

Weak* Hypertopologies with Application to Genericity of Convex Sets

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We propose a new class of hypertopologies, called here weak* hypertopologies, on the dual space $\mathcal{X}^*$ of a real or complex topological vector space $\mathcal{X}$. The most well-studied and well-known hypertopology is the one associated with the Hausdorff metric for closed sets in a complete metric space. Therefore, we study in detail its corresponding weak* hypertopology, constructed from the Hausdorff distance on the field (i.e. $\mathbb{R}$ or $\mathbb{C}$) of the vector space $\mathcal{X}$ and named here the weak*-Hausdorff hypertopology. It has not been considered so far and we show that it can have very interesting mathematical connections with other mathematical fields, in particular with mathematical logics. We explicitly demonstrate that weak* hypertopologies are very useful and natural structures by using again the weak*-Hausdorff hypertopology in order to study generic convex weak*-compact sets in great generality. We show that convex weak*-compact sets have generically a weak*-dense set of extreme points in infinite dimensions. An extension of the well-known Straszewicz theorem to Gateaux-differentiability (non necessarily Banach) spaces is also proven in the scope of this application.

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1. Introduction

Topologies for sets of closed subsets of topological spaces have been studied since the beginning of the last century. When such topologies, restricted to singletons, coincide with the original topology of the underlying space, we talk about hypertopologies and hyperspaces of closed sets. In the literature, there exist various hypertopologies and related set-convergence notions, like the Fell, Vietoris, Wijsman, proximal, Hausdorff metric, or locally finite hypertopologies, to name a few well-known examples. For a review on this field, see, e.g., [7]. Hypertopologies are important topological structures in applied mathematics, for instance to study stability of a minimization problem, as explained in [30, Chapter 8]. In this paper, we propose a new class of hypertopologies whose underlying topological space is the dual space $\mathcal{X}^*$ of a real or complex topological vector space $\mathcal{X}$, endowed with the weak* topology.

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The weak* topology of $\mathcal{X}^*$ is generally not metrizable, but locally convex and Hausdorff. Denote by $\mathbf{F}(\mathcal{X}^*)$ the set of all weak*-closed subsets of $\mathcal{X}^*$. The hypertopologies on $\mathbf{F}(\mathcal{X}^*)$ we propose here are all constructed from the following scheme: Any element $A \in \mathcal{X}$ defines a weak*-continuous linear mapping $\hat{A}$ from $\mathcal{X}^*$ to the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ of the vector space $\mathcal{X}$, in turn implying a canonical mapping $\mathbf{A}$ from $\mathbf{F}(\mathcal{X}^*)$ to the set $\mathbf{F}(\mathbb{K})$ of all closed subsets of $\mathbb{K}$. Then, providing $\mathbf{F}(\mathbb{K})$ with some hypertopology, a *weak** hypertopology on $\mathbf{F}(\mathcal{X}^*)$ is the so-called initial topology of the family $\{\mathbf{A}\}_{A \in \mathcal{X}}$.

This construction is analogous to the one of the weak* topology, which is the initial topology of the family $\{\hat{A}\}_{A \in \mathcal{X}}$. This new class of hypertopologies, called here weak* hypertopologies, does not seem to have been systematically considered in the past. The scalar topology described in [7, Section 4.3], adapted for a locally convex space like $\mathcal{X}^*$, is retrospectively a first example of a weak* hypertopology.

The most well-studied and well-known hypertopology on a complete metric space is the Hausdorff metric topology. Therefore, we study in detail the weak* hypertopology associated with the Hausdorff metric in the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ of the vector space $\mathcal{X}$. It is named here the *weak*-Hausdorff* hypertopology. To our knowledge, it has not been considered so far and we show that it can have very interesting mathematical connections with other mathematical fields, like mathematical logics:

- We use the weak*-closed convex hull operator $\overline{\text{co}}$ in order to study the weak*-Hausdorff topology of hyperspaces. Such a closure operator has certainly been considered in the past and combines (i) an algebraic (or finitary) closure operator and (ii) a topological (or Kuratowski) closure operator. It gives a first connection with mathematical logics where fascinating applications of closure operators have been developed, already by Tarski himself during the 1930's.
- We introduce the notion of immeasurable hyperspaces, by using the sets of topologically bounded weak*-closed subsets of $\mathcal{X}^*$. It is associated with an infinite collection of weak*-Hausdorff-clopen sets, which can be used to form a Boolean algebra, as is usual in mathematical logics¹. Such a study has for instance been performed in [6] for the hyperspace associated with a Stonean compact topological space and the Vietoris hypertopology.

We demonstrate that the weak*-Hausdorff hypertopology cannot distinguish a set from its weak*-closed convex hull, as it appears for other well-established hypertopologies like the scalar topology (see [7, Section 4.3]). Similar to the scalar topology, only the hyperspace $\overline{\text{co}}(\mathbf{F}(\mathcal{X}^*))$ of all $\overline{\text{co}}$ -closed, or equivalently convex weak*-closed, subsets of $\mathcal{X}^*$ is in general a Hausdorff space.

We also relate the weak*-Hausdorff convergence with the celebrated lower and upper limits of sets à la Painlevé [28, § 29] and strongly strengthen our results when the topological vector space $\mathcal{X}$ is separable. This is performed by showing that the Banach-Alaoglu theorem and the metrizability of the weak* topology for absolute polars in the dual space $\mathcal{X}^*$ of a separable space $\mathcal{X}$ are inherited by $\mathbf{F}(\mathcal{X}^*)$. This leads to a large class of metrizable weak*-Hausdorff-compact hyperspaces. Other aspects of the weak*-Hausdorff hypertopology are also studied in detail.

¹ See Stone's representation theorem for Boolean algebras.

We explicitly demonstrate that weak* hypertopologies are very useful and natural structures by using again the weak*-Hausdorff hypertopology in order to study generic convex weak*-compact sets in great generality. In fact, the present paper has been originally inspired by the fairly complicated geometrical structure of the convex weak*-compact state space of antiliminal and simple C^* -algebras. In this case, by [18, Lemma 11.2.4], the state space is known to have a weak*-dense set of extreme points. For some UHF (uniformly hyperfinite) C^* -algebras, this space can even be represented by the weak*-closure of a strictly increasing, countable, family of Poulsen simplices² [34], as explained in [11] for CAR C^* -algebras. It is interesting to know whether these disconcerting properties of *physically relevant*³ systems are purely accidental, or not.

This question is reminiscent of the celebrated Wonderland theorem, by Simon [38] in 1995, and papers [13, 14, 15, 16], by Carvalho and de Oliveira in 2016-2018, showing that apparently peculiar features of Schrödinger operators in quantum mechanics, like the existence of a purely singular spectrum, are in fact generic. See, e.g., [13]. I.e., the exceptional-looking property turn out to be the rule from the topological viewpoint. We study a similar genericity issue on convex weak*-compact sets, by using the weak*-Hausdorff hypertopology introduced by us in the present paper.

In 1959, Klee shows [27] that, for convex norm-compact sets within a Banach space, the property of having a dense set of extreme points is *generic* in infinite dimensions. More precisely, by [27, Proposition 2.1, Theorem 2.2], the set of all such convex compact subsets of an infinite-dimensional separable⁴ Banach space $\mathcal{Y}$ is generic⁵ in the complete metric space of compact convex subsets of $\mathcal{Y}$, endowed with the well-known Hausdorff metric (hyper)topology. Klee's result has been refined in 1998 by Fonf and Lindenstrauss [20, Section 4] for bounded norm-closed (but not necessarily norm-compact) convex subsets of $\mathcal{Y}$ having so-called empty quasi-interior (as a necessary condition). In this case, [20, Theorem 4.3] shows that such sets can be approximated in the Hausdorff metric topology by closed convex sets with a norm-dense set of strongly exposed points⁶. See, e.g., [21, Section 7] for a recent review on this subject.

In the present paper, we demonstrate the same genericity in the dual space $\mathcal{X}^*$, endowed with its weak*-topology, of an infinite-dimensional, separable topological vector space $\mathcal{X}$. The hypertopology used in [20, 27] is the Hausdorff metric topology, induced by the Hausdorff distance associated with the norm of a Banach space. Here, $\mathcal{X}^*$ is only endowed with the weak* topology and the Hausdorff metric (hyper)topology used in [20, 27] is naturally replaced by the weak*-Hausdorff hypertopology.

² It is a (unique, up to an affine homeomorphism, universal, homogeneous) metrizable simplex with dense extreme boundary.

³ E.g., quantum spin systems and fermions (like electrons) on lattices.

⁴ [27, Proposition 2.1, Theorem 2.2] seem to lead to the asserted property for all (possibly non-separable) Banach spaces, as claimed in [20, 21, 27]. However, [27, Theorem 1.5], which assumes the separability of the Banach space, is clearly invoked to prove the corresponding density stated in [27, Theorem 2.2]. We do not know how to remove the a priori separability condition.

⁵ That is, the complement of a meagre set, i.e., a nowhere dense set.

⁶ $x \in K$ is a *strongly exposed point* of a convex set $K \subseteq \mathcal{Y}$ when there is $f \in \mathcal{Y}^*$ satisfying $f(x) = 1$ and such that the diameter of $\{y \in K : f(y) \geq 1 - \varepsilon\}$ tends to 0 as $\varepsilon \rightarrow 0^+$. (Strongly) exposed points are extreme elements of K .

If $\mathcal{X}$ is a separable Banach space whose dual is separable, note that one can use the standard norm topology on $\mathcal{X}^*$ for continuous linear functionals and directly apply previous results [20, 27] to the separable Banach space $\mathcal{X}^*$. This is not anymore possible if one considers the weak*-topology. In particular, [20, Theorem 4.3] cannot be invoked in this specific case because, in general, weak*-compact sets do not have an empty interior, in the sense of the norm topology. Anyway, our study is done for locally convex spaces $\mathcal{X}^*$, as the dual spaces – endowed with the weak* topology – of separable, infinite-dimensional topological vector spaces $\mathcal{X}$, and considerably extends previous ones [20, 27]. We also prove similar results within weak*-closed sets of *positive* functionals of $\mathcal{X}^*$, provided some positive (convex) cone in $\mathcal{X}$ is given. We summarize our main results on infinite-dimensional convexity as follow:

- The set $\mathcal{D}_0$ of convex weak*-compact subsets with weak*-dense set of *exposed* points is weak*-Hausdorff dense in the set of all convex weak*-compact subsets of an infinite-dimensional absolute polar of a 0-neighborhood in $\mathcal{X}^*$. $\mathcal{X}$ is here a real or complex topological vector space and if $\mathcal{X}$ is additionally a separable Gateaux-differentiability space then $\mathcal{D}_0$ is also a G_δ set. Recall that a Gateaux-differentiability space $\mathcal{X}$ is defined to be a topological vector space on which every continuous convex real-valued function with a nonempty open convex subset as domain is Gateaux-differentiable on a dense subset of that domain. See [29, 33].
- Meanwhile, the set of *exposed* points is proven to be weak*-dense in the set of *extreme* points of a convex weak*-compact subset of a Gateaux-differentiability space $\mathcal{X}$. This corresponds to an extension of [33, Theorem 6.2], which only refers to Banach spaces $\mathcal{X}$. Our proof is quite direct and thus, pedagogical while being very general. This result extends the Straszewicz theorem proven in 1935 [39] by Straszewicz in $\mathbb{R}^n$, generalized by Klee in 1958 [26, Theorems (2.1), (2.3)] and in 1976 by Bair [4, Theorem 1] in arbitrary real linear space for algebraically closed convex sets with so-called finite “copointure” [5, Section II.5.1]. Note that this last condition cannot be satisfied by weak*-compact sets.
- We refine our analysis in order to get similar results for the special case of positive functionals of $\mathcal{X}^*$, by using the decomposition of equicontinuous linear functionals into positive equicontinuous components [42], as originally proven by Grotsberg and Krein [22] in 1939 for normed spaces and by Bonsall [8] in 1957 for locally convex real vector spaces.

As an application to the Banach case, we show, for an infinite-dimensional, separable and unital C^* -algebra $\mathcal{X}$, that convex weak*-compact sets of the corresponding state space, i.e., the space of positive and normalized functionals of $\mathcal{X}^*$, have generically a weak*-dense set of extreme points. This demonstrates that the fairly complicated geometrical structure of the state space of physically relevant systems is *not* accidental, but *generic* for infinite-dimensional, separable unital C^* -algebras. This is done by using a very fine weak*-type topology, namely the weak*-Hausdorff hypertopology.

To conclude, it is not so easy to highlight a particular result of this paper since various independent statements are proven along the current studies on the weak*-Hausdorff hypertopology and convex sets. Propositions 3.2, 3.3, Corollaries 3.5, 3.11, 3.13, 3.18 and 4.20 as well as Theorems 3.17, 4.5, 4.12, 4.16 are probably the most important of those results.

New mathematical concepts are also introduced in this paper: weak* hypertopologies (Definition 2.2), weak*-Hausdorff hypertopology (Definition 2.3), and immeasurable hypersets (Definition 3.1). The paper is organized as follows: Section 2 gathers all the necessary definitions to introduce in Section 2.2 the class of weak* hypertopologies as well as the important example of the weak*-Hausdorff hypertopology. We next study, in Section 3, this particular weak* hypertopology in detail. In Section 4, using the results of Section 3, we study generic convex weak*-compact sets in great generality.

2. Hyperspaces from dual spaces

2.1. Hypersets and hypermappings

($\mathcal{X}^*$): We denote by $\mathcal{X}$ a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. We define $\mathcal{O} \subseteq 2^{\mathcal{X}}$ to be the set of all 0-neighborhoods $\mathcal{U} \subseteq \mathcal{X}$. Note that the definitions of topological vector spaces found in the literature differ slightly from each other. Those differences mostly concern the Hausdorff property. Here, we use Rudin's definition [35, Section 1.6]. In this case, the space $\mathcal{X}$ is Hausdorff, by [35, Theorem 1.12].

Let $\mathcal{X}^*$ be the (topological) dual space of $\mathcal{X}$. To avoid peculiar cases, we assume throughout the paper that $\mathcal{X}^*$ *separates points on $\mathcal{X}$* . This is for instance the case for any (Hausdorff) locally convex space $\mathcal{X}$. See also [35, Exercice 5 of Chap. 3] for other examples. As explained in [35, Sections 3.8, 3.10, 3.14], recall that the $\sigma(\mathcal{X}^*, \mathcal{X})$ -topology – called the weak* topology of $\mathcal{X}^*$ – is the initial topology of the family $\{\hat{A}\}_{A \in \mathcal{X}}$ of linear mappings from $\mathcal{X}^*$ to $\mathbb{K}$, defined by

$$\hat{A}(\sigma) \doteq \sigma(A) \ , \quad \sigma \in \mathcal{X}^*, \ A \in \mathcal{X} \ . \quad (1)$$

It is, by definition, the coarsest topology on $\mathcal{X}^*$ that makes the mapping $\hat{A}$ continuous for every $A \in \mathcal{X}$. See [35, Section 3.8]. Here, the topology of $\mathcal{X}^*$ is, by default, the weak* topology. In this case, $\mathcal{X}^*$ is a (Hausdorff) locally convex space and its (topological) dual space is $\mathcal{X}$: Any element of $\mathcal{X}^{**} \equiv \mathcal{X}$ is of the form (1). See [35, Theorem 3.10]. By [35, Theorem 1.37 and Remark 1.38 (b)], the collection of all finite intersections of the sets

$$\mathcal{V}_A \doteq \{\sigma \in \mathcal{X}^* : |\sigma(A)| < 1\} \ , \quad A \in \mathcal{X}, \quad (2)$$

is a convex balanced local base of neighborhoods at $0 \in \mathcal{X}^*$ for the weak* topology. In the sequel we use the notation

$$\mathcal{V}_{\mathcal{A}_n} \doteq \bigcap_{j=1}^n \mathcal{V}_{A_j} \ , \quad n \in \mathbb{N}, \ \mathcal{A}_n \doteq (A_j)_{j=1}^n \subseteq \mathcal{X} \ , \quad (3)$$

and
$$\mathcal{C} \doteq \left\{ \mathcal{V}_{\mathcal{A}_n} : n \in \mathbb{N}, \ \mathcal{A}_n \doteq (A_j)_{j=1}^n \subseteq \mathcal{X} \right\} \quad (4)$$

for the set of cylinder sets constructed from finite sequences of $\mathcal{X}$.

(F): As is usual in the theory of hyperspaces [7], we study the set of all nonempty closed subsets of $\mathcal{X}^*$ denoted by

$$\mathbf{F}(\mathcal{X}^*) \doteq \{F \subseteq \mathcal{X}^* : F \neq \emptyset \text{ is weak* -closed}\} \ . \quad (5)$$

Similarly, $\mathbf{F}(\mathbb{K})$ is the set of nonempty closed (in the sense of the absolute value) subsets of the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The complement of any subset $\mathbf{F}_0 \subseteq \mathbf{F}(\mathcal{X}^*)$ is denoted

$$\mathbf{F}_0^c \doteq \mathbf{F}(\mathcal{X}^*) \setminus \mathbf{F}_0 . \quad (6)$$

The subset of all nonempty convex weak*-closed subsets of $\mathcal{X}^*$ is defined by

$$\mathbf{CF}(\mathcal{X}^*) \doteq \{F \in \mathbf{F}(\mathcal{X}^*) : F \text{ is convex}\} \subseteq \mathbf{F}(\mathcal{X}^*) . \quad (7)$$

There is a natural mapping from $\mathbf{F}(\mathcal{X}^*)$ to $\mathbf{CF}(\mathcal{X}^*)$: the *weak*-closed convex hull operator*, $\overline{\text{co}} : \mathbf{F}(\mathcal{X}^*) \rightarrow \mathbf{F}(\mathcal{X}^*)$, defined by

$$\overline{\text{co}}(F) \doteq \overline{\text{co}F} , \quad F \in \mathbf{F}(\mathcal{X}^*) , \quad (8)$$

where $\overline{\text{co}F}$ is the weak*-closure of the convex hull of F or, equivalently, the intersection of all weak*-closed convex subsets of $\mathcal{X}^*$ containing F . It is a *closure* (or *hull*) operator [12, Definition 5.1] since it satisfies the following properties:

- For any $F \in \mathbf{F}(\mathcal{X}^*)$, $F \subseteq \overline{\text{co}}(F)$ (extensive);
- For any $F \in \mathbf{F}(\mathcal{X}^*)$, $\overline{\text{co}}(\overline{\text{co}}(F)) = \overline{\text{co}}(F)$ (idempotent);
- For any $F_1, F_2 \in \mathbf{F}(\mathcal{X}^*)$ such that $F_1 \subseteq F_2$, $\overline{\text{co}}(F_1) \subseteq \overline{\text{co}}(F_2)$ (isotone).

Such a closure operator has certainly been used in the past. It is a composition of (i) an *algebraic* (or finitary) closure operator [12, Definition 5.4] defined by $F \mapsto \text{co}F$ with (ii) a *topological* (or Kuratowski) closure operator [24, Chapter 1, p. 43] defined by $F \mapsto \overline{F}$ on $\mathbf{F}(\mathcal{X}^*)$.

As is usual, weak*-closed subsets $F \in \mathbf{F}(\mathcal{X}^*)$ satisfying $F = \overline{\text{co}}(F)$ are, by definition, *$\overline{\text{co}}$ -closed* sets and, obviously,

$$\mathbf{CF}(\mathcal{X}^*) = \overline{\text{co}}(\mathbf{F}(\mathcal{X}^*)) \quad (9)$$

is the set of all $\overline{\text{co}}$ -closed sets. Below, the weak*-closed convex hull operator naturally appears in the study of the Hausdorff property associated with the weak*-Hausdorff hypertopology (Definition 2.3).

(B): For any cylinder $\mathcal{V} \in \mathcal{C}$, define the set of all nonempty, “ $\mathcal{V}$ -bounded”, weak*-closed subsets of $\mathcal{X}^*$ by

$$\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*) \doteq \{B \in \mathbf{F}(\mathcal{X}^*) : B \subseteq \lambda \mathcal{V} \text{ for some } \lambda \in \mathbb{R}^+\} \subseteq \mathbf{F}(\mathcal{X}^*) . \quad (10)$$

Clearly, $\mathbf{B}_{\mathcal{X}^*}(\mathcal{X}^*) = \mathbf{F}(\mathcal{X}^*)$ and

$$\mathbf{B}_{\mathcal{V}_1 \cap \mathcal{V}_2}(\mathcal{X}^*) = \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*) \cap \mathbf{B}_{\mathcal{V}_2}(\mathcal{X}^*) , \quad \mathcal{V}_1, \mathcal{V}_2 \in \mathcal{C} . \quad (11)$$

Then, the (nonempty) set of all nonempty (topologically) bounded weak*-closed subsets of $\mathcal{X}^*$ is defined by

$$\mathbf{B}(\mathcal{X}^*) \doteq \bigcap_{\mathcal{V} \in \mathcal{C}} \mathbf{B}_{\mathcal{V}}(\mathcal{X}^*) \subseteq \mathbf{F}(\mathcal{X}^*) . \quad (12)$$

See, e.g., [35, Section 1.6 and Theorem 1.37 (b)]. Let

$$\mathbf{CB}(\mathcal{X}^*) \doteq \mathbf{CF}(\mathcal{X}^*) \cap \mathbf{B}(\mathcal{X}^*) = \overline{\text{co}}(\mathbf{B}(\mathcal{X}^*)) , \quad (13)$$

where the last equality follows from the fact that the weak*-closure of the convex hull of a weak*-closed bounded set is bounded, by the triangle inequality.

If $\mathcal{X}$ is a Banach space, then $\mathbf{B}(\mathcal{X}^*)$ is nothing else than the set of all nonempty norm-bounded weak*-closed subsets of $\mathcal{X}^*$, by the uniform boundedness principle [35, Theorems 2.4 and 2.5].

(K): The set of all nonempty weak*-compact subsets of $\mathcal{X}^*$ is denoted by

$$\mathbf{K}(\mathcal{X}^*) \doteq \{K \in \mathbf{F}(\mathcal{X}^*) : K \text{ is weak}^*\text{-compact}\} \subseteq \mathbf{F}(\mathcal{X}^*) . \quad (14)$$

(Recall that in a Hausdorff space every compact set is closed.) By [35, Theorem 1.15 (b)], weak*-compact subsets of $\mathcal{X}^*$ are bounded:

$$\mathbf{K}(\mathcal{X}^*) \subseteq \mathbf{B}(\mathcal{X}^*) .$$

Since any convex weak*-compact set K satisfies $\overline{\text{co}}(K) = K$, note that

$$\mathbf{CK}(\mathcal{X}^*) \doteq \mathbf{CF}(\mathcal{X}^*) \cap \mathbf{K}(\mathcal{X}^*) \subseteq \overline{\text{co}}(\mathbf{K}(\mathcal{X}^*)) \subseteq \mathbf{CB}(\mathcal{X}^*) . \quad (15)$$

In general, as discussed for instance in [35, Section 3.18], the equality

$$\mathbf{CK}(\mathcal{X}^*) = \overline{\text{co}}(\mathbf{K}(\mathcal{X}^*))$$

does *not* hold true: For instance, take as topological $\mathbb{R}$ -vector space the space $\mathcal{X} \subseteq \mathbb{R}^{\mathbb{N}}$ of sequences $A \equiv (A_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$ that eventually vanishes, along with the supremum norm. For any $m \in \mathbb{N}$, define the continuous linear functionals $\sigma_m \in \mathcal{X}^*$ by

$$\sigma_m(A) = 2^m A_m , \quad A \equiv (A_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}} .$$

As $m \rightarrow \infty$, σ_m converges in the weak* topology to zero. In particular, the set

$$K = \{\sigma_m : m \in \mathbb{N}\} \cup \{0\} \subseteq \mathcal{X}^*$$

is weak*-compact, i.e., $K \in \mathbf{K}(\mathcal{X}^*)$, but, by [35, Theorem 2.9] with $\Gamma = \{\hat{\delta}_m\}_{m \in \mathbb{N}} \subseteq \mathcal{X}^{**}$, where $\hat{\delta}_m(\sigma) = \sigma(\delta_m)$ and $\delta_m \equiv (\delta_{m,n})_{n \in \mathbb{N}} \in \mathcal{X}$ is defined by $\delta_{m,m} = 1$ and 0 otherwise, one deduces that $\overline{\text{co}}K \notin \mathbf{K}(\mathcal{X}^*)$. This example is taken from [25, Chapter II, Section 10].

