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On the Complex Banach Conjecture

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The complex conjecture of Stefan Banach states that if V is a Banach space over the complex numbers where for some n, $1 < n < \dim(V) < \infty$, all of its n-dimensional subspaces are isometric, then V is a Hilbert space. Mikhail Gromov proved it for n even in 1967. Here, we prove it for $n \equiv 1 \bmod 4$.

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1. The main theorem

Stefan Banach [5] stated in 1932 the following general conjecture:

The Banach Conjecture. *Let V be a Banach space, real or complex, finite or infinite dimensional, all of whose n-dimensional subspaces, for some fixed integer n, $2 \leq n < \dim(V)$, are isometrically isomorphic to each other. Then, V is a Hilbert space.*

In 1959, A. Dvoretzky [8] proved a theorem from which there follows an affirmative answer to the conjecture for V real and infinite dimensional. Dvoretzky's theorem was extended in 1971 to the complex case by V. Milman [13]; settling Banach's conjecture affirmatively for the infinite dimensional case.

In 1935, Auerbach, Mazur and Ulam [4] gave a positive answer in case V is a real Banach space and $n = 2$. In 1967, M. Gromov [10] gave an affirmative answer in the case V is finite dimensional, real or complex, and n is even. In [6] the conjecture is proved for V a real finite dimensional Banach space and $n \equiv 1 \bmod 4$, except possibly when $n = 133$. Here, with an analogous approach, we give an affirmative answer in the case that V is a finite dimensional complex Banach space and $n \equiv 1 \bmod 4$.

Additionally, in the same 1967 paper, [10], Gromov proved the real Banach conjecture in codimension greater than 1 and the complex Banach conjecture in codimension greater than $2n$. Consequently, the conjecture remains unproved only when V is a complex Banach space, $n \equiv 3 \bmod 4$ and $n < \dim V < 2n$ or when V is a real Banach space $n \equiv 3 \bmod 4$ or $n = 133$, and $\dim V = n + 1$. The history behind this conjecture can be seen in [16]. It is also worthwhile to see [15] and the notes of Section 9 of [12].

A finite dimensional complex Banach space V is a Hilbert space if and only if its unit ball is a complex ellipsoid; and also, it is a Hilbert space if and only if for

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some $m > 1$ all of its m-dimensional subspaces are Hilbert spaces. Therefore, the complex Banach conjecture can be reformulated in terms of the closed unit ball $B = \{x \in V \mid \|x\| \leq 1\}$ of V, as follows:

($\bullet$) *Let $B \subset \mathbb{C}^{n+1}$, $n > 1$, be a complex symmetric convex body, all of whose sections by complex n-dimensional subspaces are complex linearly equivalent. Then, B is a complex ellipsoid.*

Indeed, if all the sections of B (the unit ball of V) by complex n-dimensional subspaces are isometric, ($\bullet$) implies that all the $(n+1)$-dimensional subspaces are Hilbert spaces, and thus that V is a Hilbert space. Therefore, to prove the isometric Banach conjecture over the complex numbers, when $n = 4k + 1$, it is enough to prove the following.

Theorem 1.1. *Let $B \subset \mathbb{C}^{n+1}$, $n \equiv 1$ mod 4, $n \geq 5$, be a complex symmetric convex body, all of whose sections by n-dimensional complex subspaces are complex linearly equivalent. Then, B is a complex ellipsoid.*

The proof of this theorem incorporates two main ingredients: one based on algebraic topology and the other on convex geometry. To state them, we first need to make precise the standard definitions involved.

A *complex hyperplane* of a vector space V over $\mathbb{C}$ is a complex codimension 1 linear subspace of V. An *affine complex hyperplane* is the translation of a complex hyperplane by some vector. A *(affine) complex hyperplane section* of a subset of a vector space is its intersection with a (affine) complex hyperplane.

Let V_1 and V_2 be two vector spaces over the complex numbers $\mathbb{C}$. We say that $K_1 \subset V_1$ is $\mathbb{C}$-*linearly equivalent* to $K_2 \subset V_2$ if there is a linear isomorphism over $\mathbb{C}$, $f : V_1 \to V_2$ (if $V_1 = V_2 = \mathbb{C}^n$, we simply write $f \in GL_n(\mathbb{C})$), such that $f(K_1) = K_2$.

A convex set $K \subset \mathbb{C}^n$ is said to be $\mathbb{C}$-*symmetric* if for every 1-dimensional complex subspace L of $\mathbb{C}^n$, $L \cap K$ is a disk centered at the origin; observe that this is equivalent to being invariant under the action of multiplication by the unitary complex numbers $\mathbb{S}^1 \subset \mathbb{C}$. For example, the unit ball of a Banach space over the complex numbers is a $\mathbb{C}$-symmetric convex body.

A *complex ellipsoid*, or a $\mathbb{C}$-*ellipsoid*, is the $\mathbb{C}$-linear image of a ball.

