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On the Construction of Maximal p-Cyclically Monotone Operators

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We deal with the construction of explicit examples of maximal p-cyclically monotone operators. To date, there is only one instance of an explicit example of a maximal 2-cyclically monotone operator that is not maximal monotone. We present a systematic way to construct this kind of examples, along with several explicit examples.

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1. Introduction

Let U, V be non-empty sets. A *multivalued operator* $T : U \rightrightarrows V$ is an application $T : U \to \mathcal{P}(V)$, that is, for $u \in U$, $T(u) \subset V$. The *domain*, *range* and *graph* of T are defined, respectively, as

$$\operatorname{dom}(T) = \big\{u \in U \ : \ T(u) \neq \emptyset\big\}, \qquad \operatorname{ran}(T) = \bigcup_{u\in U} T(u)$$
$$\text{and } \operatorname{gra}(T) = \big\{(u,v) \in U \times V \ : \ v \in T(u)\big\}.$$

A multivalued operator T is *finite* if its graph is a finite set. From now on, unless otherwise stated, we will identify multivalued operators with their graphs, so we will write $(u, v) \in T$ instead of $(u, v) \in \operatorname{gra}(T)$.

Let V be a vector space. If $A \subset V$ then $\operatorname{co}(A)$ denotes the *convex hull* of A. If $T : U \rightrightarrows V$ is a multivalued operator, the operator $T_{\mathrm{co}} : U \rightrightarrows V$ is defined as $T_{\mathrm{co}}(u) = \operatorname{co}(T(u))$, for all $u \in U$. Given two operators $T, S : U \rightrightarrows V$, $T + S$ will denote the *pointwise sum* of the operators T and S, that is,

$$(T + S)(u) = T(u) + S(u) = \{v^* + w^* \ : \ v^* \in T(u),\, w^* \in S(u)\},$$

when $u \in \operatorname{dom}(T) \cap \operatorname{dom}(S)$, and $(T + S)(u) = \emptyset$, otherwise.

Let X be a Banach space and X^* be its topological dual. The *duality product* $\langle\cdot,\cdot\rangle : X \times X^* \to \mathbb{R}$ is defined as $\langle x, x^*\rangle = x^*(x)$. Given $C \subset X$ convex, the *normal cone operator* is the operator $N_C : X \rightrightarrows X^*$ defined as

$$N_C(x) = \{x^* \in X^* \ : \ \langle y - x, x^*\rangle \leq 0,\, \forall\, y \in C\},$$

when $x \in C$ and $N_C(x) = \emptyset$, otherwise. Further, the *recession cone* of C is the set

$$0^+C = \{d \in X \ : \ \forall x \in C,\, \forall t > 0,\, x + td \in C\}.$$

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Given a cone $K \subset X$, the *polar cone* of K is the set

$$K^\circ = \{x^* \in X^* \ : \ \langle x, x^* \rangle \leq 0, \forall\, x \in K\}.$$

A multivalued operator $T : X \rightrightarrows X^*$ is called *monotone* if, for every pair (x, x^*), $(y, y^*) \in T$,

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

Moreover, T is *maximal monotone* if T is not properly contained in a monotone operator.

The notion of p-cyclical monotonicity was introduced by Rockafellar in [9] as a "midpoint" between classical monotonicity and cyclical monotonicity. A multivalued operator $T : X \rightrightarrows X^*$ is called *p-cyclically monotone*, with $p \in \mathbb{N}$, if

$$\{(x_i, x_i^*)\}_{i=0}^{p} \subset T \quad \longrightarrow \quad \sum_{i=0}^{p} \langle x_{i+1} - x_i, x_i^* \rangle \leq 0,$$

where $x_{p+1} = x_0$. An operator is called *cyclically monotone* if it is p-cyclically monotone, for all $p \in \mathbb{N}$. As in the monotone case, we can consider *maximality* for p-cyclically monotone operators: a p-cyclically monotone operator is called *maximal p-cyclically monotone*, if its graph is not properly contained in the graph of another p-cyclically monotone operator.

Note that if T is a p-cyclically monotone operator, then it is q-cyclically monotone, for all $q \leq p$. Moreover, if T is maximal p-cyclically monotone and q-cyclically monotone, for some $q > p$, then it is also maximal q-cyclically monotone. In particular, a maximal monotone operator which is p-cyclically monotone, is also maximal p-cyclically monotone. Examples of these kinds of operators are the rotation matrices [1, 6]. So it is natural to ask if a maximal p-cyclically monotone operator is also maximal monotone. The answer was given, in the negative, by Bartz, Bauschke, Borwein, Reich and Wang in [2]. Bartz et al., using Zorn's Lemma, showed the existence of a maximal 2-cyclically monotone operator that is not maximal monotone. Later on, in [3], Bauschke and Wang constructed an explicit example of such an operator. This example has also the *bizarre* property of having a non-convex closed domain (in fact, its domain is the boundary of the unit diamond $|x| + |y| = 1$.)

Construction of maximal monotone operators was addressed previously by Crouzeix, Ocaña-Anaya and Sosa [7]. See also [10]. The goal of this work is to provide a way to construct explicit examples of maximal p-cyclically monotone operators. To do this, we use the recently defined *p-cyclically monotone polar* [5]. We also provide a way to deal with the equations that arise from the definition and the p-cyclical monotonicity of this kind of examples. Although the question whether our method always provides maximal p-cyclically monotone operators remains open, we prove maximality of a certain family of examples, namely, operators that can be decomposed as the sum of a normal cone plus a union of perpendicular line segments in $X \times X^*$.

The paper is organized as follows: in section 2, we provide several technical results that are needed later. In section 3, we describe a procedure that constructs explicit examples of p-cyclically monotone operators, starting from a finite p-cyclically monotone operator. In section 4 we present four new examples: three maximal 2-cyclically monotone operators that are not maximal monotone (one with domain in $\mathbb{R}^3$) and a maximal 3-cyclically monotone operator which is not maximal 2-cyclically monotone.

2. Technical results

The p-cyclically monotone polar [5] is an extension to the p-cyclically monotone case of the well known *monotone polar* [8]. The *p-cyclically monotone polar* (or simply, p-polar) of a multivalued operator $T : X \rightrightarrows X^*$ is the operator T^{μ_p} defined via its graph as

$$T^{\mu_p} = \left\{ (x_0, x_0^*) \ : \ \sum_{i=0}^{p} \langle x_{i+1} - x_i, x_i^* \rangle \leq 0, \, \forall \{(x_i, x_i^*)\}_{i=1}^{p} \subset T \text{ with } x_{p+1} = x_0 \right\}.$$

Proposition 2.1. ([5]) *Let $T, T_i : X \rightrightarrows X^*$ be multivalued operators and $p \in \mathbb{N}$. The following hold:*

(1) $T^{\mu_{p+1}} \subset T^{\mu_p}$.