Observe that the weak*-closed convex hull operator $\overline{\text{co}}$ yields a notion of compactness, defined as follows: A set $K \in \mathbf{F}(\mathcal{X}^*)$ is $\overline{\text{co}}$ -compact if it is $\overline{\text{co}}$ -closed and each family of $\overline{\text{co}}$ -closed subsets of K which has the finite intersection property has a non-empty intersection. Compare this definition with [24, Chapter 5, Theorem 1]. The set $\mathbf{CK}(\mathcal{X}^*)$ of all nonempty convex weak*-compact sets belongs to the set

$$\mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*) \doteq \{K \in \mathbf{F}(\mathcal{X}^*) : K \text{ is } \overline{\text{co}}\text{-compact}\}$$

of $\overline{\text{co}}$ -compact sets.

Proposition 2.1. (Space of $\overline{\text{co}}$ -compact sets)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then,

$$\mathbf{CK}(\mathcal{X}^*) \subseteq \mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*) \subseteq \mathbf{CB}(\mathcal{X}^*) .$$

Proof. By [24, Chapter 5, Theorem 1], $\mathbf{CK}(\mathcal{X}^*) \subseteq \mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*)$. Now, take any $\overline{\text{co}}$ -compact element $K \in \mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*)$. If K is not bounded, then there is $A \in \mathcal{X}$ such that $\hat{A}(K) \subseteq \mathbb{K}$ is not bounded, where we recall that $\hat{A} : \mathcal{X}^* \rightarrow \mathbb{K}$ is the weak*-continuous

linear functional defined by (1). Without loss of generality, assume that $\operatorname{Re}\{\hat{A}(K)\}$ is not bounded from above. Define, for every $n \in \mathbb{N}$, the set

$$K_n \doteq \left\{ \sigma \in K : \operatorname{Re}\{\hat{A}(\sigma)\} \geq n \right\} .$$

Clearly, by convexity of K , K_n is a convex weak*-closed subset of K and the family $(K_n)_{n \in \mathbb{N}}$ has the finite intersection property, but, by construction,

$$\bigcap_{n \in \mathbb{N}} K_n = \emptyset .$$

(The intersection of preimages is the preimage of the intersection.) This contradicts the fact that K is $\overline{\text{co}}$ -compact. Therefore, $K \in \mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*)$ is bounded and, being $\overline{\text{co}}$ -compact, it is also convex. Consequently, $\mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*) \subseteq \mathbf{CB}(\mathcal{X}^*)$. $\square$

Unless $\mathcal{X}$ is a Banach space (see (22)), we do not expect the equality $\mathbf{CK}(\mathcal{X}^*) = \mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*)$ to hold true.

(U): Recall that the closed convex hull of a compact set of an infinite-dimensional topological space need not be compact and so, the set $\mathbf{K}(\mathcal{X}^*)$ of all nonempty weak*-compact subsets of $\mathcal{X}^*$ is generally not invariant under the weak*-closed convex hull operator $\overline{\text{co}}$, as explicitly demonstrated after Equation (15). This motivates the introduction of a specific, albeit still large, class of weak*-compact sets, in relation with *absolute polars* of 0-neighborhoods in $\mathcal{X}$: the absolute polar $\mathcal{U}^\circ$ of any $\mathcal{U} \in \mathcal{O}$ is defined by

$$\mathcal{U}^\circ \doteq \{ \sigma \in \mathcal{X}^* : |\sigma(A)| \leq 1 \text{ for every } A \in \mathcal{U} \} \in \mathbf{CF}(\mathcal{X}^*) , \quad (16)$$

where we recall that $\mathcal{O} \subseteq 2^{\mathcal{X}}$ is the set of all (open) 0-neighborhoods. As is well-known, it is a nonempty (balanced) convex weak*-closed set containing $0 \in \mathcal{X}^*$, but, what's more, a weak*-compact set, by the Banach-Alaoglu theorem [35, Theorem 3.15]. For any $\mathcal{U} \in \mathcal{O}$, let

$$\mathbf{U}_{\mathcal{U}}(\mathcal{X}^*) \doteq \{ U \in \mathbf{F}(\mathcal{X}^*) : U \subseteq \mathcal{U}^\circ \} \subseteq \mathbf{K}(\mathcal{X}^*) . \quad (17)$$

Clearly, one has $\mathbf{U}_{\mathcal{U}_1}(\mathcal{X}^*) \cup \mathbf{U}_{\mathcal{U}_2}(\mathcal{X}^*) \subseteq \mathbf{U}_{\mathcal{U}_1 \cap \mathcal{U}_2}(\mathcal{X}^*)$, $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{O}$.

We next denote by

$$\mathbf{U}(\mathcal{X}^*) \doteq \bigcup_{\mathcal{U} \in \mathcal{O}} \mathbf{U}_{\mathcal{U}}(\mathcal{X}^*) \subseteq \mathbf{K}(\mathcal{X}^*) \subseteq \mathbf{B}(\mathcal{X}^*) \quad (18)$$

the set of all nonempty, uniformly bounded in a 0-neighborhood, weak*-closed subsets of $\mathcal{X}^*$.

By the triangle inequality and the closure operator property of $\overline{\text{co}}$, together with the convexity and weak*-closedness of absolute polars (16), observe that the weak*-closed convex hull operator $\overline{\text{co}}$ preserves $\mathbf{U}_{\mathcal{U}}(\mathcal{X}^*)$ for every $\mathcal{U} \in \mathcal{O}$:

$$\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) \doteq \mathbf{CF}(\mathcal{X}^*) \cap \mathbf{U}_{\mathcal{U}}(\mathcal{X}^*) = \overline{\text{co}}(\mathbf{U}_{\mathcal{U}}(\mathcal{X}^*)) \subseteq \mathbf{CB}(\mathcal{X}^*) . \quad (19)$$

Hence,

$$\mathbf{CU}(\mathcal{X}^*) \doteq \mathbf{CF}(\mathcal{X}^*) \cap \mathbf{U}(\mathcal{X}^*) = \overline{\text{co}}(\mathbf{U}(\mathcal{X}^*)) \subseteq \mathbf{CK}(\mathcal{X}^*) \subseteq \mathbf{CB}(\mathcal{X}^*) . \quad (20)$$

Compare with (15).

If $\mathcal{X}$ is a Banach space then $\mathbf{B}(\mathcal{X}^*)$ equals the set of all nonempty norm-bounded weak*-closed subsets of $\mathcal{X}^*$, because, in this case, a subset of $\mathcal{X}^*$ is norm-bounded and weak*-closed iff it is weak*-compact, as is well-known. See, e.g., [7, Proposition 1.2.9]. This fact is a consequence of the uniform boundedness principle [35, Theorems 2.4 and 2.5] and the Banach-Alaoglu theorem [35, Theorem 3.15], since the absolute polar of a *norm*-closed ball of radius D in $\mathcal{X}$ is a *norm*-closed ball of radius D^{-1} in $\mathcal{X}^*$. In particular, in this situation, absolute polars can be replaced with norm-closed balls in $\mathcal{X}^*$. In fact, if $\mathcal{X}$ is a Banach space then

$$\mathbf{U}(\mathcal{X}^*) = \mathbf{K}(\mathcal{X}^*) = \mathbf{B}(\mathcal{X}^*) \quad (21)$$

is nothing else than the set of all nonempty norm-bounded weak*-closed subsets of $\mathcal{X}^*$ and, using the weak*-closed convex hull operator $\overline{\text{co}}$, we deduce that

$$\mathbf{CU}(\mathcal{X}^*) = \mathbf{CK}(\mathcal{X}^*) = \overline{\text{co}}(\mathbf{K}(\mathcal{X}^*)) = \mathbf{K}_{\overline{\text{co}}}(\mathcal{X}^*) = \mathbf{CB}(\mathcal{X}^*) , \quad (22)$$

by (13), (15), (20) and Proposition 2.1. Thus, if $\mathcal{X}$ is a Banach space, Proposition 2.1 gives an elegant abstract characterization of $\mathbf{CK}(\mathcal{X}^*)$, only expressed in terms of a closure operator, namely the weak*-closed convex hull operator. It demonstrates a first connection with other mathematical fields, in particular with mathematical logics where fascinating applications of closure operators have been developed, already by Tarski himself during the 1930's.

2.2. Weak* hypertopologies

All hypersets (i.e., sets of closed sets) in Section 2.1 can be endowed with hypertopologies. An *hypertopology* is a topology such that any net $(\sigma_j)_{j \in J}$ in the primordial space (here the dual space $\mathcal{X}^*$ or the field $\mathbb{K} = \mathbb{R}, \mathbb{C}$) converges to an element σ iff the net $(\{\sigma_j\})_{j \in J}$ converges to $\{\sigma\}$ in the corresponding hyperspace (here $\mathbf{F}(\mathcal{X}^*)$ or $\mathbf{F}(\mathbb{K})$). Recall that there are various standard hypertopologies on general sets of nonempty closed subsets of a complete metric space $(\mathcal{Y}, d)$: the Fell, Vietoris, Wijsman, proximal or locally finite hypertopologies, to name a few well-known examples. See, e.g., [7].

None of these well-known hypertopologies is used here for $\mathbf{F}(\mathcal{X}^*)$, for the weak* topology of $\mathcal{X}^*$ is generally not metrizable. However, all these hypertopologies associated with a complete metric space can be used to define, in a systematic and very natural way that is similar to the weak* topology of $\mathcal{X}^*$, a *new* class of hypertopologies on the set $\mathbf{F}(\mathcal{X}^*)$ of all nonempty weak*-closed subsets of $\mathcal{X}^*$: At any fixed $A \in \mathcal{X}$, we define the mapping $\mathbf{A} : \mathbf{F}(\mathcal{X}^*) \rightarrow \mathbf{F}(\mathbb{K})$ by

$$\mathbf{A}(F) \doteq \widehat{A}(F) \doteq \overline{\{\sigma(A) : \sigma \in F\}} , \quad F \in \mathbf{F}(\mathcal{X}^*) . \quad (23)$$

See Equation (1). For a fixed hypertopology on $\mathbf{F}(\mathbb{K})$, the associated weak* hypertopology of $\mathbf{F}(\mathcal{X}^*)$ is the coarsest topology on $\mathbf{F}(\mathcal{X}^*)$ that makes the mapping $\mathbf{A}$ continuous for every $A \in \mathcal{X}$:

Definition 2.2. (Weak* hypertopologies)

Pick some hypertopology on $\mathbf{F}(\mathbb{K})$. The associated *weak* hypertopology* τ on $\mathbf{F}(\mathcal{X}^*)$ is the initial topology of the family $\{\mathbf{A}\}_{A \in \mathcal{X}}$ of mappings from $\mathbf{F}(\mathcal{X}^*)$ to $\mathbf{F}(\mathbb{K})$ defined by (23). That is, τ is the collection of all unions of finite intersections of sets $\mathbf{A}^{-1}(V)$ with $A \in \mathcal{X}$ and V open in $\mathbf{F}(\mathbb{K})$.

By construction, τ is obviously an hypertopology on $\mathbf{F}(\mathcal{X}^*)$, keeping in mind that the topology on $\mathcal{X}^*$ is, by default, the weak* topology, i.e., the initial topology of the family $\{\hat{A}\}_{A \in \mathcal{X}}$ of linear mappings from $\mathcal{X}^*$ to $\mathbb{K}$ defined by (1).

The Fell, Vietoris, Wijsman, proximal or locally finite hypertopologies, the Hausdorff metric topology, etc., lead to various weak* hypertopologies. They belong to a new class of hypertopologies, that is, the class of weak* hypertopologies, which does not seem to have been systematically considered in the past. Note however that the scalar topology described in [7, Section 4.3], when defined, *mutatis mutandis*, for a locally convex space like $\mathcal{X}^*$, is retrospectively an example⁷ of a weak* hypertopology.

In the case that the hypertopology on $\mathbf{F}(\mathbb{K})$ is metrizable, the corresponding weak* hypertopology on $\mathbf{F}(\mathcal{X}^*)$ is the topology generated by the family of pseudometrics:

$$d^{(A)}(F_1, F_2) \doteq \mathfrak{d}(\mathbf{A}(F_1), \mathbf{A}(F_2)) \quad , \quad F_1, F_2 \in \mathbf{F}(\mathcal{X}^*), \quad A \in \mathcal{X} \quad , \quad (24)$$

where

$$\mathfrak{d} : \mathbf{F}(\mathbb{K}) \times \mathbf{F}(\mathbb{K}) \rightarrow \mathbb{R}_0^+ \cup \{\infty\} \quad (25)$$

is the metric in $\mathbf{F}(\mathbb{K})$ generating its hypertopology. (Recall also (23).) In this case, the weak* hypertopology is a uniform topology, see, e.g., [24, Chapter 6]. It is the coarsest topology on $\mathbf{F}(\mathcal{X}^*)$ that makes every mapping $\mathbf{A}$, defined by (23) for $A \in \mathcal{X}$, continuous. Equivalently, in this topology, an arbitrary net $(F_j)_{j \in J} \subseteq \mathbf{F}(\mathcal{X}^*)$ converges to F iff, for all $A \in \mathcal{X}$,

$$\lim_J d^{(A)}(F_j, F) = 0 \quad .$$

This condition defines a unique topology in $\mathbf{F}(\mathcal{X}^*)$, by [24, Chapter 2, Theorem 9]. In the context of hypertopologies, note that it is natural to consider metrics $\mathfrak{d}$ taking values in the extended positive reals $\mathbb{R}_0^+ \cup \{\infty\}$, like (26) below.

The most well-studied and well-known hypertopology associated with a metric space $(\mathcal{Y}, d)$ is the Hausdorff metric topology [7, Definition 3.2.1]. It is generated by the Hausdorff distance between two sets $Y_1, Y_2 \subseteq \mathcal{Y}$:

$$\mathfrak{d}_H(Y_1, Y_2) \doteq \max \left\{ \sup_{x_1 \in Y_1} \inf_{x_2 \in Y_2} d(x_1, x_2), \sup_{x_2 \in Y_2} \inf_{x_1 \in Y_1} d(x_1, x_2) \right\} \in \mathbb{R}_0^+ \cup \{\infty\} \quad . \quad (26)$$

In this case, the corresponding hyperspace of nonempty closed subsets of $\mathcal{Y}$ is complete iff the metric space $(\mathcal{Y}, d)$ is complete. See, e.g., [7, Theorem 3.2.4]. The Hausdorff metric topology is the hypertopology used in [20, 27], the metric d being the one associated with the norm of a separable Banach space $\mathcal{Y}$, in order to prove the density of the set of convex compact subsets of $\mathcal{Y}$ with dense extreme boundary. As a first and instructive example of a weak* hypertopology, it is thus natural to study the weak* version of the Hausdorff metric topology. It corresponds to the weak* hypertopology of Definition 2.2 with $\mathbf{F}(\mathbb{K})$ endowed with the Hausdorff metric topology:

⁷ And the unique one we are aware of.

Definition 2.3. (Weak*-Hausdorff hypertopology)

The *weak*-Hausdorff hypertopology* on $\mathbf{F}(\mathcal{X}^*)$ is the topology induced by the family of Hausdorff pseudometrics $d_H^{(A)}$ defined, for all $A \in \mathcal{X}$, by

$$d_H^{(A)}(F, \tilde{F}) = \max \left\{ \sup_{\sigma \in F} \inf_{\tilde{\sigma} \in \tilde{F}} |(\sigma - \tilde{\sigma})(A)|, \sup_{\tilde{\sigma} \in \tilde{F}} \inf_{\sigma \in F} |(\sigma - \tilde{\sigma})(A)| \right\} \in \mathbb{R}_0^+ \cup \{\infty\}, \quad (27)$$

where $F, \tilde{F} \in \mathbf{F}(\mathcal{X}^*)$.

Note that, for any $F, \tilde{F} \in \mathbf{F}(\mathcal{X}^*)$, we have

$$\begin{aligned} \sup_{\sigma \in F} \inf_{\tilde{\sigma} \in \tilde{F}} |(\sigma - \tilde{\sigma})(A)| &= \sup_{x \in \hat{A}(F)} \inf_{\tilde{x} \in \hat{A}(\tilde{F})} |x - \tilde{x}| = \sup_{x \in \hat{A}(F)} \inf_{\tilde{x} \in \mathbf{A}(\tilde{F})} |x - \tilde{x}| \\ &= \sup_{x \in \mathbf{A}(F)} \inf_{\tilde{x} \in \mathbf{A}(\tilde{F})} |x - \tilde{x}|, \end{aligned}$$

by the triangle inequality for the absolute value. (See, e.g., the arguments justifying (41).)

Definition 2.3 is equivalent to Definition 2.2 with $\mathbf{F}(\mathbb{K})$ endowed with the Hausdorff metric topology, as explained above for the more general case where $\mathbf{F}(\mathbb{K})$ is metrizable. To our knowledge, this hypertopology has not been considered so far.

Here, $\mathbf{F}(\mathcal{X}^*)$ and the subspaces $\mathbf{B}(\mathcal{X}^*)$, $\mathbf{CB}(\mathcal{X}^*)$, $\mathbf{K}(\mathcal{X}^*)$, etc., are, by default, all endowed with the weak*-Hausdorff hypertopology.

3. The weak*-Hausdorff hypertopology

3.1. Boolean algebras associated with immeasurable hyperspaces

Observe that one can only ensure that (27) is finite only if $F, \tilde{F} \in \mathbf{B}(\mathcal{X}^*) \subseteq \mathbf{F}(\mathcal{X}^*)$. We show below that the weak*-Hausdorff family of pseudometric $(d_H^{(A)})_{A \in \mathcal{X}}$ *immeasurably* separates unbounded sets from bounded ones. This motivates the following definition:

Definition 3.1. (Immeasurable hypersets)

Assume the existence of a metric $\mathfrak{d}$ satisfying (25) and generating the hypertopology in $\mathbf{F}(\mathbb{K})$. Let τ be the corresponding weak* hypertopology on $\mathbf{F}(\mathcal{X}^*)$. Two subsets $\mathbf{F}_1, \mathbf{F}_2 \subseteq \mathbf{F}(\mathcal{X}^*)$ are said to be *$\mathfrak{d}$ -immeasurable* if, for any $F_1 \in \mathbf{F}_1$ and $F_2 \in \mathbf{F}_2$, there is $A \in \mathcal{X}$ such that

$$d^{(A)}(F_1, F_2) = \infty$$

with the pseudometrics $d^{(A)}$ defined by (24). $\mathfrak{d}_H$ -immeasurable sets are named here weak*-Hausdorff-immeasurable sets. See (26).

A generally infinite collection of weak*-Hausdorff-immeasurable subspaces of $\mathbf{F}(\mathcal{X}^*)$ is given by the subspaces $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ of all nonempty, $\mathcal{V}$ -bounded, weak*-closed subsets of $\mathcal{X}^*$, defined by (10) for each cylinder $\mathcal{V} \in \mathcal{C}$ (see (4)).

Proposition 3.2. (Pairs of immeasurable subhyperspaces)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space⁸ with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For all cylinders $\mathcal{V}_1, \mathcal{V}_2 \in \mathcal{C}$ such that $\mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*) \subsetneq \mathbf{B}_{\mathcal{V}_2}(\mathcal{X}^*)$, $\mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$ and $\mathbf{B}_{\mathcal{V}_2}(\mathcal{X}^*) \setminus \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$ are weak*-Hausdorff-immeasurable. In particular, $\mathbf{B}(\mathcal{X}^*)$ and its complement $\mathbf{B}^c(\mathcal{X}^*)$, defined by (6), are weak*-Hausdorff-immeasurable.

Proof. Take any $F \in \mathbf{B}_{\mathcal{V}_2}(\mathcal{X}^*) \setminus \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$, where $\mathcal{V}_1, \mathcal{V}_2$ are two cylinders of the set $\mathcal{C}$ defined by (4) such that $\mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*) \subsetneq \mathbf{B}_{\mathcal{V}_2}(\mathcal{X}^*)$. Then, there is a net $(\sigma_j)_{j \in J} \subseteq F$ and $A \in \mathcal{X}$ such that

$$\lim_J |\sigma_j(A)| = \infty \quad \text{and} \quad \sup_{\tilde{\sigma} \in B} |\tilde{\sigma}(A)| < \infty \quad (28)$$

for any $B \in \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$. By Definition 2.3 and the triangle inequality we have, for any $B \in \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$ and $j \in J$,

$$d_H^{(A)}(F, B) \geq \inf_{\tilde{\sigma} \in B} |(\sigma_j - \tilde{\sigma})(A)| \geq |\sigma_j(A)| - \sup_{\tilde{\sigma} \in B} |\tilde{\sigma}(A)|. \quad (29)$$

By (28), it follows that $d_H^{(A)}(F, B) = \infty$, where $B \in \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$.

In other words, $\mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$ and $\mathbf{B}_{\mathcal{V}_2}(\mathcal{X}^*) \setminus \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$ are weak*-Hausdorff-immeasurable.

If $F \in \mathbf{B}^c(\mathcal{X}^*)$ then there is a cylinder $\mathcal{V}_1 \neq \mathcal{X}^*$ such that $F \notin \mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$ while any bounded set $B \in \mathbf{B}(\mathcal{X}^*)$ belongs to $\mathbf{B}_{\mathcal{V}_1}(\mathcal{X}^*)$, by Equation (12). Using the previous arguments for the special case $\mathcal{V}_2 = \mathcal{X}^* \supseteq \mathcal{V}_1$, we thus deduce that $\mathbf{B}(\mathcal{X}^*)$ and its complement $\mathbf{B}^c(\mathcal{X}^*)$ are weak*-Hausdorff-immeasurable. $\square$

Observe that $\mathbf{B}_{\mathcal{X}^*}(\mathcal{X}^*) = \mathbf{F}(\mathcal{X}^*)$ and Proposition 3.2 applied to $\mathcal{V}_2 = \mathcal{X}^*$ yield that, for each cylinder $\mathcal{V} \in \mathcal{C} \setminus \{\mathcal{X}^*\}$ such that $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*) \subsetneq \mathbf{F}(\mathcal{X}^*)$, the (nonempty) subhyperspaces $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ and its complement

$$\mathbf{B}_{\mathcal{V}}^c(\mathcal{X}^*) \doteq \mathbf{F}(\mathcal{X}^*) \setminus \mathbf{B}_{\mathcal{V}}(\mathcal{X}^*) \quad (30)$$

are weak*-Hausdorff-immeasurable, like $\mathbf{B}(\mathcal{X}^*)$ and $\mathbf{B}^c(\mathcal{X}^*)$.

Additionally, the subspaces $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$, $\mathcal{V} \in \mathcal{C}$, form a family of weak*-Hausdorff clopen sets:

Proposition 3.3. (Weak*-Hausdorff-clopen subhyperspaces)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then,

$$\mathcal{Cl} \doteq \{\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*), \mathbf{B}_{\mathcal{V}}^c(\mathcal{X}^*)\}_{\mathcal{V} \in \mathcal{C}} \quad (31)$$

is a family of weak*-Hausdorff-closed subsets of $\mathbf{F}(\mathcal{X}^*)$. In other words, $\mathcal{Cl}$ is a family of (nonempty) weak*-Hausdorff-clopen⁹ subsets of $\mathbf{F}(\mathcal{X}^*)$.

