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On k-Strong Convexity in Banach Spaces

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We introduce and study the notion of k-strong convexity in Banach spaces. It is a generalization of the notion of strong convexity first studied by Fan and Glicksberg. A Banach space is said to be k-strongly convex if it is reflexive, k-strictly convex and has the Kadec-Klee property. We use the idea of k-dimensional diameter to give several characterizations of k-strong convexity. Further, we study k-strict convexity and k-strong convexity in some products of Banach spaces. Finally, we give characterizations of k-uniform convexity that distinguish it from k-strong convexity.

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1. Introduction

Given a real Banach space X, let B_X, S_X and X^* denote the closed unit ball, the unit sphere and the dual of X respectively. For a point $x \in X$ and a nonempty set $C \subset X$, let $P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}$ where $d(x, C) = \inf\{\|x - y\| : y \in C\}$. Given $x^* \in S_{X^*}$ and $0 \leq \delta < 1$, let $S(X, x^*, \delta) = \{x \in B_X : x^*(x) \geq 1 - \delta\}$. The dimension of a nonempty convex subset C of a real linear space, denoted by $\dim(C)$, is defined to be the dimension of the subspace $\operatorname{span}(C - c)$ where $c \in C$. Throughout this article, we consider only real Banach spaces and refer to a real Banach space simply as a Banach space. A Banach space X is said to be strictly convex if for any $x_1, x_2 \in S_X$ such that $x_1 \neq x_2$,

$$\frac{\|x_1 + x_2\|}{2} < 1.$$

This is equivalent to saying that the unit sphere of X does not contain any nontrivial line-segment. Since a line segment is nothing but a convex set of dimension 1, the only nonempty convex sets in the unit sphere of a strictly convex space are singletons which are of dimension 0. In general, for $k \in \mathbb{N}$, X is said to be k-strictly convex

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[18], if for any $x_1, \dots, x_{k+1} \in S_X$ such that $x_1 - x_{k+1}, \dots, x_k - x_{k+1}$ are linearly independent (in other words, *affinely* independent $x_1, \dots, x_{k+1} \in S_X$),

$$\frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\| < 1.$$

In a k-strictly convex space, the unit sphere does not contain any k-dimensional convex set. The following theorem gives some characterizations of k-strict convexity in a Banach space.

Theorem 1.1. ([18, Lemma 4.2]) *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is k-strictly convex.*
(2) *For every nonempty convex subset C of S_X, $\dim(C) \leq k-1$.*
(3) *For every point x and nonempty convex subset C of X, $P_C(x) = \emptyset$ or $\dim(P_C(x)) \leq k-1$.*
(4) *For every $x^* \in S_{X^*}$, $S(X, x^*, 0) = \emptyset$ or $\dim(S(X, x^*, 0)) \leq k-1$.*

A Banach space X is said to be uniformly convex [3] if for every $\epsilon > 0$,

$$\inf\left\{1 - \frac{\|x_1 + x_2\|}{2} : x_1, x_2 \in B_X, \|x_1 - x_2\| \geq \epsilon\right\} > 0.$$

It is clear that every uniformly convex Banach space is strictly convex. It is also reflexive [13, Theorem 5.2.15]. A Banach space is said to have the Kadec-Klee property [13, Definition 2.5.26] if strong and weak convergence of sequences are equivalent on its unit sphere. Every uniformly convex Banach space has the Kadec-Klee property [13, Theorem 5.2.18].

A Banach space that is reflexive, strictly convex and has the Kadec-Klee property is said to be strongly convex. Strong convexity and related properties are studied in some depth in [10]. While it is known that a uniformly convex Banach space is strongly convex and it is clear that a strongly convex Banach space is strictly convex, there exist strongly convex Banach spaces which are not uniformly convex and strictly convex Banach spaces which are not strongly convex.

Example 1.2. [4, 5] Let $X = \{x \in \Pi_{n=1}^{\infty}(\mathbb{R}^n, \|\cdot\|_n) : \sum_{n=1}^{\infty} \|x(n)\|_n^2 < \infty\}$ and let $\|x\|_X = (\sum_{n=1}^{\infty} \|x(n)\|_n^2)^{\frac{1}{2}}$ for $x \in X$. Then $(X, \|\cdot\|)$ is a strongly convex Banach space that is not uniformly convex.

Example 1.3. [3, 19] For $x \in l_1$, let $\|x\| = (\|x\|_1^2 + \|x\|_2^2)^{\frac{1}{2}}$. Then $(l_1, \|\cdot\|)$ is a strictly convex Banach space. Since it is not reflexive, it is not strongly convex.

Several geometric and approximation theoretic characterizations of strong convexity are available in the literature. We list some of these in Theorems 1.4, 1.5 and 1.6.

Theorem 1.4. ([13, Theorem 5.3.17]) *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is strongly convex.*

(2) *For every nonempty convex subset C of X,*
$$\lim_{t\to d(0,C)^+} \operatorname{diam}(C\cap tB_X)=0.$$
(3) *For every $x^*\in S_{X^*}$,* $\lim_{\delta\to 0}\operatorname{diam}(S(X,x^*,\delta))=0.$

Given a point x and a nonempty set C in a Banach space, let
$$P_C(x,\delta)=\{y\in C:\|x-y\|\le d(x,C)+\delta\} \text{ for } \delta>0.$$
Let $\mathcal{C}(X)$ be the collection of all nonempty closed and convex subsets of X. For $\alpha\ge 0$ and $x^*\in S_{X^*}$, the set $\{x\in X: x^*(x)=\alpha\}$ is said to be a hyperplane of X. Let $\mathcal{H}(X)$ be the collection of all hyperplanes of X and let $\mathcal{H}_1(X)=\{H\in\mathcal{H}(X): d(0,H)=1\}$.

Theorem 1.5. ([15, Theorem 35]) *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is strongly convex.*
(2) *For every $x\in X$ and $C\in\mathcal{C}(X)$,* $\lim_{\delta\to0}\operatorname{diam}\left(P_C\left(x,\delta\right)\right)=0.$
(3) *For every $x\in X$ and $H\in\mathcal{H}(X)$,* $\lim_{\delta\to0}\operatorname{diam}\left(P_H\left(x,\delta\right)\right)=0.$
(4) *For every $H\in\mathcal{H}_1(X)$,* $\lim_{\delta\to0}\operatorname{diam}\left(P_H\left(0,\delta\right)\right)=0.$

A pair (A,B) of nonempty subsets of X is said to have property UC [9, 21] if for any sequences $\{x_n^{(1)}\},\{x_n^{(2)}\}$ in A and $\{y_n\}$ in B such that $\lim_{n\to\infty}\|x_n^{(i)}-y_n\|=dist(A,B)$ for all $i=1,2$, it is true that $\lim_{n\to\infty}\|x_n^{(1)}-x_n^{(2)}\|=0$.

Theorem 1.6. [16] *A Banach space X is strongly convex if and only if for every nonempty closed convex subset A and nonempty compact subset B of X, the pair (A,B) has property UC.*

For any $x_1,\dots,x_{k+1}\in X$, let
$$V(x_1,\dots,x_{k+1})=\sup\left\{\begin{vmatrix}1 & \cdots & 1\\ f_1(x_1) & \cdots & f_1(x_{k+1})\\ \vdots & \vdots & \vdots\\ f_k(x_1) & \cdots & f_k(x_{k+1})\end{vmatrix} : f_1,\dots,f_k\in B_{X^*}\right\}.$$
The k-dimensional volume of $\operatorname{co}\{x_1,\dots,x_{k+1}\}$ is defined to be $\frac{1}{k!}V(x_1,\dots,x_{k+1})$ [17, 20]. Note that $V(x_1,x_2)=\|x_1-x_2\|$ for any x_1,x_2. Also, $V(x_1,\dots,x_{k+1})>0$ if and only if $x_1-x_{k+1},\dots,x_k-x_{k+1}$ are linearly independent. A Banach space X is said to be *k-uniformly convex* [20] if for every $\epsilon>0$,
$$\inf\left\{1-\frac{1}{k+1}\|\sum_{i=1}^{k+1}x_i\|: x_1,\dots,x_{k+1}\in B_X, V(x_1,\dots,x_{k+1})\ge\epsilon\right\}>0.$$
Every k-uniformly convex Banach space is k-strictly convex, reflexive [20] and has the Kadec-Klee property [12]. In this article, we consider a generalization of strong convexity which we call k-strong convexity. A Banach space is said to be k-strongly convex if it is reflexive, k-strictly convex and has the Kadec-Klee property. In Section 2, we give characterizations of k-strong convexity that generalize Theorems 1.4, 1.5 and 1.6. In Section 3, we study k-strict convexity and k-strong convexity in some product spaces. Finally in Section 4, we give examples to show that for any $k\in\mathbb{N}$, k-strong convexity does not imply k-uniform convexity and k- strict convexity does not imply k-strong convexity. Further, we give characterizations of k-uniform convexity which are comparable to Theorem 1.4.