(2) *If $T_1 \subset T_2$ then $T_2^{\mu_p} \subset T_1^{\mu_p}$.*

(3) $\left(\bigcup_{i \in I} T_i \right)^{\mu_p} \subset \bigcap_{i \in I} T_i^{\mu_p}$. *(Equality holds when $p = 1$)*

(4) *The graph of T^{μ_p} is (strongly-)closed.*

(5) *T is p-cyclically monotone if, and only if, $T \subset T^{\mu_p}$.*

(6) *T is maximal p-cyclically monotone if, and only if, $T = T^{\mu_p}$.*

Lemma 2.2. *Let $T : X \rightrightarrows X^*$ be a multivalued operator and let $C = \mathrm{co}(\mathrm{dom}(T))$. Then we have for all $x_0 \in C \cap \mathrm{dom}(T^{\mu_p})$: $0^+ T^{\mu_p}(x_0) = N_C(x_0)$.*

Proof. Let $n^* \in N_C(x_0)$ and let $x_0^* \in T^{\mu_p}(x_0)$ and $t > 0$. Then, for every $\{(x_i, x_i^*)\}_{i=1}^{p} \subset T$, considering $x_{p+1} = x_0$,

$$\langle x_1 - x_0, x_0^* + tn^* \rangle + \sum_{i=1}^{p} \langle x_{i+1} - x_i, x_i^* \rangle \leq \langle x_1 - x_0, x_0^* \rangle + \sum_{i=1}^{p} \langle x_{i+1} - x_i, x_i^* \rangle \leq 0.$$

Hence $n^* \in 0^+ T^{\mu_p}(x_0)$. Conversely, consider $n^* \in 0^+ T^{\mu_p}(x_0)$, that is, we have $x_0^* + tn^* \in T^{\mu_p}(x_0)$, for all $x_0^* \in T^{\mu_p}(x_0)$ and $t > 0$. Take $x_1 \in \mathrm{dom}(T)$, $x_1^* \in T(x_1)$, and let $x_1 = x_2 = \cdots = x_p$, $x_1^* = x_2^* = \cdots = x_p^*$, thus

$$\begin{aligned} 0 &\geq \langle x_1 - x_0, x_0^* + tn^* \rangle + \sum_{i=1}^{p-1} \langle x_{i+1} - x_i, x_i^* \rangle + \langle x_0 - x_p, x_p^* \rangle \\ &= \langle x_1 - x_0, x_0^* + tn^* \rangle + \langle x_0 - x_1, x_1^* \rangle \\ &= \langle x_1 - x_0, x_0 - x_1^* \rangle + t \langle x_1 - x_0, n^* \rangle. \end{aligned}$$

If for some x_1, $\langle x_1 - x_0, n^* \rangle > 0$, then taking $t \to +\infty$ in the previous inequality would lead to a contradiction. Therefore $\langle x_1 - x_0, n^* \rangle \leq 0$, for all $x_1 \in \mathrm{dom}(T)$, and this implies that $n^* \in N_C(x_0)$. □

Proposition 2.3. *Let $T : X \rightrightarrows X^*$ be a multivalued operator, and $C = \mathrm{co}(\mathrm{dom}(T))$.*

Then $$T^{\mu_p}|_C \subset [T_{\mathrm{co}} + N_C]^{\mu_p} \subset [T_{\mathrm{co}}]^{\mu_p} \subset T^{\mu_p}.$$

In particular, for every $x \in C$, $[T_{\mathrm{co}} + N_C]^{\mu_p}(x) = [T_{\mathrm{co}}]^{\mu_p}(x) = T^{\mu_p}(x)$.

Proof. Since $T \subset T_{\rm co} \subset T_{\rm co} + N_C$, from Proposition 2.1(2),

$$[T_{\rm co} + N_C]^{\mu_p} \subset [T_{\rm co}]^{\mu_p} \subset T^{\mu_p}.$$

Now, given any $(z_0, z_0^*) \in T^{\mu_p}$, with $z_0 \in C$, we aim to prove $(z_0, z_0^*) \in (T_{\rm co} + N_C)^{\mu_p}$. Take $\{(z_j, z_j^* + n_j^*)\}_{j=1}^p \subset T_{\rm co} + N_C$, that is,

$$n_j^* \in N_C(z_j) \text{ and } z_j^* = \sum_{l^j=1}^{r_j} \lambda_{j,l^j} x_{j,l^j}^*, \forall\, j = 1, \ldots, p,$$

where $x_{j,l^j}^* \in T(z_j)$, $\lambda_{j,l^j} \geq 0$, for all $l^j = 1, \ldots, r_j$, and $\displaystyle\sum_{l^j=1}^{r_j} \lambda_{j,l^j} = 1$.
Therefore, considering $n_0^* = 0$,

$$\begin{aligned}\sum_{j=0}^{p} \langle z_{j+1} - z_j, z_j^* + n_j^* \rangle \leq \sum_{j=0}^{p} \langle z_{j+1} - z_j, z_j^* \rangle &= \sum_{j=0}^{p} \langle z_{j+1} - z_j, \sum_{l^j=1}^{r_j} \lambda_{j,l^j} x_{j,l^j}^* \rangle \\ &= \sum_{l^0=1}^{r_1} \cdots \sum_{l^p=1}^{r_p} \lambda_{0,l^0} \cdots \lambda_{p,l^p} \sum_{j=0}^{p} \langle z_{j+1} - z_j, x_{j,l^j}^* \rangle,\end{aligned}$$

where the first inequality holds since $z_0 \in C$ and $\langle z_0 - z_p, z_p^* + n_p^* \rangle \leq \langle z_0 - z_p, z_p^* \rangle$.
Note that the sum $\displaystyle\sum_{j=0}^{p} \langle z_{j+1} - z_j, x_{j,l^j}^* \rangle \leq 0$, for every fixed choice of $l^0, l^1, \ldots, l^p$.
Therefore, $T^{\mu_p}|_C \subset [T_{\rm co} + N_C]^{\mu_p}$ and the first part of the proposition follows. The second part follows immediately. □

Corollary 2.4. *Let $T : X \rightrightarrows X^*$ be a multivalued operator, $C = \mathrm{co}(\mathrm{dom}(T))$ and let N be any conic valued operator such that $N \subset N_C$. Then*

$$[T_{\rm co} + N]^{\mu_p}(x) = [T_{\rm co}]^{\mu_p}(x) = T^{\mu_p}(x).$$

Proof. It is enough to observe that, for any conic valued operator N satisfying $N \subset N_C$, it holds $T \subset T_{\rm co} \subset T_{\rm co} + N \subset T_{\rm co} + N_C$. □

Lemma 2.5. *Let $\mathcal{F} = \{(w_i, w_i^*)\}_{i=1}^n$ be a p-cyclically monotone operator, and consider the operator*

$$S = \bigcup_{i=1}^{n} \mathrm{co}\{(w_i, w_i^*), (w_{i+1}, w_{i+1}^*)\},$$

where $(w_{n+1}, w_{n+1}^*) = (w_1, w_1^*)$. If $\langle w_i - w_{i+1}, w_i^* - w_{i+1}^* \rangle = 0$ then S is p-cyclically monotone and $S^{\mu_p} = \mathcal{F}^{\mu_p}$.

Remark 2.6. The condition $\langle w_i - w_{i+1}, w_i^* - w_{i+1}^* \rangle = 0$ implies that the graph of S is a closed curve formed by consecutive perpendicular line segments in $X \times X^*$.