A *complex body of revolution* is a $\mathbb{C}$-symmetric convex body $K \subset \mathbb{C}^n$ for which there exists a 1-dimensional complex subspace L of $\mathbb{C}^n$, called its *axis of revolution*, such that for every affine complex hyperplane H orthogonal to L, we have that $H \cap K$ is either empty, a single point or a $(2n-2)$-dimensional ball centered at $H \cap L$. The *associated hyperplane of revolution* is $L^{\perp}$ (the orthogonal complement of the axis L).

Using topological methods of Lie groups, Section 2 is dedicated to prove the following.

Theorem 1.2. *Let $B \subset \mathbb{C}^{n+1}$, $n \equiv 1$ mod 4, $n \geq 5$, be a $\mathbb{C}$-symmetric convex body all of whose complex hyperplane sections are $\mathbb{C}$-linearly equivalent. Then, there exists a complex body of revolution, $K \subset \mathbb{C}^n$, with the property that every complex hyperplane section of B is $\mathbb{C}$-linearly equivalent to K.*

In Section 3, we prove the following characterization of complex ellipsoids.

Theorem 1.3. *A $\mathbb{C}$-symmetric convex body $B \subset \mathbb{C}^{n+1}$, $n \geq 4$, all of whose complex hyperplane sections are $\mathbb{C}$-linearly equivalent to a fixed complex body of revolution, is a $\mathbb{C}$-ellipsoid.*

Theorem 1.1 follows literally from Theorems 1.2 and 1.3.

2. Structure groups of the fibre bundle $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$

Denote by $GL'_n(\mathbb{C})$ the group of complex linear isomorphisms of $\mathbb{C}^n$ with determinant a positive real number. Note that if K_1 and K_2 are $\mathbb{C}$-symmetric convex bodies in $\mathbb{C}^n$ which are $\mathbb{C}$-linearly equivalent, then there is $g \in GL'_n(\mathbb{C})$ such that $g(K_1) = K_2$.

Given a $\mathbb{C}$-symmetric convex body $K \subset \mathbb{C}^n$, let $G_K := \{g \in GL'_n(\mathbb{C}) | g(K) = K\}$ be the *group of complex linear isomorphisms of K with positive real determinant*. By Lemma 1 of Gromow [10] there exists a $\mathbb{C}$-ellipsoid of minimal volume containing K centered at the origin. Suppose now that this minimal ellipsoid is a $(2n-1)$-dimensional ball. Then, every $g \in G_K$ is actually an element of $SU(n)$, because it fixes the unit ball and has determinant 1. From now on, we use the notation $G_K^0 := \{g \in SU(n) \mid g(K) = K\}$.

The link between our geometric problem and the topology is via a beautiful idea that traces back to the work of Gromov [10]. It yields the following lemma.

Lemma 2.1. *Let $B \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a $\mathbb{C}$-symmetric convex body all of whose complex hyperplane sections are $\mathbb{C}$-linearly equivalent. Then there exists a $\mathbb{C}$-symmetric convex body $K \subset \mathbb{C}^n$, with the property that every complex hyperplane section of B is $\mathbb{C}$-linearly equivalent to K and such that the structure group of the principal fibre bundle $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ can be reduced to $G_K^0 \subset SU(n)$.*

Proof. For every $x \in \mathbb{S}^{2n+1}$, let $\ell(x)$ be the complex line of $\mathbb{C}^{n+1}$ containing x and let $\ell^\perp(x)$ be the complex hyperplane of $\mathbb{C}^{n+1}$ orthogonal to $\ell(x)$. Consider

$$\eth^n = \{(x,y) \in \mathbb{S}^{2n+1} \times \mathbb{C}^{n+1} \mid y \in \ell^\perp(x)\}$$

and let $\wp : \eth^n \to \mathbb{S}^{2n+1}$ be the projection. Then $\wp$ is a n-dimensional complex vector bundle over $\mathbb{S}^{2n+1}$ with structure group $GL_n(\mathbb{C})$.

The hypothesis of the lemma imply that there is a field of the $\mathbb{C}$-symmetric convex body $B \cap \mathbb{C}^n$ in the n-vector bundle $\wp$; namely, $\{(x,y) \mid y \in B \cap \ell^\perp(x)\} \subset \eth^n$. Therefore, the structure group of the complex n-vector bundle $\wp$ can be reduced to

$$G_{B\cap\mathbb{C}^n} = \{g \in GL'_n(\mathbb{C}) | g(B \cap \mathbb{C}^n) = B \cap \mathbb{C}^n\} \subset GL_n(\mathbb{C}).$$

A good reference for the notion of reduction of the group of a fiber bundle is [17] and for its relation with the notion of field of convex bodies you may consult [14].

By Lemma 1 of Gromow [10], there exists a unique $\mathbb{C}$-ellipsoid of minimal volume, $E \subset \mathbb{C}^n$, centered at the origin and containing $B \cap \mathbb{C}^n$. Let $g_0 \in GL'_n(\mathbb{C})$ be such that $g_0(E)$ is the unit $(2n)$-ball, and let $K = g_0(B \cap \mathbb{C}^n)$. Then $G_{B\cap\mathbb{C}^n} \subset GL'_n(\mathbb{C})$ is conjugate to $G_K = G_K^0$ and therefore, the structural group of the complex n-vector bundle $\wp : \mathfrak{F}^n \to \mathbb{S}^{2n+1}$ can be reduced to G_K^0.