Proof. Let $\mathcal{V} = \mathcal{V}_{\mathcal{A}_n}$ with $\mathcal{A}_n \doteq (A_k)_{k=1}^n \subseteq \mathcal{X}$, as defined by (2)–(3). Clearly, for any $B \in \mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$, the set

$$\mathcal{F} \doteq \left\{ F \in \mathbf{F}(\mathcal{X}^*) : \forall k \in \{1, \dots, n\}, d_H^{(A_k)}(F, B) < 1 \right\}$$

⁸ Recall that all topological $\mathbb{K}$ -vector spaces $\mathcal{X}$ in this paper are Hausdorff, by [35, Theorem 1.12].

⁹ I.e., they are both open and closed in the weak*-Hausdorff hypertopology.

is a weak*-Hausdorff neighborhood of B in $\mathbf{F}(\mathcal{X}^*)$. Additionally, by definition (27) of the Hausdorff pseudometric, $\mathcal{F} \subseteq \mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$. Therefore, $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ is a weak*-Hausdorff open set. Take now a net $(B_j)_{j \in J} \subseteq \mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ converging to $F \in \mathbf{F}(\mathcal{X}^*)$ in the weak*-Hausdorff topology. In particular, for some $j \in J$,

$$d_H^{(A_k)}(F, B_j) < \infty, \quad k \in \{1, \dots, n\}.$$

Again by (27), it follows that $F \in \mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$. Hence, $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ is a weak*-Hausdorff closed set. Note that a subset of a topological space is closed iff it contains all of its limit points, by [24, Chapter 1, Theorem 5], and limit points (also named cluster or accumulation points) of a set are precisely the limits of (convergent) nets whose elements are in this set, by [24, Chapter 2, Theorem 2]. $\square$

In particular, any subspace $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$, $\mathcal{V} \in \mathcal{C}$, has empty boundary¹⁰ and thus, for infinite-dimensional topological vector spaces $\mathcal{X}$, the topological hyperspace $\mathbf{F}(\mathcal{X}^*)$ has an infinite number of connected components in the weak*-Hausdorff hypertopology. This leads to a whole collection of weak*-Hausdorff-clopen sets, which could be used to form a Boolean algebra whose lattice operations are given by the union and intersection, as is usual in mathematical logics¹¹. Such a study has been performed in [6] for the hyperspace associated with a Boolean compact topological space and the Vietoris hypertopology.

Note that the hyperspace $\mathbf{B}(\mathcal{X}^*)$ of all nonempty bounded weak*-closed subsets of $\mathcal{X}^*$ is weak*-Hausdorff-closed, the intersection of closed sets being always closed, but it is generally not weak*-Hausdorff-open, even if $\mathbf{B}(\mathcal{X}^*)$ and its complement $\mathbf{B}^c(\mathcal{X}^*)$ are weak*-Hausdorff-immeasurable. Additionally, for any fixed cylinder $\mathcal{V} \in \mathcal{C}$, observe that $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ is generally not a connected hyperspace. By (11), the weak*-Hausdorff-clopen set $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ contains (possibly infinitely many) proper weak*-Hausdorff-clopen subsets, leading to many connected components. However, the infimum (with respect to inclusion) of the family $\{\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)\}_{\mathcal{V} \in \mathcal{C}}$, that is, $\mathbf{B}(\mathcal{X}^*)$, is connected:

Proposition 3.4. ($\mathbf{B}(\mathcal{X}^*)$ as connected subhyperspace)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, the weak-Hausdorff-closed set $\mathbf{B}(\mathcal{X}^*)$ is convex and path-connected. Moreover, it is a connected component¹² of $\mathbf{F}(\mathcal{X}^*)$.*

Proof. Take any $B_0, B_1 \in \mathbf{B}(\mathcal{X}^*)$. Define the mapping f from $[0, 1]$ to $\mathbf{B}(\mathcal{X}^*)$ by

$$f(\lambda) \doteq \{(1 - \lambda)\sigma_0 + \lambda\sigma_1 : \sigma_0 \in B_0, \sigma_1 \in B_1\}, \quad \lambda \in [0, 1]. \quad (32)$$

(This already demonstrates that $\mathbf{B}(\mathcal{X}^*)$ is convex.) By Definition 2.3, for any $\lambda_1, \lambda_2 \in [0, 1]$,

$$d_H^{(A)}(f(\lambda_1), f(\lambda_2)) \leq |\lambda_2 - \lambda_1| \sup_{\sigma \in (B_0 - B_1)} |\sigma(A)|, \quad A \in \mathcal{X}.$$

Note that, for all $B_0, B_1 \in \mathbf{B}(\mathcal{X}^*)$ and $A \in \mathcal{X}$, $\sup_{\sigma \in (B_0 - B_1)} |\sigma(A)| = \infty$.

¹⁰ I.e., there is no element which is interior to neither $\mathbf{B}_{\mathcal{V}}(\mathcal{X}^*)$ nor $\mathbf{B}_{\mathcal{V}}^c(\mathcal{X}^*)$.

¹¹ See Stone's representation theorem for Boolean algebras.

¹² That is, a maximal connected subset.

So, the mapping f is a continuous function from $[0, 1]$ to $\mathbf{B}(\mathcal{X}^*)$ with $f(0) = B_0$ and $f(1) = B_1$. Therefore, $\mathbf{B}(\mathcal{X}^*)$ is path-connected. The image under a continuous mapping of a connected set is connected and, by [24, Chapter 1, Theorem 21], $\mathbf{B}(\mathcal{X}^*)$, being path-connected, is connected. In particular, $\mathbf{B}(\mathcal{X}^*)$ belongs to the connected component of any element $B \in \mathbf{B}(\mathcal{X}^*)$, denoted by $\mathcal{B}$.

For any $B \in \mathbf{B}(\mathcal{X}^*)$, define

$$\tilde{\mathcal{B}} \doteq \bigcap \{ \mathcal{F} \subseteq \mathbf{F}(\mathcal{X}^*) : \mathcal{F} \text{ is weak}^*\text{-Hausdorff-clopen and } B \in \mathcal{F} \} ,$$

the so-called pseudocomponent of B . Since a clopen is never a proper subset of a connected component, $\mathcal{B} \subseteq \tilde{\mathcal{B}}$. By Proposition 3.3, note that

$$\tilde{\mathcal{B}} \subseteq \bigcap_{\nu \in \mathcal{C}} \mathbf{B}_\nu(\mathcal{X}^*) = \mathbf{B}(\mathcal{X}^*) \subseteq \mathcal{B} .$$

This means that $\mathcal{B} = \mathbf{B}(\mathcal{X}^*)$. □

Corollary 3.5. ($\mathbf{F}(\mathcal{X}^*)$ as non-locally connected hyperspace)

Let $\mathcal{X}$ be an infinite-dimensional topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then $\mathbf{F}(\mathcal{X}^)$ is not locally connected.*

Proof. If $\mathbf{B}(\mathcal{X}^*)$ is not weak*-Hausdorff-open then $\mathbf{F}(\mathcal{X}^*)$ is not locally connected: Assume by contradiction that $\mathbf{F}(\mathcal{X}^*)$ is locally connected. Then, because of [24, Chap. 1, Problem (S), (a), p. 61], any connected component of $\mathbf{F}(\mathcal{X}^*)$ is a weak*-Hausdorff-clopen subset. This is not possible if $\mathbf{B}(\mathcal{X}^*)$ is not weak*-Hausdorff-open, because $\mathbf{B}(\mathcal{X}^*)$ is a connected component, by Proposition 3.4.

So, it remains to prove that $\mathbf{B}(\mathcal{X}^*)$ is not weak*-Hausdorff-open when $\mathcal{X}$ is infinite-dimensional. To this end, assume by contradiction that $\mathbf{B}(\mathcal{X}^*)$ is weak*-Hausdorff-open. Then, for any $B \in \mathbf{B}(\mathcal{X}^*)$, there exists $(A_k)_{k=1}^n \subseteq \mathcal{X}$ and $\varepsilon \in \mathbb{R}^+$ such that

$$\left\{ F \in \mathbf{F}(\mathcal{X}^*) : \forall k \in \{1, \dots, n\}, d_H^{(A_k)}(F, B) < \varepsilon \right\} \subseteq \mathbf{B}(\mathcal{X}^*) . \quad (33)$$

Take any
$$\sigma \in \bigcap_{k=1}^n \ker(\hat{A}_k) \setminus \{0\} \quad (34)$$

with $\hat{A}$ being defined by (1) for any $A \in \mathcal{X}$. Such an element always exists because $\mathcal{X}^*$ is infinite-dimensional, if $\mathcal{X}$ is infinite-dimensional. By (34), for any $k \in \{1, \dots, n\}$,

$$d_H^{(A_k)}(\overline{B + \mathbb{R}\sigma}, B) = 0$$

and (33) yields $\overline{B + \mathbb{R}\sigma} \in \mathbf{B}(\mathcal{X}^*)$, which is clearly not possible since $\sigma \neq 0$. Consequently, $\mathbf{B}(\mathcal{X}^*)$ is not weak*-Hausdorff-open when $\mathcal{X}$ is infinite-dimensional. □

In contrast to $\mathbf{B}(\mathcal{X}^*)$, within the set $\mathbf{B}^c(\mathcal{X}^*)$ there are possibly many connected components of $\mathbf{F}(\mathcal{X}^*)$, as one can see from Proposition 3.3.

Finally, for any 0-neighborhood $\mathcal{U} \in \mathcal{O}$, note that the hyperspace $\mathbf{U}_{\mathcal{U}}(\mathcal{X}^*)$ of all nonempty, uniformly bounded in $\mathcal{U}$, weak*-closed subsets of $\mathcal{X}^*$ defined by (17) has topological properties that are similar to the set $\mathbf{B}(\mathcal{X}^*)$:

Lemma 3.6. (Hyperconvergence of uniformly bounded near 0 sets)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Take any weak*-Hausdorff convergent net $(U_j)_{j \in J} \subseteq \mathbf{U}(\mathcal{X}^*)$ with $U_j \in \mathbf{U}_{\mathcal{U}_j}(\mathcal{X}^*)$ and $\mathcal{U}_j \in \mathcal{O}$ for $j \in J$. If

$$\mathcal{U}_\infty \doteq \bigcap_{j \in J} \mathcal{U}_j \in \mathcal{O} \quad (35)$$

then $(U_j)_{j \in J}$ converges to $U_\infty \in \mathbf{U}_{\mathcal{U}_\infty}(\mathcal{X}^*) \subseteq \mathbf{U}(\mathcal{X}^*)$.

Proof. Take any weak*-Hausdorff convergent net $(U_j)_{j \in J} \subseteq \mathbf{U}(\mathcal{X}^*)$, as stated in the lemma. Assume that the limit $U_\infty \notin \mathbf{U}_{\mathcal{U}_\infty}(\mathcal{X}^*)$ with $\mathcal{U}_\infty \in \mathcal{O}$ defined by (35). Then, there is $\sigma_\infty \in U_\infty$ and $A \in \mathcal{U}_\infty$ such that $|\sigma_\infty(A)| > 1$ and so,

$$d_H^{(A)}(U_\infty, U_j) \geq \inf_{\tilde{\sigma} \in U_j} |(\sigma_\infty - \tilde{\sigma})(A)| \geq |\sigma_\infty(A)| - \sup_{\tilde{\sigma} \in U_j} |\tilde{\sigma}(A)| > 0,$$

by Definition 2.3 and the triangle inequality. But this contradicts the fact that $(U_j)_{j \in J}$ converges to U_∞ in the weak*-Hausdorff hypertopology. Therefore we have $U_\infty \in \mathbf{U}_{\mathcal{U}_\infty}(\mathcal{X}^*)$. $\square$

Corollary 3.7. (Weak*-Hausdorff-closed subhyperspaces $\mathbf{U}_{\mathcal{U}}(\mathcal{X}^*)$)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, for any $\mathcal{U} \in \mathcal{O}$, $\mathbf{U}_{\mathcal{U}}(\mathcal{X}^*)$ is a convex, path-connected, weak*-Hausdorff-closed subset of $\mathbf{U}(\mathcal{X}^*) \subseteq \mathbf{K}(\mathcal{X}^*)$.

Proof. Convexity is obvious and path connectedness is proven by using the function (32), observing that weak*-compact subsets of $\mathcal{X}^*$ are bounded, by [35, Theorem 1.15 (b)]. By Lemma 3.6, $\mathbf{U}_{\mathcal{U}}(\mathcal{X}^*)$ is weak*-Hausdorff-closed for any fixed $\mathcal{U} \in \mathcal{O}$. $\square$

3.2. Hausdorff property and convexity

One fundamental question one shall ask regarding the hyperspace $\mathbf{F}(\mathcal{X}^*)$ is whether it is a Hausdorff space, with respect to the weak*-Hausdorff hypertopology, or not. The answer is *negative* for real Banach spaces of dimension greater than 1, as demonstrated in the next lemma by using elements of the set $\mathbf{K}(\mathcal{X}^*)$ of all nonempty weak*-compact sets defined by (14):

Lemma 3.8. (Non-weak*-Hausdorff-separable sets)

Let $\mathcal{X}$ be a topological $\mathbb{R}$ -vector space. Let us take any convex weak*-compact set $K \in \mathbf{CK}(\mathcal{X}^*)$ with weak*-path-connected weak*-closed set $\mathcal{E}(K) \subseteq K$ of extreme points¹³. Then, $\mathcal{E}(K) \in \mathbf{K}(\mathcal{X}^*)$ and $d_H^{(A)}(K, \mathcal{E}(K)) = 0$ for any $A \in \mathcal{X}$.

Proof. Let $\mathcal{X}$ be a topological $\mathbb{R}$ -vector space. Recall that any $A \in \mathcal{X}$ defines a weak*-continuous linear functional $\hat{A} : \mathcal{X}^* \rightarrow \mathbb{R}$, by Equation (1). Observe next that

$$d_H^{(A)}(K, \mathcal{E}(K)) = \max \left\{ \max_{x_1 \in \hat{A}(K)} \min_{x_2 \in \hat{A}(\mathcal{E}(K))} |x_1 - x_2|, \max_{x_2 \in \hat{A}(\mathcal{E}(K))} \min_{x_1 \in \hat{A}(K)} |x_1 - x_2| \right\}. \quad (36)$$

We obviously have the inclusions

$$\hat{A}(\mathcal{E}(K)) \subseteq \hat{A}(K) \subseteq \left[\min \hat{A}(K), \max \hat{A}(K) \right]. \quad (37)$$

¹³ Cf. the Krein-Milman theorem [35, Theorem 3.23].

By the Bauer maximum principle [10, Lemma 10.31] together with the affinity and weak*-continuity of $\hat{A}$,

$$\min \hat{A}(K) = \min \hat{A}(\mathcal{E}(K)) \quad \text{and} \quad \max \hat{A}(K) = \max \hat{A}(\mathcal{E}(K)) .$$

In particular, we can rewrite (37) as

$$\hat{A}(\mathcal{E}(K)) \subseteq \hat{A}(K) \subseteq \left[\min \hat{A}(\mathcal{E}(K)), \max \hat{A}(\mathcal{E}(K)) \right] . \quad (38)$$

Since $\mathcal{E}(K)$ is, by assumption, path-connected in the weak* topology, there is a weak*-continuous path $\gamma : [0, 1] \rightarrow \mathcal{E}(K)$ from a minimizer to a maximizer of $\hat{A}$ in $\mathcal{E}(K)$. By weak*-continuity of $\hat{A}$, it follows that

$$\left[\min \hat{A}(\mathcal{E}(K)), \max \hat{A}(\mathcal{E}(K)) \right] = \hat{A} \circ \gamma([0, 1]) \subseteq \hat{A}(\mathcal{E}(K))$$

and we infer from (38) that

$$\hat{A}(\mathcal{E}(K)) = \hat{A}(K) = \left[\min \hat{A}(K), \max \hat{A}(K) \right] = \left[\min \hat{A}(\mathcal{E}(K)), \max \hat{A}(\mathcal{E}(K)) \right] .$$

Together with (36), this last equality obviously leads to the assertion. Note that $\mathcal{E}(K) \in \mathbf{K}(\mathcal{X}^*)$ since it is, by assumption, a weak*-closed subset of the weak*-compact set K . $\square$

Corollary 3.9. (Non-Hausdorff hyperspaces)

Let $\mathcal{X}$ be a topological $\mathbb{R}$ -vector space of dimension greater than 1. Then, $\mathbf{F}(\mathcal{X}^)$, $\mathbf{B}(\mathcal{X}^*)$, $\mathbf{K}(\mathcal{X}^*)$ and $\mathbf{U}(\mathcal{X}^*)$ are all non-Hausdorff spaces.*

Proof. This corollary is a direct consequence of Lemma 3.8 by observing that the dual space of a topological $\mathbb{R}$ -vector space of dimension greater than 1 contains a two-dimensional closed disc. $\square$

In fact, the weak*-Hausdorff hypertopology cannot distinguish a set from its weak*-closed convex hull, as it also appears for other well-established hypertopologies, like the so-called scalar topology for closed sets (see [7, Section 4.3]). Similar to the scalar topology, only $\mathbf{CF}(\mathcal{X}^*)$ is a Hausdorff hyperspace. To get an intuition of this, consider the following result:

Proposition 3.10. (Separation of the weak*-closed convex hull)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Take $F_1, F_2 \in \mathbf{F}(\mathcal{X}^)$. If $d_H^{(A)}(F_1, F_2) = 0$ for all $A \in \mathcal{X}$, then $\overline{\text{co}}F_1 = \overline{\text{co}}F_2$, where $\overline{\text{co}}$ is the weak*-closed convex hull operator defined by (8).*

Proof. Pick any weak*-closed sets F_1, F_2 satisfying $d_H^{(A)}(F_1, F_2) = 0$ for all $A \in \mathcal{X}$. Let $\sigma_1 \in F_1$. By Definition 2.3, it follows that

$$\inf_{\sigma_2 \in F_2} |(\sigma_1 - \sigma_2)(A)| = 0 , \quad A \in \mathcal{X} . \quad (39)$$

Recall that the dual space $\mathcal{X}^*$ of $\mathcal{X}$ is a locally convex (Hausdorff) space in the weak* topology and its dual space is $\mathcal{X}$. By (8), $\overline{\text{co}}F_2$ is convex and weak*-closed and $\{\sigma_1\}$

is a convex weak*-compact set. If $\sigma_1 \notin \overline{\text{co}}F_2$ then we infer from the Hahn-Banach separation theorem [35, Theorem 3.4 (b)] the existence of $A_0 \in \mathcal{X}$ and $x_1, x_2 \in \mathbb{R}$ such that

$$\sup_{\sigma_2 \in \overline{\text{co}}F_2} \text{Re} \{ \sigma_2 (A_0) \} < x_1 < x_2 < \text{Re} \{ \sigma_1 (A_0) \} , \quad (40)$$

which contradicts (39) for $A = A_0$. In consequence we have $\sigma_1 \in \overline{\text{co}}F_2$ and hence, $F_1 \subseteq \overline{\text{co}}F_2$. This in turn yields $\overline{\text{co}}F_1 \subseteq \overline{\text{co}}F_2$. By switching the role of the weak*-closed sets, we thus deduce the assertion. $\square$

Corollary 3.11. ($\mathbf{CF}(\mathcal{X}^*)$ as an Hausdorff hyperspace)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, $\mathbf{CF}(\mathcal{X}^*)$ is a Hausdorff hyperspace.

Proof. This is a direct consequence of Proposition 3.10. $\square$

Note that the weak*-closed convex hull operator $\overline{\text{co}}$ is a weak*-Hausdorff continuous mapping:

Proposition 3.12. (Continuity of the weak*-closed convex hull operator)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, $\overline{\text{co}}$ is a weak*-Hausdorff continuous mapping from $\mathbf{F}(\mathcal{X}^*)$ onto $\mathbf{CF}(\mathcal{X}^*)$.

Proof. Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The surjectivity of $\overline{\text{co}}$ seen as a mapping from $\mathbf{F}(\mathcal{X}^*)$ to $\mathbf{CF}(\mathcal{X}^*)$ is obvious, by (9). Now, take any weak*-Hausdorff convergent net $(F_j)_{j \in J} \subseteq \mathbf{F}(\mathcal{X}^*)$ with limit $F_\infty \in \mathbf{F}(\mathcal{X}^*)$. Note that

$$\sup_{\sigma \in \overline{\text{co}}(F_\infty)} \inf_{\tilde{\sigma} \in \overline{\text{co}}(F_j)} |(\sigma - \tilde{\sigma})(A)| = \sup_{\sigma \in \text{co}(F_\infty)} \inf_{\tilde{\sigma} \in \overline{\text{co}}(F_j)} |(\sigma - \tilde{\sigma})(A)| , \quad A \in \mathcal{X} , \quad (41)$$

because, for any $A \in \mathcal{X}$, $j \in J$, $\sigma_1, \sigma_2 \in \overline{\text{co}}(F_\infty)$ and $\tilde{\sigma} \in \overline{\text{co}}(F_j)$,

$$|(\sigma_1 - \tilde{\sigma})(A)| - |(\sigma_2 - \tilde{\sigma})(A)| \leq |(\sigma_1 - \sigma_2)(A)| ,$$

which yields

$$\left| \inf_{\tilde{\sigma} \in \overline{\text{co}}(F_j)} |(\sigma_1 - \tilde{\sigma})(A)| - \inf_{\tilde{\sigma} \in \overline{\text{co}}(F_j)} |(\sigma_2 - \tilde{\sigma})(A)| \right| \leq |(\sigma_1 - \sigma_2)(A)|$$

for any $A \in \mathcal{X}$, $j \in J$ and $\sigma_1, \sigma_2 \in \overline{\text{co}}(F_\infty)$. Fix $n \in \mathbb{N}$, $\sigma_1, \dots, \sigma_n \in F_\infty$ and parameters $\lambda_1, \dots, \lambda_n \in [0, 1]$ such that

$$\sum_{k=1}^n \lambda_k = 1 .$$

Pick any parameter $\varepsilon \in \mathbb{R}^+$. For any $A \in \mathcal{X}$ and $k \in \{1, \dots, n\}$, we define $\tilde{\sigma}_{k,j} \in F_j$ such that

$$|(\sigma_k - \tilde{\sigma}_{k,j})(A)| \leq \inf_{\tilde{\sigma} \in F_j} |(\sigma_k - \tilde{\sigma})(A)| + \varepsilon .$$

Then, for all $j \in J$ and $\varepsilon \in \mathbb{R}^+$,

$$\inf_{\tilde{\sigma} \in \overline{\text{co}}(F_j)} \left| \left(\sum_{k=1}^n \lambda_k \sigma_k - \tilde{\sigma} \right) (A) \right| \leq \sum_{k=1}^n \lambda_k |(\sigma_k - \tilde{\sigma}_{k,j}) (A)| \leq \varepsilon + \sup_{\sigma \in F_\infty} \inf_{\tilde{\sigma} \in F_j} |(\sigma - \tilde{\sigma}) (A)| .$$

Using (41), we thus deduce that, for all $j \in J$,

$$\sup_{\sigma \in \overline{\text{co}}(F_\infty)} \inf_{\tilde{\sigma} \in \overline{\text{co}}(F_j)} |(\sigma - \tilde{\sigma}) (A)| \leq \sup_{\sigma \in F_\infty} \inf_{\tilde{\sigma} \in F_j} |(\sigma - \tilde{\sigma}) (A)| , \quad A \in \mathcal{X} . \quad (42)$$

By switching the role of F_∞ and F_j for every $j \in J$, we also arrive at the inequality

$$\sup_{\tilde{\sigma} \in \overline{\text{co}}(F_j)} \inf_{\sigma \in \overline{\text{co}}(F_\infty)} |(\sigma - \tilde{\sigma}) (A)| \leq \sup_{\tilde{\sigma} \in F_j} \inf_{\sigma \in F_\infty} |(\sigma - \tilde{\sigma}) (A)| , \quad A \in \mathcal{X} . \quad (43)$$

Since $(F_j)_{j \in J}$ converges in the weak*-Hausdorff hypertopology to F_∞ , Inequalities (42)–(43) combined with Definition 2.3 yield the weak*-Hausdorff convergence of $(\overline{\text{co}}(F_j))_{j \in J}$ to $\overline{\text{co}}(F_\infty)$. By [24, Chapter 3, Theorem 1], $\overline{\text{co}}$ is a weak*-Hausdorff continuous mapping onto $\mathbf{CF}(\mathcal{X}^*)$. $\square$

Proposition 3.12 has a direct consequence on the topological properties of hyperspaces of convex weak*-closed sets:

Corollary 3.13. (Weak*-Hausdorff-closed hyperspaces of convex sets)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

- (i) $\mathbf{CF}(\mathcal{X}^*) = \overline{\text{co}}(\mathbf{F}(\mathcal{X}^*))$ is a convex, weak*-Hausdorff-closed subset of $\mathbf{F}(\mathcal{X}^*)$.
- (ii) $\mathbf{CB}(\mathcal{X}^*) \doteq \overline{\text{co}}(\mathbf{B}(\mathcal{X}^*))$ is a convex, path-connected, weak*-Hausdorff-closed subset of $\mathbf{CF}(\mathcal{X}^*)$.
- (iii) For any $\mathcal{U} \in \mathcal{O}$, $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ is a convex, path-connected, weak*-Hausdorff-closed subset of $\mathbf{CU}(\mathcal{X}^*) \subseteq \mathbf{CK}(\mathcal{X}^*) \subseteq \mathbf{CF}(\mathcal{X}^*)$.