Proof. Define $(a_i(t), a_i^*(t)) = (1-t)(w_i, w_i^*) + t(w_{i+1}, w_{i+1}^*)$, for $t \in [0,1]$ and $i = 1, \ldots, n$. Thus, S can be rewritten as

$$S = \bigcup_{i=1}^{n} \{(a_i(t), a_i^*(t)) \;:\; t \in [0,1]\}.$$

Note that, for fixed i and t, since $\langle w_i - w_{i+1}, w_i^* - w_{i+1}^* \rangle = 0$,

$$\langle a_i(t), a_i^*(t) \rangle = \langle w_i, w_i^* \rangle + (\langle w_i, w_{i+1}^* \rangle - 2\langle w_i, w_i^* \rangle + \langle w_{i+1}, w_i^* \rangle)t \tag{1}$$

which is affine. To prove that S is p-cyclically monotone we need to verify that

$$\sup_{\{(z_j, z_j^*)\}_{j=0}^p \subset S} \sum_{j=0}^p \langle z_{j+1} - z_j, z_j^* \rangle \leq 0,$$

which is equivalent to

$$\hat{\mu} = \sup \left\{ \sum_{j=0}^p \langle a_{i_{j+1}}(t_{j+1}) - a_{i_j}(t_j), a_{i_j}^*(t_j) \rangle \ : \ i_j \in \{1, \ldots, n\}, t_j \in [0,1] \right\} \leq 0,$$

Since S is compact, the above supremum is attained, so there exist $\hat{i}_j \in \{1, \ldots, n\}$, $\hat{t}_j \in [0,1]$, $j = 0, \ldots, p$, such that

$$\hat{\mu} = \sum_{j=0}^p \langle a_{\hat{i}_{j+1}}(\hat{t}_{j+1}) - a_{\hat{i}_j}(\hat{t}_j), a_{\hat{i}_j}^*(\hat{t}_j) \rangle.$$

Fixing $\hat{i}_0, \ldots \hat{i}_p$, because of (1), the function

$$\mu(t_0, \ldots, t_p) = \sum_{j=0}^p \langle a_{\hat{i}_{j+1}}(t_{j+1}) - a_{\hat{i}_j}(t_j), a_{\hat{i}_j}^*(t_j) \rangle$$

is affine on each t_j separately. This implies that in

$$\begin{aligned} \hat{\mu} &= \max_{t_0 \in [0,1]} \cdots \max_{t_p \in [0,1]} \mu(t_0, \ldots, t_p) \\ &= \max_{t_0 \in [0,1]} \cdots \max_{t_p \in [0,1]} \sum_{j=0}^p \langle a_{\hat{i}_{j+1}}(t_{j+1}) - a_{\hat{i}_j}(t_j), a_{\hat{i}_j}^*(t_j) \rangle, \end{aligned}$$

each maximization is attained when either $t_j = 0$ or $t_j = 1$.
Therefore $(a_{\hat{i}_j}(\hat{t}_j), a_{\hat{i}_j}^*(\hat{t}_j)) \in \mathcal{F}$, for all $j = 0, \ldots, p$ and thus, $\hat{\mu} \leq 0$, since $\mathcal{F}$ is p-cyclically monotone by hypothesis.
Let $(z_0, z_0^*) \in \mathcal{F}^{\mu_p}$ and define

$$\begin{aligned} \tilde{\mu} &= \sup_{\{(z_j, z_j^*)\}_{j=1}^p \subset S} \sum_{j=0}^p \langle z_{j+1} - z_j, z_j^* \rangle \\ &= \sup \Bigg\{ \langle a_{i_1}(t_1) - z_0, z_0^* \rangle + \sum_{j=1}^{p-1} \langle a_{i_{j+1}}(t_{j+1}) - a_{i_j}(t_j), a_{i_j}^*(t_j) \rangle \\ &\qquad\qquad + \langle z_0 - a_{i_p}(t_p), a_{i_p}^*(t_p) \rangle \ : \ i_j \in \{1, \ldots, n\}, t_j \in [0,1] \Bigg\}. \end{aligned}$$

In the same way as before, the previous supremum is attained, and the values of t_j which maximize such expresion are either 0 or 1.

Therefore, for some $\{(y_i, y_i^*)\}_{i=1}^p \subset \mathcal{F}$, we have

$$\tilde{\mu} = \sum_{j=0}^{p} \langle y_{j+1} - y_j, y_j^* \rangle,$$

where $(y_0, y_0^*) = (z_0, z_0^*)$. Since $(z_0, z_0^*) \in \mathcal{F}^{\mu_p}$, we obtain $\tilde{\mu} \leq 0$, and the lemma follows. $\square$

Lemma 2.7. *Let S be a p-cyclically monotone operator, $a \in \operatorname{dom}(S^{\mu_p})$ and consider $T = (\{a\} \times S^{\mu_p}(a)) \cup S$. Then, T is also p-cyclically monotone.*

Proof. First, we will prove that T is monotone. Indeed, given $(b, b^*), (c, c^*) \in T$, let $M = \langle b - c, b^* - c^* \rangle$,

- if both $b = c = a$ then $M = 0$,
- if $b = a$ and $c \neq a$, then $M \geq 0$, since $(c, c^*) \in S$ and $(a, b^*) \in S^{\mu_p}$,
- if $b, c \neq a$ then $M \geq 0$, since S is p-cyclically monotone.

Now let $1 < r \leq p$. Assume that T is j-cyclically monotone, for every $1 \leq j < r$. We aim to prove that T is also r-cyclically monotone.

Take $\{(z_i, z_i^*)\}_{i=0}^r \subset T$. If $\{(z_i, z_i^*)\}_{i=0}^r \subset S$, then we are done. Otherwise, without loss of generality we may assume that $z_0 = a$, so $z_0^* \in S^{\mu_p}(a)$. If the remaining z_i, $i = 1, \ldots, r$ are different from a then

$$\sum_{i=0}^{r} \langle z_{i+1} - z_i, z_i^* \rangle \leq 0$$

since $(z_0, z_0^*) \in S^{\mu_p} \subset S^{\mu_r}$. Otherwise, there exists $k \neq 0$ such that $z_k = a$, and

$$\sum_{i=0}^{k-1} \langle z_{i+1} - z_i, z_i^* \rangle + \sum_{i=k}^{r} \langle z_{i+1} - z_i, z_i^* \rangle \leq 0$$

since both sums are associated to the finite sets $\{(z_i, z_i^*)\}_{i=0}^{k-1}$ and $\{(z_i, z_i^*)\}_{i=k}^{p}$ and T is already assumed to be $(k-1)$ and $(r-k)$-cyclically monotone. $\square$

The following results deal with finite operators.

Proposition 2.8. *Let $\mathcal{F} : X \rightrightarrows X^*$ be a finite multivalued operator. Given $p \geq 2$ and $z_0 \in \operatorname{dom}(\mathcal{F}^{\mu_p})$, the set $\mathcal{F}^{\mu_p}(z_0)$ is a polyhedron. Moreover,*

(1) *the graph of $\mathcal{F}^{\mu_p}$ is the solution set of at most $\#\operatorname{dom}(\mathcal{F}) \cdot \#\operatorname{ran}(\mathcal{F})$ inequalities, each one linear in terms of either z_0 or z_0^*, where $(z_0, z_0^*) \in \mathcal{F}^{\mu_p}$; and*

(2) *for any $z_0 \in \operatorname{dom}(\mathcal{F}^{\mu_p})$, the set $\mathcal{F}^{\mu_p}(z_0)$ is defined by at most $\#\operatorname{dom}(\mathcal{F})$ inequalities, each one linear in terms of z_0^*, where $z_0^* \in \mathcal{F}^{\mu_p}(z_0)$.*

Proof. By definition, $z_0^* \in \mathcal{F}^{\mu_p}(z_0)$ if, and only if,

$$\sum_{i=0}^{p} \langle z_{i+1} - z_i, z_i^* \rangle \leq 0, \ \forall \{(z_i, z_i^*)\}_{i=1}^p \subset \mathcal{F}, \tag{2}$$

considering $z_{p+1} = z_0 \in C$. This implies that $\mathcal{F}^{\mu_p}(z_0)$ is a polyhedron, as it is a finite intersection of half-spaces.