The complex n-vector bundle $\wp : \mathfrak{F}^n \to \mathbb{S}^{2n+1}$ can be reduced to $SU(n)$, yielding as associated principal bundle

$$SU(n) \hookrightarrow SU(n+1) \to SU(n+1)/SU(n) = \mathbb{S}^{2n+1}.$$

Let us denote by ξ_n this principal bundle. We have seen that $\wp$ can also be reduced to $G_K^0 \subset SU(n)$, therefore ξ_n can be reduced to G_K^0, as we wished to prove. □

Our main interest now turns naturally to study the structure groups of the principal bundle ξ_n: $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$. In particular, if $n \equiv 0 \bmod 2$, ξ_n cannot be reduced to a proper subgroup of $SU(n-1)$ [Theorem 1B of Leonard [11]]. Therefore, under the hypothesis of Lemma 2.1, G_K^0 must be $SU(n)$, and hence K must be a ball. This implies that every section of B is a complex ellipsoid and by Lemma 3.3 that B must be a complex ellipsoid. This proves the complex Banach conjecture, when n is even.

A subgroup $G \subset GL_n(\mathbb{C})$ is *reducible* if the induced action on $\mathbb{C}^n$ leaves invariant a complex k-dimensional linear subspace, $1 \le k < n$ and is *irreducible* if the induced action on $\mathbb{C}^n$ does not leave invariant a complex k-dimensional linear subspace, $1 \le k < n$.

Lemma 2.2. *Let $B \subset \mathbb{C}^{n+1}$, $n \equiv 1$ mod 4, $n \ge 5$, be a $\mathbb{C}$-symmetric convex body all of whose complex hyperplane sections are $\mathbb{C}$-linearly equivalent to a $\mathbb{C}$-symmetric convex body $K \subset \mathbb{C}^n$, such that the structure group of the principal fibre bundle ξ_n: $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ can be reduced to $G_K^0 \subset SU(n)$. Suppose G_K^0 is reducible, then G_K^0 is conjugate to a subgroup of $SU(n-1)$.*

Proof. Suppose G_K^0 leaves invariant a complex m-subspace V of $\mathbb{C}^n$; we may assume that $1 \le m \le n/2$, because $G_K^0 \subset SU(n)$ also leaves invariant the orthogonal complement of V.

The hypothesis about G_K^0 readily implies that the (real) tangent bundle $T\mathbb{S}^{2n+1}$ of the sphere $\mathbb{S}^{2n+1}$ admits a field of real $2m$-planes.

But moreover, we claim that this field of real $2m$-planes projects nicely to the tangent bundle $T\mathbb{RP}^{2n+1}$ of the $(2n+1)$-projective space $\mathbb{RP}^{2n+1}$. To see this, and using the notation and notions of the previous lemma, observe that we have well defined for every $x \in \mathbb{S}^{2n+1}$ a complex m-subspace $V_x \subset \ell^\perp(x)$ invariant under $G_{B\cap\ell^\perp(x)}$ which is conjugate to G_K^0. But $\ell^\perp(x) = \ell^\perp(-x)$ so that this field of (real) $2m$-planes is antipodally invariant and yields a corresponding field of (real) $2m$-planes over $\mathbb{RP}^{2n+1}$, as we claimed.

Since $n \equiv 1$ mod 4, by Theorem 1.1 (i) of [9], we know that the tangent bundle of the $(2n+1)$-projective space, $T\mathbb{RP}^{2n+1}$, splits into 3 trivial real line bundles and a complementary bundle that does not split. Consequently $m = 1$. That is, V has to be a complex line and therefore G_K^0 is conjugate to a subgroup of $SU(n-1)$. □

We summarize in the following lemma the results in the literature about the structure groups of the principal bundle ξ_n: $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$, needed in the sequel.

Lemma 2.3. (1) *If $n \equiv 1$ mod 4, $n \geq 5$, then the structure group of the principal bundle ξ_n: $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ reduces to $SU(n-1)$ but not to $SU(n-2)$.*

(2) *If $n \equiv 0$ mod 2, then the structure group of the principal bundle*

$$\xi_n : SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$$

does not reduce to $SU(n-1)$.

(3) *If the structure group of ξ_n: $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ reduces to a closed connected $\mathbb{C}$-irreducible* maximal *subgroup $H \subsetneq SU(n)$, then H is* simple.

(4) *If $n \geq 4$, then the structure group of ξ_n: $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ cannot be reduced to an $\mathbb{C}$-irreducible proper subgroup $G \subsetneq SU(n)$ isomorphic to $SO(k), SU(m)$ or $Sp(m)$, with $k \geq 4, m \geq 2$.*

Proof. Statements (1) and (2) follow for example from the work on the complex Stiefel manifolds of Atiyah-Todd [3] and Adams-Walker [2]. The proof of (4) follows immediately from Corollary 2.2 of Cadec-Crabb [7] and statement (3) is Theorem 3 of Leonard [11], when $G_n = SU(n)$. □

Lemma 2.4. *For all $n \equiv 1$ mod 4, $n \geq 5$, if the structure group of $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ can be reduced to $G \subset SU(n)$, then* $\dim G \geq 2n-3$.

