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Proximal Determination of Convex Functions

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We provide comparison principles for convex functions through its proximal mappings. Consequently, we prove that the norm of the proximal operator determines a convex the function up to a constant. A new characterization of Lipschitzianity in terms of the proximal operator is given.

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. By determination of a convex function $f\colon \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$, we mean a result of type "if f satisfies a given condition, then f is uniquely determined up to constant." The first determination result was proved by J. J. Moreau in Hilbert spaces (see [5, p.287]):

Theorem 1.1. (Moreau) *If $f, g \in \Gamma_0(\mathcal{H})$ are two functions such that*

$$\operatorname{prox}_f(x) = \operatorname{prox}_g(x) \quad \textit{for all } x \in \mathcal{H},$$

then f and g differ by a constant.

Moreau used the latter result to prove that the subgradients uniquely determine a convex function, which is known as an integration result. Since then, several integration results appeared for convex and nonconvex functions (see, e.g., [7, 3, 8]). In this paper, by using a recent result on the determination of convex functions [6], we provide a new determination result by showing that the norm of the proximal operator determines a convex the function up to a constant (Proposition 4.1 and Theorem 4.2). For this, we establish comparison principles for convex functions through its proximal mapping (Theorem 3.1), which is also used to obtain a new characterization of Lipschitzianity (Proposition 3.3).

The paper is organized as follows. After some preliminaries, in Section 3, we present comparison principles for convex functions in terms of its proximal operators and a new characterization of Lipschitz convex functions through proximal operators. These principles are the basis of the developments of Section 4, where it is shown that the norm of the proximal operator determines a convex function completely, up to a constant.

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2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with an inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$. We denote by $\Gamma_0(\mathcal{H})$ the set of all proper, convex and lower semicontinuous functions from $\mathcal{H}$ with values in $\mathbb{R}\cup\{+\infty\}$. For $f\in\Gamma_0(\mathcal{H})$, its Legendre-Fenchel conjugate function $f^*:\mathcal{H}\to\mathbb{R}\cup\{+\infty\}$ is given by

$$f^*(x^*)=\sup_{v\in\mathcal{H}}\{\langle x^*,v\rangle-f(v)\}.$$

It is known that $f^*\in\Gamma_0(\mathcal{H})$ and that for every $(x,x^*)\in\mathcal{H}\times\mathcal{H}$, the Legendre-Fenchel inequality holds, that is

$$f(x)+f^*(x^*)\geq\langle x^*,x\rangle.$$

For a closed set C, we denote by δ_C de indicator function of C, that is, $\delta_C(x)=0$ if $x\in C$ and $\delta_C(x)=+\infty$ if $x\neq C$. It is clear that $\delta_C\in\Gamma_0(\mathcal{H})$ if and only if C is closed and convex. Moreover, $(\delta_C)^*=\sigma_C$, where σ_C is the support function of C defined by $\sigma_C(x)=\sup_{y\in C}\langle y,x\rangle$.
For $\lambda>0$, the Moreau envelope of f of index λ is the function $f_\lambda:\mathcal{H}\to\mathbb{R}$ given by

$$f_\lambda(x):=\inf_{y\in\mathcal{H}}\left\{f(y)+\frac{1}{2\lambda}\|x-y\|^2\right\}.$$

The above infimum is attained at a unique point, $\operatorname{prox}_{\lambda f}(x)$. The mapping $\operatorname{prox}_{\lambda f}:\mathcal{H}\to\mathcal{H}$ is non-expansive and for $\lambda=1$ it is called the proximal operator, that is,

$$\operatorname{prox}_f(x)=\operatorname{argmin}_{y\in\mathcal{H}}\left\{f(y)+\frac{1}{2}\|x-y\|^2\right\}.$$

It is known that f_λ is convex, continuously differentiable on $\mathcal{H}$, and its derivative is given by

$$\nabla f_\lambda(x)=\frac{1}{\lambda}(x-\operatorname{prox}_{\lambda f}(x))\text{ for all }x\in\mathcal{H}\tag{1}$$

Moreover,
$$(f_\lambda)^*(x)=f^*(x)+\frac{\lambda}{2}\|x\|^2\text{ for all }x\in\mathcal{H}.\tag{2}$$

We refer to [1, 2] for more details about the Moreau envelope and its applications.