Let $Z = \{(z_i, z_i^*)\}_{i=1}^p \subset \mathcal{F}$. Note that the inequality in (2) is equivalent to

$$N(Z) \leq \langle z_0 - z_1, z_0^* - z_p^* \rangle, \tag{3}$$

where $N(Z) = \sum_{i=1}^{p-1} \langle z_{i+1} - z_i, z_i^* \rangle + \langle z_1 - z_p, z_p^* \rangle$. Define $\widetilde{N}(z_1, z_p^*)$ as the maximum over all the points $(z_2, z_2^*), \ldots, (z_{p-1}, z_{p-1}^*) \in \mathcal{F}$ and all $z_1^* \in \mathcal{F}(z_1)$, $z_p \in \mathcal{F}^{-1}(z_p^*)$.

Thus, $$\widetilde{N}(z_1, z_p^*) := \max \left\{ N(Z) \ : \ \begin{array}{c} \{(z_i, z_i^*)\}_{i=2}^{p-1} \subset \mathcal{F} \\ z_1^* \in \mathcal{F}(z_1),\ z_p \in \mathcal{F}^{-1}(z_p^*) \end{array} \right\} \tag{4}$$

and from (3) we deduce $\widetilde{N}(z_1, z_p^*) \leq \langle z_0 - z_1, z_0^* - z_p^* \rangle$. This is equivalent to

$$\langle z_0 - z_1, z_0^* \rangle \geq \widetilde{N}(z_1, z_p^*) + \langle z_0 - z_1, z_p^* \rangle. \tag{5}$$

Note that the last inequality only depends on z_0, z_1 and z_p^*. Therefore, for fixed z_0, any inequality generated by an arbitrary choice of $\{(z_i, z_i^*)\}_{i=1}^p$ would be exactly as, or implied by, inequality (5). Item (1) of the proposition follows by observing that there are at most $\#\operatorname{dom}(\mathcal{F})$ choices of z_1 and $\#\operatorname{ran}(\mathcal{F})$ choices of z_p^*. Note that inequality (5) is linear in terms of both z_0 and z_0^* separately.

Finally, in the same way as before, let $\widetilde{M}(z_0, z_1)$ be the maximum of (5) over all possible $z_p^* \in \operatorname{ran}(\mathcal{F})$, that is,

$$\widetilde{M}(z_0, z_1) = \max\{\widetilde{N}(z_1, z_p^*) + \langle z_0 - z_1, z_p^* \rangle \ : \ z_p^* \in \operatorname{ran}(\mathcal{F})\}. \tag{6}$$

Therefore (5) is exactly as, or implied by,

$$\widetilde{M}(z_0, z_1) \leq \langle z_0 - z_1, z_0^* \rangle. \tag{7}$$

Item *2* follows by observing that there are at most $\#\operatorname{dom}(\mathcal{F})$ choices for z_1 in (7) and that (7) is linear in terms of z_0^*. □

Remark 2.9. When the elements of $\mathcal{F}$ have small integer components, the quantities $\widetilde{N}(z_1, z_p^*)$ and $\widetilde{M}(z_0, z_1)$ can be exactly computed using any programming language. See [4, § 4] for such an implementation.

Lemma 2.10. (Farkas) *Let A be a $m \times n$ matrix and $b \in \mathbb{R}^n$. The system $Ax \geq b$ is consistent if, and only if,*

$$\forall\, y \geq 0,\ A^\top y = 0 \Longrightarrow \langle b, y \rangle \leq 0.$$

Corollary 2.11. *Let $\mathcal{F}$ and $\widetilde{M}$ be defined as in Proposition* 2.8*, and assume* $\operatorname{dom}(\mathcal{F}) = \{x_1, \ldots, x_n\}$. *Then $z_0 \in \operatorname{dom}(\mathcal{F}^{\mu_p})$ if, and only if, for all $\lambda_1, \ldots, \lambda_n \geq 0$, such that $\sum_{i=1}^n \lambda_i = 1$ and $z_0 = \sum_{i=1}^n \lambda_i x_i$, we have $\sum_{i=1}^n \lambda_i \widetilde{M}(z_0, x_i) \leq 0$.*

Proof. Let $z_0 \in X$. The system of inequalities (7) can be written in the form $A(z_0) \cdot z_0^* \geq b(z_0)$, where

$$A(z_0) = [z_0 - x_1, \ldots, z_0 - x_n]^\top, \qquad b(z_0) = (\widetilde{M}(z_0, x_1), \ldots, \widetilde{M}(z_0, x_n)).$$

Now let $y = (\lambda_1, \ldots, \lambda_n)$, with $\lambda_i \geq 0$, but not all zero, then

$$A(z_0)^\top y = \left(\sum_{i=1}^n \lambda_i\right) z_0 - \sum_{i=1}^n \lambda_i x_i = \sigma\left(z_0 - \sum_{i=1}^n \frac{\lambda_i}{\sigma} x_i\right)$$

where $\sigma = \sum_{i=1}^n \lambda_i > 0$. We thus have proved that $A(z_0)^\top y = 0$ if, and only if, $z_0 \in \operatorname{co}(\operatorname{dom}(\mathcal{F}))$ and the weighted components of y are the coefficients associated to z_0 as a convex combination of x_i's. The corollary follows by using Farkas' Lemma on system (7) and observing that

$$\langle b(z_0), y\rangle = \frac{1}{\sigma} \sum_{i=1}^n \lambda_i \widetilde{M}(z_0, x_i). \qquad \square$$

Corollary 2.12. *Let $\mathcal{F}$ be defined as in Proposition 2.8. Then*

$$\operatorname{co}(\operatorname{dom}(\mathcal{F})) \cup \operatorname{dom}(\mathcal{F}^{\mu_p}) = X.$$

3. The algorithm

From now on, X will denote a real Hilbert space and we will identify X^* with X. Let $T_0 = \{(x_i, x_i^*)\}_{i=1}^n$ be a p-cyclically monotone operator. Consider, for $k = 1, \ldots, n$, the following sequence of operators:

$$T_k = \big(\{x_k\} \times T_{k-1}^{\mu_p}(x_k)\big) \cup T_{k-1}.$$

and $T_{n+1} = T_n^{\mu_p}$. Note that, for $k = 1, \ldots, n$,

$$T_k = \bigcup_{j=1}^k \big(\{x_j\} \times T_{j-1}^{\mu_p}(x_j)\big) \cup \big\{(x_j, x_j^*)\big\}_{j=k+1}^n.$$

Lemma 2.7 implies that T_k, for each $k = 1, \ldots, n$, is p-cyclically monotone.

Remark 3.1. The operators T_k, $k = 1, \ldots, n+1$, depend not only on T_0 but also on p. We aim to eventually prove that T_{n+1} is maximal p-cyclically monotone, but even when this is the case, it might happen that it is not maximal $(p-1)$-cyclically monotone. See the example given in Section 4.3.

From now on, consider $C = \operatorname{co}(\{x_1, \ldots, x_n\})$. Note that C is convex and closed.