Proof. The proof follows closely that of Proposition 3.1 in [7]. First note that the homotopy long exact sequence

$$\cdots \to \pi_{*+1}(\mathbb{S}^{2n-1}) \to \pi_*(SU(n-1)) \to \pi_*(SU(n)) \to \pi_*(\mathbb{S}^{2n-1}) \to \dots$$

associated to the fiber bundle $SU(n-1) \hookrightarrow SU(n) \to \mathbb{S}^{2n-1}$ implies that the inclusion $SU(n-1) \hookrightarrow SU(n)$ induces isomorphisms in π_* for $0 \leq * \leq 2n-2$. Consequently the inclusion $i : SU(n-2) \hookrightarrow SU(n)$ induces isomorphisms in π_* for $0 \leq * \leq 2n-4$. Thus, from a standard homotopy lifting argument: if $\dim G \leq 2n-4$, there is a map $g : G \to SU(n-2)$ such that ig is homotopic to the inclusion $j : G \hookrightarrow SU(n)$.

Suppose furthermore, that the structure group of $SU(n-1) \hookrightarrow SU(n) \to \mathbb{S}^{2n-1}$, which we are calling ξ_n, can be reduced to $G \subset SU(n)$. This implies that the characteristic map $\chi : \mathbb{S}^{2n-2} \to SU(n)$ of the principal bundle ξ_n can be factorized through G. That is, there is a continuous map $F : \mathbb{S}^{2n-2} \to G$ such that jF is homotopic to χ. By the above paragraph, igF is homotopic to χ, which implies that the structure group of ξ_n can be reduced to $SU(n-2)$, which is a contradiction to Lemma 2.3(1). Therefore, $\dim G \geq 2n-3$. □

Lemma 2.5. *For all $n \equiv 1$ mod 4, $n \geq 5$, if the structure group of $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ can be reduced to $G \subset SU(n-1)$, then G acts transitively on $\mathbb{S}^{2n-3}$.*

Proof. We follow the ideas of Corollary 3.2 of Cadek-Crabb in [CC]. Consider the standard fibration $SU(n-2) \to SU(n-1) \xrightarrow{\pi} \mathbb{S}^{2n-3}$. If G does not act transitively on $\mathbb{S}^{2n-3}$ it means that the composition $G \overset{i}{\hookrightarrow} SU(n-1) \xrightarrow{\pi} \mathbb{S}^{2n-3}$ is not surjective, and is therefore null homotopic. Let $F : G \times I \to \mathbb{S}^{2n-3}$ be the homotopy.

Then, by the homotopy lifting property, there exists a map $\widetilde{F}$ completing the diagram

$$\begin{tikzcd} G \arrow[r, hook, "i"] \arrow[d, "I\times 0"'] & SU(n-1) \arrow[d, "\pi"] \\ G\times I \arrow[ur, "\widetilde{F}"] \arrow[r, "F"'] & \mathbb{S}^{2n-3} \end{tikzcd}$$

Commutativity of the diagram implies that $\widetilde{F}(x,1) \in SU(n-2) \subset SU(n-1)$ for every $x \in G$. Let $f : G \to SU(n-2)$ be defined by $f(x) = \widetilde{F}(x,1)$; then, up to homotopy, the following diagram commutes

$$\begin{tikzcd} \mathbb{S}^{2n} \arrow[r, "\chi_n"] & G \arrow[rr, hook, "i"] \arrow[dr, "f"'] & & SU(n-1) \\ & & SU(n-2) \arrow[ur, "j"'] & \end{tikzcd}$$

But now, precomposing $j \circ f$ with the characteristic map $\chi_n : \mathbb{S}^{2n} \to G$, yields a reduction of the structure group of $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ to $SU(n-2)$, which is a contradiction to Lemma 2.3(1). $\square$

Lemma 2.6. *Let $n \equiv 1$ mod 4, $n \geq 5$, and suppose that the structure group of the fiber bundle $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ can be reduced to a closed connected irreducible subgroup $G \subset SU(n)$. Then $G = SU(n)$.*

Proof. Assume now the structure group of $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ reduces to $G \subset SU(n)$ and G acts $\mathbb{C}$-irreducibly on $\mathbb{C}^n$ but is not all of $SU(n)$. Without loss of generality assume that G is a maximal connected, closed subgroup with this property. By Lemma 2.3(3), G is simple. By Lemma 2.3(4) and Lemma 2.4, G is a non-classical group, i.e., it is isomorphic to either $Spin_m$, or one of the 5 exceptional simple Lie groups: G_2, F_4, E_6, E_7 or E_8. Note that if G is an spin group, then its action does not factor through $SO(m)$, therefore, by Lemma 3.6 in [BHJM], G is not an spin group.

