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A Combinatorial Refinement of Generalized Jessen's Inequality

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Jessen's inequality is the functional form of Jensen's inequality for convex functions defined on an interval of real numbers. It is studied by many authors, but not always in a correct form. Jessen's inequality has a nice generalization based on totally normalised sublinear functionals. A brief overview of this topic is presented. The main result of this paper is a combinatorial improvement of the generalized Jessen's inequality. There are only a few refinements even for Jessen's inequality, our result gives a new type of refinement in a more general context. As an application we introduce and study new means.

Keywords: Jessen's inequality, discrete and integral Jensen's inequalities, subadditive functionals, means.

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1. Introduction

The functional form of Jensen's inequality for convex functions defined on an interval of $\mathbb{R}$ was given by Jessen [7]. Jessen's inequality can be formulated by linear functionals satisfying the following properties:

(C_1) Let E be a nonempty set, and let L be a subspace of the vector space of real valued functions defined on E. It is also assumed that $1_E \in L$ (for every $c \in \mathbb{R}$ the constant function $c_E : E \to \mathbb{R}$ is defined by $c_E(x) = c$).

(C_2) Let $A : L \to \mathbb{R}$ be a linear functional.

(C_3) Assume A is nonnegative that is $A(f) \geq 0$ for all nonnegative $f \in L$.

(C_4) Assume $A(1_E) = 1$.

Linear functionals satisfying (C_3) are often called isotonic (or monotonic). It is said that a linear functional is normalised (or unital) if (C_4) holds.

The set of positive integers will be denoted by $\mathbb{N}$.

A real valued function φ defined on an interval $C \subset \mathbb{R}$ is called convex if it satisfies

$$\varphi(\alpha t_1 + (1-\alpha) t_2) \leq \alpha\varphi(t_1) + (1-\alpha)\varphi(t_2)$$

for all $t_1, t_2 \in C$ and all $\alpha \in [0, 1]$.

Jessen's inequality is studied by many authors, but not always in a correct form. To understand this, we first consider the integral version of Jensen's inequality.

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Theorem 1.1. (integral Jensen's inequality, see [8]) *Let f be a real integrable function on a probability space $(E, \mathcal{A}, \mu)$ taking values in an interval $C \subset \mathbb{R}$. Then $\int_E f d\mu$ lies in C. If φ is a convex function on C such that $\varphi \circ f$ is μ-integrable, then*

$$\varphi\Big(\int_E f d\mu\Big) \leq \int_E \varphi \circ f d\mu. \tag{1}$$

In particular Jensen's inequalities can be obtained by setting the probability space as follows: E is either $\{1, \ldots, n\}$ for some $n \geq 1$ or $\mathbb{N}$, $\mathcal{A}$ is the set of all subsets of E, and $\mu := \sum_{i \in E} p_i \varepsilon_i$ where $(p_i)_{i \in E}$ represents a probability distribution (this means that $p_i \geq 0$, $i \in E$, and $\sum_{i \in E} p_i = 1$) and ε_i is the unit mass at i on $\mathcal{A}$ for all $i \in E$.

Theorem 1.2. (discrete Jensen's inequalities) *Let $C \subset \mathbb{R}$ be an interval and let $\varphi : C \to \mathbb{R}$ be a convex function.*

(a) *If $p_1, \ldots, p_n$ represent a probability distribution, and $x_1, \ldots, x_n \in C$, then*

$$\varphi\left(\sum_{i=1}^{n} p_i x_i\right) \leq \sum_{i=1}^{n} p_i \varphi(x_i). \tag{2}$$

(b) *If $p_1, p_2, \ldots$ represent a probability distribution, and $x_1, x_2, \ldots \in C$ such that the series $\sum\limits_{i=1}^{\infty} p_i x_i$ and $\sum\limits_{i=1}^{\infty} p_i \varphi(x_i)$ are absolutely convergent, then $\sum\limits_{i=1}^{\infty} p_i x_i$ lies in C and*

$$\varphi\left(\sum_{i=1}^{\infty} p_i x_i\right) \leq \sum_{i=1}^{\infty} p_i \varphi(x_i). \tag{3}$$

The following important normalised isotonic linear functional appears in Theorem 1.1.

Example 1.3. Let $(E, \mathcal{A}, \mu)$ be a *probability space.* If L means the vector space of real integrable functions on E and the linear functional A is defined on L by

$$A(f) := \int_E f d\mu,$$

then A is obviously isotonic and normalised.

Example 1.3 and Theorem 1.1 suggest the next – commonly used but not valid – properties of normalised isotonic linear functionals: Let $A : L \to \mathbb{R}$ be a normalised isotonic linear functional, and let f be a function from L taking values in an interval $C \subset \mathbb{R}$.

(P_1) Then $A(f)$ lies in C.

(P_2) If φ is a convex function on C such that $\varphi \circ f \in L$, then

$$\varphi(A(f)) \leq A(\varphi \circ f). \tag{4}$$

Actually, inequality (4) is the so-called Jessen's inequality.

A number of papers contain examples illustrating that (P_1) and (P_2) do not hold in general. If the interval C is compact and the function φ is continuous, then (P_1) and (P_2) are really true, but it is not easy to find a paper in which the proof of this assertion is complete. There is an elegant proof of this claim in [9] in a more general context, but the proof is far from elementary. A proof that is both elementary and complete may be found in [4]. Moreover, the authors work with weaker conditions than (C_1-C_4), and thus a generalization of Jessen's inequality is obtained.

To formulate the generalized Jessen's inequality, we need the following conditions:

(D_1) Let E be a nonempty set, and let L be a subspace of the vector space of real valued functions defined on E. It is also assumed that $1_E \in L$.