Proposition 3.2. *The following hold:*

(1) $T_1, \ldots, T_n$ *are p-cyclically monotone.*

(2) *For each $k = 1, \ldots, n$, $T_{k-1}^{\mu_p}(x_k)$ is a polyhedron, so it can be written as*

$$T_{k-1}^{\mu_p}(x_k) = \operatorname{co}(E_k) + N_C(x_k) \quad \textit{for some finite set } E_k.$$

(3) *Let $\mathcal{F} = \bigcup_{k=1}^n \{x_k\} \times E_k$. Then*

$$T_n = \bigcup_{k=1}^n \{x_k\} \times (\operatorname{co}(E_k) + N_C(x_k)) = \mathcal{F}_{\mathrm{co}} + N_C. \tag{8}$$

(4) $\operatorname{dom}(T_{n+1}) \subset C$.

(5) $T_{n+1}(x_k) = T_n(x_k)$, *for all $k = 1, \ldots, n$.*

(6) $T_{n+1} = \mathcal{F}^{\mu_p}\big|_C$

Proof. (1) It is an immediate consequence of Lemma 2.7.

(2) The set $T_0^{\mu_p}(x_1)$ is a polyhedron since T_0 is a finite set and by Proposition 2.8. Let $k \in \{2,\dots,n\}$ and assume that $T_{j-1}^{\mu_p}(x_j)$ is a polyhedron, for all $j < k$. Let $E_j \subset T_{j-1}^{\mu_p}(x_j)$ be a finite set such that $T_{j-1}^{\mu_p}(x_j) = \mathrm{co}(E_j) + N(x_j)$, where, in view of Lemma 2.2,

$$N(x_j) = N_C(x_j) = 0^+[T_{j-1}^{\mu_p}(x_j)], \text{ for } j = 1,\dots,k-1,$$

and consider $E_j = \{x_j^*\}$ and $N(x_j) = \{0\}$, for $j = k,\dots,n$. Thus we can write

$$T_{k-1} = \bigcup_{j=1}^{n} \{x_j\} \times (\mathrm{co}(E_j) + N(x_j)),$$

that is $T_{k-1} = F_{\mathrm{co}} + N$, where $F = \bigcup_{j=1}^{n} \{x_j\} \times E_j$ is a finite set and N is conic valued and $N \subset N_C$. By Corollary 2.4,

$$T_{k-1}^{\mu_p}(x_k) = F^{\mu_p}(x_k),$$

which is a polyhedron, since F is finite.

(3) follows from (2).

(4) Let $x \in \mathrm{dom}(T_{n+1})$ and let $x^* \in T_{n+1}(x) = T_n^{\mu_p}(x) \subset T_n^{\mu_1}(x)$. Therefore, for every (y, y^*) in T_n,

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

In particular, for any $j = 1,\dots,n$, take $t > 0$, $x_j^* \in T_n(x_j)$ and

$$n_j^* \in 0^+T_n(x_j) = 0^+T_{j-1}^{\mu_p}(x_j) = N_C(x_j),$$

thus $x_j^* + tn_j^* \in T_n(x_j)$, and

$$\langle x - x_j, x^* - (x_j^* + tn_j^*) \rangle \geq 0 \qquad \Longleftrightarrow \qquad \langle x - x_j, x^* - x_j^* \rangle \geq t\langle x - x_j, n_j^* \rangle,$$

for all $t > 0$. This implies that $\langle x - x_j, n_j^* \rangle \leq 0$, for all $n_j^* \in N_C(x_j)$, that is $x - x_j \in N_C(x_j)^\circ = T_C(x_j)$. Therefore, $x \in x_j + T_C(x_j)$, for all $j = 1,\dots,n$. Using Theorem 2.15 in [3], we conclude that

$$x \in \bigcap_{j=1}^{n} (x_j + T_C(x_j)) = C.$$

(5) Since T_n is p-cyclically monotone and $T_{n+1} = T_n^{\mu_p}$, then $T_n(x_k) \subset T_{n+1}(x_k)$. On the other hand, since $T_{k-1} \subset T_n$,

$$T_{n+1}(x_k) = T_n^{\mu_p}(x_k) \subset T_{k-1}^{\mu_p}(x_k) = T_k(x_k) = T_n(x_k).$$

(6) This is a consequence of equation (8) and Proposition 2.3. □

Remark 3.3. By (6) of the previous proposition, given $x \in C$, $T_{n+1}(x)$ is a polyhedron. Moreover, Proposition 2.8 states that $T_{n+1}(x)$ can be defined by at most n linear inequalities, when $p \geq 2$, or by at most $\#\mathcal{F}$ inequalities, when $p = 1$.

Let $s = (1,\dots,1) \in \mathbb{R}^n$ and let S_n be the simplex $S_n = \{\lambda \in \mathbb{R}^n : \lambda \geq 0, \langle \lambda, s \rangle = 1\}$.

Lemma 3.4. *Let $x_1, \dots, x_n \in X$, $C = \operatorname{co}(\{x_1, \dots, x_n\})$ and let $F : S_n \to C$ be defined as $F(\lambda) = \sum_{i=1}^{n} \lambda_i x_i$. Then F is an open map, that is, if $U \subset S_n$ is open in S_n, then $F(U)$ is open in C.*

Proof. The points in $X \times \mathbb{R}^n$, $(x_1, e_1), \dots, (x_n, e_n)$, are affinely independent, where $e_1, \dots, e_n$ are the canonical vectors in $\mathbb{R}^n$. Thus, $\hat{F} : S_n \to \operatorname{co}\{(x_i, e_i)\}$ is a bijective continuous function between compact sets, hence an homeomorphism. The lemma follows by observing that the projection $\Pi_1 : \operatorname{co}\{(x_i, e_i)\} \to C$, $\Pi_1(x, y) = x$, is open and that $F = \Pi_1 \circ \hat{F}$. $\square$

Proposition 3.5. $\operatorname{dom}(T_{n+1})$ *is closed.*

Proof. Let $z_0 \in C$, $z_0 \notin \operatorname{dom}(T_{n+1})$, and let $\mathcal{F}$ be as in Proposition 3.2(3). Using Corollary 2.11 applied to $\mathcal{F}$, there exists $\lambda_0 \in S_n$ such that $z_0 \in F(\lambda_0)$ and

$$\widetilde{\mathcal{M}}(\lambda_0) = \sum_{i=1}^{n} \lambda_i \widetilde{M}(F(\lambda_0), x_i) > 0.$$

Since $\widetilde{\mathcal{M}}$ is continuous, there exists $\delta > 0$ such that, for all $\lambda \in B(\lambda_0, \delta) \cap S_n$ we have $\widetilde{\mathcal{M}}(\lambda) > 0$. On the other hand, by Lemma 3.4, F is an open map, so there exists $\varepsilon > 0$ such that $B(z_0, \varepsilon) \cap C \subset F(B(\lambda_0, \delta) \cap S_n)$. This implies that, for all $z \in B(z_0, \varepsilon) \cap C$, there exists $\lambda \in B(\lambda_0, \delta) \cap S_n$ such that $z = F(\lambda)$ and, thus, $\widetilde{\mathcal{M}}(\lambda) > 0$, that is $z \notin \operatorname{dom}(T_{n+1})$. The proposition then follows. $\square$

4. Examples

In this section we present many different explicit examples of maximal p-cyclically monotone operators. First we show that, using the elements of the finite operator given in the proof of [2, Example 2.16] as starting points, our algorithm returns the operator given in [3]. Next we provide new examples of maximal 2 and 3-cyclically monotone operators obtained from our algorithm. The interested reader can find in [4, § 5] the comprehensive set of calculations we have done to obtain the formulas of our examples, along with implementations of the computational tools that aided us in our calculations.