To obtain our results, we need the Moreau decomposition (see [5, p. 280]).

Proposition 2.1. (Moreau decomposition) *If $f\in\Gamma_0(\mathcal{H})$, then*

$$\operatorname{prox}_f(x)+\operatorname{prox}_{f^*}(x)=x\quad\textit{for all } x\in\mathcal{H}.$$

We end this section with a comparison principle for convex functions through its gradients (see [6, Theorem 3.1]). This principle is the basis for the determination of convex functions through the norm of (sub)gradients. We refer to [6] for further results in this direction.

Proposition 2.2. *Let $f,g\in\Gamma_0(\mathcal{H})$ be two Gâteaux differentiable convex functions bounded from below such that*

$$\|\nabla f(x)\|\leq\|\nabla g(x)\|\quad\textit{for all } x\in\mathcal{H}.$$

Then, $f-\inf f\leq g-\inf g$.

3. Comparison principles

The following result is a comparison principle for convex functions.

Theorem 3.1. *Let $f, g \in \Gamma_0(\mathcal{H})$ be two functions such that*

$$\|\operatorname{prox}_f(x) - x_0\| \leq \|\operatorname{prox}_g(x) - x_0\| \quad \text{for all } x \in \mathcal{H}$$

is satisfied for some $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$. Then, $g - g(x_0) \leq f - f(x_0)$.

Proof. By virtue of Legendre-Fenchel inequality, for all $x, u \in \mathcal{H}$

$$f^*(x) + f(u) \geq \langle x, u\rangle \text{ and } g^*(x) + g(u) \geq \langle x, u\rangle.$$

Thus, if $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$, then

$$f^*(x) - \langle x, x_0\rangle \geq -f(x_0) \text{ and } g^*(x) - \langle x, x_0\rangle \geq -g(x_0),$$

which implies that the maps $x \mapsto f^*(x) - \langle x_0, x\rangle$ and $x \mapsto g^*(x) - \langle x_0, x\rangle$ are bounded from below. For $\lambda = 1$, let us consider

$$\tilde{f} = (f^* - \langle x_0, \cdot\rangle)_\lambda \text{ and } \tilde{g} = (g^* - \langle x_0, \cdot\rangle)_\lambda .$$

Then, $\tilde{f}$ and $\tilde{g}$ are $C^{1,1}$ and bounded from below functions with

$$\nabla\tilde{f}(x) = x - \operatorname{prox}_{f^* - \langle x_0, \cdot\rangle}(x) \text{ and } \nabla\tilde{g}(x) = x - \operatorname{prox}_{g^* - \langle x_0, \cdot\rangle}(x).$$

Moreover, according to Moreau's decomposition and the properties of the proximal operator, for all $x \in \mathcal{H}$

$$\begin{aligned}
\nabla\tilde{f}(x) &= x - \operatorname{prox}_{f^* - \langle x_0, \cdot\rangle}(x) = \operatorname{prox}_{(f^* - \langle x_0, \cdot\rangle)^*}(x) = \operatorname{prox}_{f(\cdot + x_0)}(x) \\
&= \operatorname{prox}_f(x + x_0) - x_0, \\
\nabla\tilde{g}(x) &= x - \operatorname{prox}_{g^* - \langle x_0, \cdot\rangle}(x) = \operatorname{prox}_{(g^* - \langle x_0, \cdot\rangle)^*}(x) = \operatorname{prox}_{g(\cdot + x_0)}(x) \\
&= \operatorname{prox}_g(x + x_0) - x_0.
\end{aligned}$$

Therefore, for all $x \in \mathcal{H}$: $\|\nabla\tilde{f}(x)\| \leq \|\nabla\tilde{g}(x)\|$.

