about the relationship between various $\mathbb{F}$ -convexities.

Problem 6.1. Which poidge-convex sets are not rt -convex?

We mentioned (and it is easily verified) that tr -convexity implies poidge-convexity.

Problem 6.2. Which poidge-convex sets are not tr -convex?

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