(D_2) Let $A : L \to \mathbb{R}$ be a sublinear functional which means that A is positive homogeneous and subadditive.

(D_3) Assume A is increasing that is $f,\, g \in L$, $f \le g$ implies $A(f) \le A(g)$.

(D_4) Assume $A(1_E) = 1$ and $A(-1_E) = -1$.

On the model of the name normalised isotonic linear functional the following names are used: sublinear functionals satisfying (D_3) are called isotonic; it is said that a sublinear functional is totally normalised if (D_4) holds.

Obviously, normalised isotonic linear functionals are totally normalised isotonic sublinear functionals too, but the converse is not true in general.

The second section has an auxiliary character: we formulate the generalized Jessen's inequality in a precise form and, for the sake of completeness, we also give a proof of it; some normalised isotonic linear functionals are given to clarify certain differences between Jessen's and Jensen's inequalities; moreover, we obtain a sufficient condition for (D_3) to be fulfilled. Some preliminary results can be found in the third section. The main result of this paper is a combinatorial improvement of the generalized Jessen's inequality which is given in the fourth section. This result essentially (but not completely) contains the refinements of the discrete and integral versions of Jensen's inequality in [5] and [6]. The proofs in [5] and [6] use the unique properties of the considered functionals substantially, and therefore sharper assertions can be obtained in the considered special cases. There are only a few refinements even for Jessen's inequality (see [3] and [11]), our result gives a new type of refinement to a more general inequality. We test the scope of theorems in the last section by applying them to some special normalised isotonic sublinear functionals and to a new mean. The idea of the introduced mean comes from the notion of quasi-arithmetic mean, and it is a correction and generalization of means studied in [2].

2. The generalized Jessen's inequality and some notes on it

The closure of a subset H of $\mathbb{R}$ is denoted by $\overline{H}$.

The following properties of convex functions are well known.

Lemma 2.1. *Let φ be a convex function on the interval $I \subset \mathbb{R}$. The endpoints of I are denoted by a and b ($a,\, b \in [-\infty, \infty]$, $a < b$). Then φ admits left and right derivatives at every inner point of I. Moreover, if $a \in I$ then the limit*

$$\varphi'(a+) := \lim_{t \to a+} \frac{\varphi(t) - \varphi(a)}{t - a} \in [-\infty, \infty[$$

exists, while, if $b \in I$, then the following limit also exists:

$$\varphi'(b-) := \lim_{t \to b-} \frac{\varphi(t) - \varphi(b)}{t - b} \in]-\infty, \infty].$$

The following result contains Jessen's inequality with maximal precision.

Theorem 2.2. (generalized Jessen's inequality) *Let A be a totally normalised isotonic sublinear functional. Let φ be a convex function on the interval $C \subset \mathbb{R}$, and let $f \in L$ such that $f(x) \in C$ for all $x \in E$. The endpoints of C are denoted by a and b ($a, b \in [-\infty, \infty]$, $a < b$). Then*

(a) *$A(f) \in \overline{C}$.*

(b) *If $A(f) \in C$, $\varphi \circ f \in L$, and φ is continuous at $A(f)$, then*

$$\varphi(A(f)) \leq A(\varphi \circ f). \tag{5}$$

Proof. (a) In case $a > -\infty$ it follows that the function $f - a_E$ is nonnegative. By (D_1), $f - a_E \in L$, and by (D_2) and (D_4),

$$0 \leq A(f - a_E) = A(f) - a,$$

which is equivalent to $A(f) \geq a$. We can prove similarly that $A(f) \leq b$ in case of $b < \infty$.

(b) The proof of this assertion is divided into two parts.

(i) Assume that φ has either right hand derivative or left hand derivative at $A(f)$. We consider only the right hand derivative, the left hand derivative can be handled in a similar way. Since φ is convex on I, we obtain

$$\varphi(t) \geq \varphi'_+(A(f))(t - A(f)) + \varphi(A(f)), \quad t \in I,$$

and therefore

$$\varphi(f(x)) \geq \varphi'_+(A(f))(f(x) - A(f)) + \varphi(A(f)), \quad x \in E,$$

or otherwise $\quad \varphi \circ f \geq \varphi'_+(A(f))(f - A(f)_E) + \varphi(A(f))_E.$

Now, apply A to both sides of this inequality. Since A is a totally normalized isotonic sublinear functional, we get

$$A(\varphi \circ f) \geq A(\varphi'_+(A(f))(f - A(f)_E) + \varphi(A(f)) \geq \varphi(A(f)).$$

(ii) By Lemma 2.1, it remains to prove the following two cases: either $A(f) = a \in I$, and $\varphi'(a+) = -\infty$, or $A(f) = b \in I$, and $\varphi'(b-) = \infty$.

We consider only the first case, the second case can be discussed similarly.

Choose a positive integer l such that the sequence $\left(a + \frac{2}{n}\right)_{n=l}^{\infty}$ belongs to the interior of I.