2. k-strong convexity

In order to geometrically characterize k-strong convexity, we use the notion of k-dimensional diameter of a subset of a Banach space X, introduced in [11]. For a subset A of a Banach space X, let

$$\operatorname{diam}_k(A) = \sup\{V(x_1, \ldots, x_{k+1}) : x_1, \ldots, x_{k+1} \in A\}.$$

The notion of k-dimensional diameter has properties very similar to that of the "classical" diameter. We begin this section by listing some of the properties of diam_k. Given a subset A of a Banach space, we denote the *closure*, the *affine hull* and the *convex hull* of A by $\overline{A}$, $\operatorname{aff}(A)$ and $\operatorname{co}(A)$ respectively.

Lemma 2.1. [11, 23] *Let X be a Banach space. For $k \in \mathbb{N}$, the map $(x_1, \ldots, x_{k+1}) \to V(x_1, \ldots, x_{k+1})$ is continuous with respect to the product topology on X^{k+1}.*

Proposition 2.2. *Let X be a Banach space. If A and B are nonempty subsets of X, then the following statements hold.*

(1) $\operatorname{diam}_k(A) \leq \operatorname{diam}_k(B)$ *whenever* $A \subseteq B$.

(2) $\operatorname{diam}_k(tA) = t^k \operatorname{diam}_k(A)$ *for* $t \geq 0$.

(3) $\operatorname{diam}_k(\overline{A}) = \operatorname{diam}_k(A)$.

(4) $\operatorname{diam}_k(\operatorname{co}(A)) = \operatorname{diam}_k(A)$.

(5) *If A is convex, then* $\operatorname{diam}_k(A) = 0$ *if and only if* $\dim(A) \leq k - 1$.

Proof. (1) and (2) easily follow from the definition of diam_k. By Lemma 2.1, we have $\operatorname{diam}_k(\overline{A}) = \operatorname{diam}_k(A)$.

Clearly for any set A, $\operatorname{diam}_k(A) \leq \operatorname{diam}_k(\operatorname{co}(A))$. Let $x_1, \ldots, x_{k+1}, y_1, y_2 \in X$ such that for some $i = 1, \ldots, k+1$ and $\lambda \in [0,1]$, $x_i = \lambda y_1 + (1-\lambda) y_2$. Using the properties of determinants, we have

$$\begin{aligned} V(x_1, \ldots, x_{k+1}) &= V(x_1, \ldots, x_{i-1}, \lambda y_1 + (1-\lambda) y_2, x_{i+1}, \ldots, x_{k+1}) \\ \leq \lambda V(x_1, \ldots, x_{i-1}, y_1, x_{i+1}, \ldots, x_{k+1}) &+ (1-\lambda) V(x_1, \ldots, x_{i-1}, y_2, x_{i+1}, \ldots, x_{k+1}). \end{aligned}$$

It follows that $\operatorname{diam}_k(\operatorname{co}(A)) \leq \operatorname{diam}_k(A)$. Hence (4) holds.

Suppose that A is a convex subset of X such that $\operatorname{diam}_k(A) = 0$. Then for any $x_1, \ldots, x_{k+1} \in A$, we have $V(x_1, \ldots, x_{k+1}) = 0$ and so $x_1 - x_{k+1}, \ldots, x_k - x_{k+1}$ are linearly dependent. If $a \in A$ and $a_1, \ldots, a_k \in A - a$, then $a_1 + a, \ldots, a_k + a, a \in A$. So, $a_1, \ldots, a_k$ are linearly dependent. Thus, $\dim(A) = \dim(\operatorname{span}(A - a)) \leq k - 1$.

Suppose that A is a convex subset of X whose dimension is at most $k-1$. Then for any $a \in A$ we have $\dim(\operatorname{span}(A - a)) \leq k - 1$. So if $x_1, \ldots, x_{k+1} \in A$, then $\dim(\operatorname{span}(A - x_{k+1})) \leq k - 1$. Therefore, $x_1 - x_{k+1}, \ldots, x_k - x_{k+1}$ are linearly dependent which proves that $V(x_1, \ldots, x_{k+1}) = 0$. Hence $\operatorname{diam}_k(A) = 0$. □

Part (5) of Proposition 2.2 enables us to rewrite Theorem 1.1 in terms of diam_k.

Theorem 2.3. *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is k-strictly convex.*

(2) *For every convex subset C of S_X,* $\operatorname{diam}_k(C) = 0$.

(3) *For every point x and nonempty convex subset C of X,* $\operatorname{diam}_k(P_C(x)) = 0$.

(4) *For every $x^* \in S_{X^*}$,* $\operatorname{diam}_k(S(X, x^*, 0)) = 0$.

An important tool that we use to generalize Theorems 1.4, 1.5 and 1.6 is the convergence of the k-dimensional diameter of a decreasing sequence of sets in a Banach space. It is well known that for a decreasing sequence $\{A_n\}$ of nonempty closed and bounded subsets of a Banach space X, $\lim_{n\to\infty} \operatorname{diam}(A_n) = 0$ if and only if $\cap_{n=1}^{\infty} A_n$ has exactly one point. Here we give some necessary and sufficient conditions for $\lim_{n\to\infty} \operatorname{diam}_k(A_n)$ to be 0 using the following result which is about the k-dimensional volumes of elements of a bounded sequence with no convergent subsequence.

Theorem 2.4. [12] *Let $\{x_n\}$ be a bounded sequence in a Banach space X which has no convergent subsequence. Then, given any positive integer k, there exists $\rho > 0$ and a subsequence $\{z_n\}$ of $\{x_n\}$ such that if $\{n_1, \ldots, n_{k+1}\}$ is any set of $k+1$ distinct positive integers, then $V(z_{n_1}, \ldots, z_{n_{k+1}}) \geq \rho$.*

Theorem 2.5. *Let $\{A_n\}$ be a decreasing sequence of nonempty closed and bounded subsets of a Banach space X and let $A_\infty = \cap_{n=1}^{\infty} A_n$. Given a positive integer k, $\lim_{n\to\infty} \operatorname{diam}_k(A_n) = 0$ if and only if $\operatorname{diam}_k(A_\infty) = 0$ and whenever $x_n \in A_n$ for $n \in \mathbb{N}$, the sequence $\{x_n\}$ has a convergent subsequence.*

Proof. Suppose that $\lim_{n\to\infty} \operatorname{diam}_k(A_n) = 0$. Since $\operatorname{diam}_k(A_\infty) \leq \operatorname{diam}_k(A_n)$ for all n, we have $\operatorname{diam}_k(A_\infty) = 0$. Let $x_n \in A_n$ for $n \in \mathbb{N}$. Given $\epsilon > 0$, choose a positive integer N such that $\operatorname{diam}_k(A_n) < \epsilon$ for $n \geq N$. Then $V\left(x_{n_1}, \ldots, x_{n_{k+1}}\right) < \epsilon$ for any $n_1, \ldots, n_{k+1} \geq N$. So, no subsequence $\{z_n\}$ of $\{x_n\}$ satisfies the condition that if $\{n_1, \ldots, n_{k+1}\}$ is any set of $k+1$ distinct positive integers, then $V(z_{n_1}, \ldots, z_{n_{k+1}}) \geq \epsilon$. By Theorem 2.4, it follows that $\{x_n\}$ has a convergent subsequence.

Suppose that $\operatorname{diam}_k(A_\infty) = 0$ and whenever $x_n \in A_n$ for $n \in \mathbb{N}$, the sequence $\{x_n\}$ has a convergent subsequence. For $n \in \mathbb{N}$, choose $x_n^{(1)}, \ldots, x_n^{(k+1)} \in A_n$ such that

$$\operatorname{diam}_k(A_n) - \frac{1}{n} < V(x_n^{(1)}, \ldots, x_n^{(k+1)}).$$

Choose a strictly increasing sequence $\{n_j\}$ of positive integers such that $\{x_{n_j}^{(i)}\}$ converges to some $y^{(i)} \in A_\infty$ for $i = 1, \ldots, k+1$. Then by Lemma 2.1, we have

$$\lim_{j\to\infty} V(x_{n_j}^{(1)}, \ldots, x_{n_j}^{(k+1)}) = V(y^{(1)}, \ldots, y^{(k+1)}) = 0$$

which proves that $\lim_{n\to\infty} \operatorname{diam}_k(A_n) = 0$. □

We now give our first set of characterizations of k-strong convexity.

Theorem 2.6. *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is k-strongly convex.*

(2) *For every nonempty convex subset C of X,*

$$\lim_{t\to d(0,C)^+} \operatorname{diam}_k(C \cap tB_X) = 0.$$

(3) *For every $x^* \in S_{X^*}$,* $\lim_{\delta\to 0} \operatorname{diam}_k(S(X, x^*, \delta)) = 0$.