Proof. By Corollary 3.11, $\mathbf{CF}(\mathcal{X}^*)$ endowed with the weak*-Hausdorff hypertopology is a Hausdorff space. Hence, by [24, Chapter 2, Theorem 3], each convergent net in this space converges in the weak*-Hausdorff hypertopology to at most one point, which, by Proposition 3.12, must be a convex weak*-closed set. Assertion (i) is thus proven. Convexity of $\mathbf{CF}(\mathcal{X}^*)$ is obvious.

To prove (ii), recall that $\mathbf{B}(\mathcal{X}^*)$ is a weak*-Hausdorff-closed set, by Proposition 3.3. Using this together with Proposition 3.12 and (13), we deduce that $\mathbf{CB}(\mathcal{X}^*)$ is also weak*-Hausdorff-closed. By Propositions 3.4 and 3.12 and the fact that the image under a continuous mapping of a path-connected space is path-connected, $\mathbf{CB}(\mathcal{X}^*)$ is also path-connected. Convexity of $\mathbf{CB}(\mathcal{X}^*)$ is obvious.

Assertion (iii) follows from Corollary 3.7 and Proposition 3.12 together with (19). In particular, the convexity of $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ is obvious. Recall also the Banach-Alaoglu theorem [35, Theorem 3.15], leading to $\mathbf{CU}(\mathcal{X}^*) \subseteq \mathbf{CK}(\mathcal{X}^*)$. $\square$

An extension of Corollary 3.13 (iii) to the set $\mathbf{CK}(\mathcal{X}^*)$ of all nonempty convex weak*-compact sets, defined by (15), is not a priori clear because the weak*-closed convex hull operator $\overline{\text{co}}$ does not necessarily maps weak*-compact sets to weak*-compact sets for general (real or complex) topological vector space $\mathcal{X}$. We do not know a

priori whether $\mathbf{K}(\mathcal{X}^*)$ and, hence, $\mathbf{CK}(\mathcal{X}^*)$, are weak*-Hausdorff-closed subset of $\mathbf{F}(\mathcal{X}^*)$. This property is at least true when $\mathcal{X}$ is a Banach space, since in this case, $\mathbf{B}(\mathcal{X}^*) = \mathbf{K}(\mathcal{X}^*)$, by the Banach-Alaoglu theorem [35, Theorem 3.15] and the uniform boundedness principle [35, Theorems 2.4 and 2.5]. See, e.g., [7, Proposition 1.2.9].

3.3. Weak*-Hausdorff hyperconvergence

It is instructive to relate weak*-Hausdorff limits of nets to lower and upper limits of sets à la Painlevé [28, § 29]: The *lower limit* of any net $(F_j)_{j \in J}$ of subsets of $\mathcal{X}^*$ is defined by

$$\text{Li}_{j \in J} F_j \doteq \left\{ \sigma \in \mathcal{X}^* : \begin{array}{l} \sigma \text{ is a weak}^* \text{ limit of a net } (\sigma_j)_{j \in J} \\ \text{with } \sigma_j \in F_j, j \succ j_0, \text{ for some } j_0 \in J \end{array} \right\}, \quad (44)$$

while its *upper limit* equals

$$\text{Ls}_{j \in J} F_j \doteq \left\{ \sigma \in \mathcal{X}^* : \begin{array}{l} \text{there is a subnet } (F_{s(i)})_{i \in I} \text{ and a net} \\ \sigma_i \rightarrow \sigma \text{ with } \sigma_i \in F_{s(i)} \text{ for all } i \in I \end{array} \right\}. \quad (45)$$

Clearly, $\text{Li}_{j \in J} F_j \subseteq \text{Ls}_{j \in J} F_j$. If $\text{Li}_{j \in J} F_j = \text{Ls}_{j \in J} F_j$ then $(F_j)_{j \in J}$ is said to be convergent to this set. See [28, § 29, I, III, VI], which however defines Li and Ls within metric spaces. This refers in the literature to the *Kuratowski* or *Kuratowski-Painlevé*¹⁴ convergence, see e.g. [30, Appendix B] and [7, Section 5.2]. By [35, Theorem 1.22], if $\mathcal{X}$ is an infinite-dimensional space, then its dual space $\mathcal{X}^*$ is not locally compact. In this case, the Kuratowski-Painlevé convergence is not topological [30, Theorem B.3.2]. See also [7, Chapter 5], in particular [7, Theorem 5.2.6 and following discussions] which relates the Kuratowski-Painlevé convergence to the so-called Fell topology.

We start by proving the weak*-Hausdorff convergence of monotonically increasing nets which are bounded from above within the subspace $\mathbf{K}(\mathcal{X}^*)$ of all nonempty weak*-compact subsets of $\mathcal{X}^*$ defined by (14).

Proposition 3.14. (Weak*-Hausdorff hyperconvergence of increasing nets)

Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then any increasing net $(K_j)_{j \in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ such that

$$K \doteq \overline{\bigcup_{j \in J} K_j} \in \mathbf{K}(\mathcal{X}^*) \subsetneq \mathbf{F}(\mathcal{X}^*) \quad (46)$$

(with respect to the weak* closure) converges in the weak*-Hausdorff hypertopology to

$$K = \overline{\text{Li}_{j \in J} K_j} = \overline{\text{Ls}_{j \in J} K_j}. \quad (47)$$

Additionally, K is the Kuratowski-Painlevé limit of $(K_j)_{j \in J}$ whenever $\mathcal{X}$ is separable.

¹⁴ The idea of upper and lower limits is due to Painlevé, as acknowledged by Kuratowski himself in [28, § 29, Footnote 1, p. 335]. We thus use the name Kuratowski-Painlevé convergence.

Proof. Let $(K_j)_{j \in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ be any increasing net, i.e., $K_{j_1} \subseteq K_{j_2}$ whenever $j_1 \prec j_2$, satisfying (46). Assume without loss of generality that $K_j \neq \emptyset$ for all $j \in J$. Because $K \in \mathbf{K}(\mathcal{X}^*)$, it is bounded, see [35, Theorem 1.15 (b)]. By the convergence of increasing bounded nets of real numbers, it follows that, for any $A \in \mathcal{X}$,

$$\lim_J \max_{\tilde{\sigma} \in K_j} \min_{\sigma \in K} |(\tilde{\sigma} - \sigma)(A)| = \sup_{j \in J} \max_{\tilde{\sigma} \in K_j} \min_{\sigma \in K} |(\tilde{\sigma} - \sigma)(A)| \leq \max_{\tilde{\sigma} \in K} \min_{\sigma \in K} |(\tilde{\sigma} - \sigma)(A)| = 0.$$

Therefore, by Definition 2.3, if

$$\limsup_J \max_{\sigma \in K} \min_{\tilde{\sigma} \in K_j} |(\tilde{\sigma} - \sigma)(A)| = 0, \quad A \in \mathcal{X}, \quad (48)$$

then the increasing net $(K_j)_{j \in J}$ converges in $\mathbf{K}(\mathcal{X}^*)$ to K . To prove (48), assume by contradiction the existence of $\varepsilon \in \mathbb{R}^+$ such that

$$\limsup_J \max_{\sigma \in K} \min_{\tilde{\sigma} \in K_j} |(\tilde{\sigma} - \sigma)(A)| \geq \varepsilon \in \mathbb{R}^+ \quad (49)$$

for some fixed $A \in \mathcal{X}$. For any $j \in J$, take $\sigma_j \in K$ such that

$$\max_{\sigma \in K} \min_{\tilde{\sigma} \in K_j} |(\tilde{\sigma} - \sigma)(A)| = \min_{\tilde{\sigma} \in K_j} |(\tilde{\sigma} - \sigma_j)(A)|. \quad (50)$$

By weak*-compactness of K , there is a subnet $(\sigma_{j_l})_{l \in L}$ converging in the weak* topology to $\sigma_\infty \in K$. Via Equation (50) and the triangle inequality, we then get that, for any $l \in L$,

$$\max_{\sigma \in K} \min_{\tilde{\sigma} \in K_{j_l}} |(\tilde{\sigma} - \sigma)(A)| \leq |(\sigma_{j_l} - \sigma_\infty)(A)| + \min_{\tilde{\sigma} \in K_{j_l}} |(\tilde{\sigma} - \sigma_\infty)(A)|.$$

By (46) and the fact that $(K_j)_{j \in J} \subseteq \mathbf{K}(\mathcal{X}^*)$ is an increasing net, it follows that

$$\lim_L \max_{\sigma \in K} \min_{\tilde{\sigma} \in K_{j_l}} |(\tilde{\sigma} - \sigma)(A)| = 0. \quad (51)$$

By the convergence of decreasing bounded nets of real numbers, note that

$$\limsup_J \max_{\sigma \in K} \min_{\tilde{\sigma} \in K_j} |(\tilde{\sigma} - \sigma)(A)| = \liminf_J \max_{\sigma \in K} \min_{\tilde{\sigma} \in K_j} |(\tilde{\sigma} - \sigma)(A)|$$

and hence, (51) contradicts (49). As a consequence, Equation (48) holds true.

In order to prove (47), observe that

$$\bigcup_{j \in J} K_j \subseteq \text{Li}_{j \in J} K_j \subseteq \text{Ls}_{j \in J} K_j \subseteq \overline{\bigcup_{j \in J} K_j}, \quad (52)$$

because the net $(K_j)_{j \in J}$ is increasing. If $\mathcal{X}$ is separable then it is well-known that the weak* topology of any compact set is metrizable [35, Theorem 3.16], see Equation (54) below. Let d be any metric generating the weak* topology on K . For any fixed $\sigma \in K$, consider the net $(\sigma_j)_{j \in J}$, where σ_j is some minimizer in K_j of $d(\sigma_j, \sigma)$. Note that a minimizer always exists because K_j is weak* compact. Clearly, this net converges in the weak* topology to $\sigma \in K$, for $(K_j)_{j \in J}$ is an increasing net. Hence, $K \subseteq \text{Li}_{j \in J} K_j$ and therefore, using (52),

$$K = \text{Li}_{j \in J} K_j = \text{Ls}_{j \in J} K_j$$

when $\mathcal{X}$ is separable. □

Non-monotonic, weak*-Hausdorff convergent nets in $\mathbf{F}(\mathcal{X}^*)$ are not trivial to study, in general. In the next proposition, we give preliminary results on limits of convergent nets.

Proposition 3.15. (Weak*-Hausdorff hypertopology vs. upper and lower limits)
Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. For any weak-Hausdorff convergent net $(F_j)_{j \in J} \subseteq \mathbf{F}(\mathcal{X}^*)$ with limit $F_\infty \in \mathbf{F}(\mathcal{X}^*)$,*

$$\text{Li}_{j \in J} F_j \subseteq \overline{\text{co}}(F_\infty)$$

and if $(F_j \equiv U_j)_{j \in J} \subseteq \mathbf{U}_{\mathcal{U}}(\mathcal{X}^)$ for some 0-neighborhood $\mathcal{U} \in \mathcal{O}$,*

$$F_\infty \equiv U_\infty \subseteq \overline{\text{co}}(\text{Ls}_{j \in J} U_j) ,$$

where we recall that $\overline{\text{co}}$ is the weak-closed convex hull operator defined by (8).*

Proof. Fix all parameters of the proposition. Assume without loss of generality that $\text{Li}_{j \in J} F_j$ is nonempty. Let $\sigma_\infty \in \text{Li}_{j \in J} F_j$, which is, by definition, the weak* limit of a net $(\sigma_j)_{j \in J}$ such that $\sigma_j \in F_j$ for all $j \succ j_0$, for some $j_0 \in J$. Then, for any $A \in \mathcal{X}$ and $j \succ j_0$,

$$\inf_{\sigma \in F_\infty} |(\sigma - \sigma_\infty)(A)| \leq |(\sigma_j - \sigma_\infty)(A)| + \inf_{\sigma \in F_\infty} \{ |(\sigma - \sigma_j)(A)| \} .$$

Taking this last inequality in the limit with respect to J and using Definition 2.3, we deduce that

$$\inf_{\sigma \in F_\infty} |(\sigma - \sigma_\infty)(A)| = 0 , \quad A \in \mathcal{X} . \quad (53)$$

If $\sigma_\infty \notin \overline{\text{co}}(F_\infty)$ then, as it is done to prove (40), we infer from the Hahn-Banach separation theorem [35, Theorem 3.4 (b)] the existence of $A_0 \in \mathcal{X}$ and $x_1, x_2 \in \mathbb{R}$ such that

$$\sup_{\sigma \in \overline{\text{co}}(F_\infty)} \text{Re} \{ \sigma(A_0) \} < x_1 < x_2 < \text{Re} \{ \sigma_\infty(A_0) \} ,$$

which contradicts (53) for $A = A_0$. As a consequence, $\sigma_\infty \in \overline{\text{co}}(F_\infty)$ and, hence, $\text{Li}_{j \in J} F_j \subseteq \overline{\text{co}}(F_\infty)$.

Assume now that $(F_j \equiv U_j)_{j \in J} \subseteq \mathbf{U}_{\mathcal{U}}(\mathcal{X}^*)$ for some 0-neighborhood $\mathcal{U} \in \mathcal{O}$ with limit $F_\infty \equiv U_\infty$. By Definition 2.3 and the Banach-Alaoglu theorem [35, Theorem 3.15], we deduce that, for any $\sigma_\infty \in U_\infty$,

$$\lim_J \inf_{\sigma \in U_j} |(\sigma - \sigma_\infty)(A)| = \lim_J \min_{\sigma \in U_j} |(\sigma - \sigma_\infty)(A)| = 0 , \quad A \in \mathcal{X} .$$

From this equality and the Banach-Alaoglu theorem [35, Theorem 3.15], for any $A \in \mathcal{X}$ and $\sigma_\infty \in U_\infty$, there is $\tilde{\sigma} \in \text{Ls}_{j \in J} U_j$ such that

$$\tilde{\sigma}(A) = \sigma_\infty(A) .$$

Consequently, one infers from the Hahn-Banach separation theorem [35, Theorem 3.4 (b)] that $\sigma_\infty \in U_\infty$ belongs to the weak*-closed convex hull of the upper limit $\text{Ls}_{j \in J} U_j$. $\square$

Corollary 3.16. (Weak*-Hausdorff hypertopology vs. upper and lower limits)
 Let $\mathcal{X}$ be a topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$, $\mathcal{U} \in \mathcal{O}$ and $U_\infty \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ be any weak*-Hausdorff limit of a convergent net $(U_j)_{j \in J} \subseteq \mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$. Then,

$$\overline{\text{Li}_{j \in J} U_j} = \overline{\text{co}}(\text{Li}_{j \in J} U_j) \subseteq U_\infty \subseteq \overline{\text{co}}(\text{Ls}_{j \in J} U_j) .$$

Proof. The assertion is an obvious application of Proposition 3.15 to the subset $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*) \subseteq \mathbf{U}_\mathcal{U}(\mathcal{X}^*)$ together with the idempotency of the weak*-closed convex hull operator $\overline{\text{co}}$. Note that $\text{Li}_{j \in J} U_j$ is in this case a convex set. $\square$

3.4. Metrizable hyperspaces

We are interested in investigating *metrizable* subspaces of $\mathbf{F}(\mathcal{X}^*)$. Metrizable topological spaces are Hausdorff, so, in the light of Corollaries 3.9 and 3.11, we restrict our analysis on subspaces of the Hausdorff hyperspace $\mathbf{CF}(\mathcal{X}^*)$ of all nonempty convex weak*-closed subsets of $\mathcal{X}^*$ defined by Equation (7).

For a separable topological $\mathbb{K}$ -vector space $\mathcal{X}$, recall that the weak* topology on any compact set $K \in \mathbf{K}(\mathcal{X}^*)$ is metrizable, see [35, Theorem 3.16]. Here, we use the following metric on K : Fix a countable dense set $(A_n)_{n \in \mathbb{N}} \subseteq \mathcal{X}$ and define

$$d(\sigma_1, \sigma_2) \doteq \sum_{n \in \mathbb{N}} \frac{2^{-n}}{1 + \max_{\sigma \in K} |\sigma(A_n)|} |(\sigma_1 - \sigma_2)(A_n)| , \quad \sigma_1, \sigma_2 \in K . \quad (54)$$

This metric is well-defined and induces the weak* topology on the weak*-compact set K . Absolute polars of $\mathcal{X}^*$ (cf. (16)) are special example of compact sets, see [35, Theorems 3.15]. We show how (54) leads to the metrizability of the weak*-Hausdorff hypertopology on the hyperspace $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ of all nonempty convex, uniformly bounded in a 0-neighborhood $\mathcal{U} \in \mathcal{O}$, weak*-closed subsets of $\mathcal{X}^*$.

Using the above metric d in (26), denote by $\mathfrak{d}_H$ the Hausdorff distance between two elements $U_1, U_2 \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$, that is¹⁵,

$$\mathfrak{d}_H(U_1, U_2) \doteq \max \left\{ \max_{\sigma_1 \in U_1} \min_{\sigma_2 \in U_2} d(\sigma_1, \sigma_2), \max_{\sigma_2 \in U_2} \min_{\sigma_1 \in U_1} d(\sigma_1, \sigma_2) \right\} . \quad (55)$$

This Hausdorff distance induces the weak*-Hausdorff hypertopology on $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$:

Theorem 3.17. (Complete metrizability of the weak*-Hausdorff hypertopology)
 Let $\mathcal{X}$ be a separable topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathcal{U} \in \mathcal{O}$.
 The family

$$\{ \{ U_2 \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*) : \mathfrak{d}_H(U_1, U_2) < r \} : r \in \mathbb{R}^+, U_1 \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*) \}$$

is a basis of the weak*-Hausdorff hypertopology of $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$. Additionally, $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ is weak*-Hausdorff-compact and completely metrizable.

¹⁵ Minima in (55) come from the compactness of sets and the continuity of d . The following maxima in (55) result from the compactness of sets and the fact that the minimum over a continuous map defines an upper semicontinuous function.

Proof. Recall that a topology is finer than a second one iff any convergent net of the first topology converges also in the second topology to the same limit. See, e.g., [24, Chapter 2, Theorems 4, 9]. We first show that the topology induced by the Hausdorff metric $\mathfrak{d}_H$ is finer than the weak*-Hausdorff hypertopology of $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ at fixed $\mathcal{U} \in \mathcal{O}$: Take any net $(U_j)_{j \in J}$ converging in $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ to U in the topology induced by the Hausdorff metric (55). Let $A \in \mathcal{X}$. By density of $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{X}$, for any $\varepsilon \in \mathbb{R}^+$, there is $n \in \mathbb{N}$ such that $(A_n - A) \in 2^{-1}\varepsilon\mathcal{U} \in \mathcal{O}$. In particular, by the definition of $\mathcal{U}^\circ$ (see (16)) and (54), for all $j \in J$,

$$d_H^{(A)}(U, U_j) \leq \varepsilon + d_H^{(A_n)}(U, U_j) \leq \varepsilon + 2^n \left(1 + \max_{\sigma \in \mathcal{U}^\circ} |\sigma(A_n)| \right) \mathfrak{d}_H(U, U_j) .$$

Thus, the net $(U_j)_{j \in J}$ converges to U also in the weak*-Hausdorff hypertopology.

Endowed with the Hausdorff metric topology, the space of closed subsets of a compact metric space is compact, by [7, Theorem 3.2.4]. In particular, by weak* compactness of absolute polars (the Banach-Alaoglu theorem [35, Theorem 3.15]), $\mathbf{U}_\mathcal{U}(\mathcal{X}^*)$ endowed with the Hausdorff metric $\mathfrak{d}_H$ is a compact hyperspace. By Corollary 3.13, $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ is closed with respect to the weak*-Hausdorff hypertopology, and thus closed with respect to the topology induced by $\mathfrak{d}_H$, because this topology is finer than the weak*-Hausdorff hypertopology, as proven above. Hence, $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ is also compact with respect to the topology induced by $\mathfrak{d}_H$. Since the weak*-Hausdorff hypertopology is a Hausdorff topology (Corollary 3.11), as is well-known [35, Section 3.8 (a)], both topologies must coincide. $\square$

Note that Theorem 3.17 is similar to the assertion [7, end of p.91]. It leads to a strong improvement of Proposition 3.14 and Corollary 3.16:

Corollary 3.18. (Weak*-Hausdorff hypertopology and Kuratowski-Painlevé convergence) *Let $\mathcal{X}$ be a separable topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathcal{U} \in \mathcal{O}$. Then any weak*-Hausdorff convergent net $(U_j)_{j \in J} \subseteq \mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ converges to the Kuratowski-Painlevé limit*

$$U_\infty = \text{Li}_{j \in J} U_j = \text{Ls}_{j \in J} U_j \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*) .$$

Proof. It is a direct consequence of Theorem 3.17 and [28, § 29, Section IX, Theorem 2]. $\square$

4. Generic hypersets in infinite dimensions

The Krein-Milman theorem [35, Theorem 3.23] tells us that any convex weak*-compact set $K \in \mathbf{CK}(\mathcal{X}^*)$ is the weak*-closure of the convex hull of the (nonempty) set $\mathcal{E}(K)$ of its extreme points:

$$K = \overline{\text{co}} \mathcal{E}(K) .$$

The set $\mathcal{E}(K)$ is also called the extreme boundary of K . We are interested in the question whether the subset of all $K \in \mathbf{CK}(\mathcal{X}^*)$ with weak*-dense set $\mathcal{E}(K)$ of extreme points is generic, or not, when the topological space $\mathcal{X}$ has infinite dimension.

As is well-known, such convex compact sets exist in infinite-dimensional topological spaces. For instance, the unit ball of any infinite-dimensional Hilbert space has a

dense extreme boundary in the weak topology. Another example is given by the celebrated Poulsen simplex constructed in 1961 [34], within the Hilbert space $\ell^2(\mathbb{N})$. In fact, a convex compact set with dense extreme boundary *is not an accident* in this case: The set of all such convex compact subsets of an infinite-dimensional separable¹⁶ Banach space $\mathcal{Y}$ is generic¹⁷ in the complete metric space of compact convex subsets of $\mathcal{Y}$, endowed with the well-known Hausdorff metric topology [7, Definition 3.2.1]. See [27, Proposition 2.1, Theorem 2.2], which have been refined in [20, Section 4]. See, e.g., [21, Section 7] for a more recent review on this subject.

In this section we demonstrate such a property in the dual space $\mathcal{X}^*$ of an infinite-dimensional, separable topological $\mathbb{K}$ -vector space $\mathcal{X}$ ($\mathbb{K} = \mathbb{R}, \mathbb{C}$), endowed with its weak*-topology. In this situation, results similar to [27, 20] can be proven in a natural way by using the weak*-Hausdorff hypertopology.