Denote the set $\{l, l+1, \ldots\}$ by $\mathbb{N}_l$. Since φ is convex,

$$\varphi(t) \geq \varphi'_+\left(a + \frac{1}{n}\right)\left(t - \left(a + \frac{1}{n}\right)\right) + \varphi\left(a + \frac{1}{n}\right), \quad t \in I, \quad n \in \mathbb{N}_l,$$

and therefore

$$\varphi(f(x)) \geq \varphi'_+\left(a+\frac{1}{n}\right)\left(f(x)-\left(a+\frac{1}{n}\right)\right)+\varphi\left(a+\frac{1}{n}\right), \quad x \in E, \quad n \in \mathbb{N}_l. \tag{6}$$

It now follows from the properties of A that

$$\begin{aligned} A(\varphi \circ f) &\geq A\left(\varphi'_+\left(a+\frac{1}{n}\right)\left(f-\left(a+\frac{1}{n}\right)_E\right)\right)+\varphi\left(a+\frac{1}{n}\right) \\ &\geq -\frac{1}{n}\varphi'_+\left(a+\frac{1}{n}\right)+\varphi\left(a+\frac{1}{n}\right), \quad n \in \mathbb{N}_l. \end{aligned} \tag{7}$$

Next we show that

$$\lim_{n\to\infty}\frac{1}{n}\varphi'_+\left(a+\frac{1}{n}\right)=0. \tag{8}$$

To this end, use the convexity of φ again to obtain

$$\frac{\varphi\left(a+\frac{1}{2n}\right)-\varphi\left(a+\frac{1}{n}\right)}{\frac{1}{2n}-\frac{1}{n}} \leq \varphi'_+\left(a+\frac{1}{n}\right) \leq \frac{\varphi\left(a+\frac{2}{n}\right)-\varphi\left(a+\frac{1}{n}\right)}{\frac{2}{n}-\frac{1}{n}}, \quad n \in \mathbb{N}_l,$$

which imply that

$$\begin{aligned} -2\left(\varphi\left(a+\frac{1}{2n}\right)-\varphi\left(a+\frac{1}{n}\right)\right) &\leq \frac{1}{n}\varphi'_+\left(a+\frac{1}{n}\right) \\ &\leq \varphi\left(a+\frac{2}{n}\right)-\varphi\left(a+\frac{1}{n}\right), \quad n \in \mathbb{N}_l. \end{aligned}$$

This estimation and the continuity of φ at a yield (8).

By using (8) and the continuity of φ at a again, it follows from (7) that

$$A(\varphi \circ f) \geq \varphi(a) = \varphi(A(f)).$$

The proof is complete. □

Remark 2.3. The result could be proved with a technique similar to the one used in [4] Theorem 3.1.

The following example shows that properties (P_1) and (P_2) are indeed not satisfied in general.

Example 2.4. Define

$$L := \left\{f :]0,\infty[\to \mathbb{R} \mid f \text{ is differentiable, } \lim_{0+} f \text{ and } \lim_{\infty} f' \text{exist and finite}\right\},$$

and $A : L \to \mathbb{R}$, $A(f) := \lim\limits_{0+} f + \lim\limits_{\infty} f'$. It is easy to check that $A : L \to \mathbb{R}$ is a normalised isotonic linear functional. The following functions belong to L:

$$f, g :]0,\infty[\to \mathbb{R}, \quad f(x) = 1 - \frac{1}{x+1}, \quad g(x) = \sqrt{x}.$$

The range of f is $]0,1[$, but $A(f) = 0 \notin]0,1[$, and hence property (P_1) does not hold.

Let $C := [0, \infty[$, and consider the convex function

$$\varphi : [0, \infty[\to \mathbb{R}, \quad \varphi(t) = \begin{cases} t^2, & t \in]0, \infty[\\ 2, & t = 0 \end{cases} .$$

The range of g is $]0, \infty[\subset C$, and $(\varphi \circ g)(x) = x$ $(x \in]0, \infty[)$. Elementary calculations show that

$$\varphi(A(g)) = \varphi(0) = 2 > 1 = A(\varphi \circ g),$$

and hence property (P_2) is not satisfied.

It is worth to note that strict inequality may hold in (5) even if A is linear and $A(f) = a \in C$ or $A(f) = b \in C$. This can never happen if A is defined by an integral as in Example 1.3. We illustrate this phenomenon by an example.

Example 2.5. Let

$$L := \left\{ f : [0, \infty[\to \mathbb{R} \mid f \text{ is differentiable, and } \lim_{\infty} f' \text{ exist and finite} \right\},$$

and define $A : L \to \mathbb{R}$, $A(f) := f(0) + \lim\limits_{\infty} f'$. It is not hard to see that A is a normalised isotonic linear functional.

Consider the interval $C := [0, \infty[$ and the continuous and convex function

$$\varphi : C \to \mathbb{R}, \quad \varphi(t) = t^2.$$

If the function $f : [0, \infty[\to \mathbb{R}$ is given by $f(x) = \sqrt{x+1} - 1$ then $f \in L$ such that $f(x) \in C$ for all $x \in [0, \infty[$, $\varphi \circ f \in L$, and

$$\varphi(A(f)) = 0 < 1 = A(\varphi \circ f).$$

Now, an additional example is given, in which an interesting normalised isotonic linear functional is constructed starting from an often used but almost trivial one.

Example 2.6. Let

$$M := \left\{ f : [0, \infty[\to \mathbb{R} \mid f \text{ is bounded and } \lim_{0+} f \text{ exists and finite} \right\},$$

and define

$$A : M \to \mathbb{R}, \quad A(f) := \lim_{0+} f. \tag{9}$$

Then A is obviously is a normalised isotonic linear functional. In this case Jessen's inequality is of no weight. Indeed, if $C \subset \mathbb{R}$ is a compact interval, $\varphi : C \to \mathbb{R}$ is a continuous (not necessarily convex) function, $f \in M$ such that $f(x) \in C$ for all $x \in [0, \infty[$, then $\varphi \circ f \in M$ and

$$\varphi(A(f)) = \varphi\left(\lim_{0+} f \right) = \lim_{0+} (\varphi \circ f) = A(\varphi \circ f).$$

Now let

$$g : [0, \infty[\to \mathbb{R}, \quad g(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases},$$

let M_1 be the vector space generated by M and g, that is

$$M_1 := \{ f + \alpha g \mid f \in M, \quad \alpha \in \mathbb{R} \},$$

and extend A to a linear functional A_1 on M_1 by $A_1(f + \alpha g) := A(f)$.