4.1. The original example of Bauschke and Wang

Consider the starting points:

$$\begin{aligned} x_1 &= (1,0), \quad x_1^* = (0,1), \qquad & x_2 &= (0,1), \quad x_2^* = (-1,0), \\ x_3 &= (-1,0), \quad x_3^* = (-1,-2), \qquad & x_4 &= (0,-1), \quad x_4^* = (0,-1). \end{aligned}$$

After the first four steps of our algorithm, we obtain the following images of $x_1, \dots, x_4$:

$$\begin{aligned} T_4(x_1) &= \{(u,v) \,:\, u + 1 \geq |v|,\, u \geq 0\}, \\ T_4(x_2) &= \{(u,v) \,:\, v \geq |u+1|\}, \\ T_4(x_3) &= \{(u,v) \,:\, u \leq -|v+1|,\, u \leq -1\}, \\ T_4(x_4) &= \{(u,v) \,:\, v \leq -|u| - 1\}. \end{aligned}$$

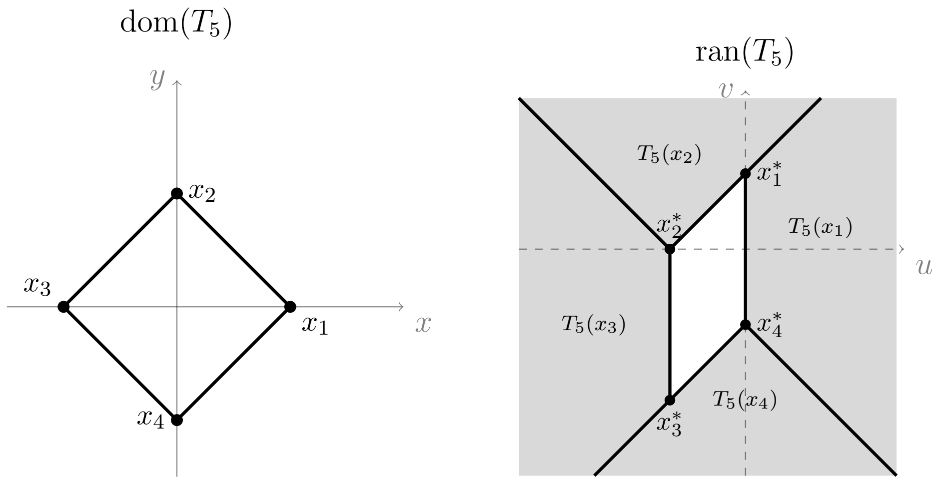

Figure 4.1: Example from Bauschke and Wang [3]

Therefore $\mathcal{F} = \{(y_i, y_i^*)\}_{i=1}^6$, as in Proposition 3.2(3), is given by

$$\begin{array}{llll} y_1 = (1,0), & y_1^* = (0,-1), & y_2 = (1,0), & y_2^* = (0,1), \\ y_3 = (0,1), & y_3^* = (-1,0), & y_4 = (-1,0), & y_4^* = (-1,0), \\ y_5 = (-1,0), & y_5^* = (-1,-2), & y_6 = (0,-1), & y_6^* = (0,-1), \end{array}$$

that is
$$\operatorname{dom}(\mathcal{F}) = \{(1,0),(0,1),(-1,0),(0,-1)\},$$
$$\operatorname{ran}(\mathcal{F}) = \{(0,1),(0,-1),(-1,0),(-1,-2)\}.$$

We now consider $z_0 = (x,y)$ and $z_0^* = (u,v)$, and follow the steps of the proof of Proposition 2.8, so we obtain the set of inequalities:

$$\begin{aligned} \max\{1-x, -x-2y-1, -y, y\} &\leq (x-1)u + yv, \\ \max\{-x, -x-2y-2, -y-1, y-1\} &\leq xu + (y-1)v, \\ \max\{-x-2, -x-2y, -y-1, y-1\} &\leq (x+1)u + yv, \\ \max\{-x-1, -x-2y-1, -y, y-2\} &\leq xu + (y+1)v. \end{aligned}$$

These inequalities were obtained and handled by Bauschke and Wang in [3]. They were able to prove that the last step operator T_5 is defined on $\{(x,y) : |x|+|y| = 1\}$ and

$$\begin{aligned} T_5(x_i) &= T_4(x_i), \quad \forall i = 1,2,3,4, \\ T_5(1-t,t) &= (-t, 1-t) + \operatorname{cone}\{(1,1)\}, \\ T_5(-t, 1-t) &= (-1,0) + \operatorname{cone}\{(-1,1)\}, \\ T_5(-1+t, -t) &= (t-1, t-2) + \operatorname{cone}\{(-1,-1)\}, \\ T_5(t, t-1) &= (0,-1) + \operatorname{cone}\{(1,-1)\}. \end{aligned}$$

See Figure 4.1 for a partial graphical representation of the domain and range of T_5.

4.2. A maximal 2-cyclically monotone operator in $\mathbb{R}^3$

Consider the starting points:

$$x_1 = (-1,-1,-1),\quad x_1^* = (-8,-8,16),\qquad x_2 = (1,0,0),\quad x_2^* = (8,12,0),$$
$$x_3 = (0,1,0),\quad x_3^* = (-12,8,0),\qquad x_4 = (0,0,1),\quad x_4^* = (0,0,16).$$

Note that $C = \operatorname{co}\{x_1, x_2, x_3, x_4\}$ is a tetrahedron in $\mathbb{R}^3$. From these starting points, the full correspondence rule of T_5 is

$$\begin{aligned}
T_5(-1,-1,-1) &= (-6,-2,14) + \operatorname{cone}\operatorname{co}\{(1,1,-3),(1,-3,1),(-3,1,1)\},\\
T_5(1,0,0) &= (-4,0,12) + \operatorname{cone}\operatorname{co}\{(1,1,1),(1,-3,1),(1,1,-3)\},\\
T_5(0,1,0) &= (-5,-1,11) + \operatorname{cone}\operatorname{co}\{(1,1,1),(-3,1,1),(1,1,-3)\},\\
T_5(0,0,1) &= (-3,-3,13) + \operatorname{cone}\operatorname{co}\{(1,1,1),(1,-3,1),(-3,1,1)\},
\end{aligned}$$

and, for all $t \in]0,1[$,

$$\begin{aligned}
T_5(t,1-t,0) &= (t-5,t-1,t+11) + \operatorname{cone}\operatorname{co}\{(1,1,-3),(1,1,1)\},\\
T_5(-t,1-2t,-t) &= (-5-t,-1-t,11+3t) + \operatorname{cone}\operatorname{co}\{(-3,1,1),(1,1,-3)\},\\
T_5(-t,-t,1-2t) &= (-3-3t,t-3,13+t) + \operatorname{cone}\operatorname{co}\{(-3,1,1),(1,-3,1)\},\\
T_5(t,0,1-t) &= (-3-t,3t-3,13-t) + \operatorname{cone}\operatorname{co}\{(1,-3,1),(1,1,1)\}.
\end{aligned}$$

Note that, although $C = \operatorname{co}\{x_i\}$ is a tetrahedron, the domain of T_5 are just the segments

$$\operatorname{dom}(T_5) = [x_2,x_3] \cup [x_3,x_1] \cup [x_1,x_4] \cup [x_4,x_2].$$

See Figure 4.2.