4.1. Infinite dimensionality of absolute polars

A large class of convex weak*-compact sets is given by absolute polars (16) of any 0-neighborhoods in $\mathcal{X}$, by the Banach-Alaoglu theorem [35, Theorem 3.15]. In fact, weak*-closed subsets of absolute polars are the main sources of weak*-compact sets in the dual space $\mathcal{X}^*$ of a real or complex topological vector space $\mathcal{X}$. Therefore, it is natural to study generic convex weak*-compact sets within some absolute polar. If the absolute polar can be embedded in a finite-dimensional subspace then there is no convex weak*-compact set with weak*-dense extreme boundary. We are thus interested in the infinite-dimensional situation: we consider absolute polars which are *infinite-dimensional*, that is, their (linear) spans are infinite-dimensional subspaces of $\mathcal{X}^*$.

Note that the infinite dimensionality of $\mathcal{X}$ does not guarantee such a property of polars in $\mathcal{X}^*$: Take for instance $\mathcal{X} = \mathcal{H}$ being any infinite-dimensional Hilbert space endowed with its weak topology and a scalar product denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}}$. For any $\mathcal{U} \in \mathcal{O}$, there are $n \in \mathbb{N}$ and $\psi_1, \dots, \psi_n \in \mathcal{H}$ such that

$$\{\varphi \in \mathcal{H} : |\langle \psi_k, \varphi \rangle_{\mathcal{H}}| < 1, k = 1, \dots, n\} \subseteq \mathcal{U} .$$

By (16), it follows that the absolute polar $\mathcal{U}^\circ$ is orthogonal to the set $\{\psi_1, \dots, \psi_n\}^\perp$, leading to

$$\mathcal{U}^\circ \subseteq \text{span} \{\psi_1, \dots, \psi_n\} .$$

In this specific case, (weak) neighborhoods are too big, implying too small absolute polars. If one takes instead the usual norm topology of the separable infinite-dimensional Hilbert space to define neighborhoods, then we obtain infinite-dimensional absolute polars, for all bounded neighborhoods, like in any infinite-dimensional Banach space.

We give a general sufficient condition on a 0-neighborhood $\mathcal{U} \in \mathcal{O}$ for the infinite-dimensionality of its absolute polar $\mathcal{U}^\circ$:

¹⁶ [27, Proposition 2.1, Theorem 2.2] seem to lead to the asserted property for all (possibly non-separable) Banach spaces, as claimed in [20, 21, 27]. However, [27, Theorem 1.5], which assumes the separability of the Banach space, is clearly invoked to prove the corresponding density stated in [27, Theorem 2.2]. We do not know how to remove the separability condition.

¹⁷ That is, the complement of a meagre set, i.e., a nowhere dense set.

Condition 4.1. (Infinite dimensionality of absolute polars)

There exists an infinite set $\{\sigma_n\}_{n \in \mathbb{N}}$ of linearly independent elements $\sigma_n \in \mathcal{X}^*$ such that $\sup |\sigma_n(\mathcal{U})| < \infty$ for every $n \in \mathbb{N}$.

Condition 4.1 obviously implies the infinite-dimensionality of the polar $\mathcal{U}^\circ$. In particular, $\mathcal{X}$ and $\mathcal{X}^*$ are infinite-dimensional, like in [20, 21, 27]. Such a condition can be satisfied within a very *large* class of topological vector spaces:

Example 1: If $\mathcal{X}$ is an infinite-dimensional Banach space then any bounded neighborhood $\mathcal{U} \in \mathcal{O}$ satisfies Condition 4.1, since it is contained in an open ball of center 0 in $\mathcal{X}$ (so that its polar $\mathcal{U}^\circ$ contains an open ball in $\mathcal{X}^*$ seen as a Banach space).

Example 2: If $\mathcal{U} \in \mathcal{O}$ is a finite-dimensional¹⁸ 0-neighborhood in $\mathcal{X}$ then, by [1, Theorem 5.110],

$$\dim \{\sigma \in \mathcal{X}^* : \sigma(\mathcal{U}) = \{0\}\} = \dim \mathcal{X}^* - \dim(\text{span} \mathcal{U}) .$$

As a consequence, $\mathcal{U}$ satisfies Condition 4.1 whenever $\mathcal{X}$ has infinite dimension. Existence of a finite-dimensional $\mathcal{U} \in \mathcal{O}$ is obviously ensured when $\mathcal{X}$ is a locally finite-dimensional space, meaning that each point of $\mathcal{X}$ has a finite-dimensional neighborhood [41, Definition 5]. A typical example of such topological vector spaces is given by the union

$$\mathcal{X} = \bigcup_{n \in \mathbb{N}} \mathcal{X}_n$$

of an increasing sequence of finite-dimensional (normed) spaces $\mathcal{X}_n$ with diverging dimension $\dim \mathcal{X}_n \rightarrow \infty$, as $n \rightarrow \infty$, whose topology has, as a 0-basis, the family of all open balls of $\mathcal{X}_n$, $n \in \mathbb{N}$, centered at 0.

Example 3: Let $\mathcal{X}$ be any vector space with *algebraic* dual space $\mathcal{X}'$, that is, the vector space of all linear functionals on $\mathcal{X}$. Assume the existence of an infinite set $\{\sigma_n\}_{n \in \mathbb{N}} \subseteq \mathcal{X}'$ of linearly independent linear functionals σ_n , $n \in \mathbb{N}$, which separates points in $\mathcal{X}$ and is pointwise uniformly bounded:

$$\sup_{n \in \mathbb{N}} |\sigma_n(A)| < \infty , \quad A \in \mathcal{X} .$$

Pick any topology on $\mathcal{X}$ such that $\{\sigma_n\}_{n \in \mathbb{N}} \subseteq \mathcal{X}^*$. If $\mathcal{U} \in \mathcal{O}$ is such that

$$\mathcal{U} \subseteq \{A \in \mathcal{X} : |\sigma_n(A)| < M(n)\}$$

for some fixed, possibly unbounded, $M : \mathbb{N} \rightarrow \mathbb{R}^+$, then $\mathcal{U}$ satisfies Condition 4.1. As an example of such a topology on $\mathcal{X}$, take the topology induced by the invariant metric

$$d_{\{\sigma_n\}_{n \in \mathbb{N}}}(A_1, A_2) \doteq \sum_{n \in \mathbb{N}} \frac{1}{M(n)} |\sigma_n(A_1 - A_2)| < \infty , \quad A_1, A_2 \in \mathcal{X} ,$$

for some fixed $M : \mathbb{N} \rightarrow \mathbb{R}^+$ with summable inverse. In this case, any $\mathcal{U} \in \mathcal{O}$ contained in some open ball satisfies Condition 4.1.

¹⁸ A subset of a vector space is finite-dimensional if its span is a finite-dimensional subspace.

4.2. Weak*-Hausdorff dense subsets of convex weak*-compact sets

We first study generic convex weak*-compact subsets of an infinite-dimensional absolute polar. More precisely, taking a 0-neighborhood $\mathcal{U} \in \mathcal{O}$ satisfying Condition 4.1, we consider the set defined by (19), that is,

$$\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) \doteq \{U \in \mathbf{CF}(\mathcal{X}^*) : U \subseteq \mathcal{U}^\circ\} \subseteq \mathbf{CK}(\mathcal{X}^*) \subseteq \mathbf{CF}(\mathcal{X}^*)$$

with $\mathcal{U}^\circ \in \mathbf{CK}(\mathcal{X}^*)$ being the absolute polar (16) of $\mathcal{U}$.

Weak*-closed subsets of absolute polars are the main source of weak*-compact sets of $\mathcal{X}^*$ and, for any $\mathcal{U} \in \mathcal{O}$, the hyperspace $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ is a weak*-Hausdorff-closed subspace of the Hausdorff hyperspace $\mathbf{CF}(\mathcal{X}^*)$, by Corollaries 3.11 and 3.13 (iii). We study the topological properties of the subset

$$\mathcal{D}_{\mathcal{U}} \doteq \left\{ U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) : U = \overline{\mathcal{E}(U)} \right\} \subseteq \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) \quad (56)$$

for any fixed 0-neighborhood $\mathcal{U} \in \mathcal{O}$, where we recall that $\mathcal{E}(U)$ is the extreme boundary of U , i.e., the (nonempty, by the Krein-Milman theorem [35, Theorem 3.23]) set of extreme points of the convex weak*-compact set U .

Note that the so-called *exposed* points are particular examples of extreme ones: a point $\sigma_0 \in C$ in a convex subset $C \subseteq \mathcal{X}^*$ is *exposed* if there is $A \in \mathcal{X}$ such that the real part of the weak*-continuous functional $\hat{A} : \sigma \mapsto \sigma(A)$ from $\mathcal{X}^*$ to $\mathbb{K} = \mathbb{R}, \mathbb{C}$ (cf. (1)) takes its *unique* maximum on C at $\sigma_0 \in C$. Considering exposed points instead of general extreme points is technically convenient. So, we denote by $\mathcal{E}_0(U) \subseteq \mathcal{E}(U)$ the set of all exposed points of any element $U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ and define

$$\mathcal{D}_{0,\mathcal{U}} \doteq \left\{ U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) : U = \overline{\mathcal{E}_0(U)} \right\} \subseteq \mathcal{D}_{\mathcal{U}} \subseteq \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) \quad (57)$$

for any fixed 0-neighborhood $\mathcal{U} \in \mathcal{O}$.

In order to have the weak*-Hausdorff density of $\mathcal{D}_{0,\mathcal{U}} \subseteq \mathcal{D}_{\mathcal{U}}$ in $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$, the absolute polar $\mathcal{U}^\circ$ must be infinite-dimensional, that is, its (linear) span $\mathcal{U}^\circ$ is an infinite-dimensional subspace of $\mathcal{X}^*$. This refers to Condition 4.1 on the 0-neighborhood $\mathcal{U} \subseteq \mathcal{X}$. Then, like in the proof of [20, Theorem 4.3] and in contrast with [27], we design elements of $\mathcal{D}_{0,\mathcal{U}}$ that approximate $U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ by using a procedure that is very similar to the construction of the Poulsen simplex [34]. Note however that Poulsen used the existence of orthonormal bases in infinite-dimensional Hilbert spaces¹⁹. Here, the Hahn-Banach separation theorem [35, Theorem 3.4 (b)] replaces the orthogonality property coming from the Hilbert space structure. In all previous results [27, 20] on the density of convex compact sets with dense extreme boundary, the norm topology is used, while the primordial topology is here the weak* topology. In this context, the metrizability of weak* and weak*-Hausdorff topologies on absolute polars is pivotal. See Theorem 3.17.

We give now the precise assertion along with its proof:

¹⁹ In [34], Poulsen uses the Hilbert space $\ell^2(\mathbb{N})$ to construct his example of a convex compact set (in fact a simplex) with dense extreme boundary.

Theorem 4.2. (Weak*-Hausdorff density of $\mathcal{D}_{0,\mathcal{U}}$)

Let $\mathcal{X}$ be a separable topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $\mathcal{U} \in \mathcal{O}$ satisfy Condition 4.1. Then, $\mathcal{D}_{0,\mathcal{U}} \subseteq \mathcal{D}_{\mathcal{U}}$ is a weak*-Hausdorff dense subset of $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$.

Proof. Let $\mathcal{X}$ be a separable (infinite-dimensional) topological $\mathbb{K}$ -vector space ($\mathbb{K} = \mathbb{R}, \mathbb{C}$) and fix once and for all $\mathcal{U} \in \mathcal{O}$ satisfying Condition 4.1, as well as a convex weak*-compact subset $U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$. The construction of convex weak*-compact sets in $\mathcal{D}_{0,\mathcal{U}}$ approximating U is done in several steps:

Step 0: Let d be the metric defined by (54) and generating the weak* topology on the absolute polar $\mathcal{U}^\circ$. Then, for any $\varepsilon \in \mathbb{R}^+$, there is a finite set $\{\omega_j\}_{j=1}^{n_\varepsilon} \subseteq U$, $n_\varepsilon \in \mathbb{N}$, such that

$$U \subseteq \bigcup_{j=1}^{n_\varepsilon} B(\omega_j, \varepsilon) , \quad (58)$$

where $B(\omega, r) \doteq \{\sigma \in \mathcal{U}^\circ : d(\omega, \sigma) < r\} \subseteq \mathcal{U}^\circ$ (59)

denotes the weak*-open ball of radius $r \in \mathbb{R}^+$, centered at $\omega \in \mathcal{U}^\circ$. We then define the convex weak*-compact set

$$U_0 \doteq \text{co}\{\omega_1, \dots, \omega_{n_\varepsilon}\} \subseteq \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon}\} . \quad (60)$$

By (58), note that $\mathfrak{d}_H(U, U_0) \leq \varepsilon$, (61)

where $\mathfrak{d}_H$ is the Hausdorff distance associated with the metric d , as defined by (55).

Step 1: Observe that the absolute polar $\mathcal{U}^\circ$ is weak*-separable, by its weak* compactness [35, Theorem 3.15] and its metrizability [35, Theorem 3.16]. Take any weak*-dense countable set $\{\varrho_{0,k}\}_{k \in \mathbb{N}}$ of U_0 . By Condition 4.1 together with a simple rescaling argument, note also the existence of an infinite set of linearly independent, continuous linear functionals within $\mathcal{U}^\circ$. As a consequence, there is $\sigma_1 \in \mathcal{U}^\circ \setminus \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon}\}$ satisfying

$$\sup |\sigma_1(\mathcal{U})| = 1 . \quad (62)$$

As in the proof of Proposition 3.10, recall that $\mathcal{X}^*$, endowed with the weak* topology, is a locally convex space with $\mathcal{X}$ being its dual space. Since $\{\sigma_1\}$ is a convex weak*-compact set and $\text{span}\{\omega_1, \dots, \omega_{n_\varepsilon}\}$ is convex and weak*-closed [35, Theorem 1.42], we infer from the Hahn-Banach separation theorem [35, Theorem 3.4 (b)] the existence of $A_1 \in \mathcal{X}$ such that

$$\sup \{\text{Re}\{\sigma(A_1)\} : \sigma \in \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon}\}\} < \text{Re}\{\sigma_1(A_1)\} .$$

Since $\text{span}\{\omega_1, \dots, \omega_{n_\varepsilon}\}$ is a linear space, observe that

$$\text{Re}\{\sigma(A_1)\} = 0 , \quad \sigma \in \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon}\} . \quad (63)$$

Additionally, by rescaling $A_1 \in \mathcal{X}$, we can assume without loss of generality that

$$\text{Re}\{\sigma_1(A_1)\} = 1 . \quad (64)$$

Let

$$\omega_{n_\varepsilon+1} \doteq (1 - \lambda_1) \varpi_1 + \lambda_1 \sigma_1 , \text{ with } \lambda_1 \doteq \min\{1, 2^{-2}\varepsilon\} , \varpi_1 \doteq \varrho_{0,1} \in U_0 . \quad (65)$$

In contrast to the proof of [20, Theorem 4.3], we use a convex combination to automatically ensure that $\omega_{n_\varepsilon+1} \in \mathcal{U}^\circ$, by convexity of the absolute polar $\mathcal{U}^\circ$. By (54), the inequality $\lambda_1 \leq 2^{-2}\varepsilon$ yields

$$d(\omega_{n_\varepsilon+1}, \varpi_1) \leq 2^{-1}\varepsilon . \quad (66)$$

Define the new convex weak*-compact set

$$U_1 \doteq \text{co}\{\omega_1, \dots, \omega_{n_\varepsilon+1}\} \subseteq \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon+1}\} .$$

Observe that $\omega_{n_\varepsilon+1}$ is an exposed point of U_1 , by (63) and (64). By (55), (60) and (66), note also that $\mathfrak{d}_H(U_0, U_1) \leq 2^{-1}\varepsilon$, which, by the triangle inequality and (61), yields

$$\mathfrak{d}_H(U, U_1) \leq (1 + 2^{-1})\varepsilon \quad (67)$$

for an arbitrary (but previously fixed) $\varepsilon \in \mathbb{R}^+$.

Step 2: Take any weak* dense countable set $\{\varrho_{1,k}\}_{k \in \mathbb{N}}$ of U_1 . By Condition 4.1, there is $\sigma_2 \in \mathcal{U}^\circ \setminus \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon+1}\}$ with

$$\sup |\sigma_2(\mathcal{U} \cup \{A_1\})| \leq \min\{1, 2^{-1}\lambda_1\} . \quad (68)$$

As before, we deduce from the Hahn-Banach separation theorem [35, Theorem 3.4 (b)] the existence of $A_2 \in \mathcal{X}$ such that

$$\text{Re}\{\sigma_2(A_2)\} = 1 \quad \text{and} \quad \text{Re}\{\sigma(A_2)\} = 0 , \quad \sigma \in \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon+1}\} . \quad (69)$$

Let

$$\omega_{n_\varepsilon+2} \doteq (1 - \lambda_2)\varpi_2 + \lambda_2\sigma_2 , \quad \text{with} \quad \lambda_2 \doteq \min\{1, 2^{-3}\varepsilon\} , \quad \varpi_2 \doteq \varrho_{1,1} \in U_1 . \quad (70)$$

In this case, similar to inequality (66),

$$d(\omega_{n_\varepsilon+2}, \varpi_2) \leq 2^{-2}\varepsilon . \quad (71)$$

Define the new convex weak*-compact set

$$U_2 \doteq \text{co}\{\omega_1, \dots, \omega_{n_\varepsilon+2}\} \subseteq \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon+2}\} .$$

By (69), $\omega_{n_\varepsilon+2}$ is an exposed point of U_2 , but it is not obvious that the exposed point $\omega_{n_\varepsilon+1}$ of U_1 is still an exposed point of U_2 , with respect to $A_1 \in \mathcal{X}$. This property is a consequence of

$$\text{Re}\{\omega_{n_\varepsilon+2}(A_1)\} = (1 - \lambda_2)\text{Re}\{\varpi_2(A_1)\} + \lambda_2\text{Re}\{\sigma_2(A_1)\} < \text{Re}\{\omega_{n_\varepsilon+1}(A_1)\} = \lambda_1 ,$$

(see (63), (65) and (70)), which holds true because of Equation (68). By (55), (67) and (71) together with the triangle inequality,

$$\mathfrak{d}_H(U, U_2) \leq (1 + 2^{-1} + 2^{-2})\varepsilon$$

for an arbitrary (but previously fixed) $\varepsilon \in \mathbb{R}^+$.

Step $n \rightarrow \infty$: We now iterate the above procedure, ensuring, at each step $n \geq 3$, that the addition of the element

$$\omega_{n_\varepsilon+n} \doteq (1 - \lambda_n)\varpi_n + \lambda_n\sigma_n , \quad \text{with} \quad \lambda_n \doteq \min\{1, 2^{-(n+1)}\varepsilon\} , \quad (72)$$

in order to define the convex weak*-compact set

$$U_n \doteq \text{co} \{ \omega_1, \dots, \omega_{n_\varepsilon+n} \} \subseteq \text{span} \{ \omega_1, \dots, \omega_{n_\varepsilon+n} \} , \quad (73)$$

does not destroy the property of the elements $\omega_{n_\varepsilon+1}, \dots, \omega_{n_\varepsilon+n-1}$ being exposed. To this end, for any $n \geq 2$, we choose $\sigma_n \in \mathcal{U}^\circ \setminus \text{span} \{ \omega_1, \dots, \omega_{n_\varepsilon+n-1} \}$ such that

$$\sup |\sigma_n(\mathcal{U} \cup \{A_1\} \cup \dots \cup \{A_{n-1}\})| \leq \min \{ 1, 2^{-1}\lambda_1, \dots, 2^{-1}\lambda_{n-1} \} . \quad (74)$$

Here, for any integer $n \geq 2$ and $j \in \{1, \dots, n-1\}$, $A_j \in \mathcal{X}$ satisfies

$$\text{Re} \{ \sigma_j(A_j) \} = 1 \quad \text{and} \quad \text{Re} \{ \sigma(A_j) \} = 0 , \quad \sigma \in \text{span} \{ \omega_1, \dots, \omega_{n_\varepsilon+j-1} \} . \quad (75)$$

We also have to conveniently choose $\varpi_n \in U_{n-1}$ in order to get the asserted weak* density. Like in the proof of [20, Theorem 4.3] the sequence $(\varpi_n)_{n \in \mathbb{N}}$ is chosen such that

$$\{ \varpi_n \}_{n \in \mathbb{N}} = \{ \varrho_{n,k} \}_{n \in \mathbb{N}_0, k \in \mathbb{N}}$$

and all the functionals $\varrho_{n,k}$ appear infinitely many times in the sequence $(\varpi_n)_{n \in \mathbb{N}}$. In this case, we obtain a weak*-dense set $\{ \omega_n \}_{n \in \mathbb{N}}$ in the convex weak*-compact set

$$U_\infty \doteq \overline{\text{co} \{ \{ \omega_n \}_{n \in \mathbb{N}} \}} \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*) , \quad (76)$$

which, by construction, satisfies

$$\mathfrak{d}_H(U, U_\infty) \leq \sum_{n=0}^{\infty} 2^{-n} \varepsilon = 2\varepsilon$$

for an arbitrary (but previously fixed) $\varepsilon \in \mathbb{R}^+$.