Since every $f \in M$ has right hand limit at 0, it is not hard to think that $f+\alpha g \in M_1$, $f+\alpha g \geq 0$ implies that $A(f) = \lim\limits_{0+} f \geq 0$, and therefore A_1 is nonnegative on M_1. We have shown that A_1 is a normalised isotonic linear functional. Jessen's inequality is already interesting by considering A_1. Indeed, let

$$\varphi : [-1, 1] \to \mathbb{R}, \quad \varphi(x) = x^2.$$

Then φ is convex, $g(x) \in [-1, 1]$ for all $x \in [0, \infty[$, and $\varphi \circ g = 1_{[0,\infty[} \in M_1$, and hence Jessen's inequality gives

$$\varphi(A_1(g)) = 0 < 1 = A\left(1_{[0,\infty[}\right) = A_1(\varphi \circ g).$$

In fact $A_1 : M_1 \to \mathbb{R}$ can be extended to a nonnegative linear functional on the vector space

$$L := \{f : [0, \infty[\to \mathbb{R} \mid f \text{ is bounded}\},$$

(see e.g. [1]) and thus we obtain a new normalised isotonic linear functional essentially different from the usual ones.

Remark 2.7. Consider the normalised isotonic linear functional A defined in (9).

(a) It is easy to check that for all $r \in [-1, 1]$ we can also extend A to a nonnegative linear functional A_1 on M_1 by $A_1(f + \alpha g) := A(f) + \alpha r$.

(b) A direct extension of either A or A_1 to a nonnegative linear functional on L can be done by applying the Hahn-Banach theorem with the subadditive and positive homogeneous functional

$$p : L \to \mathbb{R}, \quad p(f) = \limsup_{0+} f.$$

(c) The functional A (or any of its extensions) cannot be defined by an integral as in Example 1.3.

Let $A : L \to \mathbb{R}$ be a totally normalised isotonic sublinear functional. Then it is obvious that

(D_5) For all $f \in L$ with $|f(x)| \leq 1$ $(x \in E)$, we have $|A(f)| \leq 1$.

It has been proved in [9] that (D_2) and (D_4-D_5) imply (D_3) when L is the vector space of real valued continuous functions on a compact Hausdorff space. We show that this implication also holds if the functions in L are bounded.

Lemma 2.8. *Assume* (D_1)–(D_2) *and* (D_4)–(D_5). *If the functions in L are bounded, then* (D_3) *is also satisfied.*

Proof. Assume $f \in L$ and $f \leq 0_E$. Since f is bounded, $m := -\inf f \in [0, \infty[$. The subadditivity of A implies that

$$A(f) \leq A(-m_E) + A(f + m_E) = -m + A(f + m_E). \tag{10}$$

Since $0_E \leq f + m_E \leq m_E$, the condition ($D_5$) yields that $A(f + m_E) \leq m$, and thus by (10), $A(f) \leq 0$. Due to the subadditivity of A, it follows from this that if $f, g \in L$ and $f \leq g$, then

$$A(f) - A(g) \leq A(f - g) \leq 0.$$

The proof is complete. □

Finally, it is illustrated by a simple example that the boundedness condition cannot be omitted from the conditions of the previous result.

Example 2.9. Denote the identical function on $[0,\infty[$ by $id_{[0,\infty[}$. Let

$$L := \left\{ a_{[0,\infty[} + b \cdot id_{[0,\infty[} \mid a, b \in \mathbb{R} \right\},$$

and $A : L \to \mathbb{R}$, $A\left(a_{[0,\infty[} + b \cdot id_{[0,\infty[}\right) := a + |b|$. It is easy to check that (D_1)–(D_2) and (D_4)–(D_5) are satisfied without the boundedness of the functions in L. Moreover, A is also nonnegative. Since

$$-id_{[0,\infty[} \le 0_{[0,\infty[}, \quad A\left(-id_{[0,\infty[}\right) = 1 > 0,$$

it can be seen that A is not isotonic.

3. Preliminary results

We shall use the following result for double series.

Theorem 3.1. *(see [10]) Let $a(i,j) \in \mathbb{R}$ $((i,j) \in \mathbb{N} \times \mathbb{N})$. If any of the two sums*

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} |a(i,j)| \right), \quad \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} |a(i,j)| \right)$$

is finite, then both of the series

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a(i,j) \right), \quad \sum_{j=1}^{\infty} \left(\sum_{i=1}^{\infty} a(i,j) \right)$$

are absolutely convergent and both have the same sum.

The positive part f^+ and the negative part f^- of a real valued function f are defined in the usual way.

We need another result for the proofs.

Lemma 3.2. *Assume (D_1), and assume further that for all $f \in L$ the function $|f|$ also belongs to L. If $f \in L$ taking values in an interval $C \subset \mathbb{R}$ and φ is a convex function on C such that $\varphi \circ f \in L$, then there exists a convex function ψ on C such that $|\varphi| \le \psi$ and $\psi \circ f \in L$ too.*

Proof. Under the conditions on L, it is a Stone vector lattice, and therefore $g \in L$ implies $|g| \in L$, $g^+ \in L$ and $g^- \in L$.

Along with the function φ the function φ^+ is also convex. Since $\varphi \circ f \in L$, the function $\varphi^+ \circ f = (\varphi \circ f)^+$ belongs to L too.

The convexity of φ on C shows that there is an affine function $l : C \to \mathbb{R}$, $l(t) = at+b$ for which $\varphi(t) \ge at + b, \quad t \in C$. Then

$$\varphi^-(t) = \max(-\varphi(t), 0) \le \max(-at - b, 0) = l^-(t) \le |at + b| = |l(t)|, \quad t \in C.$$

Since $f \in L$, $|l| \circ f$ is also an element of L.

