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Sharp Bernstein Inequalities for Polynomials on a Real Hilbert Space

Juan Fernández-Sánchez
Grupo de Investigación de Teoría de Cópulas y Aplicaciones, Universidad de Almería, Spain
jfsjufesa@gmail.es

Gustavo A. Muñoz-Fernández, Daniel L. Rodríguez-Vidanes
Juan B. Seoane-Sepúlveda
Dep. de Análisis Matemático y Matemática Aplicada and Inst. de Matemática Interdisciplinar, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, Spain
gustavo_fernandez@mat.ucm.es, darodr05@ucm.es, jseoane@mat.ucm.es

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We search for extensions of several sharp Bernstein type inequalities already known for polynomials on the real line, to the case of polynomials on a real Hilbert space.

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1. Introduction and preliminaries

The study of the magnitude of the derivatives of a polynomial has drawn the attention of a great many mathematicians since the end of the 19^{th} century. D. Mendeleev (Fig. 1.2), inventor of the periodic table, posed and answered (see [17]) the following seemingly easy problem: If $P(x) = ax^2 + bx + c$ is a quadratic polynomial such that $|P(x)|$ is bounded by 1 in a given interval $[\alpha, \beta]$, how large can $|P'(x)|$ be in $[\alpha, \beta]$? Using an appropriate change of variable, the interval $[\alpha, \beta]$ can be assumed to be $[-1, 1]$. Mendeleev proved that if $|P(x)|$ is bounded by 1 in $[-1, 1]$, then $|P'(x)| \leq 4$ for all $x \in [-1, 1]$, and 4 is sharp in the sense that equality is attained for $P(x) = 1 - 2x^2$ at $x = \pm 1$. This result was generalized by V. A. Markov in a paper published in 1889 ([16]) for polynomials of arbitrary degree:

Theorem 1.1. (A. A. Markov, 1889) *If P is a polynomial of degree at most $n \in \mathbb{N}$ such that $|P(x)|$ is bounded by 1 in $[-1, 1]$, then $|P'(x)| \leq n^2$ for all $x \in [-1, 1]$. Equality is attained at the end points of $[-1, 1]$ for the n-th Chebyshev polynomial of the first kind, defined by $T_n(x) = \cos(n \arccos x)$ for $x \in [-1, 1]$.*

A further generalization of Medeleev original problem was published by V. A. Markov, brother of A. A. Markov (Fig. 1.1), who found a sharp estimate on the k-th derivative of a polynomial of arbitrary degree. The notation $P^{(k)}$ will be used to denote the k-th derivative of P from now on:

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Theorem 1.2. (V. A. Markov, 1892) *If P is a polynomial of degree at most $n \in \mathbb{N}$ such that $|P(x)|$ is bounded by 1 in $[-1,1]$, then*

$$|P^{(k)}(x)| \leq |T_n^{(k)}(\pm 1)| = \frac{n^2(n^2-1)\cdots(n^2-(k-1)^2)}{1\cdot 3\cdots(2k-1)},$$

for all $x \in [-1,1]$. Obviously, equality is attained at the end points of $[-1,1]$ for T_n.

Although Markov's estimates are sharp at $x = \pm 1$, they can be improved for any fixed point in $(-1,1)$. If $x \in \mathbb{R}$ is fixed, then we define $\mathcal{B}_{n,k}(x)$ as the best (smallest) constant in the following inequality:

$$|P^{(k)}(x)| \leq \mathcal{B}_{n,k}(x) \cdot \|P\|, \tag{1}$$

for every polynomial P of degree at most n. For simplicity, we set $\mathcal{B}_{n,1} = \mathcal{B}_n$ for all $n \in \mathbb{N}$. The following estimate on $\mathcal{B}_n(x)$ was obtained by S. Bernstein (Fig. 1.2) in 1912 ([6, 7]):

Theorem 1.3. (S. Bernstein) *If P is a polynomial of degree at most n such that $|P(x)|$ is bounded by 1, then*

$$|P'(x)| \leq \frac{n}{\sqrt{1-x^2}},$$

for every $x \in (-1,1)$. In other words,

$$\mathcal{B}_n(x) \leq \frac{n}{\sqrt{1-x^2}} \tag{2}$$

in $(-1,1)$.

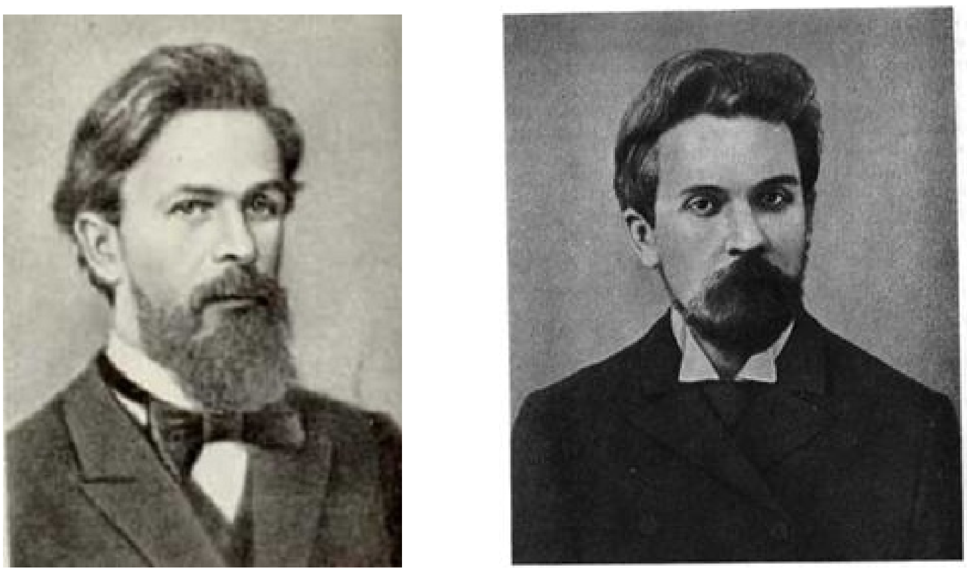

Figure 1.1: A. A. Markov (1856–1922) and V. A. Markov (1871–1897).

Pointwise estimates for higher derivatives are also known. For instance we have the following result from 1938 due to Duffin and Schaeffer ([9]):

Theorem 1.4. (R. Duffin & A. C. Schaeffer) *If P is a polynomial of degree at most n such that $|P(x)|$ is bounded by 1 in $[-1,1]$ and $x \in (-1,1)$, then for $1 \leq k \leq n$ we have*

$$|P^{(k)}(x)| \leq \sqrt{\left[T_n^{(k)}(x)\right]^2 + \left[S_n^{(k)}(x)\right]^2},$$

where $S_n(x) = \sin(n \arccos x)$, for all $x \in [-1,1]$. Thus

$$\mathcal{B}_{n,k}(x) \leq \mathcal{M}_{n,k}(x), \tag{3}$$

where $\mathcal{M}_{n,k}(x) := \sqrt{\left[T_n^{(k)}(x)\right]^2 + \left[S_n^{(k)}(x)\right]^2}$ in $(-1,1)$.

The previous estimate generalizes Theorem 1.3 in the sense that (2) coincides with (3) for $k=1$. Also, from Theorem 1.2 we deduce that $\mathcal{B}_{n,k}(x) \leq |T^{(k)}(\pm 1)|$ for all $x \in [-1,1]$. Inequality (3) improves the previous estimate in most of the interval $[-1,1]$, although it is not good in any neighborhood of ± 1. Interestingly, Duffin and Schaeffer used Theorem 1.4 to provide an alternative (and simpler) proof of V. A. Markov's estimate, Theorem 1.2.