Finally, by Lemma 2.4, dim$G \geq 2n-3$, hence to rule out the exceptional groups, one can simply check (e.g., in Wikipedia) the following table in which we list the smallest complex irreducible representations for them, and the smallest complex irreducible representation congruent to 1 mod 4 is in boldface, verifying that in all cases, dim$G \leq 2n-4$. $\square$

Group	G_2	F_4	E_6	E_7	E_8
dim G	14	52	78	133	248
Irreps	7	26	27	56	248
	14	52	78	**133**	3875
	27	**273**	351	⋮	⋮
	64	⋮	**2925**	⋮	**1763125**
	77	⋮	⋮	⋮	⋮

As a corollary of Lemmas 2.2 and 2.6, we have Theorem 1.2.

Proof of Theorem 1.2. By Lemma 2.1, there exists $K \subset \mathbb{C}^n$, a $\mathbb{C}$-symmetric convex body with the property that every complex hyperplane section of B is $\mathbb{C}$-linearly equivalent to K and such that the structure group of the principal fibre bundle $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ can be reduced to $G_K^0 \subset SU(n)$. If $G_K^0 \subset SU(n)$ is irreducible, then we may assume without loss of generality that G_K^0 is a maximal connected irreducible subgroup of $SU(n)$ and therefore, by Lemma 2.6, $G_K^0 = SU(n)$. Consequently, K is a ball.

If on the contrary G_K^0 leaves invariant a nontrivial subspace, by Lemma 2.2, $G_K^0 \subset SU(n-1)$ and hence by Lemma 2.5, G_K^0 acts transitively on $\mathbb{S}^{2n-3}$. This immediately implies that K is a complex body of revolution, as we wished. □

3. Complex bodies of revolution

We will call $K \subset \mathbb{C}^n$ a $\mathbb{C}$-*linear body of revolution*, if it is the image of a complex body of revolution under a $\mathbb{C}$-linear isomorphism. Thus, it is a $\mathbb{C}$-symmetric convex body that comes equipped with an *axis of revolution*, L, which is a complex line, and a *hyperplane of revolution*, H, which is a complementary hyperplane (but not necessarily orthogonal) to L, and it satisfies that all its sections with affine hyperplanes H' parallel to H are either empty, a point or a $\mathbb{C}$-ellipsoid centered at L and homothetic to the $\mathbb{C}$-ellipsoid $H \cap K$.

The main ingredient for the proof of Theorem 1.3 is the following.

Theorem 3.1. *Let $B \subset \mathbb{C}^{n+1}$ be a $\mathbb{C}$-symmetric convex body, $n \geq 4$, all of whose complex hyperplane sections are $\mathbb{C}$-linear bodies of revolution. Then, one of the complex hyperplane sections is a $\mathbb{C}$-ellipsoid.*

The main bulk of this section is devoted to prove Theorem 3.1. For that purpose, we need six lemmas concerning $\mathbb{C}$-linear bodies of revolution and $\mathbb{C}$-ellipsoids. We start with the latter.

Lemma 3.2. *A $\mathbb{C}$-symmetric ellipsoid is a $\mathbb{C}$-ellipsoid.*

Proof. We need to recall some well known facts about *ellipsoids*, by which we mean real ellipsoids thought of as convex bodies.

Let $E \subset \mathbb{R}^n$ be a n-dimensional ellipsoid centered at the origin. For every k-dimensional subspace, $H \subset \mathbb{R}^n$ with $1 \leq k < n$, there exists a complementary $(n-k)$-dimensional subspace, L of $\mathbb{R}^n$, called its *polar subspace with respect to E*, such that

$$\partial E \cap L = \{x \in \mathbb{R}^n \mid H + x \text{ is a k-dimensional plane tangent to } \partial E \text{ at } x\}$$

(this set is called the *shadow boundary of E in the direction H*). Moreover, H is the polar subspace of L with respect to E, and the section $L \cap E$ is a $(n-k)$-dimensional ellipsoid with the following property: for every $(n-k)$-plane L', parallel to L, the corresponding section $L' \cap E$ is either the empty set, a point in H or an ellipsoid homothetic to $L \cap E$ and centered at H. For more about shadow boundaries see Section 1.12 of [12].

Let $K \subset \mathbb{C}^n$ be a $\mathbb{C}$-symmetric ellipsoid. We will prove that there is a linear isomorphism $g \in GL(n, \mathbb{C})$ such that $g(K)$ is a ball, by induction on the dimension n. Clearly, the statement is true for $n = 1$. Suppose it is true for dimension $n - 1$, we shall prove it for dimension n.

Assume the diameter of K is h an let $[-u, u]$ be a diameter of K, let L be the unique complex line containing the vector u. By hypothesis $D = L \cap K$ is a disk centered at the origin all of whose diameters are also diameters of K. This implies that the polar to L with respect to E is the complex hyperplane, H, orthogonal to L. Then, for every affine complex line L' orthogonal to H and touching $\mathrm{int}(K)$, the section $L' \cap K$ is a disk with center at H.