Hence, by virtue of Proposition 2.2,

$$\tilde{f} \leq \tilde{g} + \inf \tilde{f} - \inf \tilde{g} = \tilde{g} - f(x_0) + g(x_0),$$

where we have used that

$$\inf \tilde{f} = \inf(f^* - \langle x_0, \cdot\rangle) = -f^{**}(x_0) = -f(x_0)$$

and
$$\inf \tilde{g} = \inf(g^* - \langle x_0, \cdot\rangle) = -g^{**}(x_0) = -g(x_0).$$

Then, by conjugation, we obtain that

$$(\tilde{g})^* \leq (\tilde{f})^* - f(x_0) + g(x_0).$$

Then, due to (2),

$$\begin{aligned}
(\tilde{f})^*(x) &= (f^* - \langle x_0, \cdot\rangle)^*(x) + \frac{1}{2}\|x\|^2 = f(x + x_0) + \frac{1}{2}\|x\|^2, \\
(\tilde{g})^*(x) &= (g^*(x) - \langle x_0, \cdot\rangle)^* + \frac{1}{2}\|x\|^2 = g(x + x_0) + \frac{1}{2}\|x\|^2.
\end{aligned}$$

Hence, $g(x + x_0) \leq f(x + x_0) - f(x_0) + g(x_0)$, which ends the proof. $\square$

The following proposition provides an example of application of Theorem 3.1.

Proposition 3.2. *Let $\ell \geq 0$ and $g \in \Gamma_0(\mathcal{H})$ such that g^* is bounded from below and*

$$\|x\| - \ell \leq \| \operatorname{prox}_g(x)\| \quad \text{for all } x \in \mathcal{H}.$$

Then, $g - g(0) \leq \ell\| \cdot \|$. Moreover, if $\ell \equiv 0$, then g is constant.

Proof. Indeed, if $f = \ell\| \cdot \|$, then

$$\operatorname{prox}_f(x) = \left(1 - \frac{\ell}{\max\{\|x\|, \ell\}}\right) x,$$

and f^* is bounded from below with $\inf f^* = 0$. Thus, following Theorem 3.1, we have $g - g(0) \leq \ell\| \cdot \|$. Finally, if $\ell = 0$, then g is a constant function (a convex function which is bounded from above is constant). □

The following result gives a Lipschitzianity characterization for a convex function.

Proposition 3.3. *Let $f\colon \mathcal{H} \to \mathbb{R}$ be a convex and lower semicontinuous function. Then, f is ℓ-Lipschitz if and only*

$$\|x\| - \ell \leq \| \operatorname{prox}_f(x+y) - y\| \text{ for all } x, y \in \mathcal{H}. \tag{3}$$

Proof. On the one hand, if f is ℓ-Lipschitz, then for all x, y

$$\|x\| - \ell \leq \|x + y - \operatorname{prox}_f(x+y)\| + \| \operatorname{prox}_f(x+y) - y\| \leq \ell + \| \operatorname{prox}_f(x+y) - y\|,$$

where we have used that $x + y - \operatorname{prox}_f(x+y) \in \partial f(\operatorname{prox}_f(x+y)) \subset \ell\,\mathbb{B}$.

On the other hand, assume that (3) holds and fix $y \in \mathcal{H}$. Let us consider the functions $h := f(\cdot + y)$ and $g = \ell\| \cdot \|$. Then, for all $x \in \mathcal{H}$

$$\operatorname{prox}_h(x) = \operatorname{prox}_f(x+y) - y \text{ and } \operatorname{prox}_g(x) = \left(1 - \frac{\ell}{\max\{\|x\|, \ell\}}\right) x.$$

Moreover, since $\operatorname{dom}(f) = \mathcal{H}$, h^* is bounded from below and $\inf h^* = -f(y)$. Therefore, for all $x \in \mathcal{H}$

$$\| \operatorname{prox}_g(x)\| \leq \|x\| - \ell \leq \| \operatorname{prox}_f(x+y) - y\| = \| \operatorname{prox}_h(x)\|. \tag{4}$$

By virtue of Theorem 3.1, we obtain that

$$f(x+y) \leq \ell\|x\| + \inf g^* - \inf h^*.$$

Finally, since $\inf g^* = 0$ and $\inf h^* = -f(y)$, we get that

$$f(x+y) \leq f(y) + \ell\|x\|,$$

which implies that f is ℓ-Lipschitz. □

4. Determination of convex functions

Since then, several integration results appeared In this section, we present the main finding of the paper; that is, the norm of the proximal operator determines a convex function up to a constant. The following two results extends Theorem 1.1.