Proof. (1) $\Rightarrow$ (2):
Let C be a nonempty convex subset of a k-strongly convex Banach space X with $d(0, C) = r$. Since it is always true that $\lim_{t\to 0^+} \operatorname{diam}_k (C \cap tB_X) = 0$, it is enough to consider the case $r > 0$, and since

$$\lim_{t\to r^+} \operatorname{diam}_k(C \cap tB_X) = \lim_{t\to 1^+} r^k \operatorname{diam}_k(\tfrac{1}{r}C \cap tB_X),$$

it is sufficient to prove for the case $r = 1$. Let $C_n = \overline{C} \cap (1 + \frac{1}{n})B_X$ for $n \in \mathbb{N}$ and $C_\infty = \cap_{n=1}^{\infty} C_n$. It is enough to prove that $\lim_{n\to\infty} \operatorname{diam}_k(C_n) = 0$. Since C_∞ is contained in S_X and X is k-strictly convex, we have $\operatorname{diam}_k(C_\infty) = 0$. Let $x_n \in C_n$ for $n \in \mathbb{N}$. Since $1 \le \|x_n\| \le 1 + \frac{1}{n}$, we have $\lim_{n\to\infty} \|x_n\| = 1$. Since X is reflexive, $\{x_n\}$ has a subsequence $\{x_{n_j}\}$ which converges weakly to some $x \in C_\infty$. By the Kadec-Klee property of X, it follows that $\lim_{j\to\infty} x_{n_j} = x$. Applying Theorem 2.5, we have $\lim_{n\to\infty} \operatorname{diam}_k(C_n) = 0$.

(2) $\Rightarrow$ (3):
For every nonempty convex subset C of X, let $\lim_{t\to d(0,C)^+} \operatorname{diam}_k (C \cap tB_X) = 0$. Let $x^* \in S_{X^*}$. It is enough to prove that $\lim_{n\to\infty} \operatorname{diam}_k(S(X, x^*, \frac{1}{n})) = 0$. Let $x_n \in S(X, x^*, \frac{1}{n})$ for $n \in \mathbb{N}$. Consider the set $C = \{x \in X : x^*(x) \ge 1\}$. It is a convex set with $d(0, C) = 1$. Let $C_n = C \cap \left(1 + \frac{1}{n}\right) B_X$ for $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \operatorname{diam}_k(C_n) = 0$ and $\cap_{n=1}^{\infty} C_n = S(X, x^*, 0) = \cap_{n=1}^{\infty} S\left(X, x^*, \frac{1}{n}\right)$. If for $n \in \mathbb{N}$

$$y_n = \left(1 + \frac{1}{n}\right) \frac{x_{n+1}}{\|x_{n+1}\|},$$

then $y_n \in C_n$. By Theorem 2.5, $\operatorname{diam}_k(S(X, x^*, 0)) = 0$ and $\{y_n\}$ has a convergent subsequence. Since $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$, it follows that $\{x_n\}$ has a convergent subsequence and using Theorem 2.5 again, we have $\lim_{n\to\infty} \operatorname{diam}_k(S(X, x^*, \frac{1}{n})) = 0$.

(3) $\Rightarrow$ (1):
For every $x^* \in S_{X^*}$, let $\lim_{\delta\to 0} \operatorname{diam}_k (S(X, x^*, \delta)) = 0$. Then by Theorem 2.5, $S(X, x^*, 0)$ is nonempty and $\operatorname{diam}_k(S(X, x^*, 0)) = 0$. Thus, X is reflexive and k-strictly convex. Let $\{x_n\}$ be a sequence in S_X converging weakly to $x \in S_X$. To prove that $\{x_n\}$ is convergent, it is enough to prove that $\{x_n\}$ has a convergent subsequence. Choose $x^* \in S_{X^*}$ such that $x^*(x) = 1$. Then $\lim_{n\to\infty} x^*(x_n) = 1$ and so there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \in S(X, x^*, \frac{1}{j})$ for all $j \in \mathbb{N}$. By Theorem 2.5, the sequence $\{x_{n_j}\}$ has a convergent subsequence. Hence X has the Kadec-Klee property. $\square$

Before obtaining the other characterizations of k-strong convexity, we need some definitions. A pair (A, B) of nonempty subsets of a Banach space X is said to have property k-UC [11] if for any sequences $\{x_n^{(1)}\}, \ldots, \{x_n^{(k+1)}\}$ in A and $\{y_n\}$ in B such that $\lim_{n\to\infty} \|x_n^{(i)} - y_n\| = dist(A, B)$ for $i = 1, \ldots, k+1$, it is true that $\lim_{n\to\infty} V(x_n^{(1)}, \ldots, x_n^{(k+1)}) = 0$. A sequence $\{x_n\}$ in a nonempty subset C of X is called a minimizing sequence for x in C if $\lim_{n\to\infty} \|x - x_n\| = d(x, C)$. A nonempty subset C of X is said to be k-strongly Chebyshev at x if $\operatorname{diam}_k(P_C(x)) = 0$ and every minimizing sequence for x in C has a convergent subsequence.

Theorem 2.7. [8] *A Banach space X is reflexive and has the Kadec-Klee property if and only if given any nonempty closed convex subset C of X and $x \in X$, every minimizing sequence for x in C has a convergent subsequence.*

Lemma 2.8. *Let C be a nonempty closed subset of a Banach space X and let $x \in X$. Then the following statements are equivalent.*

(1) *C is k-strongly Chebyshev at x.*

(2) *$(C, \{x\})$ has property k-UC.*

(3) $\lim_{\delta \to 0} \operatorname{diam}_k(P_C(x, \delta)) = 0$.

Proof. (1) $\Rightarrow$ (2):

Suppose that C is k-strongly Chebyshev at x. Let $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \ldots, \{x_n^{(k+1)}\}$ be sequences in C such that $\lim_{n\to\infty} \|x - x_n^{(i)}\| = d(x, C)$ for $i = 1, \ldots, k+1$. It is sufficient to prove that a subsequence of $\{V(x_n^{(1)}, \ldots, x_n^{(k+1)})\}$ converges to 0 as $n \to \infty$. Choose a strictly increasing sequence $\{n_j\}$ of positive integers such that the sequences $\{x_{n_j}^{(1)}\}, \ldots, \{x_{n_j}^{(k+1)}\}$ are convergent with respective limits $y^{(1)}, \ldots, y^{(k+1)}$. Since $y^{(1)}, \ldots, y^{(k+1)} \in P_C(x)$, we have $V(y^{(1)}, \ldots, y^{(k+1)}) = 0$. By Lemma 2.1, it follows that $\lim_{j\to\infty} V(x_{n_j}^{(1)}, \ldots, x_{n_j}^{(k+1)}) = 0$.

(2) $\Rightarrow$ (3): Suppose that the pair $(C, \{x\})$ has property k-UC. For $n \in \mathbb{N}$, choose $x_n^{(1)}, \ldots, x_n^{(k+1)} \in P_C\left(x, \frac{1}{n}\right)$ such that $\operatorname{diam}_k(P_C(x, \frac{1}{n})) - \frac{1}{n} < V(x_n^{(1)}, \ldots, x_n^{(k+1)})$. Since $\lim_{n\to\infty} \|x - x_n^{(i)}\| = d(x, C)$ for $i = 1, ..., k+1$ we have

$$\lim_{n\to\infty} V(x_n^{(1)}, ..., x_n^{(k+1)}) = 0.$$

Therefore, $\lim_{n\to\infty} \operatorname{diam}_k(P_C(x, \frac{1}{n})) = 0$. Hence $\lim_{\delta\to 0} \operatorname{diam}_k(P_C(x, \delta)) = 0$.

(3) $\Rightarrow$ (1): Suppose that $\lim_{\delta\to 0} \operatorname{diam}_k(P_C(x, \delta)) = 0$. Let $\{x_n\}$ be a sequence in C such that we have $\lim_{n\to\infty} \|x - x_n\| = d(x, C)$. Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \in P_C(x, \frac{1}{j})$ for all $j \in \mathbb{N}$. By Theorem 2.5, it follows that $\{x_{n_j}\}$ has a convergent subsequence and $\operatorname{diam}_k(P_C(x)) = 0$. □

By the Hahn-Banach separation theorem, if C is a nonempty convex subset of a Banach space X with $d(0, C) = 1$, then there exists $x^* \in S_{X^*}$ such that $x^*(x) \geq 1$ for all $x \in C$. We use the following lemma about the hyperplane $H = \{x \in X : x^*(x) = 1\}$ to prove some approximation theoretic characterizations of k-strong convexity.

Lemma 2.9. [15, Lemma 8] *Suppose that C is a nonempty convex subset of a Banach space X with $d(0, C) = 1$. Let $x^* \in S_{X^*}$ such that $x^*(x) \geq 1$ for all $x \in C$ and let $H = \{x \in X : x^*(x) = 1\}$. If $x_n \in P_C\left(0, \frac{1}{n}\right)$ for $n \in \mathbb{N}$, then there exist $\delta_n > 0$ and $y_n \in P_H(0, \delta_n)$ for $n \in \mathbb{N}$ such that $\lim_{n\to\infty} \delta_n = 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Theorem 2.10. *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is k-strongly convex.*

(2) *For every $x \in X$ and $C \in \mathcal{C}(X)$, $\lim_{\delta\to 0} \operatorname{diam}_k(P_C(x, \delta)) = 0$.*

(3) *For every $x \in X$ and $H \in \mathcal{H}(X)$, $\lim_{\delta\to 0} \operatorname{diam}_k(P_H(x, \delta)) = 0$.*

(4) *For every $H \in \mathcal{H}_1(X)$, $\lim_{\delta\to 0} \operatorname{diam}_k(P_H(0, \delta)) = 0$.*

Proof. (1) $\Rightarrow$ (2): This follows from Theorem 2.7 and Lemma 2.8.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4): Trivial.