V. Voronovskaja developed a procedure based on functional analysis to calculate $\mathcal{B}_n(x)$ in the 30's of the 20^{th} century (see [24]). These ideas were adapted later by Gusev [14] to obtain $\mathcal{B}_{n,k}(x)$ for all k with $1 \leq k \leq n$. However, both Gusev and Voronovskaja's results are not completely explicit. Using techniques from convex analysis, G. Araújo, G. A. Muñoz, D. L. Rodríguez and J. B. Seoane have proved recently (see [2]) a formula for $\mathcal{B}_2$, $\mathcal{B}_3$ and $\mathcal{B}_{3,2}$:

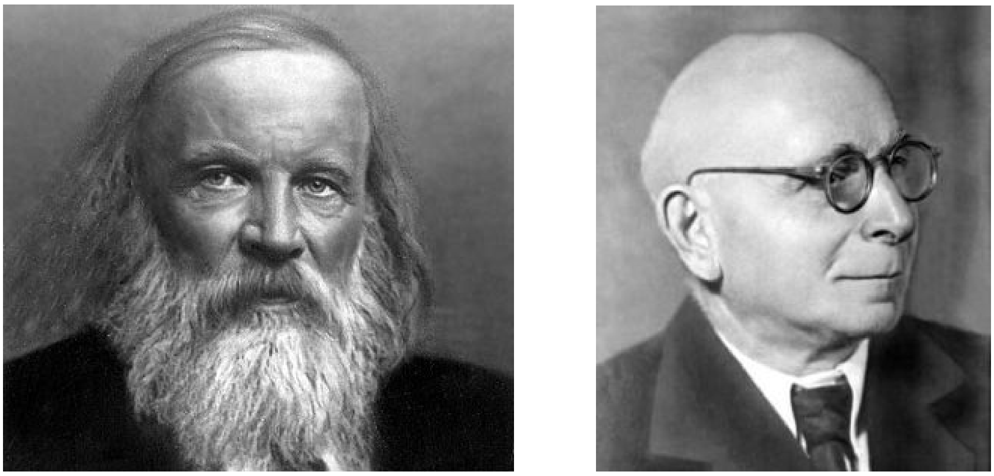

Figure 1.2: D. Mendeleev (1834–1907) and S. Bernstein (1880–1968).

Theorem 1.5. *All possible Benstein functions for $n = 2, 3$ are given by*

$$\mathcal{B}_2(x) = \begin{cases} \frac{1}{1-|x|} & \text{if } |x| \leq \frac{1}{2}, \\ 4|x| & \text{if } \frac{1}{2} \leq |x| \leq 1, \end{cases} \tag{4}$$

$$\mathcal{B}_3(x)=\begin{cases}3(1-4x^2) & \textit{if } |x|\le \frac{\sqrt{7}-2}{6},\\ \dfrac{7\sqrt{7}+10}{9(|x|+1)} & \textit{if } \frac{\sqrt{7}-2}{6}\le |x|\le \frac{2\sqrt{7}-1}{9},\\ \dfrac{-16x^3}{(1-9x^2)(1-x^2)} & \textit{if } \frac{2\sqrt{7}-1}{9}\le |x|\le \frac{1+2\sqrt{7}}{9},\\ \dfrac{7\sqrt{7}-10}{9(1-|x|)} & \textit{if } \frac{1+2\sqrt{7}}{9}\le |x|\le \frac{\sqrt{7}+2}{6},\\ 3(4x^2-1) & \textit{if } |x|\ge \frac{\sqrt{7}+2}{6},\end{cases} \tag{5}$$

and

$$\mathcal{B}_{3,2}(x)=\begin{cases}\dfrac{4}{1-9x^2} & \textit{if } |x|\le \frac{1}{9},\\ \dfrac{32}{9(|x|-1)^2} & \textit{if } \frac{1}{9}\le |x|\le \frac{1}{3},\\ 24|x| & \textit{if } |x|\ge \frac{1}{3}.\end{cases} \tag{6}$$

A list of extremal polynomials, that is, polynomials for which $\mathcal{B}_2(x_0)$, $\mathcal{B}_3(x_0)$ or $\mathcal{B}_{3,2}(x_0)$ are attained for a given x_0, can be found in [2].

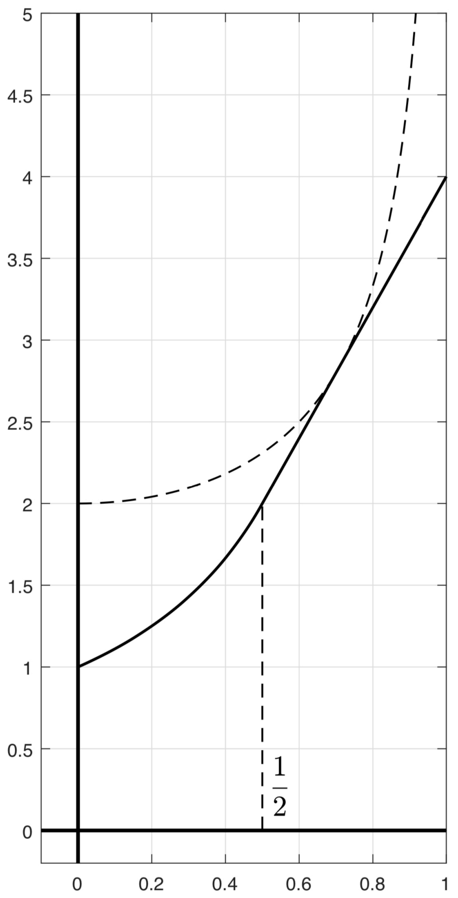

Figure 1.3: Sketch of $\mathcal{B}_2(x)$ (solid line) compared to Bernstein's estimate $2/\sqrt{1-x^2}$ (dashed line).

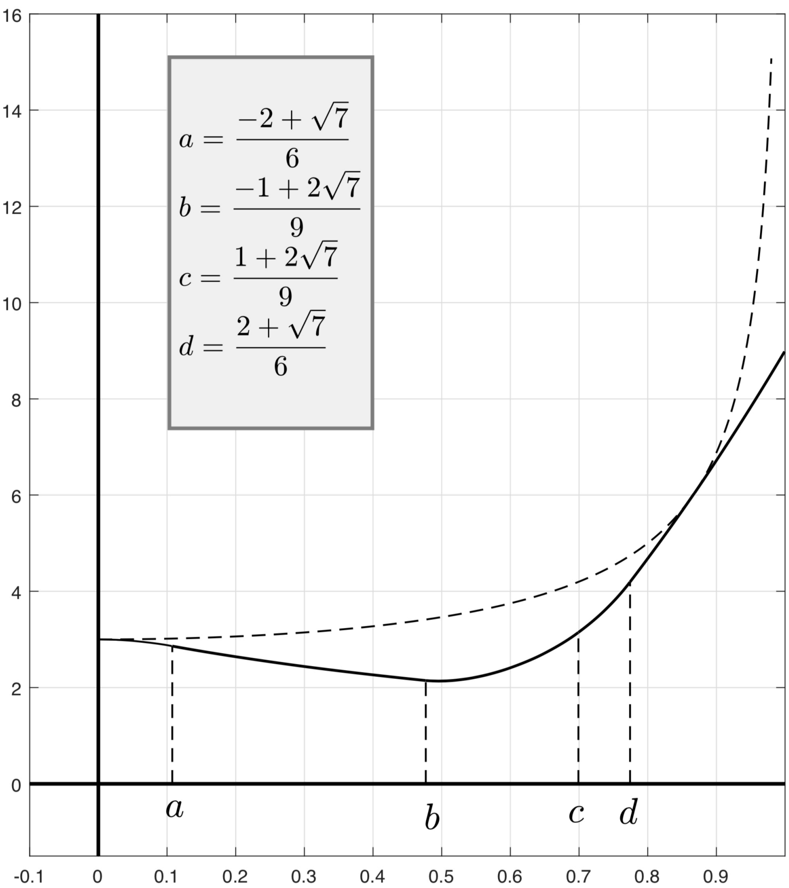

Figure 1.4: Sketch of $\mathcal{B}_3(x)$ (solid line) compared to Bernstein's estimate $3/\sqrt{1-x^2}$ (dashed line).