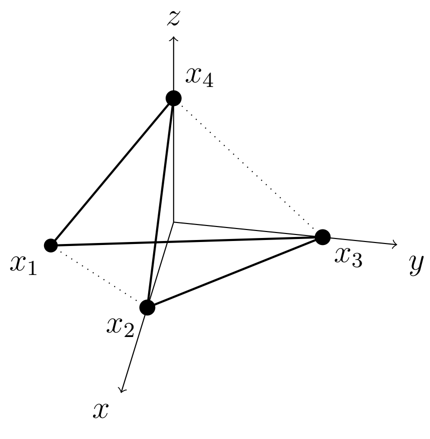

Figure 4.2: Domain of a maximal 2-cyclically monotone operator in $\mathbb{R}^3$

From the correspondence rule of T_5, we can conclude that the graph of T_5 is a closed curve formed by consecutive perpendicular line segments in $\mathbb{R}^6$ and the extremes of such lines segments form a 2-cyclically monotone operator. From Lemma 2.5, T_5 is 2-cyclically monotone and, since $T^5 = T_4^{\mu_2}$, T^5 is maximal 2-cyclically monotone.

4.3. A maximal 3-cyclically monotone operator

Consider the starting points:

$$x_1 = (1,0),\ x_1^* = (1,1), \quad x_2 = (1,1),\ x_2^* = (0,2), \quad x_3 = (0,1),\ x_3^* = (-1,1),$$
$$x_4 = (-1,0),\ x_4^* = (-1,-1), \quad x_5 = (0,-1),\ x_5^* = (1,-1).$$

In this case $\operatorname{dom}(T_6) = [x_1, x_2] \cup [x_2, x_3] \cup [x_3, x_4] \cup [x_4, x_5] \cup [x_5, x_1]$, and

$$\begin{aligned}
T_6(1,0) &= \operatorname{co}\{(0,0),(2,2)\} + \operatorname{cone}\operatorname{co}\{(1,-1),(1,0)\},\\
T_6(1,1) &= (0,2) + \operatorname{cone}\operatorname{co}\{(1,0),(0,1)\},\\
T_6(0,1) &= \operatorname{co}\{(0,2),(-1,1)\} + \operatorname{cone}\operatorname{co}\{(0,1),(-1,1)\},\\
T_6(-1,0) &= \operatorname{co}\{(-1,1),(-1,-1)\} + \operatorname{cone}\operatorname{co}\{(-1,1),(-1,-1)\},\\
T_6(0,-1) &= (0,0) + \operatorname{cone}\operatorname{co}\{(-1,-1),(1,-1)\},
\end{aligned}$$

and, for every $t \in]0,1[$,

$$\begin{aligned}
T_6(1,t) &= (2-2t, 2) + \operatorname{cone}\{(1,0)\},\\
T_6(1-t,1) &= (0,2) + \operatorname{cone}\{(0,1)\},\\
T_6(-t,1-t) &= (-1,1) + \operatorname{cone}\{(-1,1)\},\\
T_6(t-1,-t) &= (t-1,t-1) + \operatorname{cone}\{(-1,-1)\},\\
T_6(t,t-1) &= (0,0) + \operatorname{cone}\{(1,-1)\}.
\end{aligned}$$

See Figure 4.3 for a graphical representation of the domain and range of T_6.

Note that T_6 can be written as $T_6 = S + N_C$, where the graph of S is a closed curve formed by consecutive perpendicular line segments on $\mathbb{R}^4$. From Lemma 2.5, the operator T_6 is 3-cyclically monotone and, since $T_6 = T_5^{\mu_3}$, T_6 is maximal 3-cyclically monotone. On the other hand, also from Lemma 2.5 and Proposition 2.3,

$$(0,0) \in \mathcal{F}^{\mu_2}|_C = S^{\mu_2}|_C = (S + N_C)^{\mu_2} = T_6^{\mu_2}.$$

However $(0,0) \notin T_6$. Therefore, T_6 is not maximal 2-cyclically monotone.

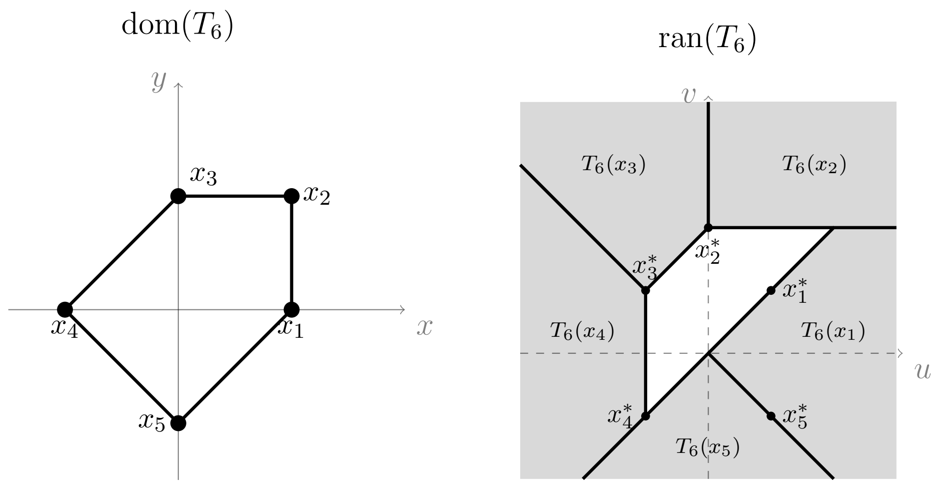

Figure 4.3: A maximal 3-cyclically monotone operator

4.4. A perturbation of Bauschke and Wang's example

Consider the starting points:

$$\begin{aligned} x_1 = (1,0),\ x_1^* = (0,1), \quad x_2 = (0,1),\ x_2^* = (-1,0), \\ x_3 = (-1,0),\ x_3^* = (-2,-2), \quad x_4 = (0,-1),\ x_4^* = (0,-1). \end{aligned}$$

These points are the same as in Section 4.1, but considering $x_3^* = (-2,-2)$ instead of $(-1,-2)$. From these starting points we obtain

$$\operatorname{dom}(T_5) = \{(0,0)\} \cup [x_1,x_2] \cup [x_2,x_3] \cup [x_3,x_4] \cup [x_4,x_1],$$

together with the correspondence rule:

$$\begin{aligned} T_5(1,0) &= \operatorname{co}\{(-1/2,-1/2),(-1/2,1/2)\} + \operatorname{cone}\operatorname{co}\{(1,-1),(1,1)\}, \\ T_5(0,1) &= \operatorname{co}\{(-1,0),(-2,0)\} + \operatorname{cone}\operatorname{co}\{(1,1),(-1,1)\}, \\ T_5(-1,0) &= \operatorname{co}\{(-3/2,-1/2),(-3/2,-3/2)\} + \operatorname{cone}\operatorname{co}\{(-1,1),(-1,-1)\}, \\ T_5(0,-1) &= \operatorname{co}\{(-1,-1),(0,-1)\} + \operatorname{cone}\operatorname{co}\{(-1,-1),(1,-1)\}, \\ T_5(0,0) &= (-1,-1/2), \end{aligned}$$

and, for all $t \in]0,1[$,

$$T_5(1-t,t) = \begin{cases} (-t-1/2,-t+1/2) + \operatorname{cone}\{(1,1)\}, & \text{if } t \le 1/2, \\ (-1,0) + \operatorname{cone}\{(1,1)\}, & \text{if } t > 1/2, \end{cases}$$

$$T_5(-t,1-t) = \begin{cases} (t-2,-t) + \operatorname{cone}\{(-1,1)\}, & \text{if } t \le 1/2, \\ (-3/2,-1/2) + \operatorname{cone}\{(-1,1)\}, & \text{if } t > 1/2, \end{cases}$$

$$T_5(t-1,-t) = \begin{cases} (t-3/2,t-3/2) + \operatorname{cone}\{(-1,-1)\}, & \text{if } t \le 1/2, \\ (-1,-1) + \operatorname{cone}\{(-1,-1)\}, & \text{if } t > 1/2, \end{cases}$$

$$T_5(t,t-1) = \begin{cases} (-t,t-1) + \operatorname{cone}\{(1,-1)\}, & \text{if } t \le 1/2, \\ (-1/2,-1/2) + \operatorname{cone}\{(1,-1)\}, & \text{if } t > 1/2. \end{cases}$$

See Figure 4.4 for a graphical representation of the domain and range.