Using that the function $|l|$ is convex, $|\varphi| = \varphi^+ + \varphi^-$, and the sum of two convex function is also convex, it follows from the above that $\psi := \varphi^+ + |l|$ can be chosen. The proof is complete. $\square$

4. A combinatorial refinement of the generalized Jessen's inequality

We need the following hypotheses:

(H_1) Assume a totally normalised isotonic sublinear functional $A : L \to \mathbb{R}$ is given, that is (D_1-D_4) are satisfied. Further, it is assumed that for all $f \in L$ the function $|f|$ also belongs to L (in this case L is a Stone vector lattice).

(H_2) Let the index set I denote either $\{1, \ldots, n\}$ for some $n \geq 1$ or $\mathbb{N}$. Let the index set J denote either $\{1, \ldots, k\}$ for some $k \geq 1$ or $\mathbb{N}$.

(H_3) Let $(\lambda_j)_{j\in J}$ represent a positive probability distribution which means that $\lambda_j > 0$ $(j \in J)$, and $\sum_{j\in J} \lambda_j = 1$. For each $j \in J$ let π_j be a permutation of the set I.

(H_4) Suppose we are given a sequence $\mathfrak{A}_I = (A_i)_{i\in I}$ of isotonic sublinear functionals $A_i : L \to \mathbb{R}$ with $A_i(1_E) = -A_i(-1_E) > 0$ for all $i \in I$ and $\sum\limits_{i\in I} A_i = A$.

From now on the phrase closed interval always means an interval in $\mathbb{R}$ which is a closed (not necessarily compact) set.

The main result of this paper is the following refinement of the generalized Jessen's inequality.

Theorem 4.1. *Assume* (H_1)–(H_4). *Let $C \subset \mathbb{R}$ be a closed interval, and $\varphi : C \to \mathbb{R}$ be a continuous convex function. Let $f \in L$ taking values in C such that $\varphi \circ f \in L$. Then*

$$\varphi(A(f)) \leq C_{funct} = C_{funct}(f, \varphi, \lambda, \pi, \mathfrak{A}_I)$$

$$:= \sum_{i\in I} \left(\sum_{j\in J} \lambda_j A_{\pi_j(i)}(1_E) \right) \varphi \left(\frac{\sum\limits_{j\in J} \lambda_j A_{\pi_j(i)}(f)}{\sum\limits_{j\in J} \lambda_j A_{\pi_j(i)}(1_E)} \right) \leq A(\varphi \circ f). \tag{11}$$

Proof. In this proof we don't distinguish between finite and infinite sums, their common name is series. A series with finite number of reals has a finite evaluation, and hence it is obviously considered (absolutely) convergent.

We begin with three simple remarks.

For each $i \in I$ the functional A_i is sublinear and isotonic, as well as L is a vector lattice, and hence

$$-A_i(|g|) \leq A_i(-|g|) \leq A_i(g) \leq A_i(|g|), \quad g \in L, \quad i \in I,$$

which implies

$$|A_i(g)| \leq A_i(|g|), \quad g \in L, \quad i \in I. \tag{12}$$

This yields

$$\sum_{i\in I} |A_i(g)| \leq \sum_{i\in I} A_i(|g|) = A(|g|), \quad g \in L,$$

that is, the series $\sum_{i\in I} A_i(g)$ is absolutely convergent, and therefore

$$\sum_{i\in I} A_{\pi_j(i)}(g) = \sum_{i\in I} A_i(g), \quad j \in J \tag{13}$$

because π_j is a permutation of the set I for each $j \in J$.

Since the functionals A_i $(i \in I)$ are isotonic, $\sum\limits_{i\in I} A_i = A$, and π_j is a permutation of the set I for each $j \in J$, we have for every nonnegative $g \in L$ that

$$0 \leq A_{\pi_j(i)}\left(g\right) \leq A\left(g\right), \quad i \in I, \quad j \in J. \tag{14}$$

We proceed in three steps.

(a) We first study the convergence of some series.

(a_1) By using (12) and (14), we obtain that

$$\sum_{j\in J} \lambda_j \left|A_{\pi_j(i)}\left(f\right)\right| \leq \sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(|f|\right) \leq A\left(|f|\right) \sum_{j\in J} \lambda_j = A\left(|f|\right) < \infty,$$

and hence the series

$$\sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(f\right) \tag{15}$$

is absolutely convergent for each $i \in I$. As a special case, the series

$$\sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(1_E\right)$$

is also convergent (absolutely) with positive sum for each $i \in I$.

(a_2) Due to Lemma 3.2, there exists a convex function ψ on C such that

$$|\varphi| \leq \psi \text{ and } \psi \circ f \in L. \tag{16}$$

Since the functional $g \to \dfrac{1}{A_{\pi_j(i)}\left(1_E\right)} A_{\pi_j(i)}\left(g\right)$, $g \in L$, is totally normalised, isotonic and sublinear for all $i \in I$ and $j \in J$, the generalized Jessen's inequality and (14) imply that

$$\begin{aligned}
&\sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(1_E\right) \left|\varphi\left(\frac{1}{A_{\pi_j(i)}\left(1_E\right)} A_{\pi_j(i)}\left(f\right)\right)\right| \\
&\leq \sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(1_E\right) \psi\left(\frac{1}{A_{\pi_j(i)}\left(1_E\right)} A_{\pi_j(i)}\left(f\right)\right) \\
&\leq \sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(\psi \circ f\right) \leq A\left(\psi \circ f\right) \sum_{j\in J} \lambda_j = A\left(\psi \circ f\right), \quad i \in I,
\end{aligned} \tag{17}$$

which shows that the series

$$\sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(1_E\right) \varphi\left(\frac{1}{A_{\pi_j(i)}\left(1_E\right)} A_{\pi_j(i)}\left(f\right)\right) \tag{18}$$

and

$$\sum_{j\in J} \lambda_j A_{\pi_j(i)}\left(1_E\right) \psi\left(\frac{1}{A_{\pi_j(i)}\left(1_E\right)} A_{\pi_j(i)}\left(f\right)\right) \tag{19}$$

are absolutely convergent for each $i \in I$.