The study of polynomials in (real or complex) Banach spaces and its interaction with many fields within mathematics has been the object of deep study by many mathematicians for more than a century [1, 3, 4, 8, 10, 11, 12, 13, 18, 19].

In this paper we study generalizations of the formulas (4), (5) and (6) for polynomials on a real Hilbert space. Our results are attained by applying an idea already used in the past (see for instance [20, 21, 22]) that allows us to interpret an infinite dimensional problem as a problem in the real line. The proofs of this section rely heavily on the inner product structure within a Hilbert space and use the explicit form of the Bernstein function provided in Section 3 for $n = 2, 3$. For this reason, we tend to believe that our generalizations are not valid in a non-Euclidean space and that a better understanding of the one dimensional Bernstein functions $\mathcal{B}_{n,k}$ for $n \geq 4$ could also be used to generalize the one dimensional estimates to Hilbert spaces.

For convenience we recall the basic definitions needed to discuss polynomials on a real Banach space E. A continuous n-homogeneous polynomial on E is a mapping $P : E \to \mathbb{R}$ such that there exists a continuous symmetric n-linear form $L : E^n \to \mathbb{R}$ for which $P(x) = L(x, \dots, x)$ for all $x \in E$. In this case we put $P = \widehat{L}$. We define

$$\|P\| = \sup\{|P(x)| : \ \|x\| \leq 1\}.$$

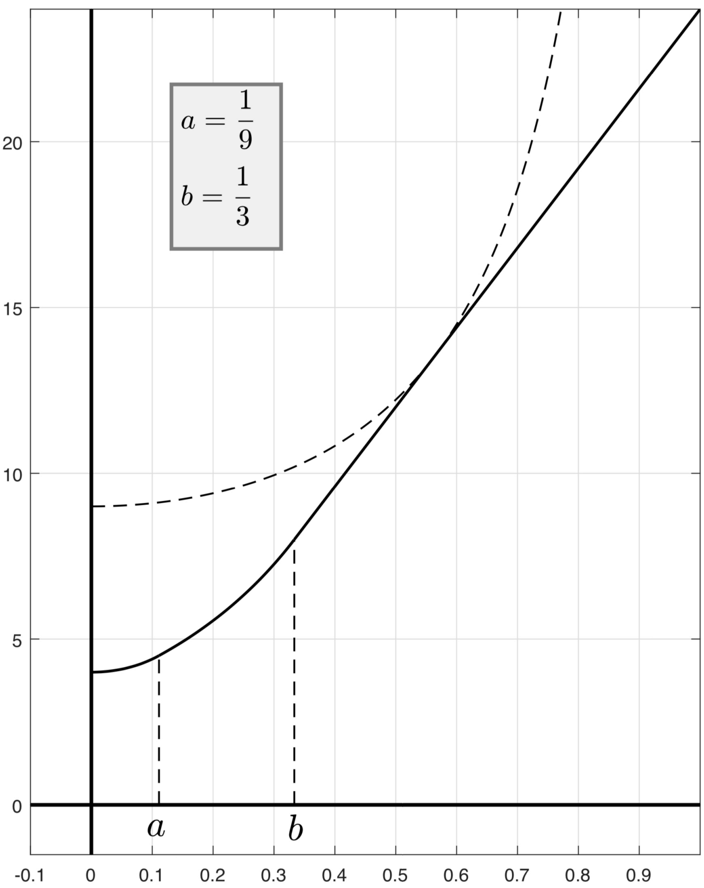

Figure 1.5: Sketch of $\mathcal{B}_{3,2}(x)$ (solid line) compared to the Duffin and Schaeffer's estimate $\mathcal{M}_{3,2}(x) = \frac{3\sqrt{-8x^2+9}}{\sqrt{(1-x^2)^3}}$ (dashed line).

If $L : E^n \to \mathbb{R}$ is a continuous n-linear mapping we define

$$\|L\| = \sup\{|L(x_1, \ldots, x_n)| : \ \|x_1\| \leq 1, \ldots, \|x_n\| \leq 1\}.$$

We let $\mathcal{P}(^nE)$, $\mathcal{L}(^nE)$ and $\mathcal{L}^s(^nE)$ denote, respectively, the Banach spaces of all continuous n-homogeneous polynomials, the continuous n-linear mappings and the continuous symmetric n-linear mappings on E endowed with the norms defined above. More generally, a map $P : E \to \mathbb{R}$ is a *continuous polynomial of degree* $\leq n$ if

$$P = P_0 + P_1 + \ldots + P_n,$$

where $P_k \in \mathcal{P}(^kE)$ $(1 \leq k \leq n)$, and $P_0 : E \to \mathbb{R}$ is a constant function. We use the notation $\mathcal{P}_n(E)$ to denote the space of all the continuous polynomials of degree $\leq n$ on E endowed with the norm

$$\|P\| = \sup\{|P(x)| : \ \|x\| \leq 1\}.$$

If $P \in \mathcal{P}_n(E)$, then P is an analytic function and we represent its Fréchet derivatives at $x \in E$ by $D^kP(x)$. Recall that $D^kP(x)$ would be, in fact, a symmetric k-linear form on E^k, whose associated k-homogeneous polynomial will be represented by $\widehat{D}^kP(x)$.

For a modern exposition on polynomials in Banach spaces we refer the reader to [8] (see, also, [5, §4] for a recent reference on a new approach to the topic).

2. Bernstein's functions in Hilbert spaces

In this section we search for possible generalizations of (4) and (5) for polynomials on a Hilbert space. If E is a real Banach space and $x \in E$ is fixed, then we define $\mathcal{B}_{n,k}(x,E)$ as the best (smallest) constant in the following inequality:

$$\|\widehat{D}^k P(x)\| \leq \mathcal{B}_{n,k}(x,E) \cdot \|P\|, \tag{7}$$

for every polynomial $P \in \mathcal{P}_n(E)$. For simplicity, we set $\mathcal{B}_{n,1}(x,E) = \mathcal{B}_n(x,E)$ for all $n \in \mathbb{N}$.

Notice that Y. Sarantopoulos ([23]) proved in 1991 that for every real Banach space E and $x \in E$ with $\|x\| < 1$ we have $\mathcal{B}_n(x,E) \leq \frac{n}{\sqrt{1-\|x\|^2}}$. Also, L. A. Harris proved in 2010 ([15]) that in any Banach space E, $\mathcal{B}_n(x,E) = T_n(\|x\|)$ for all $x \in E$ with $\|x\| \geq 1$. The estimates that will be provided in this section are done by restricting adequately a polynomial on a real Hilbert space to a polynomial in one variable. The way we will perform this restriction is taken from [20]. Since the restriction is key in our proof, we are going to describe the procedure for completeness.