Step $n = \infty$: It remains to verify that $\omega_{n_\varepsilon+j}$, $j \in \mathbb{N}$, are exposed points of U_∞ , whence $U_\infty \in \mathcal{D}$. By (72) with $\varpi_n \in U_{n-1}$ (see (73)), for each natural number $n \geq j+1$, there are $\alpha_{n,j-1}^{(j)}, \dots, \alpha_{n,n}^{(j)} \in [0, 1]$ and $\rho_n^{(j)} \in \text{co} \{ \omega_1, \dots, \omega_{n_\varepsilon+j-1} \}$ such that

$$\alpha_{n,j-1}^{(j)} + \alpha_{n,j}^{(j)} + \sum_{k=j+1}^n \alpha_{n,k}^{(j)} \lambda_k = 1 \quad (77)$$

and

$$\omega_{n_\varepsilon+n} = \alpha_{n,j-1}^{(j)} \rho_n^{(j)} + \alpha_{n,j}^{(j)} \omega_{n_\varepsilon+j} + \sum_{k=j+1}^n \alpha_{n,k}^{(j)} \lambda_k \sigma_k .$$

Additionally, define $\alpha_{n,k}^{(j)} \doteq 1$ for all natural numbers $k \geq n$ while $\alpha_{n,k}^{(j)} \doteq 0$ for $k \in \mathbb{N}_0$ such that $k \leq j-2$. Using (74), (75) and (77), at fixed $j \in \mathbb{N}$, we thus obtain that

$$\begin{aligned} \text{Re} \{ \omega_{n_\varepsilon+n}(A_j) \} &= \alpha_{n,j}^{(j)} \text{Re} \{ \omega_{n_\varepsilon+j}(A_j) \} + \sum_{k=j+1}^n \alpha_{n,k}^{(j)} \lambda_k \text{Re} \{ \sigma_k(A_j) \} \\ &\leq \alpha_{n,j}^{(j)} \lambda_j + \sum_{k=j+1}^n \alpha_{n,k}^{(j)} \lambda_k (2^{-1} \lambda_j) \leq \lambda_j \left(1 - 2^{-1} \sum_{k=j+1}^n \alpha_{n,k}^{(j)} \lambda_k \right) \end{aligned} \quad (78)$$

for any $n \geq j + 1$, while, for any natural number $n \leq j - 1$,

$$\operatorname{Re} \{\omega_{n_\varepsilon+n}(A_j)\} = 0 ,$$

using (75). Fix $j \in \mathbb{N}$ and let $\omega_\infty \in U_\infty$ be a solution to the variational problem

$$\max_{\sigma \in U_\infty} \operatorname{Re} \{\sigma(A_j)\} = \operatorname{Re} \{\omega_\infty(A_j)\} \geq \operatorname{Re} \{\omega_{n_\varepsilon+j}(A_j)\} = \lambda_j . \quad (79)$$

(U_∞ is weak*-compact, by weak*-compactness of $\mathcal{U}^\circ \supseteq U_\infty$.) By weak*-density of $\{\omega_n\}_{n \in \mathbb{N}}$ in U_∞ , there is a sequence $(\omega_{n_\varepsilon+n_l})_{l \in \mathbb{N}}$ converging to ω_∞ in the weak* topology. Since U_j is weak*-compact and $\alpha_{n,k}^{(j)} \in [0, 1]$ for all $k \in \mathbb{N}_0$ and $n, j \in \mathbb{N}$, by a standard argument with a so-called diagonal subsequence, we can choose the sequence $(n_l)_{l \in \mathbb{N}}$ such that $(\rho_{n_l}^{(j)})$ weak*-converges to $\rho_\infty^{(j)} \in U_{j-1}$, and $(\alpha_{n_l,k}^{(j)})_{l \in \mathbb{N}}$ has a limit $\alpha_{\infty,k}^{(j)} \in [0, 1]$ for any fixed $k \in \mathbb{N}_0$ and $j \in \mathbb{N}$. Using (72), (78) and the inequality

$$\sum_{k=j+1}^{n_l} \alpha_{n_l,k}^{(j)} \lambda_k \leq \varepsilon \sum_{k=j+1}^{\infty} 2^{-(k+1)} = 2^{-(j+1)} \varepsilon$$

together with Lebesgue's dominated convergence theorem, we thus arrive at

$$\operatorname{Re} \{\omega_\infty(A_j)\} = \lim_{l \rightarrow \infty} \operatorname{Re} \{\omega_{n_\varepsilon+n_l}(A_j)\} \leq \lambda_j \left(1 - 2^{-1} \sum_{k=j+1}^{\infty} \lambda_k \lim_{l \rightarrow \infty} \alpha_{n_l,k}^{(j)} \right) .$$

Because of (79), it follows that

$$\alpha_{\infty,k}^{(j)} \doteq \lim_{l \rightarrow \infty} \alpha_{n_l,k}^{(j)} = 0 , \quad k \in \{j+1, \dots, \infty\} . \quad (80)$$

As absolute polars are weak* compact [35, Theorem 3.15], for any $A \in \mathcal{X}$, the continuous function $\hat{A}$ defined by (1) satisfies

$$\sup |\hat{A}(\mathcal{U}^\circ)| < \infty .$$

Therefore, by (72), (77) and Lebesgue's dominated convergence theorem, for all $A \in \mathcal{X}$,

$$\lim_{l \rightarrow \infty} \omega_{n_\varepsilon+n_l}(A) = \lim_{l \rightarrow \infty} \alpha_{n_l,j-1}^{(j)} \rho_{n_l}^{(j)}(A) + \omega_{n_\varepsilon+j}(A) \lim_{l \rightarrow \infty} \alpha_{n_l,j}^{(j)} + \sum_{k=j+1}^{\infty} \lambda_k \sigma_k(A) \lim_{l \rightarrow \infty} \alpha_{n_l,k}^{(j)} ,$$

which combined with (80) implies that

$$\lim_{l \rightarrow \infty} \omega_{n_\varepsilon+n_l}(A) = \alpha_{\infty,j-1}^{(j)} \rho_\infty^{(j)}(A) + \alpha_{\infty,j}^{(j)} \omega_{n_\varepsilon+j}(A) ,$$

where $\rho_\infty^{(j)} \in U_{j-1}$, $\alpha_{\infty,j-1}^{(j)}, \alpha_{\infty,j}^{(j)} \in [0, 1]$ and $\alpha_{\infty,j-1}^{(j)} + \alpha_{\infty,j}^{(j)} = 1$. Hence, the sequence $(\omega_{n_\varepsilon+n_l})_{l \in \mathbb{N}}$ weak* converges to an element of U_j . (Recall that U_j is defined by (73) for $n = j \in \mathbb{N}$.) Since $\omega_{n_\varepsilon+j}$ is by construction the unique maximizer of

$$\max_{\sigma \in U_j} \operatorname{Re} \{\sigma(A_j)\} = \operatorname{Re} \{\omega_{n_\varepsilon+j}(A_j)\}$$

and (79) holds true with $\omega_\infty \in U_j$, we deduce that $\omega_\infty = \omega_{n_\varepsilon+j}$, which is thus an exposed point of U_∞ for any $j \in \mathbb{N}$. $\square$

Recall now that $\mathbf{CU}(\mathcal{X}^*)$ is the set of all nonempty, uniformly bounded in a 0-neighborhood, weak*-closed subsets of $\mathcal{X}^*$ defined by (20), that is,

$$\mathbf{CU}(\mathcal{X}^*) \doteq \bigcup_{U \in \mathcal{O}} \mathbf{CF}(\mathcal{X}^*) \cap \mathbf{U}_U(\mathcal{X}^*) \doteq \bigcup_{U \in \mathcal{O}} \mathbf{CU}_U(\mathcal{X}^*) .$$

Provided there is one 0-neighborhood in $\mathcal{X}$ satisfying Condition 4.1, Theorem 4.2 directly implies the weak*-Hausdorff density in $\mathbf{CU}(\mathcal{X}^*)$ of the sets $\mathcal{D}_0$ and $\mathcal{D}$, where

$$\mathcal{D}_0 \doteq \bigcup_{U \in \mathcal{O}} \mathcal{D}_{0,U} \subseteq \mathcal{D} \doteq \bigcup_{U \in \mathcal{O}} \mathcal{D}_U \quad (81)$$

are the sets of all $U \in \mathbf{CU}(\mathcal{X}^*)$ with weak*-dense exposed, respectively extreme, boundary (see (56) and (57)):

Corollary 4.3. (Weak*-Hausdorff density of $\mathcal{D}_0$)

Let $\mathcal{X}$ be a separable topological $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and assume the existence of one 0-neighborhood in $\mathcal{X}$ satisfying Condition 4.1. Then, $\mathcal{D}_0 \subseteq \mathcal{D}$ is a weak*-Hausdorff dense subset of $\mathbf{CU}(\mathcal{X}^*)$.

Proof. By (16), observe that, for any $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{O}$,

$$\mathcal{U}_1 \cap \mathcal{U}_2 \in \mathcal{O} \quad \text{and} \quad \mathcal{U}_1^\circ \cup \mathcal{U}_2^\circ \subseteq (\mathcal{U}_1 \cap \mathcal{U}_2)^\circ .$$

Using this together with Theorem 4.2, one deduces the assertion. $\square$

Our proof of Theorem 4.2 differs in several important aspects from the one of [20, Theorem 4.3], even if it has the same general structure, inspired by Poulsen's construction [34], as already mentioned. To be more precise, as compared to the proof of [20, Theorem 4.3], *Step 0* is new and is a direct consequence of the weak*-compactness of U , a property not assumed in [20, Theorem 4.3]. *Step 1* to *Step* $n \rightarrow \infty$ are similar to what is done in [20], but with the essential difference that convex combinations are used to produce new (strongly) exposed points and the required bounds on $\{\lambda_n, \sigma_n\}_{n \in \mathbb{N}}$ are very different. Compare Equations (72) and (74) with the bounds on v_1, v_2, v_3 given in [20, p. 27-29], at parameters $r_1(t), r_2(t), r_3(t) = 1$. In particular, there is no norm on $\mathcal{X}^*$ ($\mathcal{X}$ is not necessarily a Banach space and, in any case, $\mathcal{X}^*$ is endowed with the weak* topology) and we use estimates on convex combinations that completely differ from what is done in [20, Theorem 4.3]. This corresponds to *Step* $n = \infty$.

Note that [20, Theorem 4.3] shows the density of convex norm-closed sets with dense set of *strongly* exposed points. A strongly exposed point σ_0 in some convex set $C \subseteq \mathcal{X}^*$ is an exposed point for some $\hat{A} \in \mathcal{X}$ with the additional property that any minimizing net of the real part of $\hat{A}$ (cf. (1)) has to converge to σ_0 in the weak* topology²⁰. Note that such a minimizing net is weak*-convergent with the

²⁰ One should not mistake the notion of strongly exposed points discussed here for the notion of weak* strongly exposed points of [33, Definition 5.8], where $\mathcal{X}$ is always a Banach space and a weak* strongly exposed point is a (weak*) exposed point with the additional property that any minimizing net of the real part of $\hat{A}$ has to converge to σ_0 in the norm topology of $\mathcal{X}^*$.

exposed point σ_0 being its limit, by weak* continuity of $\hat{A}$. If $C = K$ is weak*-compact, this yields that any minimizing net converges to σ_0 in the weak* topology. In other words, any exposed point is *automatically* strongly exposed in all convex weak*-compact sets $K \in \mathbf{CK}(\mathcal{X}^*) \supseteq \mathbf{CU}(\mathcal{X}^*)$.

4.3. Extension of the Straszewicz Theorem

In this section, we study the relations between the set $\mathcal{D}_{\mathcal{U}}$ of convex weak*-compact sets with weak*-dense set of *extreme* points and the set $\mathcal{D}_{0,\mathcal{U}}$ of convex weak*-compact sets with weak*-dense set of *exposed* points, for any fixed 0-neighborhood $\mathcal{U} \in \mathcal{O}$. See (56) and (57). In fact, we give a very general condition on the topological vector space $\mathcal{X}$ leading to the equality $\mathcal{D}_{\mathcal{U}} = \mathcal{D}_{0,\mathcal{U}}$ for all $\mathcal{U} \in \mathcal{O}$. This result is used in Section 4.4.

Such a study is reminiscent of the Straszewicz theorem: In 1935, Straszewicz proves [39] that the set of exposed points of a convex compact space of a finite-dimensional space ($\mathbb{R}^n$) is *dense* in the set of extreme points. See, e.g., [1, Theorem 7.89]. An extension of this result²¹ to convex (locally) norm-compact (closed) subsets of an infinite-dimensional normed space was performed by Klee in 1958, see [26, Theorems (2.1), (2.3)]. In 1976, Bair [4, Theorem 1] proves the Straszewicz theorem in an arbitrary real vector space for algebraically closed convex sets with so-called finite “copointure”, see [5, Section II.5.1]. This last condition cannot be satisfied by weak*-compact sets. In fact, such studies on dual spaces $\mathcal{X}^*$ have been performed by Larman and Phelps in [29] for special Banach spaces $\mathcal{X}$, named *Gateaux-differentiability space* [33, Definition 6.1]. For topological vector spaces, it means the following:

Definition 4.4. (Gateaux-differentiability space)

A *Gateaux-differentiability space* $\mathcal{X}$ is a topological vector space on which every continuous convex real-valued function with a nonempty open convex subset as domain is Gateaux-differentiable on a dense set in that domain.

Compare with [33, Definition 6.1]. Recall that a *weak Asplund* space is a topological vector space on which every continuous convex real-valued function with a nonempty open convex subset as domain is Gateaux-differentiable on a *generic* set in that domain. See, e.g., [33, Definition 1.22] for the Banach case or [37, p. 203] in the general case. Phelps explains in [33, p. 95] that the new space class of Definition 4.4 is obviously “formally larger than the class of weak Asplund spaces, but in some ways is a more natural object of study.” All these spaces are reminiscent of the celebrated Mazur theorem [31, Satz 2] (see also [10, Theorem 10.44]) proven for separable real Banach spaces.

If the Banach space $\mathcal{X}$ is a Gateaux-differentiability space then, by [33, Theorem 6.2], every convex weak*-compact set $K \in \mathbf{CK}(\mathcal{X}^*)$ is the weak*-closed convex hull of its exposed points. By the Milman theorem [10, Theorem 10.13] it follows that the set $\mathcal{E}_0(K)$ of exposed points is weak*-dense in the set $\mathcal{E}(K)$. [33, Theorem 6.2] refers to Banach spaces and we give here another extension of the Straszewicz

²¹ There is also a result [2] of Asplund in 1963 generalizing the Straszewicz theorem to so-called *k*-exposed and *k*-extreme points in the finite-dimensional space $\mathbb{R}^n$.

theorem to all (possibly non-Banach) Gateaux-differentiability spaces, because the weak*-density of the set of exposed points is an important ingredient in the next subsection. Our proof is quite direct and thus, relatively pedagogical while being very general:

Theorem 4.5. (Extension of the Straszewicz theorem - I)

Let $\mathcal{X}$ be a Gateaux-differentiability $\mathbb{K}$ -vector space with $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then, for any convex weak*-compact set $K \in \mathbf{CK}(\mathcal{X}^*)$, the set of exposed points of K is weak*-dense in $\mathcal{E}(K)$.

Proof. Fix all parameters of the theorem. Assume without loss of generality that $\mathcal{X}$ is a $\mathbb{C}$ -vector space. (The case $\mathbb{K} = \mathbb{R}$ is even slightly simpler.) Denote by $\mathcal{E}_0(K) \subseteq \mathcal{E}(K)$ the set of all exposed points of K . Assume that

$$\overline{\text{co}}(\mathcal{E}_0(K)) \subsetneq K, \quad (82)$$

i.e., there is an element $\sigma_0 \in K \setminus \overline{\text{co}}(\mathcal{E}_0(K))$. We thus infer from the Hahn-Banach separation theorem [35, Theorem 3.4 (b)] the existence of $A_0 \in \mathcal{X}$ such that

$$\max \{ \text{Re} \{ \sigma(A_0) \} : \sigma \in \overline{\text{co}}(\mathcal{E}_0(K)) \} < \text{Re} \{ \sigma_0(A_0) \}. \quad (83)$$

Since K is weak*-compact,

$$h(A) \doteq \max_{\sigma \in K} \text{Re} \{ \hat{A}(\sigma) \}, \quad A \in \mathcal{X}, \quad (84)$$

defines a continuous mapping $h : \mathcal{X} \rightarrow \mathbb{R}$. Observe that h is a convex function because it is the maximum of a family of linear functions. Recall that a tangent $\mathbb{R}$ -linear functional $dh(A)$ at $A \in \mathcal{Y}$ satisfies

$$h(A+B) - h(A) \geq [dh(A)](B), \quad B \in \mathcal{X},$$

by definition. Any maximizer $\sigma_A \in K$ of the variational problem (84) yields a (continuous) tangent ($\mathbb{R}$ -linear) functional:

$$h(A+B) - h(A) \geq \text{Re} \{ \sigma_A(B) \}, \quad B \in \mathcal{X}.$$

Therefore, if there is a unique continuous tangent ($\mathbb{R}$ -linear) functional $dh(A)$ at $A \in \mathcal{Y}$, then the solution $\sigma_A \in K$ of the variational problem (84) is unique:

$$\sigma_A(B) = [dh(A)](B) + i [dh(A)](-iB), \quad B \in \mathcal{X},$$

by $\mathbb{C}$ -linearity of σ_A . In particular, σ_A is, in this case, an exposed point of K . By Definition 4.4, there is a net $(\sigma_j)_{j \in J}$ of exposed points of K as well as a net $(A_j)_{j \in J} \in \mathcal{X}$ converging to A_0 such that

$$\text{Re} \{ \sigma_j(A_j) \} = \max_{\sigma \in K} \text{Re} \{ \hat{A}_j(\sigma) \}, \quad j \in J. \quad (85)$$

By taking any maximizer of $\text{Re} \{ \hat{A}_0(\sigma) \}$ over $\sigma \in K$, note that

$$\liminf_J \text{Re} \{ \sigma_j(A_j) \} = \liminf_J \max_{\sigma \in K} \text{Re} \{ \hat{A}_j(\sigma) \} \geq \max_{\sigma \in K} \text{Re} \{ \hat{A}_0(\sigma) \} \quad (86)$$

while, by compactness of K , we can assume without loss of generality that $(\sigma_j)_{j \in J}$ converges to some $\sigma_\infty \in K$. Now, by the Banach-Steinhaus theorem [35, Theorem 2.5] together with (85)–(86), it follows that

$$\begin{aligned} \operatorname{Re}\{\sigma_\infty(A_0)\} &= \lim_J \operatorname{Re}\{\sigma_j(A_0)\} = \lim_J \operatorname{Re}\{\sigma_j(A_j)\} \\ &= \lim_J \max_{\sigma \in K} \operatorname{Re}\{\hat{A}_j(\sigma)\} = \max_{\sigma \in K} \operatorname{Re}\{\hat{A}_0(\sigma)\}. \end{aligned}$$

In fact, the second equality above, i.e.,

$$\lim_J \operatorname{Re}\{\sigma_j(A_0)\} = \lim_J \operatorname{Re}\{\sigma_j(A_j)\}, \quad (87)$$

results from the Banach-Steinhaus theorem: The family $(\hat{A}_j)_{j \in J}$, as defined by (1) for $A = A_j$, $j \in J$, is a collection of continuous linear mappings from $\mathcal{X}^*$ to $\mathbb{K}$ and the set $\{\hat{A}_j(\sigma) : j \in J\}$ is bounded for any $\sigma \in \mathcal{X}^*$, by convergence of $(A_j)_{j \in J}$ to A_0 . By [35, Theorem 2.5], $(\hat{A}_j)_{j \in J}$ is equicontinuous and Equation (87) follows. As a consequence, by (83), there is an exposed point outside $\overline{\operatorname{co}}(\mathcal{E}_0(K))$, which contradicts (82). Therefore, $\overline{\operatorname{co}}(\mathcal{E}_0(K)) = K$ and, by the Milman theorem [10, Theorem 10.13], $\mathcal{E}(K) \subseteq \overline{\mathcal{E}_0(K)}$. $\square$

Corollary 4.6. (Extension of the Straszewicz theorem - II)

Let $\mathcal{X}$ be a Gateaux-differentiability $\mathbb{K}$ -vector space ($\mathbb{K} = \mathbb{R}, \mathbb{C}$). Then, for any $\mathcal{U} \in \mathcal{O}$, $\mathcal{D}_{0,\mathcal{U}} = \mathcal{D}_{\mathcal{U}}$ with $\mathcal{D}_{\mathcal{U}}$ and $\mathcal{D}_{0,\mathcal{U}}$ being respectively defined by (56) and (57).

Proof. The assertion is an obvious consequence of Theorem 4.5. $\square$

Obviously, Theorem 4.5 and Corollary 4.6 hold true for *all* weak Asplund spaces. Well-known examples of such spaces are separable Baire spaces: Recall that a topological space is a Baire space if every non-empty open subset of this space is non-meager. See, e.g., [36, Prerequisites B.9]. Both completely metrizable spaces and locally compact Hausdorff spaces are Baire spaces, by Baire's theorem (also known as the category theorem) [35, Theorem 2.2]. In particular, Banach spaces are very specific Baire spaces. By [37, Theorem 2.1], the Mazur theorem can be extended to all *separable* topological vector spaces which are Baire spaces. We thus obtain the following corollary:

Corollary 4.7. (Extension of the Straszewicz theorem – separable case)

If the topological vector space $\mathcal{X}$ is a separable Baire space then Theorem 4.5 and Corollary 4.6 hold true.

Proof. Combine [37, Theorem 2.1] with Theorem 4.5 and Corollary 4.6. $\square$

Separability is not a necessary condition in Corollary 4.7. In nonseparable Banach space theory, a pivotal role is played by the so-called *weakly compactly generated* Banach spaces $\mathcal{X}$, meaning that $\mathcal{X}$ is the closed linear span of a weakly compact subset. See [43, Definition 3.1] or [33, Definition 2.41]. Weakly compactly generated Banach spaces have been proven to be weak Asplund spaces²² [3, Theorem 2]. We thus obtain the following assertion:

²² In fact, Asplund proves that if $\mathcal{X}$ admits an equivalent norm which has a strictly convex dual norm then $\mathcal{X}$ is a weak Asplund space. See [33, Corollary 2.39].

Corollary 4.8. (Extension of the Straszewicz theorem – nonseparable case)

If $\mathcal{X}$ is a weakly compactly generated Banach space then Theorem 4.5 and Corollary 4.6 hold true.

Proof. Combine [3, Theorem 2] with Theorem 4.5 and Corollary 4.6. $\square$

For more details on weak Asplund spaces, see for instance [33, 19].

In 1979, Larman and Phelps in [29] raised the question whether every Gateaux-differentiability space is a weak Asplund space. Known examples [17, 40, 19] of individual convex continuous functions that are Gateaux-differentiable on a dense, *but non-residual*, subset of their domain suggest that a Gateaux-differentiability space is not necessarily a weak Asplund space. A first answer to Larman and Phelps's question has been given in 2006, in [32] where a Gateaux-differentiability space $\mathcal{X}$ that is *not* a weak Asplund space is constructed.

This shows that Theorem 4.5 and Corollary 4.6 are very general, probably *optimum*, results on the weak*-density of the set of exposed points in the extreme boundary of a convex weak*-compact set $K \in \mathbf{CK}(\mathcal{X}^*)$. If $\mathcal{X}$ is a Banach space, note that [33, Theorem 6.2] already tells us that a convex weak*-compact set $K \in \mathbf{CK}(\mathcal{X}^*)$ is the weak*-closed convex hull of its exposed points *iff* $\mathcal{X}$ is a Gateaux-differentiability space. This equivalence holds probably true for more general (not necessarily Banach) topological vector spaces.

4.4. G_δ subsets of convex weak*-compact sets

Recall that

$$\mathbf{CU}_\mathcal{U}(\mathcal{X}^*) \doteq \mathbf{CF}(\mathcal{X}^*) \cap \mathbf{U}_\mathcal{U}(\mathcal{X}^*) \doteq \{U \in \mathbf{CF}(\mathcal{X}^*) : U \subseteq \mathcal{U}^\circ\} .$$

See (16)–(17) and (19). Because of Corollary 3.13 (iii), it is a weak*-Hausdorff-closed space. If $\mathcal{X}$ is separable then it is even weak*-Hausdorff-compact and completely metrizable, by Theorem 3.17. From the Banach-Alaoglu theorem [35, Theorem 3.15], every $U \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ is a convex weak*-compact set and is thus the weak*-closure of the convex hull of the (nonempty) set $\mathcal{E}(U)$ of its extreme points (cf. the Krein-Milman theorem [35, Theorem 3.23]).

Having all this information in mind together with the Straszewicz theorem (Corollary 4.6), we are in a position to show that, for general separable topological vector spaces $\mathcal{X}$ and each $\mathcal{U} \in \mathcal{O}$, the subset

$$\mathcal{D}_\mathcal{U} \doteq \left\{ U \in \mathbf{CU}_\mathcal{U}(\mathcal{X}^*) : U = \overline{\mathcal{E}(U)} \right\} \subseteq \mathbf{CU}_\mathcal{U}(\mathcal{X}^*) ,$$

already defined in (56), is a G_δ subset of $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$ endowed with the weak*-Hausdorff hypertopology:

Theorem 4.9. ($\mathcal{D}_\mathcal{U}$ as a G_δ set)

Let $\mathcal{X}$ be a separable Gateaux-differentiability space (Definition 4.4). Then, for any $\mathcal{U} \in \mathcal{O}$, $\mathcal{D}_\mathcal{U}$ is a G_δ subset of $\mathbf{CU}_\mathcal{U}(\mathcal{X}^*)$.

Proof. Let $\mathcal{X}$ be a separable Gateaux-differentiability space. Assume without loss of generality that $\mathcal{X}$ is a $\mathbb{C}$ -vector space. (The case $\mathbb{K} = \mathbb{R}$ is even simpler.)