By induction we have that $H \cap K$ is a $\mathbb{C}$-ellipsoid. Therefore, using a $\mathbb{C}$-linear isomorphism, we may assume that $H \cap K$ is a $(2n-2)$-dimensional ball of diameter h. To conclude the proof of the lemma, we prove that K is a ball.

Let λ be a real line subspace contained in H and let Δ be the 3-dimensional real subspace generated by λ and L. Since $(L+x) \cap K$ is a disk with center at λ for every $x \in \lambda \cap \mathrm{int}(K)$, $\Delta \cap K$ is a real ellipsoid of revolution with axis the line λ. Since the three axis of this ellipsoid are equal, this implies, that $\Delta \cap K$ is a 3-dimensional ball with center at the origin. Since this holds for every real 3-dimensional subspace containing L, we have that K is a ball, as we wished. □

Lemma 3.3. *Let $B \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a $\mathbb{C}$-symmetric convex body, all of whose complex hyperplane sections are $\mathbb{C}$-ellipsoids. Then B is a $\mathbb{C}$-ellipsoid.*

Proof. We prove that B is an ellipsoid. Then, by Lemma 3.2, B is a $\mathbb{C}$-ellipsoid.

Consider that $\mathbb{C}^n = \mathbb{R}^{2n}$. By Theorem 2.12.2 of [12], it is enough to prove that every real two dimensional subspace intersects B in an ellipse. Let Π be a two dimensional real plane generated by $\{v_1, v_2\}$. If Π is a complex line, $\Pi \cap B$ is a ball, so assume it is not. Let L_i be the complex line containing v_i, $i = 1, 2$. Consequently, Π is contained in the complex plane P generated by $\{L_1, L_2\}$. By hypothesis, $P \cap B$ is a section of an ellipsoid and hence is itself an ellipsoid. This implies that $\Pi \cap B$ is an elipse. Therefore, B is an ellipsoid. □

Note that every complex line through the origin is an axis of revolution of a ball centered at the origin and consequently, every complex hyperplane is a hyperplane of revolution of an ellipsoid centered at the origin.

Lemma 3.4. *A $\mathbb{C}$-linear body of revolution $K \subset \mathbb{C}^n$, $n \geq 3$, admitting two different hyperplanes of revolution, is a $\mathbb{C}$-ellipsoid.*

Proof. By Lemma 1 of Gromow [10], let E be the unique $\mathbb{C}$-ellipsoid of minimal volume centered at the origin containing K and we may suppose, without loss of generality, that E is the unit ball. Since every symmetry of K is a symmetry of the unit ball, our hypothesis now imply that K is a complex body of revolution with two different axis of revolution. Let L_1 and L_2 be two different complex lines and let G_1 and G_2 be the complex rotation groups around the axis L_1 and L_2, respectively; they are both conjugate to $SU(n-1)$. Suppose G is a compact subgroup of $SU(n)$ that contains both G_1 and G_2. We shall prove that the action of G in $\mathbb{S}^{2n-1}$ is transitive. If this is so, and both L_1 and L_2 are axis of revolution of K, then $G_K^0 = \{g \in SU(n) \mid g(K) = K\}$, which is compact because K is a compact convex body, would act transitively on ∂K and K would be a ball.

Let P be the complex plane generated by L_1 and L_2 and let π_1, π_2 and π_0 be the orthogonal projections onto L_1, L_2 and P, respectively. Furthermore, let $D = P \cap \operatorname{int}(\mathbb{B})$, where $\mathbb{B}$ is the unit ball of $\mathbb{C}^n$. Consider the set

$$U = \pi_0^{-1}(D) \cap \mathbb{S}^{2n-1}\,.$$

Note that U is an open connected dense subset of $\mathbb{S}^{2n-1}$ because $\mathbb{S}^{2n-1} \backslash U = P \cap \mathbb{S}^{2n-1}$ is a 3-sphere contained in $\mathbb{S}^{2n-1}$, and since $n \geq 3$, its (topological) codimension is at least 2.

Let $x \in U$. Our purpose is to construct an open neighborhood W of x in U such that W is contained in the orbit $G \cdot x$ of x under the action of G in $\mathbb{S}^{2n-1}$. This will be enough to prove the lemma because U is a connected open dense subset of $\mathbb{S}^{2n-1}$.

Let $H_1 = \pi_1^{-1}(\pi_1(x))$. It is the complex affine hyperplane orthogonal to L_1 and passing through x, so that $G_1 \cdot x = H_1 \cap \mathbb{S}^{2n-1}$. Let $W_1 = H_1 \cap D$. It is an open disk in an affine line parallel to the line $L_1^\perp \cap P$, and observe that restricted to this affine line $(H_1 \cap P)$, the map π_2 is a complex affine isomorphism onto L_2 because $L_1 \neq L_2$. So that $W_2 = \pi_2(W_1)$ is an open subset of $L_2 \cap D$ that contains $\pi_2(x)$.

Let $W = \pi_2^{-1}(W_2) \cap U$. It is an open neighborhood of x in U. We are left to prove that W is contained in the orbit $G \cdot x$.