(4) $\Rightarrow$ (1): Let C be a nonempty convex subset of X with $d(0,C)=1$. By Theorem 2.6, it is enough to prove that $\lim_{n\to\infty}\operatorname{diam}_k(C\cap(1+\frac{1}{n})B_X)=0$. Choose $x^*\in S_{X^*}$ such that $C\subseteq\{x\in X: x^*(x)\geq 1\}$. Then $C\cap(1+\frac{1}{n})B_X=P_C(0,\frac{1}{n})$.

Let $x_n\in C\cap\left(1+\frac{1}{n}\right)B_X$ for $n\in\mathbb{N}$ and let $H=\{x\in X: x^*(x)=1\}$. Then $d(0,H)=1$, $C\cap B_X\subseteq P_H(0)$ and by Lemma 2.9, there exist $\{\delta_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty}\delta_n=0$, $y_n\in P_H(0,\delta_n)$ and $\lim_{n\to\infty}\|x_n-y_n\|=0$. Since $\lim_{n\to\infty}P_H(0,\delta_n)=0$, $\operatorname{diam}_k(P_H(0))=0$ and $\{y_n\}$ has a convergent subsequence by Theorem 2.5. So $\operatorname{diam}_k(C\cap B_X)=0$ and $\{x_n\}$ has a convergent subsequence. Again by Theorem 2.5, we have $\lim_{n\to\infty}\operatorname{diam}_k\left(C\cap\left(1+\frac{1}{n}\right)B_X\right)=0$. Hence X is k-strongly convex. $\square$

Theorem 2.11. *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is k-strongly convex.*

(2) *If A is a nonempty closed convex subset and B is a nonempty compact subset of X, then (A,B) has property k-UC.*

Proof. (1) $\Rightarrow$ (2):
Suppose that X is k-strongly convex. Let A be a nonempty closed convex subset of X and let B be a nonempty compact subset of X. If $d(A,B)=0$, then by Lemma 2.1, (A,B) has property k-UC. Suppose that $d(A,B)>0$. Let $\{x_n^{(1)}\},\ldots,\{x_n^{(k+1)}\}$ be sequences in A and let $\{y_n\}$ be a sequence in B such that

$$\lim_{n\to\infty}\|x_n^{(1)}-y_n\|=\cdots=\lim_{n\to\infty}\|x_n^{(k+1)}-y_n\|=d(A,B).$$

It is sufficient to prove that a subsequence of $\{V(x_n^{(1)},\ldots,x_n^{(k+1)})\}$ converges to 0. Since X is reflexive and B is compact, there exists a strictly increasing sequence $\{n_j\}$ of positive integers such that $\{x_{n_j}^{(i)}\}$ converges weakly to some $x^{(i)}\in A$ for $i=1,\ldots,k+1$ and $\lim_{j\to\infty}y_{n_j}=y$ for some $y\in B$. By the Kadec-Klee property of X, we have $\lim_{j\to\infty}x_{n_j}^{(i)}-y_{n_j}=x^{(i)}-y$ and therefore $\lim_{j\to\infty}x_{n_j}^{(i)}=x^{(i)}\in P_A(y)$ for $i=1,\ldots,k+1$. Since X is k-strictly convex, $\operatorname{diam}_k(P_A(y))=0$. By Lemma 2.1, $\lim_{j\to\infty}V(x_{n_j}^{(1)},\ldots,x_{n_j}^{(k+1)})=0$. Thus, the pair (A,B) has property k-UC.

(2) $\Rightarrow$ (1):
Suppose that for any closed convex subset A and compact subset B of X, (A,B) has property k-UC. Then for any closed convex subset A of X, $(A,\{x\})$ has property k-UC for all $x\in X$. By Lemma 2.8, A is k-strongly Chebyshev at every $x\in X$. This fact combined with Theorem 2.7 proves that X is k-strongly convex. $\square$

3. k-strict convexity and k-strong convexity in product spaces

Given $d\in\mathbb{N}$, a norm $\|\cdot\|_E$ on $\mathbb{R}^d$ is said to be monotone if for $s_1,\ldots,s_d,t_1,\ldots,t_d\in\mathbb{R}$, we have $\|(s_1,\ldots,s_d)\|_E\leq\|(t_1,\ldots,t_d)\|_E$ whenever $|s_i|\leq|t_i|$ for $i=1,\ldots,d$. It is said to be strictly monotone if $\|(s_1,\ldots,s_d)\|_E<\|(t_1,\ldots,t_d)\|_E$ whenever $|s_i|\leq|t_i|$ for $i=1,\ldots,d$ and $|s_j|<|t_j|$ for some $j\in\{1,\ldots,d\}$. Given Banach spaces $X_1,\ldots,X_d$, let $F_{(x_1,\ldots,x_d)}=(\|x_1\|,\ldots,\|x_d\|)$ and $\|(x_1,\ldots,x_d)\|=\|F_{(x_1,\ldots,x_d)}\|_E$ for $(x_1,\ldots,x_d)\in X_1\otimes\cdots\otimes X_d$. The resulting product Banach space is said to be the product of $X_1,\ldots,X_d$ over $(\mathbb{R}^d,\|\cdot\|_E)$ and is denoted by $(X_1\otimes\cdots\otimes X_d)_E$.

The standard norm $\|\cdot\|_p$ is a monotone norm on $\mathbb{R}^d$ for $1 \leq p \leq \infty$ and is strictly monotone for $1 \leq p < \infty$. The product of $X_1, \ldots, X_d$ over $(\mathbb{R}^d, \|\cdot\|_p)$ is denoted by $(X_1 \otimes \cdots \otimes X_d)_p$ for $1 \leq p \leq \infty$.

In this section, we study n-strict convexity and n-strong convexity where $n \in \mathbb{N}$, in product spaces of this type. We prove results for products of two spaces for the sake of convenience. These can be extended to arbitrary finite products either by a similar method or by induction. Let $\|(\alpha,\beta)\|_{E^*} = \sup\{\alpha s + \beta t : \|(s,t)\|_E = 1\}$ for $(\alpha,\beta) \in \mathbb{R}^2$. It is well known that the dual of $(\mathbb{R}^2, \|\cdot\|_E)$ is isometrically isomorphic to $(\mathbb{R}^2, \|\cdot\|_{E^*})$ under the map $f \to (f(e_1), f(e_2))$. We further note that the maps $x \to \frac{1}{\|(1,0)\|_E}(x,0)$ and $y \to \frac{1}{\|(0,1)\|_E}(0,y)$ are linear isometries, from X and Y, onto $(X \otimes \langle 0\rangle)_E$ and $(\langle 0\rangle \otimes Y)_E$ respectively, where $\langle 0\rangle$ denotes the trivial subspace of either of the spaces.

Theorem 3.1. [1] *A norm $\|\cdot\|_E$ on $\mathbb{R}^2$ is monotone if and only if for any $(s,t) \in \mathbb{R}^2$, $\|(s,t)\|_E = \|(|s|,|t|)\|_E$.*

Theorem 3.2. [1] *A norm $\|\cdot\|_E$ on $\mathbb{R}^2$ is monotone if and only if the norm $\|\cdot\|_{E^*}$ is monotone.*

The following result is observed in [14]. We give a proof for the sake of completeness.

Theorem 3.3. [14] *Let X, Y be Banach spaces and let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^2$. For $\Phi \in (X \otimes Y)^*_E$, let $\Phi_X(x) = \Phi(x,0)$ for all $x \in X$ and $\Phi_Y(y) = \Phi(0,y)$ for all $y \in Y$. Then $\Phi \to (\Phi_X, \Phi_Y)$ is a linear isometry from $(X \otimes Y)^*_E$ onto $(X^* \otimes Y^*)_{E^*}$.*

Proof. Clearly for $\Phi \in (X \otimes Y)^*_E$, we have $\Phi_X \in X^*$, $\Phi_Y \in Y^*$ and $\Phi \to (\Phi_X, \Phi_Y)$ is a linear map. For any $f \in X^*$ and $g \in Y^*$, let $\Phi(x,y) = f(x) + g(y)$ for all $(x,y) \in (X \otimes Y)_E$. Then $\Phi \in (X \otimes Y)^*_E$ and $f = \Phi_X$ and $g = \Phi_Y$. Thus $\Phi \to (\Phi_X, \Phi_Y)$ maps $(X \otimes Y)^*_E$ onto $(X^* \otimes Y^*)_{E^*}$.

Let $\Phi \in (X \otimes Y)^*_E$, $f = \Phi_X$ and $g = \Phi_Y$. Then, for $(x,y) \in (X \otimes Y)_E$, $|\Phi(x,y)| = |f(x) + g(y)| \leq \|f\|\|x\| + \|g\|\|y\| \leq \|(\|f\|, \|g\|)\|_{E^*}\|(\|x\|, \|y\|)\|_E$. So, $\|\Phi\| \leq \|(\|f\|, \|g\|)\|_{E^*}$.