For any $\mathcal{U} \in \mathcal{O}$, we use the metric d defined by (54) and generating the weak* topology on the absolute polar $\mathcal{U}^\circ$ defined by (16). For any $\mathcal{U} \in \mathcal{O}$, recall that we denote by $B(\omega, r) \subseteq \mathcal{U}^\circ$ the weak*-open ball of radius $r \in \mathbb{R}^+$ centered at $\omega \in \mathcal{U}^\circ$, defined by (59). Fix once and for all $\mathcal{U} \in \mathcal{O}$. Then, for any $m \in \mathbb{N}$, let $\mathcal{F}_{\mathcal{U},m}$ be the set of all nonempty convex weak*-compact subsets $U \subseteq \mathcal{U}^\circ$ such that $B(\omega, 1/m) \cap \mathcal{E}(U) = \emptyset$ for some $\omega \in U$, i.e.,

$$\mathcal{F}_{\mathcal{U},m} \doteq \{U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) : \exists \omega \in U, B(\omega, 1/m) \cap \mathcal{E}(U) = \emptyset\} \subseteq \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) . \quad (88)$$

Recall that $\mathcal{E}(U)$ is the nonempty set of extreme points of U . Now, observe that the complement of $\mathcal{D}_{\mathcal{U}}$ in $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ equals

$$\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) \setminus \mathcal{D}_{\mathcal{U}} = \bigcup_{m \in \mathbb{N}} \mathcal{F}_{\mathcal{U},m} . \quad (89)$$

Therefore, $\mathcal{D}_{\mathcal{U}}$ is a G_δ subset of $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ if $\mathcal{F}_{\mathcal{U},m}$ is a weak*-Hausdorff-closed set for any $m \in \mathbb{N}$.

By Theorem 3.17, the weak*-Hausdorff hypertopology of $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ is metrizable and $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ is a weak*-Hausdorff-closed set (see Corollary 3.13 (iii)). So, fix $m \in \mathbb{N}$ and take any sequence $(U_n)_{n \in \mathbb{N}} \subseteq \mathcal{F}_{\mathcal{U},m}$ converging with respect to the weak*-Hausdorff hypertopology to $U_\infty \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$. For any $n \in \mathbb{N}$, there is $\omega_n \in U_n$ such that $B(\omega_n, 1/m) \cap \mathcal{E}(U_n) = \emptyset$. By metrizability and weak* compactness of the absolute polar $\mathcal{U}^\circ$ and Corollary 3.18, there is a subsequence $(\omega_{n_k})_{k \in \mathbb{N}}$ converging to some $\omega_\infty \in U_\infty$. Assume that, for some $\varepsilon \in (0, 1/m)$, there is $\sigma_\infty \in \mathcal{E}(U_\infty)$ such that

$$d(\omega_\infty, \sigma_\infty) \leq \frac{1}{m} - \varepsilon .$$

Recall meanwhile that $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) \subseteq \mathbf{CK}(\mathcal{X}^*)$. Then, by weak*-density of the set of exposed points in $\mathcal{E}(U_\infty)$ (Theorem 4.5), we can assume without loss of generality that σ_∞ is an exposed point. In particular, there is $A \in \mathcal{X}$ such that

$$\max_{\sigma \in U_\infty} \operatorname{Re}\{\hat{A}(\sigma)\} = \hat{A}(\sigma_\infty) , \quad (90)$$

with σ_∞ being the *unique* maximizer in U_∞ and where $\hat{A}$ is the mapping $\sigma \mapsto \sigma(A)$ from $\mathcal{X}^*$ to $\mathbb{C}$ defined by (1). Consider now the sets

$$\mathcal{M}_n \doteq \left\{ \tilde{\sigma} \in U_n : \max_{\sigma \in U_n} \operatorname{Re}\{\hat{A}(\sigma)\} = \hat{A}(\tilde{\sigma}) \right\} , \quad n \in \mathbb{N} .$$

By linearity and weak*-continuity of the function $\hat{A}$, together with the weak*-compactness of U_n , the set $\mathcal{M}_n$ is a convex weak*-compact subset of U_n for any $n \in \mathbb{N}$. In fact, $\mathcal{M}_n$ is a (weak*-closed) face²³ of U_n and thus, any extreme point of $\mathcal{M}_n$ belongs to $\mathcal{E}(U_n)$.

²³ It means that if $\sigma \in \mathcal{M}_n$ is a finite convex combination of elements $\sigma_j \in U_n$ then all $\sigma_j \in \mathcal{M}_n$.

So, pick any extreme point $\sigma_n \in \mathcal{E}(U_n)$ of $\mathcal{M}_n$ for each $n \in \mathbb{N}$. Since

$$\begin{aligned} \max_{\sigma \in U_n} \operatorname{Re}\{\hat{A}(\sigma)\} - \max_{\tilde{\sigma} \in U_\infty} \operatorname{Re}\{\hat{A}(\tilde{\sigma})\} &= \max_{\sigma \in U_n} \min_{\tilde{\sigma} \in U_\infty} \operatorname{Re}\{\hat{A}(\sigma - \tilde{\sigma})\} \\ &\leq \max_{\sigma \in U_n} \min_{\tilde{\sigma} \in U_\infty} |(\sigma - \tilde{\sigma})(A)| , \end{aligned}$$

$$\begin{aligned} \text{and} \quad \max_{\tilde{\sigma} \in U_\infty} \operatorname{Re}\{\hat{A}(\tilde{\sigma})\} - \max_{\sigma \in U_n} \operatorname{Re}\{\hat{A}(\sigma)\} &= \max_{\tilde{\sigma} \in U_\infty} \min_{\sigma \in U_n} \operatorname{Re}\{\hat{A}(\tilde{\sigma} - \sigma)\} \\ &\leq \max_{\tilde{\sigma} \in U_\infty} \min_{\sigma \in U_n} |(\sigma - \tilde{\sigma})(A)| , \end{aligned}$$

we deduce from Definition 2.3 and the weak*-Hausdorff convergence of $(U_n)_{n \in \mathbb{N}}$ to U_∞ that

$$\lim_{n \rightarrow \infty} \operatorname{Re}\{\hat{A}(\sigma_n)\} = \lim_{n \rightarrow \infty} \max_{\sigma \in U_n} \operatorname{Re}\{\hat{A}(\sigma)\} = \max_{\sigma \in U_\infty} \operatorname{Re}\{\hat{A}(\sigma)\} = \hat{A}(\sigma_\infty) .$$

Therefore, keeping in mind the convergence of the subsequence $(\omega_{n_k})_{k \in \mathbb{N}}$ towards $\omega_\infty \in U_\infty$, there is a subsequence $(\sigma_{n_{k(l)}})_{l \in \mathbb{N}}$ of $(\sigma_{n_k})_{k \in \mathbb{N}}$ (itself being a subsequence of $(\sigma_n)_{n \in \mathbb{N}}$) converging to σ_∞ , as it is the *unique* maximizer of (90) and $\hat{A}$ is weak*-continuous. Since, for any $l \in \mathbb{N}$,

$$\begin{aligned} d(\sigma_{n_{k(l)}}, \omega_{n_{k(l)}}) &\leq d(\sigma_\infty, \omega_\infty) + d(\omega_\infty, \omega_{n_{k(l)}}) + d(\sigma_{n_{k(l)}}, \sigma_\infty) \\ &\leq \frac{1}{m} - \varepsilon + d(\omega_\infty, \omega_{n_{k(l)}}) + d(\sigma_{n_{k(l)}}, \sigma_\infty) \end{aligned}$$

with $\varepsilon \in (0, 1/m)$ and $\sigma_n \in \mathcal{E}(U_n)$ for $n \in \mathbb{N}$, we thus arrive at a contradiction. Therefore, $U_\infty \in \mathcal{F}_{\mathcal{U}, m}$. This means that $\mathcal{F}_{\mathcal{U}, m}$ is a weak*-Hausdorff-closed set for any $m \in \mathbb{N}$ and hence, the countable union (89) is a F_σ set with complement being $\mathcal{D}_{\mathcal{U}}$. The assertion follows, as the complement of an F_σ set is a G_δ set. $\square$

Corollary 4.10. ($\mathcal{D}_{\mathcal{U}}$ as a G_δ set)

If the topological vector space $\mathcal{X}$ is a separable Baire space then, for any $\mathcal{U} \in \mathcal{O}$, $\mathcal{D}_{\mathcal{U}}$ is a G_δ subset of $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^)$.*

Proof. Combine [37, Theorem 2.1] with Theorem 4.9. $\square$

To conclude this section, recall now that $\mathbf{CU}(\mathcal{X}^*)$ is the set of all nonempty, uniformly bounded in a 0-neighborhood, weak*-closed subsets of $\mathcal{X}^*$ defined by (20), that is,

$$\mathbf{CU}(\mathcal{X}^*) \doteq \bigcup_{\mathcal{U} \in \mathcal{O}} \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) .$$

By considering the special case of completely metrizable topological vector spaces $\mathcal{X}$, which are separable Baire spaces, one can also prove that the subset

$$\mathcal{D} \doteq \bigcup_{\mathcal{U} \in \mathcal{O}} \mathcal{D}_{\mathcal{U}}$$

of all $U \in \mathbf{CU}(\mathcal{X}^*)$ with weak*-dense extreme boundary (see (56)) is also a G_δ subset of $\mathbf{CU}(\mathcal{X}^*)$, endowed with the weak*-Hausdorff hypertopology:

Corollary 4.11. ($\mathcal{D}$ as a G_δ set)

If $\mathcal{X}$ is a separable, completely metrizable topological vector space then $\mathcal{D}$ is a G_δ subset of $\mathbf{CU}(\mathcal{X}^*)$.

Proof. By assumption, there is a metric $d_{\mathcal{X}}$ generating the topology on $\mathcal{X}$ and, since $\mathcal{U}_1 \subseteq \mathcal{U}_2$ yields $\mathcal{U}_2^\circ \subseteq \mathcal{U}_1^\circ$ (see (16)), we observe that

$$\mathbf{CU}(\mathcal{X}^*) = \bigcup_{D \in \mathbb{N}} \mathbf{CU}_{B(0, D^{-1})}(\mathcal{X}^*) \quad \text{and} \quad \mathbf{CU}(\mathcal{X}^*) \setminus \mathcal{D} = \bigcup_{D, m \in \mathbb{N}} \mathcal{F}_{B(0, D^{-1}), m}, \quad (91)$$

where $\mathcal{F}_{\mathcal{U}, m}$ is defined by (88) for any $\mathcal{U} \in \mathcal{O}$ and

$$B(0, R) \doteq \{A \in \mathcal{X} : d_{\mathcal{X}}(0, A) < R\} \in \mathcal{O} \quad (92)$$

is the open ball of radius $R \in \mathbb{R}^+$ in $\mathcal{X}$. Any completely metrizable topological vector space is a Gateaux-differentiability space, by [37, Theorem 2.1]. As shown in the proof of Theorem 4.9, $\mathcal{F}_{\mathcal{U}, m}$ is thus a weak*-Hausdorff-closed set for any $m \in \mathbb{N}$ and $\mathcal{U} \in \mathcal{O}$. By (91), the assertion follows. $\square$

4.5. Generic convex weak*-compact sets

Outcomes of Sections 4.2 and 4.4 for separable, infinite-dimensional topological vector spaces directly yield, as stated in the next theorem, that the convex weak*-compact sets with weak*-dense extreme boundary are generic convex weak*-compact sets, answering the main question raised in the introduction of Section 4. Recall that $\mathcal{D}_{\mathcal{U}}$ and $\mathcal{D}$ are the sets of convex weak*-compact sets with weak*-dense extreme boundary, respectively defined by (56) and (81).

Theorem 4.12. (Generic convex weak*-compact sets for topological spaces)

- (i) Let $\mathcal{X}$ be a separable Gateaux-differentiability space (Definition 4.4) and a 0-neighborhood $\mathcal{U} \in \mathcal{O}$ satisfying Condition 4.1. Then, $\mathcal{D}_{\mathcal{U}}$ is a weak*-Hausdorff-dense G_δ subset of $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$.
- (ii) Let $\mathcal{X}$ be a separable, completely metrizable space and assume the existence of some 0-neighborhood in $\mathcal{X}$ satisfying Condition 4.1. Then $\mathcal{D}$ is a weak*-Hausdorff-dense G_δ subset of $\mathbf{CU}(\mathcal{X}^*)$.

Proof. In order to prove the first statement (i), combine Theorem 4.2 with Theorem 4.9. Assertion (ii) is a direct consequence of Corollaries 4.3 and 4.11. $\square$

As a matter of fact, the Hausdorff metric topology is very fine, as compared to various standard hypertopologies (apart from the Vietoris²⁴ hypertopology). Consequently, the weak*-Hausdorff hypertopology can be seen as a very fine, weak*-type, topology on $\mathbf{CU}(\mathcal{X}^*)$. It means that the density of the subset of all convex weak*-compact sets with weak*-dense set of extreme points stated in Theorem 4.12 is a very strong property. The study of generic convex weak*-compact sets can also be performed within special weak*-closed subsets of the dual space $\mathcal{X}^*$. An important example is given by so-called *positive* continuous functionals.

²⁴ Vietoris and Hausdorff metric topologies are not comparable.

In order to present this example, we consider a topological $\mathbb{R}$ -vector space $\mathcal{X}$. As is usual, $\mathcal{X}^+ \subseteq \mathcal{X}$ is said to be a *convex cone* in $\mathcal{X}$ if, for all $\lambda \in \mathbb{R}^+$ and $A_1, A_2 \in \mathcal{X}^+$, $\lambda A_1 \in \mathcal{X}^+$ (cf. cone) and $A_1 + A_2 \in \mathcal{X}^+$ (cf. convex). Such a $\mathcal{X}^+$ is of course a convex set in the usual sense. Recall that any convex cone $\mathcal{X}^+ \subseteq \mathcal{X}$ naturally defines a preorder relation²⁵ $\preceq$ in $\mathcal{X}$ as follows: $A_1 \preceq A_2$ iff $A_2 - A_1 \in \mathcal{X}^+$. This preorder is compatible²⁶ with the *real* vector space structure of $\mathcal{X}$. Given such a preorder, intervals in $\mathcal{X}$ are defined to be the sets

$$[A_1, A_2] \doteq \{A \in \mathcal{X} : A_1 \preceq A \preceq A_2\} \subseteq \mathcal{X}, \quad A_1, A_2 \in \mathcal{X}. \quad (93)$$

A linear functional on $\mathcal{X}$ is defined to be *positive* if its values on $\mathcal{X}^+$ are non-negative. The set of all continuous positive functionals is denoted by

$$\mathcal{X}^{*,+} \doteq \bigcap_{A \in \mathcal{X}^+} \{\sigma \in \mathcal{X}^* : \sigma(A) \geq 0\} \subseteq \mathcal{X}^*.$$

Being the intersection of weak*-closed sets, $\mathcal{X}^{*,+}$ is weak*-closed. Therefore, we consider the hyperspaces defined, for any 0-neighborhood $\mathcal{U} \in \mathcal{O}$, by

$$\begin{aligned} \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+}) &\doteq \{U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*) : U \subseteq \mathcal{X}^{*,+}\} \\ \text{and} \quad \mathbf{CU}(\mathcal{X}^{*,+}) &\doteq \bigcup_{\mathcal{U} \in \mathcal{O}} \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+}). \end{aligned} \quad (94)$$

Similar to $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$ (Corollary 3.13 (iii)), $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+})$ is a weak*-Hausdorff-closed space, at least under the separability condition:

Lemma 4.13. (Complete metrizability of hyperspaces for positive functionals)
Let $\mathcal{X}$ be a separable topological $\mathbb{R}$ -vector space. Then, for any $\mathcal{U} \in \mathcal{O}$, $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{,+})$ is a weak*-Hausdorff-closed subset of the compact and completely metrizable hyperspace $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$.*

Proof. Fix any 0-neighborhood $\mathcal{U} \in \mathcal{O}$. By (94) and Theorem 3.17, $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+})$ belongs to the weak*-Hausdorff-compact and completely metrizable hyperspace $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^*)$. By Corollary 3.18 and because $\mathcal{X}^{*,+}$ is a weak*-closed set, $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+})$ is weak*-Hausdorff-closed. $\square$

Using (56), (57) and (81), we naturally define the subsets

$$\mathcal{D}_{\mathcal{U}}^+ \doteq \mathcal{D}_{\mathcal{U}} \cap \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+}) \quad \text{and} \quad \mathcal{D}_{0,\mathcal{U}}^+ \doteq \mathcal{D}_{0,\mathcal{U}} \cap \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+}) \subseteq \mathcal{D}_{\mathcal{U}}^+$$

for any $\mathcal{U} \in \mathcal{O}$, as well as

$$\mathcal{D}^+ \doteq \mathcal{D} \cap \mathbf{CU}(\mathcal{X}^{*,+}) \quad \text{and} \quad \mathcal{D}_0^+ \doteq \mathcal{D}_0 \cap \mathbf{CU}(\mathcal{X}^{*,+}) \subseteq \mathcal{D}^+. \quad (95)$$

Then, similar to Theorem 4.2 and Corollary 4.3, we then get the following density property within the set of positive continuous functionals:

²⁵ I.e., a binary relation that is reflexive and transitive, but not necessarily antisymmetric.

²⁶ That is, if $A_1 \preceq A_2$ then, for any $\alpha \in \mathbb{R}^+$ and $A \in \mathcal{X}$, $\alpha A_1 \preceq \alpha A_2$ and $A_1 + A \preceq A_2 + A$.

Theorem 4.14. (Weak*-Hausdorff density of $\mathcal{D}_{0,\mathcal{U}}^+$)

Let $\mathcal{X}$ be a separable locally convex $\mathbb{R}$ -vector space with $\mathcal{X}^+ \subseteq \mathcal{X}$ being a convex cone. Assume that, for all $\mathcal{U} \in \mathcal{O}$, there is an interval $[A_1, A_2] \subseteq \mathcal{X}$ and some $\tilde{\mathcal{U}} \in \mathcal{O}$ such that $\tilde{\mathcal{U}} \subseteq [A_1, A_2] \subseteq \mathcal{U}$. Then, for any $\mathcal{U} \in \mathcal{O}$ satisfying Condition 4.1 and $U \in \mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+})$, there is $\tilde{\mathcal{U}} \in \mathcal{O}$ with $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ and a sequence $(U_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}_{0,\tilde{\mathcal{U}}}^+ \subseteq \mathcal{D}_{\tilde{\mathcal{U}}}^+$ converging to U in the weak*-Hausdorff topology.

Proof. In order to show the assertion, it suffices to reproduce the proof of Theorem 4.2, with the addition of one essential ingredient: the decomposition of *equicontinuous* linear functionals into positive *equicontinuous* components [42], as originally proven by Grosberg and Krein [22] for normed spaces and by Bonsall [8] for locally convex $\mathbb{R}$ -vector spaces. For any $\mathcal{U} \in \mathcal{O}$, this means that there is $\tilde{\mathcal{U}} \in \mathcal{O}$ with $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ such that *every* continuous linear functional $\sigma \in \mathcal{U}^\circ$ can be decomposed as

$$\sigma = \rho_1 - \rho_2, \quad \rho_1, \rho_2 \in \mathcal{X}^{*,+} \cap \tilde{\mathcal{U}}^\circ. \quad (96)$$

In order to prove this assertion, all conditions of the theorem are necessary, except Condition 4.1 and the separability of $\mathcal{X}$. Note in particular that the existence of arbitrarily small neighborhoods of the origin which are intervals, as assumed here, is a necessary and sufficient condition to obtain the decomposition of equicontinuous linear functionals in any real locally convex vector space, as shown in [42].

Then, at *Step 1* of the proof of Theorem 4.2, because of (96), there is

$$\sigma_1 \in (\tilde{\mathcal{U}}^\circ \setminus \text{span}\{\omega_1, \dots, \omega_{n_\varepsilon}\}) \cap \mathcal{X}^{*,+}.$$

So, we proceed by taking σ_1 as a positive continuous functional in $\tilde{\mathcal{U}}^\circ$ instead of a general functional of $\mathcal{U}^\circ$. One then iterates the arguments, as explained in the proof of Theorem 4.2, always taking a positive continuous functional $\sigma_n \in \mathcal{X}^{*,+} \cap \tilde{\mathcal{U}}^\circ$ appearing in the positive decomposition (96) (ρ_1 or ρ_2) of a continuous functional $\sigma \in \mathcal{U}^\circ$, as already explained. In doing so, we ensure that the convex weak*-compact set U_∞ of Equation (76) belongs to $\mathbf{CU}_{\tilde{\mathcal{U}}}(\mathcal{X}^{*,+})$. Of course, the neighborhood $\mathcal{U}$ in Equations (62), (68) and (74) in the proof of Theorem 4.2 has to be replaced by $\tilde{\mathcal{U}}$. $\square$

Observe that the last theorem is not a direct consequence of Theorem 4.2 because the complement of $\mathcal{D}_{0,\mathcal{U}}^+$ is generally open and dense in $\mathcal{D}_{0,\mathcal{U}}$.

Corollary 4.15. (Weak*-Hausdorff density of $\mathcal{D}_0^+$)

Assume conditions of Theorem 4.14 together with the existence of one 0-neighborhood satisfying Condition 4.1. Then, $\mathcal{D}_0^+ \subseteq \mathcal{D}^+$ is a weak*-Hausdorff dense subset of $\mathbf{CU}(\mathcal{X}^{*,+})$.

Proof. The intersection of any 0-neighborhood $\mathcal{U} \in \mathcal{O}$ with the 0-neighborhood satisfying Condition 4.1 leads to a new 0-neighborhood satisfying Condition 4.1. In particular, we can assume without loss of generality that any element $U \in \mathbf{CU}(\mathcal{X}^{*,+})$ belongs to some $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+})$ with $\mathcal{U} \in \mathcal{O}$ satisfying Condition 4.1. The assertion is then a direct consequence of Theorem 4.14. $\square$

By Lemma 4.13 combined with Theorem 3.17 and Corollary 3.18, note that the proof of Theorem 4.9 can be straightforwardly adapted in order to prove that, for any 0-neighborhood $\mathcal{U} \in \mathcal{O}$, $\mathcal{D}_{\mathcal{U}}^+$ is a G_δ subset of $\mathbf{CU}_{\mathcal{U}}(\mathcal{X}^{*,+})$, provided $\mathcal{X}$ is a separable Gateaux-differentiability space (Definition 4.4). If $\mathcal{X}$ is a separable, completely metrizable topological vector space then $\mathcal{D}^+$ is a G_δ subset of $\mathbf{CU}(\mathcal{X}^{*,+})$. Cf. proof of Corollary 4.11. Using this together with Corollary 4.15, one directly gets:

Theorem 4.16. (Generic convex weak*-compact sets – positive functionals)
Assume the conditions of Corollary 4.15 with $\mathcal{X}$ being a completely metrizable topological vector space. Then, $\mathcal{D}^+$ is a weak-Hausdorff-dense G_δ subset of $\mathbf{CU}(\mathcal{X}^{*,+})$.*

Proof. See proofs of Theorem 4.9 and Corollary 4.11 together with Theorem 3.17, Corollary 3.18, Lemma 4.13 and Corollary 4.15. We omit the details. $\square$

Such results for sets of positive functionals is, e.g., useful in the context of separable C^* -algebras, as explained in Section 4.6, because closed balls centered at the origin in the (real) Banach space of all self-adjoint elements of such algebras are intervals.

4.6. Application to separable Banach spaces

If $\mathcal{X}$ is a Banach space then recall that $\mathbf{B}(\mathcal{X}^*)$ equals the set of all nonempty norm-bounded weak*-closed subsets of $\mathcal{X}^*$ and a set in $\mathcal{X}^*$ is norm-bounded and weak*-closed iff it is weak*-compact. See, e.g., [7, Proposition 1.2.9]. In particular, in this situation, absolute polars can be replaced with norm-closed balls in $\mathcal{X}^*$ and

$$\mathbf{CU}(\mathcal{X}^*) = \mathbf{CK}(\mathcal{X}^*) = \mathbf{CB}(\mathcal{X}^*) \quad (97)$$

is a convex, path-connected, weak*-Hausdorff-closed subset of $\mathbf{CF}(\mathcal{X}^*)$, by Equation (22) and Corollary 3.13 (ii). In this case, Theorem 4.12 (ii) can be rephrased as follows:

Theorem 4.17. (Generic convex weak*-compact sets)
Let $\mathcal{X}$ be an infinite-dimensional separable Banach space. Then, the set $\mathcal{D}$ of all nonempty convex weak-compact sets K with a weak*-dense set $\mathcal{E}(K)$ of extreme points is a weak*-Hausdorff-dense G_δ subset of the weak*-Hausdorff-closed space $\mathbf{CK}(\mathcal{X}^*) = \mathbf{CB}(\mathcal{X}^*)$.*

Proof. If $\mathcal{X}$ is an infinite-dimensional separable Banach space, then any open ball of center 0 in $\mathcal{X}$ satisfies Condition 4.1. Therefore, the assertion follows from Theorem 4.12 (ii). $\square$

As a consequence, $\mathcal{D}$ is generic in the hyperspace $\mathbf{CK}(\mathcal{X}^*)$, that is, the complement of a meagre set, i.e., a nowhere dense set. In other words, $\mathcal{D}$ is of second category in $\mathbf{CK}(\mathcal{X}^*)$.