Let H be a real Hilbert space endowed with the norm $\|\cdot\|$ induced by the inner product $\langle\cdot,\cdot\rangle$ and $x, y \in H$ so that $0 < \|x\| < 1$ and $\|y\| = 1$. In the following B_H and S_H will represent, respectively, the closed unit ball and the unit sphere of H. Consider the straight line passing through x and parallel to y as in Figure 2.1. This line meets S_H at the points $x_1 = x + t_1 y$ and $x_2 = x + t_2 y$, with

$$t_1 = -\langle x,y\rangle^2 - \sqrt{\langle x,y\rangle^2 + 1 - \|x\|^2},$$
$$t_2 = -\langle x,y\rangle^2 + \sqrt{\langle x,y\rangle^2 + 1 - \|x\|^2}.$$

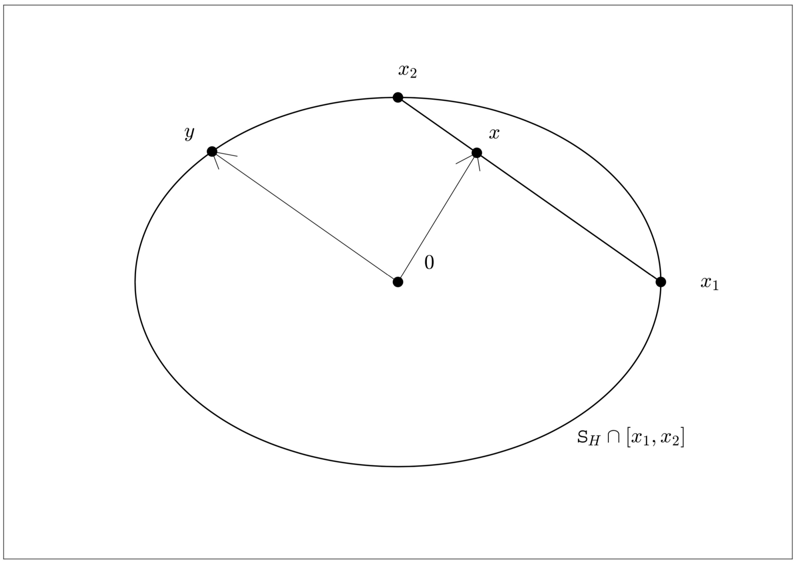

Figure 2.1: All polynomials on H are restricted to the segment with endpoints x_1 and x_2 constructed as in this figure.

Observe that t_1 and t_2 are the solutions of the quadratic equation in t given by $\|x + ty\|^2 = 1$. Obviously, the segment $[x_1, x_2]$ can be parametrized by the mapping

$[t_1, t_2] \ni t \mapsto x + ty \in [x_1, x_2]$ and is entirely contained in B_H. Now, by rescaling and shifting $[t_1, t_2]$ one can see easily that $[-1, 1] \ni t \mapsto x + \alpha(t - \beta)y \in [x_1, x_2]$, where

$$\alpha = \sqrt{1 - \|x\|^2 + \langle x, y\rangle^2} \quad \text{and} \quad \beta = \frac{\langle x, y\rangle}{\alpha}, \tag{8}$$

is another parametrization of $[x_1, x_2]$. Hence, if $P \in \mathcal{P}_n(H)$ then the one real variable polynomial Q defined by

$$Q(t) = P(x + \alpha(t - \beta)y)$$

satisfies $\|Q\| \le \|P\|$.

Theorem 2.1. *Let H be a real Hilbert space, $P \in \mathcal{P}_2(H)$ with $\|P\| \le 1$, $x \in \mathsf{B}_H \backslash \mathsf{S}_H$, and $y \in \mathsf{S}_H$. Then*

$$|DP(x)y| \le \begin{cases} \frac{\sqrt{1-\|x\|^2+\langle x,y\rangle^2}+|\langle x,y\rangle|}{1-\|x\|^2} & \textit{if } |\langle x, y\rangle| \le \sqrt{\frac{1-\|x\|^2}{3}}, \\ \frac{4|\langle x,y\rangle|}{1-\|x\|^2+\langle x,y\rangle^2} & \textit{if } |\langle x, y\rangle| \ge \sqrt{\frac{1-\|x\|^2}{3}}. \end{cases}$$

Proof. Fix $x \in H$ with $0 < \|x\| < 1$. Let $y \in \mathsf{S}_H$ arbitrary and consider the polynomial $Q(t) = P(x + \alpha(t - \beta)y)$ with α, β as in (8). Now applying (4) to Q we have

$$\alpha|DP(x + \alpha(t - \beta)y)y| = |Q'(t)| \le \mathcal{B}_2(t),$$

for every $t \in [-1, 1]$. Observe that $\beta \in [-1, 1]$ and that $|\beta| \le 1/2$ is equivalent to $|\langle x, y\rangle| \le \sqrt{\frac{1-\|x\|^2}{3}}$. Then

$$|DP(x)y| \le \frac{1}{\alpha}\mathcal{B}_2(\beta) = \frac{1}{\alpha}\begin{cases} \frac{1}{1-|\beta|} & \text{if } |\beta| \le \frac{1}{2}, \\ 4|\beta| & \text{if } \frac{1}{2} \le |\beta| \le 1, \end{cases}$$

$$= \begin{cases} \frac{\sqrt{1-\|x\|^2+\langle x,y\rangle^2}+|\langle x,y\rangle|}{1-\|x\|^2} & \text{if } |\langle x, y\rangle| \le \sqrt{\frac{1-\|x\|^2}{3}}, \\ \frac{4|\langle x,y\rangle|}{1-\|x\|^2+\langle x,y\rangle^2} & \text{if } |\langle x, y\rangle| \ge \sqrt{\frac{1-\|x\|^2}{3}}. \end{cases}$$

□

Now it is possible to obtain a pointwise estimate on the norm of the derivative in Hilbert spaces.

Corollary 2.2. *Let H be a real Hilbert space and $P \in \mathcal{P}_2(H)$ with $\|P\| \le 1$. If $\|x\| \le \sqrt{2}/2$, then*

$$\|DP(x)\| \le \begin{cases} \frac{1}{1-\|x\|} & \textit{if } \|x\| \le \frac{1}{2}, \\ 4\|x\| & \textit{if } \frac{1}{2} \le \|x\| \le \frac{\sqrt{2}}{2}. \end{cases} \tag{9}$$

The estimate is sharp.

Proof. Since $y \in \mathsf{S}_H$, the maximum value attained by $|\langle x, y\rangle|$ is $\|x\|$. Thus, if the inequality $|\langle x, y\rangle| \leq \sqrt{\frac{1-\|x\|^2}{3}}$ is satisfied for every $y \in \mathsf{S}_H$, then so is the inequality $\|x\| \leq \sqrt{\frac{1-\|x\|^2}{3}}$, which is equivalent to $\|x\| \leq \frac{1}{2}$.

Assume first that $\|x\| \leq \frac{1}{2}$. Let us define the function

$$f(z) = \frac{\sqrt{1-K^2+z^2}+z}{1-K^2}$$

on the interval $(0, K]$ where $0 < K \leq \frac{1}{2}$. Notice that f is strictly increasing on $(0, K]$. Therefore, the function

$$\frac{\sqrt{1-\|x\|^2+\langle x, y\rangle^2}+|\langle x, y\rangle|}{1-\|x\|^2}$$

is also strictly increasing in $|\langle x, y\rangle|$ on the interval $(0, \|x\|]$. Thus, the maximum is attained at $\|x\|$. Since $\|x\| \leq \frac{1}{2}$, we have $|\langle x, y\rangle| \leq \sqrt{\frac{1-\|x\|^2}{3}}$ for any $y \in \mathsf{S}_H$. Then,

$$|DP(x)y| \leq \frac{\sqrt{1-\|x\|^2+\langle x, y\rangle^2}+|\langle x, y\rangle|}{1-\|x\|^2} \leq \frac{1}{1-\|x\|},$$

for any $y \in \mathsf{S}_H$. Now let us define the function

$$g(z) = \frac{4z}{1-K^2+z^2}$$

on the interval $\left(\sqrt{\frac{1-K^2}{3}}, K\right]$, where $\frac{1}{2} < K < 1$. Notice that g is not strictly monotone on $\left(\sqrt{\frac{1-K^2}{3}}, K\right]$ for arbitrary $K \in \left(\frac{1}{2}, 1\right)$. In fact, g is strictly increasing if, and only if, $K \leq \sqrt{1-K^2}$, which is equivalent to $K \leq \frac{\sqrt{2}}{2}$. Thus, if we restrict g to the interval $(K, 1)$ with $K \geq \frac{\sqrt{2}}{2}$, then g is an strictly decreasing function.