Let $(s,t) \in \mathbb{R}^2$. Given $\epsilon \in (0,1)$, choose $x \in X$, $y \in Y$ such that

$$\|x\| = |s|, \quad \epsilon|s| < \frac{f}{\|f\|}(x), \quad \|y\| = |t|, \quad \epsilon|t| < \frac{g}{\|g\|}(y).$$

Now,
$$\begin{aligned} |\|f\|s + \|g\|t| \leq \|f\||s| + \|g\||t| &< \frac{1}{\epsilon}(f(x) + g(y)) = \frac{1}{\epsilon}\Phi(x,y) \\ &\leq \frac{1}{\epsilon}\|\Phi\|\|(x,y)\| = \frac{1}{\epsilon}\|\Phi\|\|(s,t)\|_E. \end{aligned}$$

Thus, $|\|f\|s + \|g\|t| \leq \|\Phi\|\|(s,t)\|_E$ for all $(s,t) \in \mathbb{R}^2$. So we have the inequality $\|(\|f\|, \|g\|)\|_{E^*} \leq \|\Phi\|$. Hence $\|\Phi\| = \|(\|\Phi_X\|, \|\Phi_Y\|)\|_{E^*} = \|(\Phi_X, \Phi_Y)\|$. □

From Theorem 3.3, it is clear that each element of $(X \otimes Y)^*_E$ can be identified with an element $(f,g) \in (X^* \otimes Y^*)_{E^*}$ with the definition $(f,g)(x,y) = f(x) + g(y)$ for all $(x,y) \in (X \otimes Y)_E$.

Theorem 3.4. *Let X, Y be Banach spaces and let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^2$. The product space $(X \otimes Y)_E$ is reflexive if and only if X and Y are reflexive.*

Proof. If $(X \otimes Y)_E$ is reflexive, then since X and Y are isomorphic to the closed subspaces, $(X \otimes \langle 0 \rangle)_E$ and $(\langle 0 \rangle \otimes Y)_E$ respectively, X and Y are reflexive.

From Theorem 3.3, it is clear that the space $(X^{**} \otimes Y^{**})_{E^{**}}$ represents $(X \otimes Y)_E^{**}$. Define $J_{(x,y)}(f, g) = (f, g)(x, y)$ for all $(x, y) \in (X \otimes Y)_E$ and $(f, g) \in (X^* \otimes Y^*)_{E^*}$. Similarly define J_x and J_y for $x \in X$ and $y \in Y$ respectively. Let $F \in X^{**}$ and $G \in Y^{**}$. If X and Y are reflexive, then $F = J_x$ and $G = J_y$ for some $x \in X$ and $y \in Y$. Then, $(F, G)(f, g) = J_x(f) + J_y(g) = f(x) + g(y)$ for all $(f, g) \in (X^* \otimes Y^*)_{E^*}$ which shows that $(F, G) = J_{(x,y)}$. □

The following theorem can be proved in a similar manner.

Theorem 3.5. *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces and let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^d$. The product space $(X_1 \otimes \cdots \otimes X_d)_E$ is reflexive if and only if X_i is reflexive for $i = 1, \ldots, d$.*

Theorem 3.6. [7, 14] *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces and let $\|\cdot\|_E$ be a strictly monotone norm on $\mathbb{R}^d$. The product space $(X_1 \otimes \cdots \otimes X_d)_E$ has the Kadec-Klee property if and only if X_i has the Kadec-Klee property for $i = 1, \ldots, d$.*

We now establish some results on n-strict convexity and n-strong convexity in some product spaces. If X, Y are Banach spaces and W is a subspace of $X \otimes Y$, let $W_X = \{x \in X : (x, y) \in W\}$ and $W_Y = \{y \in Y : (x, y) \in W\}$. The following lemma is useful here and also in the next section.

Lemma 3.7. [24] *Let $x_1, \ldots, x_{k+1} \in B_X$.*

(1) *For any $w \in \operatorname{co}\{x_1, \ldots, x_{k+1}\}$, we have*

$$\frac{1 - \|w\|}{k+1} \leq 1 - \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i\|.$$

(2) *For any $w \in B_X$ and $w \in \operatorname{aff}\{x_1, \ldots, x_{k+1}\}$, we have*

$$\frac{(1 - \|w\|) V(x_1, \ldots, x_{k+1})}{(k+1)^{\frac{k+3}{2}}} \leq 1 - \frac{1}{k+1} \|\sum_{i=1}^{k+1} x_i\|.$$

Theorem 3.8. *Let X, Y be Banach spaces and let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^2$. If $(X \otimes Y)_E$ is n-strictly convex for some $n \in \mathbb{N}$, then there exist $m, k \in \mathbb{N}$ such that X is m-strictly convex and Y is k-strictly convex and $m + k - 1 = n$.*

Proof. Suppose that $(X \otimes Y)_E$ is n-strictly convex for some $n \in \mathbb{N}$. Since $(X \otimes \langle 0 \rangle)_E$ and $(\langle 0 \rangle \otimes Y)_E$ are subspaces of $(X \otimes Y)_E$, these spaces are n-strictly convex. Since X and Y are isometrically isomorphic to $(X \otimes \langle 0 \rangle)_E$ and $(\langle 0 \rangle \otimes Y)_E$ respectively, X and Y are both n-strictly convex. Let k be the smallest positive integer such that Y is k-strictly convex and let $m = n - k + 1$.

In case $k = 1$, then $m = n$ and we showed that X is n-strictly convex. If $k > 1$, then let $r = \frac{1}{\|(1,1)\|_E}$. Let $x_1, \ldots, x_{m+1} \in X$ such that $V(x_1, \ldots, x_{m+1}) > 0$ and $\|x_i\| = r$ for $i = 1, \ldots, m+1$. Choose $y_1, \ldots, y_k \in Y$ such that $V(y_1, \ldots, y_k) > 0$ and

$\|y_1\| = \cdots = \|y_k\| = \frac{1}{k}\|\sum_{i=1}^{k} y_i\| = r$. Let $(u_i, v_i) = (x_i, y_k)$ for $i = 1, \ldots, m+1$ and $(u_{m+i+1}, v_{m+i+1}) = (x_{m+1}, y_i)$ for $i = 1, \ldots, k-1$. Then we have $(u_i, v_i) \in S_{(X\otimes Y)_E}$ for $i = 1, \ldots, n+1$ and $\{(u_i, v_i) - (u_{m+1}, v_{m+1}) : i = 1, \ldots, n+1, i \neq m+1\}$ is a linearly independent set. Therefore,

$$\|\left(\tfrac{1}{n+1}\|\textstyle\sum_{i=1}^{n+1} u_i\|, \tfrac{1}{n+1}\|\sum_{i=1}^{n+1} v_i\|\right)\|_E = \tfrac{1}{n+1}\|\textstyle\sum_{i=1}^{n+1}(u_i, v_i)\| < 1.$$

Note that $\frac{1}{n+1}\sum_{i=1}^{n+1} u_i = \frac{1}{n+1}(\sum_{i=1}^{m+1} x_i + (k-1)x_{m+1}) \in \operatorname{co}\{x_1, \ldots, x_{m+1}\}$ and $\frac{1}{n+1}\sum_{i=1}^{n+1} v_i = \frac{1}{n+1}(\sum_{i=1}^{k-1} y_i + (m+1)y_k) \in \operatorname{co}\{y_1, \ldots, y_k\}$. By Lemma 3.7 we have $\frac{1}{n+1}\|\sum_{i=1}^{n+1} v_i\| = r$. But then, by the choice of r, $\frac{1}{n+1}\|\sum_{i=1}^{n+1} u_i\| < r$. Another application of Lemma 3.7 gives $\frac{1}{m+1}\|\sum_{i=1}^{m+1} x_i\| < r$ which proves that X is m-strictly convex. □

If $\|\cdot\|_E$ is monotone and $(\mathbb{R}^2, \|\cdot\|_E)$ is strictly convex, then the converse holds.

Lemma 3.9. *Let X, Y be Banach spaces. Let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^2$ and let $(\mathbb{R}^2, \|\cdot\|_E)$ be strictly convex. For $k \in \mathbb{N}$, let $(x_1, y_1), \ldots, (x_{k+1}, y_{k+1}) \in S_{(X\otimes Y)_E}$. If $\frac{1}{k+1}\|\sum_{i=1}^{k+1}(x_i, y_i)\| = 1$, then $\|x_1\| = \cdots = \|x_{k+1}\| = \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\|$ and $\|y_1\| = \cdots = \|y_{k+1}\| = \frac{1}{k+1}\|\sum_{i=1}^{k+1} y_i\|$.*

Proof. Let $z_1, \ldots, z_{k+1} \in S_{(X\otimes Y)_E}$ and $z_i = (x_i, y_i)$ for $i \in \{1, \ldots, k+1\}$. Suppose that $\frac{1}{k+1}\|\sum_{i=1}^{k+1} z_i\| = 1$. Then, by Lemma 3.7, $\|F_w\|_E = \|w\| = 1$ for all $w \in \operatorname{co}\{z_1, \ldots, z_{k+1}\}$. So, for any $u, v \in \operatorname{co}\{z_1, \ldots, z_{k+1}\}$, $\|F_u\|_E = \|F_v\|_E = \|F_{\frac{1}{2}(u+v)}\|_E = 1$. Also, $1 = \|F_{\frac{1}{2}(u+v)}\|_E \leq \frac{1}{2}\|F_u + F_v\|_E \leq \frac{1}{2}(\|F_u\|_E + \|F_v\|_E) \leq 1$. Thus, $\frac{1}{2}\|F_u + F_v\|_E = 1$. Since $(\mathbb{R}^2, \|\cdot\|_E)$ is strictly convex, we must have $F_u = F_v$. It follows that $F_{z_1} = \cdots = F_{z_{k+1}} = F_{\frac{1}{k+1}\sum_{i=1}^{k+1} z_i}$. So, $\|x_1\| = \cdots = \|x_{k+1}\| = \frac{1}{k+1}\|\sum_{i=1}^{k+1} x_i\|$ and $\|y_1\| = \cdots = \|y_{k+1}\| = \frac{1}{k+1}\|\sum_{i=1}^{k+1} y_i\|$. □