(a_3) Taking into account $|\varphi \circ f| \in L$ and (12), and then (13) and $\sum\limits_{i \in I} A_i = A$, we get

$$\sum_{j \in J} \left(\sum_{i \in I} \lambda_j \left| A_{\pi_j(i)} (\varphi \circ f) \right| \right) \leq \sum_{j \in J} \left(\sum_{i \in I} \lambda_j A_{\pi_j(i)} (|\varphi \circ f|) \right) \tag{20}$$

$$= \sum_{j \in J} \left(\lambda_j \sum_{i \in I} A_{\pi_j(i)} (|\varphi \circ f|) \right) = \sum_{j \in J} \left(\lambda_j \sum_{i \in I} A_i (|\varphi \circ f|) \right)$$

$$= A (|\varphi \circ f|) \sum_{j \in J} \lambda_j = A (|\varphi \circ f|), \tag{21}$$

and therefore Theorem 3.1 yields that both of the series

$$\sum_{i \in I} \left(\sum_{j \in J} \lambda_j A_{\pi_j(i)} (\varphi \circ f) \right), \quad \sum_{j \in J} \left(\sum_{i \in I} \lambda_j A_{\pi_j(i)} (\varphi \circ f) \right)$$

are absolutely convergent and both have the same sum. According to this, an analogous procedure to that used to obtain (21) from the second expression in (20) ($\varphi \circ f$ instead of $|\varphi \circ f|$) can be applied, and we get

$$\sum_{i \in I} \left(\sum_{j \in J} \lambda_j A_{\pi_j(i)} (\varphi \circ f) \right) = \sum_{j \in J} \left(\sum_{i \in I} \lambda_j A_{\pi_j(i)} (\varphi \circ f) \right) = A (\varphi \circ f). \tag{22}$$

When choosing φ as the identical function on C we have that the series

$$\sum_{i \in I} \left(\sum_{j \in J} \lambda_j A_{\pi_j(i)} (f) \right), \quad \sum_{j \in J} \left(\sum_{i \in I} \lambda_j A_{\pi_j(i)} (f) \right) \tag{23}$$

are absolutely convergent and both have the same sum $A(f)$.

(a_4) It has already been proved that series (15) and (19) are absolutely convergent. This makes it clear that the discrete Jensen's inequality can be applied to

$$\psi \left(\frac{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)} (f)}{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)} (1_E)} \right), \quad i \in I,$$

(see (16) for the properties of the function ψ), and we obtain

$$\sum_{i \in I} \left(\sum_{j \in J} \lambda_j A_{\pi_j(i)} (1_E) \right) \left| \varphi \left(\frac{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)} (f)}{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)} (1_E)} \right) \right| \leq C_{funct} (f, \psi, \lambda, \pi, \mathfrak{A}_I)$$

$$= \sum_{i \in I} \left(\sum_{j \in J} \lambda_j A_{\pi_j(i)} (1_E) \right) \psi \left(\frac{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)} (1_E) \frac{1}{A_{\pi_j(i)}(1_E)} A_{\pi_j(i)} (f)}{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)} (1_E)} \right)$$

$$\leq \sum_{i \in I} \left(\sum_{j \in J} \lambda_j A_{\pi_j(i)} (1_E) \psi \left(\frac{1}{A_{\pi_j(i)} (1_E)} A_{\pi_j(i)} (f) \right) \right).$$

By using the middle inequality in (17) and then (22) with $\psi \circ f$ instead of $\varphi \circ f$, we find that

$$C_{funct}\left(f, \psi, \lambda, \pi, \mathfrak{A}_I\right) \leq \sum_{i \in I}\left(\sum_{j \in J} \lambda_j A_{\pi_j(i)}\left(\psi \circ f\right)\right) = A\left(\psi \circ f\right),$$

and hence the series defining $C_{funct}\left(f, \varphi, \lambda, \pi, \mathfrak{A}_I\right)$ is absolutely convergent. This shows also that $C_{funct}\left(f, \varphi, \lambda, \pi, \mathfrak{A}_I\right)$ is well defined.

(b) Now the proof of the second inequality in (11) follows a method similar to the one used in (a$_4$) (the derivation can be applied to φ instead of ψ, an we refer to (18) instead of (19)).

(c) Since $A\left(1_E\right) = 1$, it follows from (22) (let φ be the identical function on C and $f := 1_E$) that

$$\sum_{i \in I}\left(\sum_{j \in J} \lambda_j A_{\pi_j(i)}\left(1_E\right)\right) = A\left(1_E\right) = 1. \tag{24}$$

It now follows from the absolute convergence of $C_{funct}\left(f, \varphi, \lambda, \pi, \mathfrak{A}_I\right)$ (see (a$_4$)) and the absolute convergence of the first series in (23) that the discrete Jensen's inequality can be applied, and it implies that

$$C_{funct} \geq \varphi\left(\sum_{i \in I}\left(\sum_{j \in J} \lambda_j A_{\pi_j(i)}\left(f\right)\right)\right).$$

Another application of (23) gives that

$$\varphi\left(\sum_{i \in I}\left(\sum_{j \in J} \lambda_j A_{\pi_j(i)}\left(f\right)\right)\right) = \varphi\left(A\left(f\right)\right),$$

which is the first inequality in (11).