In the Banach case, the weak*-Hausdorff hypertopology on $\mathbf{CK}(\mathcal{X}^*)$ is finer than the scalar topology [7, Section 4.3] restricted to weak*-closed sets. The linear topology on the set of nonempty closed convex subsets is the supremum of the scalar and Wijsman topologies. Since the Wijsman topology [7, Definition 2.1.1] requires a metric space, one has to use the norm on $\mathcal{X}^*$ and the linear topology is not comparable with the weak*-Hausdorff hypertopology. If one uses the metric (54)

generating the weak* topology on balls of $\mathcal{X}^*$ for a separable Banach space $\mathcal{X}$, then the Wijsman and linear topologies for norm-closed balls of $\mathcal{X}^*$ are coarser than the weak*-Hausdorff hypertopology, by Theorem 3.17. In fact, as already mentioned, the weak*-Hausdorff hypertopology can be seen as a very fine, weak*-type, topology on $\mathbf{CK}(\mathcal{X}^*)$ and the density of the subset of all convex weak* compact sets with weak*-dense set of extreme points stated in Theorem 4.17 is a very strong property.

Meanwhile, in the Banach case, the weak*-Hausdorff density property within the set of positive continuous functionals, as stated in Theorems 4.14 and 4.16, can be strengthened. In order to present these outcomes, by fixing a convex cone $\mathcal{X}^+ \subseteq \mathcal{X}$ in a real Banach space $\mathcal{X}$ and using the usual norm $\|\cdot\|_{\mathcal{X}^*}$ on continuous functionals of $\mathcal{X}^*$, we define the spaces

$$\mathbf{CK}(\mathcal{X}^{*,+}) \doteq \{K \in \mathbf{CK}(\mathcal{X}^*) : K \subseteq \mathcal{X}^{*,+}\} \quad (98)$$

$$\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+}) \doteq \{K \in \mathbf{CK}(\mathcal{X}^{*,+}) : \forall \sigma \in K, \|\sigma\|_{\mathcal{X}^*} \leq 1\} \quad (99)$$

$$\mathbf{CK}_1(\mathcal{X}^{*,+}) \doteq \{K \in \mathbf{CK}(\mathcal{X}^{*,+}) : \forall \sigma \in K, \|\sigma\|_{\mathcal{X}^*} = 1\} \quad (100)$$

as well as the set

$$\mathcal{D}^+ \doteq \left\{K \in \mathbf{CK}(\mathcal{X}^{*,+}) : K = \overline{\mathcal{E}(K)}\right\} \quad (101)$$

of all nonempty convex weak*-compact sets of $\mathcal{X}^{*,+}$ with weak*-dense extreme boundary. Compare with (95). Additionally, define

$$E_{\mathcal{X}} \doteq \{\sigma \in \mathcal{X}^{*,+} : \|\sigma\|_{\mathcal{X}^*} = 1\},$$

the set of positive normalized functionals of $\mathcal{X}^*$. We arrive at the following result:

Theorem 4.18. (Generic convex weak*-compact sets – positive functionals)

Let $\mathcal{X}$ be an infinite-dimensional separable real Banach space with $\mathcal{X}^+ \subseteq \mathcal{X}$ being a convex cone. Assume that, for any open ball $B(0, R)$ of radius $R \in \mathbb{R}^+$, centered at $0 \in \mathcal{X}$, there is an interval $[A_1, A_2] \subseteq \mathcal{X}$ (see (93)) and some $r \in \mathbb{R}^+$ such that $B(0, r) \subseteq [A_1, A_2] \subseteq B(0, R)$. Then, one has:

- (i) $\mathcal{D}^+$ is a weak*-Hausdorff-dense G_δ subset of the weak*-Hausdorff-closed space $\mathbf{CK}(\mathcal{X}^{*,+})$.
- (ii) $\mathcal{D}^+ \cap \mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$ is a weak*-Hausdorff-dense G_δ subset of the weak*-Hausdorff-compact and completely metrizable space $\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$.
- (iii) If $E_{\mathcal{X}} \in \mathbf{CK}_1(\mathcal{X}^{*,+})$ then $\mathcal{D}^+ \cap \mathbf{CK}_1(\mathcal{X}^{*,+})$ is a weak*-Hausdorff-dense G_δ subset of the weak*-Hausdorff-compact and completely metrizable space $\mathbf{CK}_1(\mathcal{X}^{*,+})$.

Proof. First of all, note that $\mathbf{CK}(\mathcal{X}^{*,+})$ belongs to the weak*-Hausdorff-closed hyperspace $\mathbf{CK}(\mathcal{X}^*)$, by (97) and Corollary 3.13 (ii). By Corollary 3.18 and because $\mathcal{X}^{*,+}$ is a weak*-closed set, $\mathbf{CK}(\mathcal{X}^{*,+})$ is also weak*-Hausdorff-closed. By Lemma 4.13,

$$\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+}) = \mathbf{CU}_{B(0,1)}(\mathcal{X}^*) \cap \mathbf{CK}(\mathcal{X}^{*,+})$$

is a weak*-Hausdorff-compact and completely metrizable space. By Corollary 3.18 and the weak*-closedness of $E_{\mathcal{X}} \in \mathbf{CK}_1(\mathcal{X}^{*,+})$, $\mathbf{CK}_1(\mathcal{X}^{*,+})$ is also a weak*-Hausdorff-closed subspace of $\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$, and is thus weak*-Hausdorff-compact and completely metrizable. Now, we prove Assertions (i)–(iii):

(i): The first assertion is simply Theorem 4.16 applied to the Banach case, keeping in mind that all open balls of center 0 in $\mathcal{X}$ satisfy Condition 4.1 when $\mathcal{X}$ is an infinite-dimensional separable Banach space.

(ii): In the proof of Theorem 4.9, redefine the set $\mathcal{F}_{u,m}$ of Equation (88) by

$$\mathcal{F}_m \doteq \{K \in \mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+}) : \exists \omega \in K, B(\omega, 1/m) \cap \mathcal{E}(K) = \emptyset\} . \quad (102)$$

Then, by following the arguments of the proof of Theorem 4.9, it follows that $\mathcal{D}^+ \cap \mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$ is a G_δ subset of $\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$, because $\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$ is weak*-Hausdorff-closed and any separable real Banach space $\mathcal{X}$ is a Gateaux-differentiability space, by the Mazur theorem [10, Theorem 10.44]. It remains to prove the asserted density. This follows from a straightforward adaptation of the proof of Theorem 4.14: Use the same arguments with the additional condition that $\|\sigma_n\|_{\mathcal{X}^*} \leq 1$.

(iii): Recall that, by assumption, $E_{\mathcal{X}}$ is convex and weak*-compact. We thus proceed in the same way as in the proof of Assertion (ii). The proof of Theorem 4.9 can be adapted to this situation, by redefining the set $\mathcal{F}_{u,m}$ of Equation (88): Replace $\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$ with $\mathbf{CK}_1(\mathcal{X}^{*,+})$ in (102). Hence, $\mathcal{D}^+ \cap \mathbf{CK}_1(\mathcal{X}^{*,+})$ is a G_δ subset of $\mathbf{CK}_1(\mathcal{X}^{*,+})$. Like in Assertion (ii), in order to prove the asserted density, we employ a straightforward adaptation of the proof of Theorem 4.14: In addition to the condition $\|\sigma_n\|_{\mathcal{X}^*} \leq 1$ used in the proof of Assertion (ii), we also replace the definition of $\omega_{n_\varepsilon+n}$ in (72) by

$$\omega_{n_\varepsilon+n} \doteq (1 - \lambda_n \|\sigma_n\|_{\mathcal{X}^*}) \varpi_n + \lambda_n \sigma_n \in E_{\mathcal{X}}$$

at any step $n \in \mathbb{N}$. □

Note that Theorem 4.18 (iii) does *not* directly follow from (i)–(ii) because the complement of $\mathbf{CK}_1(\mathcal{X}^{*,+})$ in either $\mathbf{CK}(\mathcal{X}^{*,+})$ or $\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$ is, in general, open and dense in the weak*-Hausdorff topology.

The assumptions of Theorem 4.18 are particularly relevant for C^* -algebra: Let $\mathcal{X}$ be a separable unital C^* -algebra, that is, a *complex* Banach algebra having a unit $\mathbf{1} \in \mathcal{X}$ for its product and endowed with an antilinear involution $A \mapsto A^*$ such that

$$(AB)^* = B^*A^* \quad \text{and} \quad \|A^*A\|_{\mathcal{X}} = \|A\|_{\mathcal{X}}^2, \quad A, B \in \mathcal{X} .$$

The *real* Banach (sub)space of all self-adjoint elements of $\mathcal{X}$ is denoted by

$$\mathcal{X}^{\mathbb{R}} \doteq \{A \in \mathcal{X} : A = A^*\} .$$

Recall that $\mathbf{1} \in \mathcal{X}^{\mathbb{R}}$ and any element $A \in \mathcal{X}$ can be decomposed as

$$A = \operatorname{Re}\{A\} + i\operatorname{Im}\{A\} \quad \text{with} \quad \operatorname{Re}\{A\} = \frac{A + A^*}{2} \in \mathcal{X}^{\mathbb{R}}, \quad \operatorname{Im}\{A\} = \frac{A - A^*}{2i} \in \mathcal{X}^{\mathbb{R}} .$$

Therefore, any continuous linear functional $\sigma \in \mathcal{X}^*$ is uniquely determined by its values on the real Banach space $\mathcal{X}^{\mathbb{R}}$ and any $\sigma \in (\mathcal{X}^{\mathbb{R}})^*$ can be identified with an element of $\mathcal{X}^*$. As is well-known, such functionals are the so-called hermitian elements²⁷ of $\mathcal{X}^*$, which forms the *real* vector space

$$(\mathcal{X}^*)^{\mathbb{R}} \doteq \bigcap_{A \in \mathcal{X}} \left\{ \sigma \in \mathcal{X}^* : \sigma(A^*) = \overline{\sigma(A)} \right\} \subseteq \mathcal{X}^* . \quad (103)$$

²⁷ That is, functionals $\sigma \in \mathcal{X}^*$ satisfying $\sigma(A^*) = \overline{\sigma(A)}$ for any $A \in \mathcal{X}$.

The mapping from $(\mathcal{X}^{\mathbb{R}})^*$ to $(\mathcal{X}^*)^{\mathbb{R}}$ is denoted by $\Uparrow$ and, for any $\sigma \in (\mathcal{X}^{\mathbb{R}})^*$,

$$\Uparrow(\sigma)(A) \doteq \sigma(\operatorname{Re}\{A\}) + i\sigma(\operatorname{Im}\{A\}) , \quad A \in \mathcal{X} . \quad (104)$$

This mapping is a (real) linear homeomorphism, $(\mathcal{X}^{\mathbb{R}})^*$ and $(\mathcal{X}^*)^{\mathbb{R}}$ being endowed with the weak* topology²⁸. It naturally induces a mapping, again denoted by $\Uparrow$, from $\mathbf{CK}((\mathcal{X}^{\mathbb{R}})^*)$ to

$$\mathbf{CK}((\mathcal{X}^*)^{\mathbb{R}}) \doteq \{K \in \mathbf{CK}(\mathcal{X}^*) : K \subseteq (\mathcal{X}^*)^{\mathbb{R}}\} ,$$

which is again a homeomorphism:

Lemma 4.19. (Homeomorphism between $\mathbb{C}$ - and $\mathbb{R}$ -structures in C^* -algebras)
Let $\mathcal{X}$ be a separable unital C^ -algebra. $\Uparrow$ is a homeomorphism from $\mathbf{CK}((\mathcal{X}^{\mathbb{R}})^*)$ to $\mathbf{CK}((\mathcal{X}^*)^{\mathbb{R}})$, both spaces being endowed with the weak*-Hausdorff topology.*

Proof. Any element of $(\mathcal{X}^*)^{\mathbb{R}}$, i.e., any hermitian continuous linear functional on $\mathcal{X}$, can be pushed forward, through the restriction mapping, to an element of $(\mathcal{X}^{\mathbb{R}})^*$, i.e., a continuous linear functional on $\mathcal{X}^{\mathbb{R}}$. This obviously yields a weak*-Hausdorff continuous bijective mapping $\sqsupset$ from $\mathbf{CK}((\mathcal{X}^*)^{\mathbb{R}})$ to $\mathbf{CK}((\mathcal{X}^{\mathbb{R}})^*)$. Observe also that the union of any weak*-Hausdorff convergent net $(K_j)_{j \in J} \subseteq \mathbf{CK}((\mathcal{X}^*)^{\mathbb{R}})$ is norm-bounded. To prove this, one uses an argument by contradiction and the uniform boundedness principle (see, e.g., [35, Theorems 2.4 and 2.5]). Therefore, we restrict without loss of generality our study on

$$\mathbf{CK}_{B(0,R)}((\mathcal{X}^*)^{\mathbb{R}}) \doteq \mathbf{CU}_{B(0,R)}(\mathcal{X}^*) \cap \mathbf{CK}((\mathcal{X}^*)^{\mathbb{R}})$$

at fixed $R \in \mathbb{R}^+$. Here, $B(0, R) \subseteq \mathcal{X}$ is the open ball of radius $R \in \mathbb{R}^+$ centered at 0, as defined by (92). Since $\mathcal{X}$ is separable, we infer from Theorem 3.17 that $\mathbf{CU}_{B(0,R)}(\mathcal{X}^*)$ is weak*-Hausdorff-compact which, combined with Corollary 3.18, in turn implies that $\mathbf{CK}_{B(0,R)}((\mathcal{X}^*)^{\mathbb{R}})$ is weak*-Hausdorff-closed and, hence, weak*-Hausdorff-compact. $\sqsupset$ is thus a weak*-Hausdorff continuous bijective mapping from the weak*-Hausdorff-compact space $\mathbf{CK}_{B(0,R)}((\mathcal{X}^*)^{\mathbb{R}})$ to $\mathbf{CK}_{B(0,R)}((\mathcal{X}^{\mathbb{R}})^*)$. Consequently, the inverse of $\sqsupset$ is also weak*-continuous. This inverse is nothing else than the mapping $\Uparrow$, which is thus a homeomorphism. $\square$

The fact that $\Uparrow$ defined on $(\mathcal{X}^{\mathbb{R}})^*$ by (104) is a homeomorphism is obvious, but as a mapping on $\mathbf{CK}((\mathcal{X}^{\mathbb{R}})^*)$, this property, asserted in Lemma 4.19, is a priori not clear at all. It would be obvious if one used the hypertopology (Definition 2.2) induced by the family of pseudometrics $\tilde{d}_H^{(A)}$ defined, for all $A \in \mathcal{X}$ and $F, \tilde{F} \in \mathbf{F}(\mathcal{X}^*)$, by

$$\begin{aligned} & \tilde{d}_H^{(A)}(F, \tilde{F}) \\ & \doteq \max \left\{ \sup_{\sigma \in F} \inf_{\tilde{\sigma} \in \tilde{F}} |\operatorname{Re}\{(\sigma - \tilde{\sigma})(A)\}| , \sup_{\tilde{\sigma} \in \tilde{F}} \inf_{\sigma \in F} |\operatorname{Re}\{(\sigma - \tilde{\sigma})(A)\}| \right\} \in \mathbb{R}^+ \cup \{\infty\} . \end{aligned}$$

Compare with Definition 2.3: This hypertopology is *coarser* than the weak*-Hausdorff hypertopology, because $|\operatorname{Re}\{z\}| \leq |z|$ for any $z \in \mathbb{C}$. Note that the inequality $|z| \leq |\operatorname{Re}\{z\}| + |\operatorname{Im}\{z\}|$ for $z \in \mathbb{C}$ is not really useful to prove Lemma 4.19 because of the combination of suprema and infima in Definition 2.3. To show Lemma 4.19 we use the separability of the C^* -algebra and abstract compactness arguments, like in the proof of Theorem 3.17.

²⁸ The same holds true with respect to the norm topology for continuous linear functionals.

Meanwhile, it is also well-known [18, Proposition 1.6.1] that

$$\mathcal{X}^+ \equiv (\mathcal{X}^{\mathbb{R}})^+ \doteq \{AA^* : A \in \mathcal{X}\} \subseteq \mathcal{X}^{\mathbb{R}} \subseteq \mathcal{X}$$

is a closed convex cone, which is additionally pointed²⁹. In other words, the cone $\mathcal{X}^+$ is the set of positive elements of the C^* -algebra $\mathcal{X}$ [18, Definition 1.6.5]. It yields a partial order³⁰ $\preceq$ in $\mathcal{X}$ (and not only a preorder), and in $\mathcal{X}^{\mathbb{R}}$, which is compatible with the real vector space structure of $\mathcal{X}$. The weak*-closed set of all positive (continuous) linear functionals is denoted by

$$\mathcal{X}^{*,+} \doteq \bigcap_{A \in \mathcal{X}^+} \{\sigma \in \mathcal{X}^* : \sigma(A) \geq 0\} \subseteq (\mathcal{X}^*)^{\mathbb{R}}.$$

It belongs to the set of hermitian elements of $\mathcal{X}^*$, by [18, Proposition 2.1.5]. Using this set, we define by (98)–(101), respectively, the spaces $\mathbf{CK}(\mathcal{X}^{*,+})$, $\mathbf{CK}_{\leq 1}(\mathcal{X}^{*,+})$, $\mathbf{CK}_1(\mathcal{X}^{*,+})$ as well as $\mathcal{D}^+$. Then, one obtains an analogous version of Theorem 4.18 for separable unital C^* -algebras:

Corollary 4.20. (Generic convex weak*-compact sets – C^* -algebras)

Let $\mathcal{X}$ be a infinite-dimensional separable unital C^ -algebra. Then, Assertions (i)–(iii) of Theorem 4.18 hold true.*

Proof. First, since $\mathcal{X}$ is a (unital) C^* -algebra,

$$\|A\|_{\mathcal{X}} = \max_{\sigma \in \mathcal{X}^{*,+} : \|\sigma\|_{\mathcal{X}^*} = 1} |\sigma(A)|, \quad A \in \mathcal{X}^{\mathbb{R}},$$

and, by the functional calculus in C^* -algebra [18, Section 1.5] combined with [18, Proposition 1.6.1], for all $A \in \mathcal{X}^{\mathbb{R}}$ and $R \in \mathbb{R}^+$,

$$0 \leq \|A\|_{\mathcal{X}} \leq R \quad \text{iff} \quad -R\mathbf{1} \preceq A \preceq R\mathbf{1}.$$

In particular, the closure of any open ball $B(0, R)$ of radius $R \in \mathbb{R}^+$ centered at $0 \in \mathcal{X}$ in $\mathcal{X}^{\mathbb{R}}$ is the interval $[-R\mathbf{1}, R\mathbf{1}] \subseteq \mathcal{X}^{\mathbb{R}}$ (see (93)). By [18, Proposition 2.1.9], note additionally that

$$E_{\mathcal{X}} \doteq \{\sigma \in \mathcal{X}^{*,+} : \|\sigma\|_{\mathcal{X}^*} = 1\} = \{\sigma \in \mathcal{X}^{*,+} : \sigma(\mathbf{1}) = 1\} \in \mathbf{CK}_1(\mathcal{X}^{*,+}). \quad (105)$$

We can thus apply Theorem 4.18 for the real Banach space $\mathcal{X}^{\mathbb{R}}$ to get Assertions (i)–(iii) with

$$(\mathcal{X}^{\mathbb{R}})^{*,+} = \mathbf{1}^{-1}(\mathcal{X}^{*,+})$$

being the set of all positive (continuous) linear functionals of $(\mathcal{X}^{\mathbb{R}})^*$. By Lemma 4.19, the mapping $\mathbf{1}$ from $(\mathcal{X}^{\mathbb{R}})^*$ to $(\mathcal{X}^*)^{\mathbb{R}} \subseteq \mathcal{X}^*$ defined by (104) yields a weak*-Hausdorff homeomorphism between $\mathbf{CK}((\mathcal{X}^{\mathbb{R}})^*)$ and $\mathbf{CK}((\mathcal{X}^*)^{\mathbb{R}})$. Therefore, Assertions (i)–(iii) also hold true when we replace $(\mathcal{X}^{\mathbb{R}})^{*,+}$ with $\mathcal{X}^{*,+} \subseteq (\mathcal{X}^*)^{\mathbb{R}}$. $\square$

²⁹ That is, the only element $A \in (\mathcal{X}^{\mathbb{R}})^+$ such that $-A, A \in (\mathcal{X}^{\mathbb{R}})^+$ is $A = 0$.

³⁰ I.e., a binary relation that is reflexive, transitive and antisymmetric.

If $\mathcal{X}$ is a separable unital C^* -algebra then the set $E_{\mathcal{X}}$ of all positive and normalized linear functionals of $\mathcal{X}^*$ (see (105)) is the so-called *state space* in the algebraic formulation of quantum mechanics. It is weak*-closed and thus a weak*-compact subset of the unit ball of $\mathcal{X}^*$, by the Banach-Alaoglu theorem [35, Theorem 3.15]. Since $E_{\mathcal{X}}$ is also convex, $E_{\mathcal{X}} \in \mathbf{CK}_1(\mathcal{X}^{*,+})$. For antiliminal³¹ and simple³² C^* -algebras, $E_{\mathcal{X}} \in \mathcal{D}^+ \cap \mathbf{CK}_1(\mathcal{X}^{*,+})$, by [18, Lemma 11.2.4]. Important examples of such C^* -algebras are the (even subalgebra of the) CAR C^* -algebras for (non-relativistic) fermions on the lattice. Quantum-spin systems, i.e., infinite tensor products of copies of some elementary finite dimensional matrix algebra, referring to a spin variable, are also important examples. They are, for instance, widely used in quantum information theory as well as in condensed matter physics. In all these physical situations, the corresponding (non-commutative) C^* -algebra $\mathcal{X}$ is separable and $E_{\mathcal{X}}$ is thus a metrizable weak*-compact convex set. It is *not* a simplex [9, Example 4.2.6], but

$$E_{\mathcal{X}} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{P}_n} \quad (106)$$

is the weak*-closure of the union of a strictly increasing sequence

$$(\mathfrak{P}_n)_{n \in \mathbb{N}} \subseteq \mathcal{D}^+ \cap \mathbf{CK}_1(\mathcal{X}^{*,+})$$

of Poulsen [34] simplices. Equation (106) is a consequence of well-known results (see, e.g., [23, 10]) and we give its complete proof in [11]. In other words, by Proposition 3.14, $E_{\mathcal{X}}$ is the *weak*-Hausdorff limit* of the increasing sequence $(\mathfrak{P}_n)_{n \in \mathbb{N}}$ within $\mathcal{D}^+ \cap \mathbf{CK}_1(\mathcal{X}^{*,+})$. Together with Corollary 4.20, this demonstrates the amazing structural richness of the state space $E_{\mathcal{X}}$ for infinite-dimensional quantum systems.

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³¹ A C^* -algebra $\mathcal{X}$ is antiliminal if the zero ideal is its only liminal closed two-sided ideal. A C^* -algebra $\mathcal{X}$ is called liminal if, for every irreducible representation π of $\mathcal{X}$ and each $A \in \mathcal{X}$, $\pi(A)$ is compact.

³² A C^* -algebra $\mathcal{X}$ is simple if the only closed two-sided ideals of $\mathcal{X}$ are the trivial sets $\{0\}$ and $\mathcal{X}$.

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