Given $y \in W$, let $H_2 = \pi_2^{-1}(\pi_2(y))$, so that $G_2 \cdot y = H_2 \cap \mathbb{S}^{2n-1}$. Consider the affine subspace $\Gamma = H_1 \cap H_2$ of dimension $n - 2 > 0$. By construction, H_2 intersects W_1 in a point, so that Γ touches the interior of the unit ball $\mathbb{B}$. Therefore, $\Gamma \cap \mathbb{S}^{2n-1} = (G_1 \cdot x) \cap (G_2 \cdot y)$ is not empty. This implies that $G \cdot x = G \cdot y$, so that $y \in G \cdot x$, and hence $W \subset G \cdot x$. □

Lemma 3.5. *Every complex hyperplane section $\Gamma \cap K$ of a $\mathbb{C}$-linear body of revolution $K \subset \mathbb{C}^n$, $n \geq 3$, is a $\mathbb{C}$-linear body of revolution. Furthermore, if H is the complex hyperplane of revolution of K, then either $\Gamma = H$ or $\Gamma \cap H$ is a hyperplane of revolution of $\Gamma \cap K$.*

Proof. Without loss of generality, we may assume that K is a complex body of revolution; that is, if its axis of revolution is the complex line L, then $H = L^\perp$ is the corresponding hyperplane of revolution and we have that $H \cap K$ is a ball centered at the origin.

Assume $\Gamma \neq H$. We will prove that $K_1 = \Gamma \cap K$ is a complex body of revolution in Γ with hyperplane of revolution $H_1 = \Gamma \cap H$. Let L_1 be the complex line orthogonal to H_1 in Γ; it will be the axis of K_1.

Given $H_1' \subset \Gamma$ parallel to H_1, we have to consider the intersection $H_1' \cap K_1 = H_1' \cap K$.

Let H' be the affine hyperplane of $\mathbb{C}^n$ parallel to H that contains H_1'. By hypothesis, we have that $H' \cap K$ is either empty, a point or a ball (in H') centered at L. Therefore, its intersection with H_1' (a hyperplane of H') is either empty, a point or a ball. By construction, in the two last cases, the point or the center of the ball lies in L_1;

indeed, the plane generated by L and L_1 is orthogonal to H_1. Therefore, $K_1 \subset \Gamma$ is a complex body of revolution as we wished. □

Lemma 3.6. *Let $K \subset \mathbb{C}^n$ be a $\mathbb{C}$-linear body of revolution with axis of revolution L, $n \geq 3$. Suppose $\Gamma \subset \mathbb{C}^n$ is a complex hyperplane containing L for which $\Gamma \cap K$ is a $\mathbb{C}$-ellipsoid. Then K is a $\mathbb{C}$-ellipsoid*

Proof. First, we may assume that K is a complex body of revolution with axis of revolution L, hyperplane of revolution $H = L^\perp$ and such that $H \cap K$ is the unit ball in H. By hypothesis and Lemma 3.5, $\Gamma \cap K$ is a $\mathbb{C}$-ellipsoid and a complex body of revolution with axis of revolution L. Using a $\mathbb{C}$-linear map which is the identity on H and a dilatation on L, we may assume $\Gamma \cap K$ is a unit ball centered at the origin; so that $\Gamma \cap K = \Gamma \cap \mathbb{B}$, where $\mathbb{B} \subset \mathbb{C}^n$ is the unit ball. Our purpose is to prove that $K = \mathbb{B}$ to conclude the proof.

For every affine hyperplane H' parallel to H that touches the interior of K, we have that both $H' \cap K$ and $H' \cap \mathbb{B}$ are concentric balls. Furthermore, they have the same radius because their boundaries have non empty intersection (in Γ). Consequently, $H' \cap K = H' \cap \mathbb{B}$ and hence $K = \mathbb{B}$, as we wished. □

From now on, let $B \subset \mathbb{C}^{n+1}$ be a $\mathbb{C}$-symmetric convex body, all of whose complex hyperplane sections are non $\mathbb{C}$-elliptical, $\mathbb{C}$-linear bodies of revolution. First of all note that the assignation of the axis of revolution in every complex hyperplane is continuos. The proof is analogous to the proof of Lemma 2.7 of [6]. Therefore, there is a field of complex lines inside the complex n-vector bundle $\wp : \mathfrak{F}^n \to \mathbb{S}^{2n+1}$ and this implies that the structure group of its principal bundle

$$\xi_n : SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$$

reduces to $SU(n-1)$. Consequently, we may assume that n is odd, otherwise, if n is even, this contradicts Lemma 2.3(2).

For every complex line $\ell \subset \mathbb{C}^{n+1}$ denote by $\ell^\perp$ the complex hyperplane subspace of $\mathbb{C}^{n+1}$ orthogonal to ℓ. Furthermore, by Lemma 3.4, denote by L_ℓ the unique axis of revolution of $\ell^\perp \cap B$ and by H_ℓ the corresponding complex $(n-1)$-dimensional subspace of revolution of $\ell^\perp \cap B$. Note that the complex line L_ℓ contains the origin because the center of the $\mathbb{C}$-ellipsoid $H_\ell \cap B$ is the origin.