Assume now that $\frac{1}{2} < \|x\| \leq \frac{\sqrt{2}}{2}$. In this case, the function $\frac{4|\langle x,y\rangle|}{1-\|x\|^2+\langle x,y\rangle^2}$ is strictly increasing in $|\langle x, y\rangle|$ on the interval $\left(\sqrt{\frac{1-\|x\|^2}{3}}, \|x\|\right]$, which implies that the maximum is attained at $\|x\|$. Thus, if $y \in \mathsf{S}_H$ is such that $|\langle x, y\rangle| \geq \sqrt{\frac{1-\|x\|^2}{3}}$, then

$$|DP(x)y| \leq \frac{4|\langle x, y\rangle|}{1-\|x\|^2+\langle x, y\rangle^2} \leq 4\|x\|.$$

On the other hand, since f is strictly increasing on the interval $\left(0, \sqrt{\frac{1-K^2}{3}}\right]$, where $\frac{1}{2} < K \leq \frac{\sqrt{2}}{2}$, we have that the maximum of f is attained at $\sqrt{\frac{1-K^2}{3}}$. Therefore, since the function

$$\frac{\sqrt{1-\|x\|^2+\langle x, y\rangle^2}+|\langle x, y\rangle|}{1-\|x\|^2}, \quad \text{where } |\langle x, y\rangle| \leq \sqrt{\frac{1-\|x\|^2}{3}},$$

is strictly increasing in $|\langle x,y\rangle|$, we have that $\frac{\sqrt{1-\|x\|^2+\langle x,y\rangle^2}+|\langle x,y\rangle|}{1-\|x\|^2}$ attains its maximum at $\sqrt{\frac{1-\|x\|^2}{3}}$. Thus, if $y\in \mathsf{S}_H$ is such that $|\langle x,y\rangle|\le\sqrt{\frac{1-\|x\|^2}{3}}$, then

$$|DP(x)y|\le\frac{\sqrt{1-\|x\|^2+\langle x,y\rangle^2}+|\langle x,y\rangle|}{1-\|x\|^2}\le\sqrt{\frac{3}{1-\|x\|^2}}.$$

Hence, if $\frac{1}{2}<\|x\|\le\frac{\sqrt{2}}{2}$, then, for any $y\in\mathsf{S}_H$,

$$|DP(x)y|\le\max\left\{4\|x\|,\sqrt{\frac{3}{1-\|x\|^2}}\right\}=4\|x\|. \qquad \square$$

Remark 2.3. A direct application of Theorem 1.5 with $\frac{\sqrt{2}}{2}<\|x\|$ does not provide any substantial improvement of previously known estimates on $\|DP(x)\|$.

Indeed, the function $\frac{4|\langle x,y\rangle|}{1-\|x\|^2+\langle x,y\rangle^2}$ is strictly decreasing in $|\langle x,y\rangle|$ on the interval $(\sqrt{1-\|x\|^2},\|x\|]$, which means that the maximum is attained at $\sqrt{1-\|x\|^2}$. Thus, if $y\in\mathsf{S}_H$ is such that $|\langle x,y\rangle|\ge\sqrt{1-\|x\|^2}$, then

$$|DP(x)y|\le\frac{4|\langle x,y\rangle|}{1-\|x\|^2+\langle x,y\rangle^2}\le\frac{2}{\sqrt{1-\|x\|^2}}.$$

By using the same arguments as above, if $y\in\mathsf{S}_H$ is such that $|\langle x,y\rangle|\le\sqrt{\frac{1-\|x\|^2}{3}}$, then

$$|DP(x)y|\le\frac{\sqrt{1-\|x\|^2+\langle x,y\rangle^2}+|\langle x,y\rangle|}{1-\|x\|^2}\le\sqrt{\frac{3}{1-\|x\|^2}}.$$

Analogously if $y\in\mathsf{S}_H$ is such that $\sqrt{\frac{1-\|x\|^2}{3}}\le|\langle x,y\rangle|\le\sqrt{1-\|x\|^2}$, then

$$|DP(x)y|\le\frac{4|\langle x,y\rangle|}{1-\|x\|^2+\langle x,y\rangle^2}\le\frac{2}{\sqrt{1-\|x\|^2}}.$$

Hence, if $\frac{\sqrt{2}}{2}<\|x\|<1$, then

$$|DP(x)y|\le\max\left\{\frac{2}{\sqrt{1-\|x\|^2}},\sqrt{\frac{3}{1-\|x\|^2}}\right\}=\frac{2}{\sqrt{1-\|x\|^2}},$$

for any $y\in\mathsf{S}_H$, from which $\|DP(x)\|\le\frac{2}{\sqrt{1-\|x\|^2}}$ follows. This was already proved by Y. Sarantopoulos ([23]).

Remark 2.4. Observe that for any real Hilbert space H, Theorem 1.5 and Corollary 2.2, show that

$$\mathcal{B}_2(x,H)=\mathcal{B}_2(\|x\|),$$

for every $x\in\mathsf{B}_H$ with $\|x\|\le\frac{\sqrt{2}}{2}$. Futhermore, we also know that $\mathcal{B}_2(x,H)=\mathcal{B}_2(\|x\|)$ if $\|x\|\ge 1$.

A natural question that arises here is whether $\mathcal{B}_2(x,H) = \mathcal{B}_2(\|x\|)$ is true as well if $\sqrt{2}/2 < \|x\| < 1$. The authors have numerical evidence showing that the formula $\mathcal{B}_2(x,H) = \mathcal{B}_2(\|x\|)$ could be true for all x even in the case that H is not a Hilbert space. For instance, if we consider the space $\mathcal{P}_2(\ell_\infty^2)$, i.e., the space of the polynomials on $\mathbb{R}^2$ of degree at most 2 endowed with the sup norm on the unit square $[-1,1]^2$, then a simple Monte Carlo type experiment shows that the formula $\mathcal{B}_2(x,\ell_\infty^2) = \mathcal{B}_2(\|x\|)$ could be true for every $x \in \mathsf{B}_{\ell_\infty^2}$. We are using the standard notation ℓ_∞^2 to represent $\mathbb{R}^2$ endowed with the norm $\|(x,y)\|_\infty = \max\{|x|,|y|\}$.