The proof is complete. □

5. Applications

First some interesting totally normalised isotonic sublinear functionals are given.

Example 5.1. (a) Let the index set I denote either $\{1, \ldots, n\}$ for some $n \geq 1$ or $\mathbb{N}$, and let $\left(p_i\right)_{i \in I}$ represent a probability distribution. Let L be the vector space of all bounded real valued functions on the nonempty set E.

(a$_1$) Let $\left(E_i\right)_{i \in I}$ be a sequence of nonempty subsets of E. Define the functional $A : L \to \mathbb{R}$ by

$$A\left(f\right) := \sum_{i \in I} p_i \sup_{E_i} f.$$

(a$_2$) Let E be a metric space, and let $\left(a_i\right)_{i \in I}$ be a sequence of accumulation points of E. Define the functional $A : L \to \mathbb{R}$ by

$$A\left(f\right) := \sum_{i \in I} p_i \limsup_{a_i} f.$$

(a_3) Let $(A_i)_{i\in I}$ be a sequence of totally normalised isotonic sublinear functionals on L. Define the functional $A : L \to \mathbb{R}$ by

$$A(f) := \sum_{i\in I} p_i A_i(f)$$

(b) Let L be the vector space of all bounded real valued functions on the nonempty set E.

(b_1) Let $\mathfrak{A}$ be a nonempty set of totally normalised isotonic sublinear functionals on L. Define the functional $A : L \to \mathbb{R}$ by

$$A(f) := \sup_{B\in\mathfrak{A}} B(f).$$

(b_2) Let (A_i) be a net of totally normalised isotonic sublinear functionals on L. Define the functional $A : L \to \mathbb{R}$ by

$$A(f) := \limsup_{i} A_i(f).$$

(c) Let $(E, \mathcal{A}, \mu)$ be a probability space, let L be a subspace of integrable real valued functions on E such that $1_E \in L$, and let $B : L \to \mathbb{R}$ be a totally normalised isotonic sublinear functional. Define the functional $A : L \to \mathbb{R}$ by

$$A(f) := \int_E \max\left(B(f)_E, f\right) d\mu.$$

It is not hard to see that each functional defined above is a totally normalised isotonic sublinear functional.

Special cases of examples (b_2) and (c) can be found in [9].

The following applications of our main result are motivated by the previous examples.

Proposition 5.2. *Assume* (H_2) *and* (H_3). *Let $(p_i)_{i\in I}$ represent a positive probability distribution, and let L be the vector space of all bounded real valued functions on the nonempty set E. Let $C \subset \mathbb{R}$ be a closed interval, and $\varphi : C \to \mathbb{R}$ be a continuous convex function. Let $f \in L$ taking values in C such that $\varphi \circ f \in L$.*

(a) *If $(A_i)_{i\in I}$ is a sequence of totally normalised isotonic sublinear functionals on L, then*

$$\varphi\left(\sum_{i\in I} p_i A_i(f)\right) \leq \sum_{i\in I}\left(\sum_{j\in J} \lambda_j p_{\pi_j(i)}\right) \varphi\left(\frac{\sum\limits_{j\in J} \lambda_j p_{\pi_j(i)} A_{\pi_j(i)}(f)}{\sum\limits_{j\in J} \lambda_j p_{\pi_j(i)}}\right)$$

$$\leq \sum_{i\in I} p_i A_i(\varphi \circ f).$$

(b) *If* $(E_i)_{i\in\Lambda}$ *is a sequence of nonempty subsets of* E*, then*

$$\varphi\left(\sum_{i\in I}p_i\sup_{E_i}f\right)\leq\sum_{i\in I}\left(\sum_{j\in J}\lambda_jp_{\pi_j(i)}\right)\varphi\left(\frac{\sum\limits_{j\in J}\lambda_jp_{\pi_j(i)}\sup\limits_{E_{\pi_j(i)}}(f)}{\sum\limits_{j\in J}\lambda_jp_{\pi_j(i)}}\right)$$

$$\leq\sum_{i\in I}p_i\sup_{E_i}(\varphi\circ f).$$

(c) *If* E *is a metric space, and* $(a_i)_{i\in I}$ *is a sequence of accumulation points of* E*, then*

$$\varphi\left(\sum_{i\in I}p_i\limsup_{a_i}f\right)\leq\sum_{i\in I}\left(\sum_{j\in J}\lambda_jp_{\pi_j(i)}\right)\varphi\left(\frac{\sum\limits_{j\in J}\lambda_jp_{\pi_j(i)}\limsup\limits_{a_{\pi_j(i)}}(f)}{\sum\limits_{j\in J}\lambda_jp_{\pi_j(i)}}\right)$$

$$\leq\sum_{i\in I}p_i\limsup_{a_i}(\varphi\circ f).$$

Proof. Apply Theorem 4.1 to the functionals defined in Examples 5.1 (a_3), (a_1) and (a_2), respectively. □

Now we introduce and study some new means.

The range of a function f is denoted by R_f.

Let $C\subset\mathbb{R}$ be an interval, and let $q:C\to\mathbb{R}$ be a continuous and strictly monotone function. If $(X,\mathcal{A},\mu)$ is a probability space, and $f:X\to C$ is a function such that $q\circ f$ is μ-integrable on X, then

$$M_q(f,\mu):=q^{-1}\left(\int_X q\circ f d\mu\right)$$

is called the *quasi-arithmetic mean* (integral q-mean) of f.