Theorem 3.10. *Let X, Y be Banach spaces. Let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^2$ and let $(\mathbb{R}^2, \|\cdot\|_E)$ be strictly convex. If X is m-strictly convex and Y is k-strictly convex, then $(X \otimes Y)_E$ is $(m+k-1)$-strictly convex.*

Proof. Let $m, k \in \mathbb{N}$. Suppose that the product space $(X \otimes Y)_E$ is not $(m+k-1)$-strictly convex. Let $z_i = (x_i, y_i) \in S_{(X\otimes Y)_E}$ for $i = 1, ..., m+k$ such that we have $V(z_1, \ldots, z_{m+k}) > 0$ and $\frac{1}{m+k}\|\sum_{i=1}^{m+k} z_i\| = 1$. If $W = \operatorname{span}\{z_i - z_{m+k} : i = 1, \ldots, m+k-1\}$, then $W_X = \operatorname{span}\{x_i - x_{m+k} : i = 1, \ldots, m+k-1\}$ and $W_Y = \operatorname{span}\{y_i - y_{m+k} : i = 1, \ldots, m+k-1\}$. If $\dim(W_X) \leq m-1$ and $\dim(W_Y) \leq k-1$, then $\dim(W_X \otimes W_Y) \leq m+k-2$. But $W \subset W_X \otimes W_Y$ and $\dim(W) = m+k-1$. So $\dim(W_X) \geq m$ or $\dim(W_Y) \geq k$.

Suppose that $\dim(W_X) \geq m$. Let $(a_{m+1}, b_{m+1}) = z_{m+k}$. Then there exist $(a_1, b_1), ..., (a_m, b_m) \in \{z_1, \ldots, z_{m+k-1}\}$ such that $V(a_1 \ldots, a_{m+1}) > 0$. Now by Lemma 3.7, $\frac{1}{m+1}\|\sum_{i=1}^{m+1}(a_i, b_i)\| = 1$ and so by Lemma 3.9, we have $\|a_1\| = \cdots = \|a_{m+1}\| = \frac{1}{m+1}\|\sum_{i=1}^{m+1} a_i\|$. This shows that X is not m-strictly convex. Similarly, if $\dim(W_Y) \geq k$, then it follows that Y is not k-strictly convex. Thus, if X is m-strictly convex and Y is k-strictly convex, then $(X \otimes Y)_E$ is $(m+k-1)$-strictly convex. □

For arbitrary $d \in \mathbb{N}$, we have the following result.

Theorem 3.11. *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces. Let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^d$ and let $(\mathbb{R}^d, \|\cdot\|_E)$ be strictly convex. The space $(X_1 \otimes \cdots \otimes X_d)_E$ is n-strictly convex for some $n \in \mathbb{N}$ if and only if there exist $k_1, \ldots, k_d \in \mathbb{N}$ with $n = k_1 + \cdots + k_d - d + 1$ such that X_i is k_i-strictly convex for $i = 1, \ldots, d$.*

Proof. If $(X_1 \otimes \cdots \otimes X_d)_E$ is n-strictly convex, then each of the spaces $X_1, \ldots, X_d$ is n-strictly convex. Let k_j be the smallest positive integer such that X_j is k_j-strictly convex for $j = 2, \ldots, d$ and let $k_1 = n - (k_2 + \cdots + k_d) + d - 1$. Let $r > 0$ such that for $(1, ..., 1) \in \mathbb{R}^d$, $r\|(1, ..., 1)\|_E = 1$. Let $x_{(1,1)}, ..., x_{(1,k_1+1)} \in X_1$ such that $\|x_{(1,1)}\| = \cdots = \|x_{(1,k_1+1)}\| = r$. For $j = 2, \ldots, d$, if $k_j = 1$, let $x_{(j,1)} \in X_j$ such that $\|x_{(j,1)}\| = r$ and if $k_j > 1$, let $x_{(j,1)}, \ldots, x_{(j,k_j)} \in X_j$ such that $V(x_{(j,1)}, \ldots, x_{(j,k_j)}) > 0$ and $\|x_{(j,1)}\| = \cdots = \|x_{(j,k_j)}\| = \frac{1}{k_j}\|\sum_{i=1}^{k_j} x_{(j,i)}\| = r$.
Define $z_1, \ldots, z_{n+1} \in (X_1 \otimes \cdots \otimes X_d)_E$ such that

$$\begin{array}{lcllllll}
z_1 & = & \big(x_{(1,1)} & x_{(2,k_2)} & x_{(3,k_3)} & \cdots & x_{(d,k_d)}\big), \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_{(k_1+1)} & = & \big(x_{(1,k_1+1)} & x_{(2,k_2)} & x_{(3,k_3)} & \cdots & x_{(d,k_d)}\big), \\
z_{(k_1+1)+1} & = & \big(x_{(1,k_1+1)} & x_{(2,1)} & x_{(3,k_3)} & \cdots & x_{(d,k_d)}\big), \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_{(k_1+1)+(k_2-1)} & = & \big(x_{(1,k_1+1)} & x_{(2,k_2-1)} & x_{(3,k_3)} & \cdots & x_{(d,k_d)}\big), \\
z_{(k_1+1)+(k_2-1)+1} & = & \big(x_{(1,k_1+1)} & x_{(2,k_2)} & x_{(3,1)} & \cdots & x_{(d,k_d)}\big), \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_{(k_1+1)+(k_2-1)+(k_3-1)} & = & \big(x_{(1,k_1+1)} & x_{(2,k_2)} & x_{(3,k_3-1)} & \cdots & x_{(d,k_d)}\big),
\end{array}$$

and so on upto

$$\begin{array}{lcllllll}
z_{(k_1+1)+(k_2-1)+\cdots+(k_{d-1}-1)+1} & = & \big(x_{(1,k_1+1)} & x_{(2,k_2)} & x_{(3,k_3)} & \cdots & x_{(d,1)}\big), \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
z_{n+1} & = & \big(x_{(1,k_1+1)} & x_{(2,k_2)} & x_{(3,k_3)} & \cdots & x_{(d,k_d-1)}\big).
\end{array}$$

Then proceed as in Theorem 3.8.
The proof of the converse is similar to that of Theorem 3.10. □

In particular we have the following known result.

Corollary 3.12. [6, 22] *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces. Let $\|\cdot\|_E$ be a monotone norm on $\mathbb{R}^d$ and let $(\mathbb{R}^d, \|\cdot\|_E)$ be strictly convex. The space $(X_1 \otimes \cdots \otimes X_d)_E$ is strictly convex if and only if X_i is strictly convex for $i = 1, \ldots, d$.*

We have the following results on n-strong convexity in product spaces based on Theorems 3.4, 3.6 and 3.11.

Theorem 3.13. *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces. Let $\|\cdot\|_E$ be a strictly monotone norm on $\mathbb{R}^d$ and let $(\mathbb{R}^d, \|\cdot\|_E)$ be strictly convex. The space $(X_1 \otimes \cdots \otimes X_d)_E$ is n-strongly convex for some $n \in \mathbb{N}$ if and only if there exist $k_1, \ldots, k_d \in \mathbb{N}$ with $n = k_1 + \cdots + k_d - d + 1$ such that X_i is k_i-strongly convex for $i = 1, \ldots, d$.*

Corollary 3.14. *For $d \in \mathbb{N}$, let $X_1, \dots, X_d$ be Banach spaces and let $1 < p < \infty$. The space $(X_1 \otimes \cdots \otimes X_d)_p$ is n-strongly convex for some $n \in \mathbb{N}$ if and only if there exist $k_1, \dots, k_d \in \mathbb{N}$ with $n = k_1 + \cdots + k_d - d + 1$ such that X_i is k_i-strongly convex for $i = 1, \dots, d$.*

In the case of $\|\cdot\|_E = \|\cdot\|_1$, we have slightly different results. It can be easily verified that if X, Y are any Banach spaces, $(X \otimes Y)_1$ is never strictly convex. We know from [2] that $(X \otimes Y)_1$ is 2-uniformly convex if and only if X and Y are uniformly convex. For any $n > 1$, we have the following result.