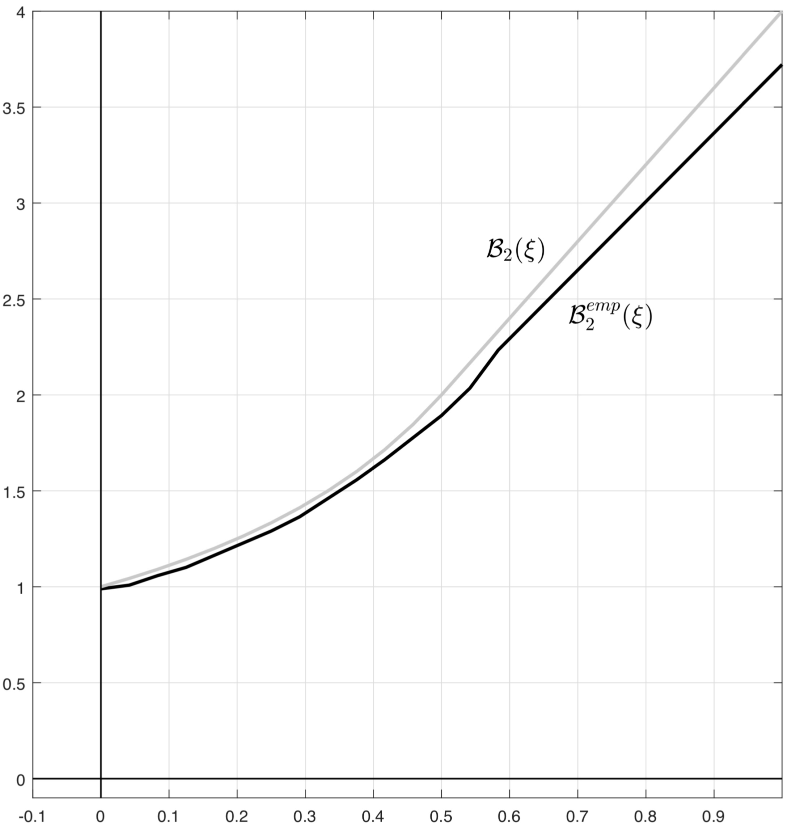

Figure 2.2: Representation of $\mathcal{B}_2(\xi)$ (gray) compared to the empiric value of $\mathcal{B}_2(\xi,\ell_\infty^2)$, namely $\mathcal{B}_2^{\mathrm{emp}}(\xi,\ell_\infty^2)$, calculated by using a Monte Carlo type experimet (black) with 20,000 normalized random polynomials and a mesh of 2500 points in $[-1,1]^2$.

This experiment has been implemented with MATLAB in the following terms: We have considered sets of n normalized random polynomials in $\mathcal{P}_2(\ell_\infty^2)$ and a fixed mesh M in $[-1,1]^2$ consisting of m^2 uniformly distributed points. Then, if $M_\infty = \{\|(x,y)\|_\infty : (x,y) \in M\}$, for each $\xi \in M_\infty$ we have calculated the maximum of $\|DP(x,y)\|$, where P ranges over all the random polynomials and (x,y) runs through all the points $(x,y) \in M$ such that $\|(x,y)\|_\infty = \xi$. Observe that here $\|DP(x,y)\|$ is the norm of $DP(x,y)$ as a linear form on ℓ_∞^2, i.e., the ℓ_1-norm of $\nabla P(x,y)$. This maximum will be denoted by $\mathcal{B}_2^{\mathrm{emp}}(\xi,\ell_\infty^2)$. All the times the experiment was run with different choices of n and m we obtained the relationship $\mathcal{B}_2(\xi) \geq \mathcal{B}_2^{\mathrm{emp}}(\xi,\ell_\infty^2)$ for all $\xi \in M_\infty$, as expected if the formula $\mathcal{B}_2(x,\ell_\infty^2) = \mathcal{B}_2(\|x\|)$ were true for every $x \in \mathsf{B}_{\ell_\infty^2}$. In Figure 2.2 we have represented $\mathcal{B}_2(\xi)$ and $\mathcal{B}_2^{\mathrm{emp}}(\xi,\ell_\infty^2)$ for $n = 20,000$ and $m = 50$.

Problem 2.5. (i) *Does there exist a real Hilbert space of dimension* > 1 *such that* $\mathcal{B}_2(x,H) > \mathcal{B}_2(\|x\|)$ *for every* $x \in \mathsf{B}_H$ *with* $\frac{\sqrt{2}}{2} < \|x\| < 1$*?*

(ii) *Let* $\overline{\mathcal{B}_2}(x)\colon \left[\frac{\sqrt{2}}{2}, 1\right) \to \mathbb{R}$ *be defined by* $\overline{\mathcal{B}_2}(x) := \frac{2}{\sqrt{1-x^2}}$*, does there exist a sequence* $\{H_k\}_{k\in\mathbb{N}}$ *of real Hilbert spaces such that the sequence of functions*

$$\left\{\mathcal{B}_2(x,H_k) \restriction \left\{x \in \mathsf{B}_{H_k}\colon \frac{\sqrt{2}}{2} \leq \|x\| < 1\right\}\right\}_{k\in\mathbb{N}}$$

converges pointwise to $\overline{\mathcal{B}_2}$*?*

We study now the first derivative of polynomials of degree at most 3 on a real Hilbert space.

Theorem 2.6. *Let* H *be a real Hilbert space,* $P \in \mathcal{P}_3(H)$ *with* $\|P\| \leq 1$*,* $x \in \mathsf{B}_H \setminus \mathsf{S}_H$ *and* $y \in \mathsf{S}_H$*. Then*

$$|DP(x)y| \leq \begin{cases} \dfrac{3(1-\|x\|^2-3\langle x,y\rangle^2)}{(1-\|x\|^2+\langle x,y\rangle^2)^{\frac{3}{2}}} & \text{if } |\langle x,y\rangle| \leq \frac{a\sqrt{1-\|x\|^2}}{\sqrt{1-a^2}}, \\ \dfrac{7\sqrt{7}+10}{9\left(\sqrt{1-\|x\|^2+\langle x,y\rangle^2}+|\langle x,y\rangle|\right)} & \text{if } \frac{a\sqrt{1-\|x\|^2}}{\sqrt{1-a^2}} \leq |\langle x,y\rangle| \leq \frac{b\sqrt{1-\|x\|^2}}{\sqrt{1-b^2}}, \\ \dfrac{-16\langle x,y\rangle^3}{(1-\|x\|^2-8\langle x,y\rangle^2)(1-\|x\|^2)} & \text{if } \frac{b\sqrt{1-\|x\|^2}}{\sqrt{1-b^2}} \leq |\langle x,y\rangle| \leq \frac{c\sqrt{1-\|x\|^2}}{\sqrt{1-c^2}}, \\ \dfrac{7\sqrt{7}-10}{9\left(\sqrt{1-\|x\|^2+\langle x,y\rangle^2}-|\langle x,y\rangle|\right)} & \text{if } \frac{c\sqrt{1-\|x\|^2}}{\sqrt{1-c^2}} \leq |\langle x,y\rangle| \leq \frac{d\sqrt{1-\|x\|^2}}{\sqrt{1-d^2}}, \\ -\dfrac{3(1-\|x\|^2-3\langle x,y\rangle^2)}{(1-\|x\|^2+\langle x,y\rangle^2)^{\frac{3}{2}}} & \text{if } |\langle x,y\rangle| \geq \frac{d\sqrt{1-\|x\|^2}}{\sqrt{1-d^2}}, \end{cases}$$

where $a = \frac{\sqrt{7}-2}{6}$*,* $b = \frac{2\sqrt{7}-1}{9}$*,* $c = \frac{2\sqrt{7}+1}{9}$ *and* $d = \frac{\sqrt{7}+2}{6}$*.*