Proposition 4.1. *Let $f, g \in \Gamma_0(\mathcal{H})$ be two functions such that we have for some $x_0 \in \operatorname{dom} f \cap \operatorname{dom} g$*

$$\| \operatorname{prox}_f(x) - x_0\| = \| \operatorname{prox}_g(x) - x_0\| \quad \text{for all } x \in \mathcal{H}.$$

Then, $f - f(x_0) = g - g(x_0)$.

The following result summarizes several determination principles for convex functions.

Theorem 4.2. *Let $f, g \in \Gamma_0(\mathcal{H})$ be two functions such that f^* and g^* are bounded from below. Then, the following assertions are equivalent:*

(i) *For all $x \in \mathcal{H}$, $\| \operatorname{prox}_f(x)\| = \| \operatorname{prox}_g(x)\|$.*

(ii) *For all $x \in \mathcal{H}$, $f(x) = g(x) + \inf f^* - \inf g^*$.*

(iii) *For all $x \in \mathcal{H}$ we have $\partial f(x)^\circ = \partial g(x)^\circ$, where $\partial f(x)^\circ = \operatorname{Proj}_{\partial f(x)}(0)$ and $\partial g(x)^\circ = \operatorname{Proj}_{\partial g(x)}(0)$.*

(iv) *For all $x \in \mathcal{H}$, $\partial f(x) = \partial g(x)$.*

(v) *For all $x \in \mathcal{H}$, $\operatorname{prox}_f(x) = \operatorname{prox}_g(x)$.*

Proof. (i)⇒(ii) follows from Theorem 3.1.

(ii)⇒(iii) is trivial.

(iii)⇒ (iv) follows from [4, Corollaire 2.2].

(iv)⇒(v) follows from the formula $\operatorname{prox}_f(\cdot) = (I + \partial f)^{-1}(\cdot)$.

Finally, (v)⇒(i) is trivial. □

The following example shows that the hypotheses for the implication (i)⇒(ii) are sharp.

Example 4.3. We consider $f = \delta_{\{x_1\}}$ and $g = \delta_{\{x_2\}}$, where $x_1 \neq x_2$ and $\|x_1\| = \|x_2\|$. Then, for all $x \in \mathcal{H}$

$$\| \operatorname{prox}_f(x)\| = \|x_1\| = \|x_2\| = \| \operatorname{prox}_g(x)\|.$$

However, $f^* = \langle x_1, \cdot\rangle$ and $g^* = \langle x_2, \cdot\rangle$ are not bounded from below.

Theorem 4.2 allow us to obtain the following characterization of support functions.

Corollary 4.4. *Let C be a nonempty, closed and convex set containing* 0. *Then, $f \in \Gamma_0(\mathcal{H})$ satisfies*

$$\| \operatorname{prox}_f(x)\| = d_C(x) \text{ for all } x \in \mathcal{H} \tag{5}$$

if and only if f is the support of C up to a constant.

Proof. Indeed, on the one hand, if f is the support of C up to a constant, then $\operatorname{prox}_f = \operatorname{prox}_{\sigma_C}$, which implies (5). On the other hand, if (5) holds, then f^* is bounded from below.

Moreover, by Proposition 2.1,

$$\|\operatorname{prox}_f(x)\| = d_C(x) = \|x - \operatorname{proj}_C(x)\| = \|\operatorname{prox}_{\sigma_C}(x)\| \text{ for all } x \in \mathcal{H},$$

where we have used that $(\delta_C)^* = \sigma_C$. Therefore, by Theorem 4.2, f is the support of C up to a constant. $\square$

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