Theorem 3.15. *Let X, Y be Banach spaces. If $(X \otimes Y)_1$ is n-strictly convex for some $n > 1$, then there exist $m, k \in \mathbb{N}$ such that X is m-strictly convex, Y is k-strictly convex and $n = m + k$.*

Proof. Suppose that $(X \otimes Y)_1$ is n-strictly convex for some $n > 1$. Then, the spaces X and Y are both n-strictly convex. Let k be the smallest positive integer such that Y is k-strictly convex and let $m \in \mathbb{N}$. If $k = 1$, then choose $y_1 \in Y$ such that $\|y_1\| = \frac{m}{m+1}$. If $k > 1$, then choose $y_1, \dots, y_k \in Y$ such that $V(y_1, \dots, y_k) > 0$ and $\|y_1\| = \cdots = \|y_k\| = \frac{1}{k}\|\sum_{i=1}^{k} y_i\| = \frac{m}{m+1}$. For $x_1, \dots, x_{m+1} \in S_X$, let

$$z_j = \Big(\frac{1}{m+1}x_j, \frac{1}{k}\sum_{i=1}^{k} y_i\Big) \qquad \text{for } j \in \{1, \dots, m+1\}$$

and $$z_{m+j+1} = \Big(0, \frac{(m+1)}{m}y_j\Big) \qquad \text{for } j \in \{1, \dots, k\}.$$

Then $z_1, \dots, z_{m+k+1} \in S_{(X \otimes Y)_1}$ and

$$\begin{aligned} \|\sum_{i=1}^{m+k+1} z_i\| &= \frac{1}{m+1}\|\sum_{i=1}^{m+1} x_i\| + (m+1)\|\frac{1}{k}\sum_{i=1}^{k} y_i + \frac{1}{m}\sum_{i=1}^{k} y_i\| \\ &= \frac{1}{m+1}\|\sum_{i=1}^{m+1} x_i\| + m + k. \end{aligned}$$

Since $\frac{1}{k}\|\sum_{i=1}^{k} \frac{m+1}{m} y_i\| = 1$, by Lemma 3.7, we have $d(0, \operatorname{aff}\{\frac{m+1}{m}y_1, ..., \frac{m+1}{m}y_k\}) = 1$. So, $\frac{1}{k}\sum_{i=1}^{k} y_i \notin \operatorname{aff}\{\frac{m+1}{m}y_1, \dots, \frac{m+1}{m}y_k\}$ and therefore the elements

$$\tfrac{m+1}{m}y_1 - \tfrac{1}{k}\textstyle\sum_{i=1}^{k} y_i, \dots, \tfrac{m+1}{m}y_k - \tfrac{1}{k}\sum_{i=1}^{k} y_i$$

are linearly independent. If $\lambda_1, \dots, \lambda_m, \lambda_{m+2}, \dots, \lambda_{m+k+1} \in \mathbb{R}$ such that

$$\sum_{j=1}^{m} \lambda_j (z_j - z_{m+1}) + \sum_{j=1}^{k} \lambda_{m+j+1}(z_{m+j+1} - z_{m+1}) = 0,$$

then $\lambda_{m+2} = \cdots = \lambda_{m+k+1} = 0$ which leaves us with $\sum_{j=1}^{m} \lambda_j (x_j - x_{m+1}) = 0$. Thus, if $x_1 - x_{m+1}, ..., x_m - x_{m+1}$ are linearly independent, then $z_1 - z_{m+1}, ..., z_m - z_{m+1}, z_{m+2} - z_{m+1}, ..., z_{m+k+1} - z_{m+1}$ are also linearly independent.

Suppose that X is not m-strictly convex. Then there exist $x_1, \dots, x_{m+1} \in S_X$ such that $V(x_1, \dots, x_{m+1}) > 0$ and $\frac{1}{m+1}\|\sum_{i=1}^{m+1} x_i\| = 1$. If $z_1, \dots, z_{m+k+1}$ are defined as above, then we have $z_1, \dots, z_{m+k+1} \in S_{(X \otimes Y)_1}$, $V(z_1, \dots, z_{m+k+1}) > 0$ and

$\frac{1}{m+k+1}\|\sum_{i=1}^{m+k+1} z_i\| = 1$ which implies $(X \otimes Y)_1$ is not $(m+k)$-strictly convex. But if $m = n-k$, then $(X \otimes Y)_1$ is $(m+k)$-strictly convex and therefore X must be m-strictly convex. □

Theorem 3.16. *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces. If $(X_1 \otimes \cdots \otimes X_d)_1$ is n-strictly convex for some $n \in \mathbb{N}$, then there exist $k_1, \ldots, k_d \in \mathbb{N}$ with $n = k_1 + \cdots + k_d$ such that X_i is k_i-strictly convex for $i = 1, \ldots, d$.*

Proof. Since $(X_1 \otimes \cdots \otimes X_{d+1})_1 = ((X_1 \otimes \cdots \otimes X_d)_1 \otimes X_{d+1})_1$, the proof follows by Theorem 3.15 and induction. □

The converse of the above result is also true. In fact, we show that it is true for any strictly monotone norm $\|\cdot\|_E$ on $\mathbb{R}^d$ by using the following theorem.

Theorem 3.17. [18, Lemma 4.2] *A Banach space X is k-strictly convex if and only if for any linearly independent $x_1, \ldots, x_{k+1} \in X$, we have*

$$\|\sum_{i=1}^{k+1} x_i\| < \sum_{i=1}^{k+1} \|x_i\|.$$

Theorem 3.18. *Let X, Y be Banach spaces and let $\|\cdot\|_E$ be a strictly monotone norm on $\mathbb{R}^2$. If X is m-strictly convex and Y is k-strictly convex, then $(X \otimes Y)_E$ is $(m+k)$-strictly convex.*

Proof. Let $(x_1, y_1), \ldots, (x_{m+k+1}, y_{m+k+1}) \in S_{(X \otimes Y)_E}$ be such that

$$\mathrm{span}\{(x_1, y_1) - (x_{m+k+1}, y_{m+k+1}), \ldots, (x_{m+k}, y_{m+k}) - (x_{m+k+1}, y_{m+k+1})\}$$

is $(m+k)$-dimensional. If $(x_1, y_1), \ldots, (x_{m+k+1}, y_{m+k+1})$ are linearly dependent, then 0 must be in the affine hull of $(x_1, y_1), \ldots, (x_{m+k+1}, y_{m+k+1})$. By Lemma 3.7 we get

$$\frac{1}{m+k+1}\|\sum_{i=1}^{m+k+1}(x_i, y_i)\| < 1.$$

On the other hand, suppose that $(x_1, y_1), \ldots, (x_{m+k+1}, y_{m+k+1})$ are linearly independent. If $W = \mathrm{span}\{(x_1, y_1), \ldots, (x_{m+k+1}, y_{m+k+1})\}$, then

$$W_X = \mathrm{span}\{x_i : i = 1, \ldots, m+k+1\} \quad \text{and} \quad W_Y = \mathrm{span}\{y_i : i = 1, \ldots, m+k+1\}.$$

If $\dim(W_X) \le m$ and $\dim(W_Y) \le k$, then $\dim(W_X \otimes W_Y) < m+k+1$. But this cannot happen as $W \subset W_X \otimes W_Y$. Hence $\dim(W_X) > m$ or $\dim(W_Y) > k$. Suppose that $\dim(W_X) > m$. Choose $(a_1, b_1), \ldots, (a_{m+1}, b_{m+1}) \in \{(x_1, y_1), \ldots, (x_{m+k+1}, y_{m+k+1})\}$ such that $a_1, \ldots, a_{m+1}$ are linearly independent. By the m-strict convexity of X and Theorem 3.17, we have $\|\sum_{i=1}^{m+1} a_i\| < \sum_{i=1}^{m+1}\|a_i\|$ and by Lemma 3.7,

$$\frac{1}{m+k+1}\left(1 - \frac{1}{m+1}\|\sum_{i=1}^{m+1}(a_i, b_i)\|\right) \le 1 - \frac{1}{m+k+1}\|\sum_{i=1}^{m+k+1}(x_i, y_i)\|.$$

Rearranging and using the strict monotonicity of $\|\cdot\|_E$, we have

$$\frac{1}{m+k+1}\|\sum_{i=1}^{m+k+1}(x_i, y_i)\| \le \frac{m+k+\frac{1}{m+1}\left\|\left(\|\sum_{i=1}^{m+1} a_i\|, \|\sum_{i=1}^{m+1} b_i\|\right)\right\|_E}{m+k+1}$$

$$
\begin{aligned}
&< \frac{m+k+\frac{1}{m+1}\left\|\left(\sum_{i=1}^{m+1}\|a_i\|, \sum_{i=1}^{m+1}\|b_i\|\right)\right\|_E}{m+k+1} \\
&\leq \frac{m+k+\frac{1}{m+1}\sum_{i=1}^{m+1}\|(\|a_i\|, \|b_i\|)\|_E}{m+k+1} \\
&= \frac{m+k+\frac{1}{m+1}\sum_{i=1}^{m+1}\|(a_i, b_i)\|}{m+k+1} \leq 1.
\end{aligned}
$$

Thus, $(X \otimes Y)_E$ is $(m+k)$-strictly convex. □

Theorem 3.19. *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces and let $\|\cdot\|_E$ be a strictly monotone norm on $\mathbb{R}^d$. If X_i is k_i-strictly convex for $i = 1, \ldots, d$, then $(X_1 \otimes \cdots \otimes X_d)_E$ is $(k_1 + \cdots + k_d)$-strictly convex.*

Proof. The proof is similar to that of Theorem 3.18. □

Theorem 3.20. *For $d \in \mathbb{N}$, let $X_1, \ldots, X_d$ be Banach spaces.*
The space $(X_1 \otimes \cdots \otimes X_d)_1$ is n-strongly convex for some $n \in \mathbb{N}$ if and only if there exist $k_1, \ldots, k_d \in \mathbb{N}$ with $n = k_1 + \cdots + k_d$ such that X_i is k_i-strongly convex for $i = 1, \ldots, d$.