Proof. Take x with $0 < \|x\| < 1$ and consider $y \in \mathsf{S}_H$. Now, as above, we apply the one dimensional estimate (5) to the polynomial $Q(t) = P(x + \alpha(t-\beta)y)$ with α, β as in (8) to obtain:

$$\alpha|DP(x+\alpha(t-\beta)y)y| = |Q'(t)| \leq \mathcal{B}_3(t),$$

for every $t \in [-1,1]$. In particular, since $\beta \in [-1,1]$, we have

$$|DP(x)y| \leq \frac{1}{\alpha}\mathcal{B}_3(\beta) = \frac{1}{\alpha}\begin{cases} 3(1-4\beta^2) & \text{if } |\beta| \leq \frac{\sqrt{7}-2}{6}, \\ \dfrac{7\sqrt{7}+10}{9(|\beta|+1)} & \text{if } \frac{\sqrt{7}-2}{6} \leq |\beta| \leq \frac{2\sqrt{7}-1}{9}, \\ \dfrac{-16x^3}{(1-9\beta^2)(1-\beta^2)} & \text{if } \frac{2\sqrt{7}-1}{9} \leq |\beta| \leq \frac{2\sqrt{7}+1}{9}, \\ \dfrac{7\sqrt{7}-10}{9(1-|\beta|)} & \text{if } \frac{2\sqrt{7}+1}{9} \leq |\beta| \leq \frac{\sqrt{7}+2}{6}, \\ 3(4\beta^2-1) & \text{if } |\beta| \geq \frac{\sqrt{7}+2}{6}. \end{cases}$$

$$= \begin{cases} \dfrac{3(1-\|x\|^2-3\langle x,y\rangle^2)}{(1-\|x\|^2+\langle x,y\rangle^2)^{\frac{3}{2}}} & \text{if } |\langle x,y\rangle| \le \frac{a\sqrt{1-\|x\|^2}}{\sqrt{1-a^2}}, \\ \dfrac{7\sqrt{7}+10}{9\left(\sqrt{1-\|x\|^2+\langle x,y\rangle^2}+|\langle x,y\rangle|\right)} & \text{if } \frac{a\sqrt{1-\|x\|^2}}{\sqrt{1-a^2}} \le |\langle x,y\rangle| \le \frac{b\sqrt{1-\|x\|^2}}{\sqrt{1-b^2}}, \\ \dfrac{-16\langle x,y\rangle^3}{(1-\|x\|^2-8\langle x,y\rangle^2)(1-\|x\|^2)} & \text{if } \frac{b\sqrt{1-\|x\|^2}}{\sqrt{1-b^2}} \le |\langle x,y\rangle| \le \frac{c\sqrt{1-\|x\|^2}}{\sqrt{1-c^2}}, \\ \dfrac{7\sqrt{7}-10}{9\left(\sqrt{1-\|x\|^2+\langle x,y\rangle^2}-|\langle x,y\rangle|\right)} & \text{if } \frac{c\sqrt{1-\|x\|^2}}{\sqrt{1-c^2}} \le |\langle x,y\rangle| \le \frac{d\sqrt{1-\|x\|^2}}{\sqrt{1-d^2}}, \\ -\dfrac{3(1-\|x\|^2-3\langle x,y\rangle^2)}{(1-\|x\|^2+\langle x,y\rangle^2)^{\frac{3}{2}}} & \text{if } |\langle x,y\rangle| \ge \frac{d\sqrt{1-\|x\|^2}}{\sqrt{1-d^2}}, \end{cases}$$

where $a = \frac{\sqrt{7}-2}{6}$, $b = \frac{2\sqrt{7}-1}{9}$, $c = \frac{2\sqrt{7}+1}{9}$ and $d = \frac{\sqrt{7}+2}{6}$. □

Remark 2.7. Let H be a real Hilbert space and $P \in \mathcal{P}_3(H)$ with $\|P\| \le 1$. If we fix x with $\|x\| < 1$ in the estimate of Theorem 2.6 and maximize the functions appearing there in terms of $\langle x,y\rangle$, with $y \in \mathsf{S}_H$, it is posible to obtain the following pointwise estimate on $\|DP(x)\|$:

$$\|DP(x)\| \le \begin{cases} \frac{3}{\sqrt{1-\|x\|^2}} & \text{if } \|x\| \le \frac{2\sqrt{7}+1}{9}, \\ \frac{(7\sqrt{7}+10)\sqrt{1+\frac{2\sqrt{7}+1}{9}}}{9\sqrt{1-\frac{2\sqrt{7}+1}{9}}\sqrt{1-\|x\|^2}} & \text{if } \frac{2\sqrt{7}+1}{9} < \|x\|. \end{cases}$$

We spare the details. Observe that the bound $\frac{3}{\sqrt{1-\|x\|^2}}$ was already proved previously by Y. Sarantopoulos ([23]).

On the other hand, one might be tempted to think, for instance, that $\mathcal{B}_3(x,H) = \mathcal{B}_3(\|x\|)$ for every $x \in H$. However, this is not the case if H has dimension $n > 1$. Indeed, assume by way of contradiction that $\mathcal{B}_3(\|x\|) = \mathcal{B}_3(x,\mathbb{R}^n)$, where $n > 1$. Then, given $(x_{0,1},\ldots,x_{0,n}) \in \mathbb{R}^n$ with $\sqrt{x_{0,1}^2+\cdots+x_{0,n}^2} \in \left(0,\frac{\sqrt{7}-2}{6}\right)$, we would have

$$\|DP_n(x_{0,1},\ldots,x_{0,n})\| \le 3\left[1-4(x_{0,1}^2+\cdots+x_{0,n}^2)\right]$$

for every $P_n \in \mathcal{P}_3(\mathbb{R}^n)$ with $\|P_n\| = 1$. Fix $x_{0,1} \in \left(0,\frac{\sqrt{7}-2}{6}\right)$ and choose small enough $x_{0,2},\ldots,x_{0,n}$ in $\mathbb{R}\setminus\{0\}$ such that $\sqrt{x_{0,1}^2+\cdots+x_{0,n}^2} \in \left(0,\frac{\sqrt{7}-2}{6}\right)$. By Theorem 1.5, take $P \in \mathcal{P}_3(\mathbb{R})$ with $\|P\| = 1$ such that $\|DP(x_{0,1})\| = 3(1-4x_{0,1}^2)$. Now consider the polynomial $\overline{P}_n(x_1,\ldots,x_n) = P(x_1)$, which satisfies $\|\overline{P}_n\| = \|P\| = 1$. Then,

$$\|D\overline{P}_n(x_{0,1},y_1,\ldots,y_{n-1})\| = 3(1-4x_{0,1}^2),$$

for every $y_1,\ldots,y_{n-1} \in \mathbb{R}$. In particular

$$\|D\overline{P}_n(x_{0,1},x_{0,2},\ldots,x_{0,n})\| = 3(1-4x_{0,1}^2) > 3\left[1-4(x_{0,1}^2+x_{0,2}^2+\cdots+x_{0,n}^2)\right],$$

which is a contradiction.

Finally we provide a pointwise estimate on the norm of the second derivative of a polynomial of degree at most 3 on a real Hilbert space.

Theorem 2.8. *Let H be a real Hilbert space, $P\in\mathcal{P}_3(H)$ with $\|P\|\le 1$ and $x\in\mathsf{B}_H\setminus\mathsf{S}_H$. Then*

$$\|\widehat{D}^2P(x)\| \le \begin{cases} \dfrac{4}{1-9\|x\|^2} & \text{if } \|x\|\le\frac{1}{9},\\ \dfrac{32}{9(1-\|x\|)^2} & \text{if } \frac{1}{9}\le\|x\|\le\frac{1}{3},\\ 24\|x\| & \text{if } \frac{1}{3}\le\|x\|\le\frac{\sqrt{3}}{3},\\ \dfrac{16}{\sqrt{3}(1-\|x\|^2)} & \text{if } \frac{\sqrt{3}}{3}<\|x\|. \end{cases} \tag{10}$$

The estimate is obviously sharp for every $x\in\mathsf{B}_H$ with $\|x\|\le\frac{\sqrt{3}}{3}$.