Lemma 3.7. *Let $B \subset \mathbb{C}^{n+1}$, $n \geq 4$, be a $\mathbb{C}$-symmetric convex body, all of whose complex hyperplane sections are non $\mathbb{C}$-elliptical, $\mathbb{C}$-linear bodies of revolution. Suppose ℓ_1 and ℓ_2 are two different complex lines with the property that $L_{\ell_2} \subset \ell_1^\perp$. Then*

$$H_{\ell_2} \cap \ell_1^\perp = H_{\ell_1} \cap \ell_2^\perp = H_{\ell_2} \cap H_{\ell_1}.$$

Proof. Consider $\ell_1^\perp \cap \ell_2^\perp$, the complex $(n-1)$-dimensional subspace of $\ell_2^\perp$. By hypothesis, $\ell_2^\perp \cap B$ is a non $\mathbb{C}$-elliptical, $\mathbb{C}$-linear body of revolution with axis of revolution L_{ℓ_2}. Therefore, since $L_{\ell_2} \subset \ell_1^\perp \cap \ell_2^\perp$, we have that $\ell_1^\perp \cap \ell_2^\perp \cap B$ is an $\mathbb{C}$-linear body of revolution with corresponding axis of revolution L_{ℓ_2}. Furthermore, by Lemma 3.6, $\ell_2^\perp \cap B$ is not a $\mathbb{C}$-ellipsoid. Moreover, the unique hyperplane of revolution of $\ell_1^\perp \cap \ell_2^\perp \cap B$ is $\ell_1^\perp \cap H_{\ell_2}$.

On the other hand, $\ell_1^\perp \cap \ell_2^\perp$ is a complex $(n-1)$-dimensional subspace of $\ell_1^\perp$. Note that $\ell_1^\perp \cap \ell_2^\perp \neq H_{\ell_1}$, otherwise $\ell_1^\perp \cap \ell_2^\perp \cap B = H_{\ell_1} \cap B$ would be an ellipsoid contradicting our previous assumption. Since $\ell_1^\perp \cap B$ is a non $\mathbb{C}$-elliptical, $\mathbb{C}$-linear body of revolution and $\ell_1^\perp \cap \ell_2^\perp \neq H_{\ell_1}$, then by Lemma 3.5, $\ell_1^\perp \cap \ell_2^\perp \cap B$ is a non $\mathbb{C}$-elliptical, $\mathbb{C}$-linear body of revolution with corresponding subspace of revolution $H_{\ell_1} \cap \ell_2^\perp$. Consequently, by Lemma 3.4, since $n-1 \geq 3$, we have that $H_{\ell_2} \cap \ell_1^\perp = H_{\ell_1} \cap \ell_2^\perp$. □

Proof of Theorem 3.1. Exactly as in the proof of Lemma 2.7 of [6], the assignation $\ell \to L_\ell \subset \ell^\perp$ is a continuous function on ℓ. If n is even, there is a field of complex lines inside the complex n-vector bundle $\wp : \eth^n \to \mathbb{S}^{2n+1}$ and this implies that the structure group of its principal bundle ξ_n: $SU(n) \hookrightarrow SU(n+1) \to \mathbb{S}^{2n+1}$ reduces to $SU(n-1)$. Consequently, we may assume that n is odd, $n \geq 5$, otherwise, if n is even, this contradicts Lemma 2.3(2).

We start proving that there are two different complex lines ℓ_1 and ℓ_2 such that

$$L_{\ell_2} \subset H_{\ell_1}.$$

For that purpose, let L be a complex line in H_{ℓ_1} and denote by $L^\perp$ the complex hyperplane of $\mathbb{C}^{n+1}$ orthogonal to L. Denote by $\Pi : \mathbb{C}^{n+1} \to L^\perp$ the orthogonal projection along L. Of course, Π is a linear map over the complex numbers $\mathbb{C}$.

For every complex line $\ell \subset L^\perp$, consider $L_\ell \subset \ell^\perp$. Suppose $L_\ell \neq L$, for every $\ell \subset L^\perp$. If this is so, $\Pi(L_\ell)$ is a complex line through the origin in $L^\perp$, ortogonal to ℓ. Furthermore, as in the proof of Lemma 2.7 of [6], the assignation $\ell \to \Pi(L_\ell)$ is a continuos function on ℓ. Since $n-1$ is even, this is a contradiction to Lemma 2.3(2). This implies the existence of ℓ_2 such that $L = L_{\ell_2} \subset H_{\ell_1}$.

Note now that this is a contradiction to Lemma 3.7 because clearly $L_{\ell_2} \subset \ell_1^\perp$, hence

$$L_{\ell_2} \subset H_{\ell_1} \cap \ell_2^\perp = H_{\ell_2} \cap \ell_1^\perp \subset H_{\ell_2}$$

which is imposible. □

Finally, from Theorem 3.1 and Lemma 3.3, Theorem 1.3 follows immediately; which, in turn, completes the proof of Theorem 1.1.

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