4. k-uniform convexity

Using results in section 3, we slightly modify examples 1.2 and 1.3 to obtain examples that illustrate that k-strong convexity does not imply k-uniform convexity and k-strict convexity does not imply k-strong convexity for any $k \in \mathbb{N}$. We also use the following theorem.

Theorem 4.1. [20] *For any $k \in \mathbb{N}$, every k-uniformly convex Banach space has an equivalent norm under which it is uniformly convex.*

Example 4.2. Let $(X, \|\cdot\|_X)$ be as in Example 1.2. For $k \in \mathbb{N}$, let $Y = (\mathbb{R}^k, \|\cdot\|_\infty)$. Since Y is finite-dimensional it is reflexive and has the Kadec-Klee property. In particular, since Y is k-dimensional, it is k-strictly convex [24]. So by Theorem 3.20, $(X \otimes Y)_1$ is $(k+1)$-strongly convex. The space $(X, \|\cdot\|_X)$ does not have an equivalent norm under which it is uniformly convex [4]. So by Theorem 4.1, $(X, \|\cdot\|_X)$ is not n-uniformly convex for any $n \in \mathbb{N}$ and therefore $(X \otimes Y)_1$ is not n-uniformly convex, for any $n \in \mathbb{N}$.

Example 4.3. Let $X = (l_1, \|\cdot\|)$ be as defined in Example 1.3 and $Y = (\mathbb{R}^k, \|.\|_\infty)$. Then $(X \otimes Y)_1$ is $(k+1)$-strictly convex by Theorem 3.18 but not $(k+1)$-strongly convex as it is not reflexive.

We conclude this article with a characterization of k-uniform convexity which can be compared with the characterizations of k-strong convexity as in Theorem 2.6. The following result is known.

Theorem 4.4. [15, Theorem 34] *Let X be a Banach space. Then the following statements are equivalent.*

(1) *X is uniformly convex.*

(2) $\lim_{\delta \to 0} \sup\{\mathrm{diam}(S(X, x^*, \delta) : x^* \in S_{X^*}\} = 0$.

An easy consequence of Lemma 3.7 is the following characterization of k-uniform convexity.

Theorem 4.5. *Let X be a Banach space. Then X is k-uniformly convex if and only if* $\inf\{1 - d(0, \mathrm{co}\{x_1, \ldots, x_{k+1}\}) : x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \geq \epsilon\} > 0$ *for every $\epsilon > 0$.*

Theorem 4.6. *Let X be a Banach space and let $\mathcal{C}_1$ be the collection of all convex subsets of X such that $d(0, C) = 1$. Then the following statements are equivalent.*

(1) *X is k-uniformly convex.*

(2) $\lim_{\delta \to 0} \sup\{\mathrm{diam}_k(S(X, x^*, \delta) : x^* \in S_{X^*}\} = 0$.

(3) $\lim_{t \to 1^+} \sup\{\mathrm{diam}_k(C \cap tB_X) : C \in \mathcal{C}_1\} = 0$.

Proof. (1) $\Rightarrow$ (2):
Suppose that X is k-uniformly convex. For $\epsilon > 0$, choose $\delta \in (0, 1)$ such that for any $x_1, \ldots, x_{k+1} \in B_X$ with $V(x_1, \ldots, x_{k+1}) \geq \epsilon$,

$$\frac{1}{k+1} \| \sum_{i=1}^{k+1} x_i \| < 1 - \delta.$$

Let $x^* \in S_{X^*}$. Since $S(X, x^*, \delta)$ is a convex set, for any $x_1, \ldots, x_{k+1} \in S(X, x^*, \delta)$, we have

$$1 - \delta \leq \frac{1}{k+1} x^*(\sum_{i=1}^{k+1} x_i) \leq \frac{1}{k+1} \| \sum_{i=1}^{k+1} x_i \|$$

which shows that $V(x_1, \ldots, x_{k+1}) < \epsilon$. Thus $\mathrm{diam}_k(S(X, x^*, \delta)) \leq \epsilon$ for all $x^* \in S_{X^*}$. This proves that $\lim_{\delta \to 0} \sup\{\mathrm{diam}_k(S(X, x^*, \delta) : x^* \in S_{X^*}\} = 0$.

(2) $\Rightarrow$ (3):
Suppose that $\lim_{\delta \to 0} \sup\{\mathrm{diam}_k(S(X, x^*, \delta) : x^* \in S_{X^*}\} = 0$. For $\epsilon > 0$, choose $\delta \in (0, 1)$ such that $\mathrm{diam}_k(S(X, x^*, \delta)) < \epsilon$ for all $x^* \in S_{X^*}$. Let $1 < t < \frac{1}{1-\delta'}$ where $\delta' = \frac{\delta}{k+1}$. For any $C \in \mathcal{C}_1$, if $x_1, \ldots, x_{k+1} \in C \cap tB_X$, then

$$1 \leq \|x_1\|, \ldots, \|x_{k+1}\|, \frac{1}{k+1} \| \sum_{i=1}^{k+1} x_i \| \leq t.$$

Choose $x^* \in S_{X^*}$ such that $\quad x^*(\sum_{i=1}^{k+1} x_i) = \| \sum_{i=1}^{k+1} x_i \|.$

Then $$1 - \delta' < \frac{1}{t} \leq \frac{1}{t(k+1)} \| \sum_{i=1}^{k+1} x_i \| = \frac{1}{t(k+1)} x^*(\sum_{i=1}^{k+1} x_i).$$

Since $\frac{1}{t} x^*(x_i) \leq 1$ for $i = 1, \ldots, k+1$, it follows that

$$1 - \delta < \frac{1}{t} x^*(x_i) \text{ for } i = 1, \ldots, k+1.$$

Thus, $\frac{1}{t} x_1, \ldots, \frac{1}{t} x_{k+1} \in S(X, x^*, \delta)$. Since $\mathrm{diam}_k(S(X, x^*, \delta)) < \epsilon$, we have

$$\frac{1}{t^k} V(x_1, \ldots, x_{k+1}) < \epsilon.$$

Thus $\dfrac{1}{t^k}\operatorname{diam}_k(C \cap tB_X) \le \epsilon$ for any $C \in \mathcal{C}_1$ when $t \in \left(1, \frac{1}{1-\delta'}\right)$.

This proves that, $\lim\limits_{t\to 1^+} \sup\left\{\dfrac{1}{t^k}\operatorname{diam}_k(C \cap tB_X) : C \in \mathcal{C}_1\right\} = 0$, and therefore,

$$\lim_{t\to 1^+} \sup\{\operatorname{diam}_k(C \cap tB_X) : C \in \mathcal{C}_1\} = 0.$$

(3) ⇒ (1):
Suppose that $\lim_{t\to 1^+} \sup\{\operatorname{diam}_k(C \cap tB_X) : C \in \mathcal{C}_1\} = 0$. Then

$$\lim_{t\to 1^+} \sup\left\{\frac{1}{t^k}\operatorname{diam}_k(C \cap tB_X) : C \in \mathcal{C}_1\right\} = 0.$$

Let $\epsilon > 0$ and $x_1, \ldots, x_{k+1} \in B_X$ such that we have $V(x_1, \ldots, x_{k+1}) \ge \epsilon$. Choose $\delta \in (0, 1)$ such that if $t \in \left(1, \frac{1}{1-\delta}\right)$, then $\frac{1}{t^k}\operatorname{diam}_k(C \cap tB_X) < \epsilon$ for $C \in \mathcal{C}_1$. If $d(0, \operatorname{co}\{x_1, \ldots, x_{k+1}\}) = 0$, then we have $\delta < 1 = 1 - d(0, \operatorname{co}\{x_1, \ldots, x_{k+1}\})$. If $d(0, \operatorname{co}\{x_1, \ldots, x_{k+1}\}) > 0$, let

$$t = \frac{1}{d(0, \operatorname{co}\{x_1, \ldots, x_{k+1}\})} \text{ and } C = t\operatorname{co}\{x_1, \ldots, x_{k+1}\}.$$

Then $C \in \mathcal{C}_1$. Since $C \subseteq tB_X$, we have

$$\operatorname{diam}_k(C \cap tB_X) = \operatorname{diam}_k(C) = t^k V(x_1, \ldots, x_{k+1}) \ge t^k\epsilon.$$

Therefore, $t \notin \left(1, \frac{1}{1-\delta}\right)$. Since by Theorem 2.5, $\operatorname{diam}_k(C \cap B_X) = 0$ we have $t \ne 1$. So we must have $t \ge \frac{1}{1-\delta}$. Thus, $1 - d(0, \operatorname{co}\{x_1, \ldots, x_{k+1}\}) \ge \delta$. So,

$$\inf\{1 - d(0, \operatorname{co}\{x_1, \ldots, x_{k+1}\}) : x_1, \ldots, x_{k+1} \in B_X, V(x_1, \ldots, x_{k+1}) \ge \epsilon\} \ge \delta > 0.$$

By Theorem 4.5, X is k-uniformly convex. □

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