Based on the notion of quasi-arithmetic mean, a new mean generated by a normalised isotonic linear functional is introduced in [2]: Assume a normalised isotonic linear functional $A:L\to\mathbb{R}$ is given. Let $C:=\,]a,b[$, $-\infty\leq a<b\leq\infty$, and let $q:C\to\mathbb{R}$ be a continuous and strictly monotone function such that $q\circ f\in L$ for some $f\in L$. Then define

$$M_q(f,A):=q^{-1}(A(q\circ f)). \tag{25}$$

This definition is not precise, the mean (25) does not always exist. For example, let $C=\,]0,\infty[$, $q:\,]0,\infty[\,\to\mathbb{R}$, $q(x)=1-\frac{1}{1+x^2}$, let A be the normalised isotonic linear functional defined in Example 2.4, and let $f:\,]0,\infty[\,\to\mathbb{R}$, $f(x)=\sqrt{x}$. Then q is continuous and strictly increasing, $R_q=\,]0,1[$, f and $q\circ f$ belong to L, but in spite of these (see Example 2.4)

$$A(q\circ f)=0\notin R_q.$$

In the following considerable generalization of the mean (25) the interval C is not necessarily open, the function f does not necessarily belong to L, a totally normalised isotonic sublinear functional is considered and under the conditions it exists.

Definition 5.3. Let a totally normalised isotonic sublinear functional $A : L \to \mathbb{R}$ be given. Let $C \subset \mathbb{R}$ be an interval, $q : C \to \mathbb{R}$ be a continuous and strictly monotone function, and *let* $f : E \to \mathbb{R}$ *take values in* C such that $\overline{q\left(]\inf f, \sup f[\right)} \subset R_q$ and $q \circ f \in L$. Define

$$M_q\left(f, A\right) := q^{-1}\left(A\left(q \circ f\right)\right). \tag{26}$$

Remark 5.4. Assume the conditions of the previous definition are satisfied. Let $\alpha := \inf f \in [-\infty, \infty[$ and $\beta := \sup f \in]-\infty, \infty]$.

(a) Since C is an interval and $R_f \subset C$, $]\alpha, \beta[\subset C$. The continuity of q implies that $q\left(]\alpha, \beta[\right)$ is also an interval. Due to the strict monotonicity of q, we have

$$q\left(]\alpha, \beta[\right) = \begin{cases}]\lim_{\alpha+} q, \lim_{\beta+} q[\,, & \text{if } q \text{ is increasing} \\ ]\lim_{\beta+} q, \lim_{\alpha+} q[\,, & \text{if } q \text{ is decreasing} \end{cases}. \tag{27}$$

Since $\overline{q\left(]\alpha, \beta[\right)}$ is a closed interval containing $R_{q \circ f}$, Theorem 2.2 (a) and the inclusion $\overline{q\left(]\alpha, \beta[\right)} \subset R_q$ yield that

$$A\left(q \circ f\right) \in \overline{q\left(]\alpha, \beta[\right)} \subset R_q, \tag{28}$$

and therefore the definition (26) is correct.

(b) The number $M_q\left(f, A\right)$ is really a mean in the following sense: Suppose $\alpha \in \mathbb{R}$. By using (27) and (28), it is easy to think that $M_q\left(f, A\right) \geq \alpha$. We can get similarly that if $\beta \in \mathbb{R}$, then $M_q\left(f, A\right) \leq \beta$.

Further, we introduce some other means related to the formula C_{funct}.

Definition 5.5. Assume (H_1–H_4). Let $C \subset \mathbb{R}$ be a closed interval, $q, r : C \to \mathbb{R}$ be continuous and strictly monotone functions, and *let* $f \in E \to \mathbb{R}$ *take values in* C such that $\overline{q\left(]\inf f, \sup f[\right)} \subset R_q$, $q \circ f \in L$ and $\overline{r\left(]\inf f, \sup f[\right)} \subset R_r$, $r \circ f \in L$. Then we define the following mean of f with respect to C_{funct}:

$$M_{q,r}\left(f, A, \lambda, \pi, \mathfrak{A}_I\right)$$

$$:= q^{-1}\left(\sum_{i \in I}\left(\sum_{j \in J} \lambda_j A_{\pi_j(i)}\left(1_E\right)\right) q \circ r^{-1}\left(\frac{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)}\left(r \circ f\right)}{\sum\limits_{j \in J} \lambda_j A_{\pi_j(i)}\left(1_E\right)}\right)\right).$$

In the next result the introduced means are compared.

Proposition 5.6. *Assume the conditions in Definition* 5.5 *hold. If either* $q \circ r^{-1}$ *is convex and* q *is strictly increasing, or* $q \circ r^{-1}$ *is concave and* q *is strictly decreasing, then*

$$M_r\left(f, A\right) \leq M_{q,r}\left(\varphi, \mu, \lambda, \pi, \mathfrak{A}_I\right) \leq M_q\left(f, A\right),$$

while if either $r \circ q^{-1}$ *is convex and* r *is strictly decreasing, or* $r \circ q^{-1}$ *is concave and* r *is strictly increasing, then*

$$M_q\left(f, A\right) \leq M_{r,q}\left(\varphi, \mu, \lambda, \pi, \mathfrak{A}_I\right) \leq M_r\left(f, A\right).$$

Proof. We consider only the case when $q \circ r^{-1}$ is convex and q is strictly increasing. By applying Theorem 4.1 with $\varphi := q \circ r^{-1}$ and with $r \circ f$ instead of f, we obtain

$$q \circ r^{-1} (A(r \circ f))$$

$$\leq \sum_{i \in I} \left(\sum_{j \in J} \lambda_j A_{\pi_j(i)} (1_E) \right) q \circ r^{-1} \left(\frac{\sum_{j \in J} \lambda_j A_{\pi_j(i)} (r \circ f)}{\sum_{j \in J} \lambda_j A_{\pi_j(i)} (1_E)} \right) \leq A(q \circ f),$$

and this implies the result since q^{-1} is strictly increasing.

The proof is complete. □

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