Note that the operator T_5 can be written as

$$T_5 = (S + N_C) \cup \{((0,0),(-1,-1/2))\},$$

where the graph of S consists of consecutive line segments in $\mathbb{R}^4$. From Lemma 2.5 we know that S is 2-cyclically monotone. Also from Lemma 2.5 we have $\mathcal{F}^{\mu_2} = S^{\mu_2}$. Since $((0,0),(-1,-1/2)) \in \mathcal{F}^{\mu_2}$, we conclude that T_5 is also 2-cyclically monotone. Therefore T_5 is maximal 2-cyclically monotone.

Remark 4.1. It is possible to verify that if we use $x_5 = (0,0)$, $x_5^* = (-1,-1/2)$ as an additional starting point, we would have obtained the same operator.

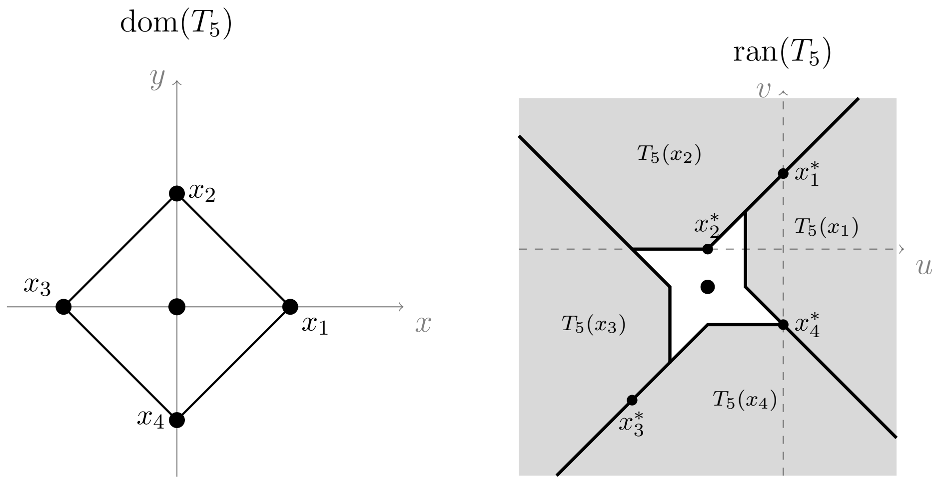

Figure 4.4: Perturbation of Bauschke and Wang's example

4.5. Another bizarre maximal 2-cyclically monotone operator

Consider the starting points:

$$x_1 = (0,0),\ x_1^* = (-1,-1/2), \quad x_2 = (1,0),\ x_2^* = (0,1), \quad x_3 = (0,1),\ x_3^* = (-1,0),$$
$$x_4 = (-1,0),\ x_4^* = (-2,2), \quad x_5 = (0,-1),\ x_5^* = (0,-1).$$

In this case, the full correspondence rule of T_6 is

$$\begin{aligned}
T_6(0,0) &= \mathrm{co}\{(-1,-1),(-1,0)\},\\
T_6(1,0) &= \mathrm{co}\{(0,-1),(0,1)\} + \mathrm{cone}\{(1,-1),(1,1)\},\\
T_6(0,1) &= \mathrm{co}\{(-1,0),(-2,0)\} + \mathrm{cone}\{(1,1),(-1,1)\},\\
T_6(-1,0) &= \mathrm{co}\{(-2,0),(-2,-2)\} + \mathrm{cone}\{(-1,1),(-1,-1)\},\\
T_6(0,-1) &= \mathrm{co}\{(-1,-1),(0,-1)\} + \mathrm{cone}\{(-1,-1),(1,-1)\}.
\end{aligned}$$

and, for all $t \in]0,1[$,

$$\begin{aligned}
T_6(1-t,t) &= \begin{cases} (-2t,1-2t), & \text{if } t \in]0,1/2[,\\ (-1,0), & \text{if } t \in [1/2,1[, \end{cases}\\
T_6(-t,1-t) &= (-2,0),\\
T_6(-1+t,-t) &= \begin{cases} (2t-2,2t-2), & \text{if } t \in]0,1/2[,\\ (-1,-1), & \text{if } t \in [1/2,1[, \end{cases}\\
T_6(t,t-1) &= (0,-1).
\end{aligned}$$

In addition, for $\alpha \in]-1,1[\setminus\{0\}$,

$$T_6(0,\alpha) = \begin{cases} [-\alpha-1,-1] \times \{0\}, & \text{if } \alpha \in]0,1[,\\ [-1,-\alpha-1] \times \{-1\}, & \text{if } \alpha \in]-1,0[, \end{cases}$$

and
$$T_6(x,y) = \begin{cases} (-1,0), & \text{if } 0 < x \le y < 1-x,\\ (-1,-1), & \text{if } -1-x < y \le x < 0. \end{cases}$$

See Figure 4.5 for a partial graph of the domain and range of T_6.

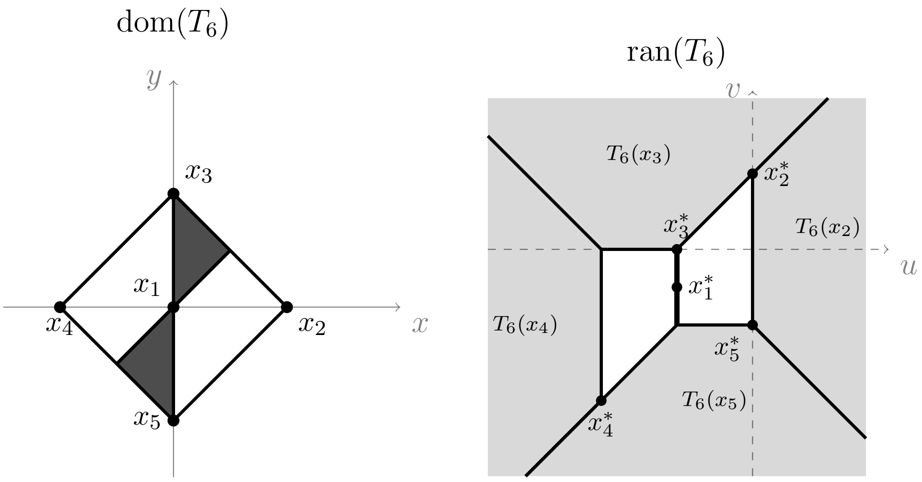

Figure 4.5: Yet another maximal 2-cyclically monotone operator

It is possible to verify that T_6 is maximal 2-cyclically monotone, see [4, § 5.5.1] for a computational proof of this fact.

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