Proof. As in the previous two proofs, we just need to prove (10) for x satisfying $0<\|x\|<1$. Consider again $y\in\mathsf{S}_H$ and the polynomial $Q(t)=P(x+\alpha(t-\beta)y)$ with α,β as in (8). Applying the one dimensional estimate (6) to Q we have

$$\alpha^2|\widehat{D}^2P(x+\alpha(t-\beta)y)y| = |Q''(t)| \le \mathcal{B}_{3,2}(t),$$

for every $t\in[-1,1]$, from which

$$|\widehat{D}^2P(x)y| \le \frac{1}{\alpha^2}\mathcal{B}_{3,2}(\beta) = \frac{1}{\alpha^2}\begin{cases} \dfrac{4}{1-9\beta^2} & \text{if } |\beta|\le\frac{1}{9},\\ \dfrac{32}{9(1-|\beta|)^2} & \text{if } \frac{1}{9}\le|\beta|\le\frac{1}{3},\\ 24|\beta| & \text{if } |\beta|\ge\frac{1}{3}, \end{cases}$$

$$= \begin{cases} \dfrac{4}{1-\|x\|^2-8\langle x,y\rangle^2} & \text{if } |\langle x,y\rangle|\le\frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}},\\ \dfrac{32}{9\left(\sqrt{1-\|x\|^2+\langle x,y\rangle^2}-|\langle x,y\rangle|\right)^2} & \text{if } \frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}}\le|\langle x,y\rangle|\le\frac{\sqrt{1-\|x\|^2}}{2\sqrt{2}},\\ \dfrac{24|\langle x,y\rangle|}{(1-\|x\|^2+\langle x,y\rangle^2)^{3/2}} & \text{if } |\langle x,y\rangle|\ge\frac{\sqrt{1-\|x\|^2}}{2\sqrt{2}}. \end{cases}$$

Following the proofs of Theorems 2.1 and 2.6, if for every $y\in\mathsf{S}_H$ we have that $|\langle x,y\rangle|\le\frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}}$, then $\|x\|\le\frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}}$ if, and only if, $\|x\|\le\frac{1}{9}$.

Assume first that $\|x\|\le\frac{1}{9}$ and let us define the function

$$f(z)=\frac{4}{1-K^2-8z^2},$$

on $(0,K)$, where $0<K\le\frac{1}{9}$.

Since f is strictly increasing on $(0, K)$ for every $K \in \left(0, \frac{1}{9}\right]$, we have that the function $\frac{4}{1-\|x\|^2-8\langle x,y\rangle^2}$ is strictly increasing in $|\langle x,y\rangle|$ on $(0, \|x\|]$. The latter implies that $\frac{4}{1-\|x\|^2-8\langle x,y\rangle^2}$ attains its maximum at $\|x\|$. Hence, we have, for any $y \in \mathsf{S}_H$,

$$|DP(x)y| \leq \frac{4}{1-\|x\|^2-8\langle x,y\rangle^2} \leq \frac{4}{1-9\|x\|^2}.$$

On the other hand, if there exist $y \in \mathsf{S}_H$ such that

$$\frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}} \leq |\langle x,y\rangle| \leq \frac{\sqrt{1-\|x\|^2}}{2\sqrt{2}},$$

then $\frac{1}{9} \leq \|x\| \leq \frac{1}{3}$.

Assume that $\frac{1}{9} \leq \|x\| \leq \frac{1}{3}$ and let us define the function

$$g(z) = \frac{32}{9\left(\sqrt{1-K^2+z^2}-z\right)^2},$$

on $\left(\frac{\sqrt{1-K^2}}{4\sqrt{5}}, K\right]$, where $\frac{1}{9} < K \leq \frac{1}{3}$. Notice that g is an strictly increasing function on $\left(\frac{\sqrt{1-K^2}}{4\sqrt{5}}, K\right]$ for every $K \in \left(\frac{1}{9}, \frac{1}{3}\right]$. Thus, the function

$$\frac{32}{9\left(\sqrt{1-\|x\|^2+\langle x,y\rangle^2}-|\langle x,y\rangle|\right)^2}$$

is strictly increasing in $|\langle x,y\rangle|$ on $\frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}} \leq |\langle x,y\rangle| \leq \|x\|$. Hence, if $y \in \mathsf{S}_H$ is such that $\frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}} \leq |\langle x,y\rangle| \leq \|x\|$, then

$$|DP(x)y| \leq \frac{32}{9\left(\sqrt{1-\|x\|^2+\langle x,y\rangle^2}-|\langle x,y\rangle|\right)^2} \leq \frac{32}{9(1-\|x\|)^2}.$$

Furthermore, since f defined on $\left(0, \frac{\sqrt{1-K^2}}{4\sqrt{5}}\right]$, where $\frac{1}{9} < K \leq \frac{1}{3}$ is also strictly increasing, then

$$|DP(x)y| \leq \frac{4}{1-\|x\|^2-8\langle x,y\rangle^2} \leq \frac{40}{9(1-\|x\|^2)},$$

for any $y \in \mathsf{S}_H$ with $|\langle x,y\rangle| \leq \frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}}$. Thus,

$$|DP(x)y| \leq \max\left\{\frac{40}{9(1-\|x\|^2)}, \frac{32}{9(1-\|x\|)^2}\right\} = \frac{32}{9(1-\|x\|)^2},$$

for any $y \in \mathsf{S}_H$.

Finally assume that $\|x\| > \frac{1}{3}$ and let us define

$$h(z) = \frac{24z}{(1 - K^2 + z^2)^{3/2}},$$

on $\left(\frac{\sqrt{1-K^2}}{2\sqrt{2}}, K\right]$, where $\frac{1}{3} < K < 1$. Notice that h is strictly increasing if, and only if, $K \leq \frac{\sqrt{1-K^2}}{\sqrt{2}}$, which is equivalent to $K \leq \frac{\sqrt{3}}{3}$. Thus, assume first that $\frac{1}{3} < \|x\| \leq \frac{\sqrt{3}}{3}$. In this case, the function $\frac{24|\langle x,y\rangle|}{(1-\|x\|^2+\langle x,y\rangle^2)^{3/2}}$ is strictly increasing in $|\langle x, y\rangle|$ on $\left(\frac{\sqrt{1-\|x\|^2}}{2\sqrt{2}}, \|x\|\right]$. Hence, if $y \in \mathsf{S}_H$ is such that $\frac{\sqrt{1-\|x\|^2}}{2\sqrt{2}} < |\langle x, y\rangle|$ and there does not exits $y \in \mathsf{S}_H$ such that $\frac{\sqrt{1-\|x\|^2}}{\sqrt{2}} < |\langle x, y\rangle|$, then

$$|DP(x)y| \leq \frac{24|\langle x, y\rangle|}{(1 - \|x\|^2 + \langle x, y\rangle^2)^{3/2}} \leq 24\|x\|.$$

Furthermore, since f and g are strictly increasing on $\left(0, \frac{\sqrt{1-K^2}}{4\sqrt{5}}\right]$ and $\left(\frac{\sqrt{1-K^2}}{4\sqrt{5}}, \frac{\sqrt{1-K^2}}{2\sqrt{2}}\right]$, respectively, where $K \in \left(\frac{1}{3}, \frac{\sqrt{3}}{3}\right]$, we have

$$|DP(x)y| \leq \frac{4}{1 - \|x\|^2 - 8\langle x, y\rangle^2} \leq \frac{40}{9(1 - \|x\|^2)},$$

for any $y \in \mathsf{S}_H$ with $|\langle x, y\rangle| \leq \frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}}$, and

$$|DP(x)y| \leq \frac{32}{9\left(\sqrt{1 - \|x\|^2 + \langle x, y\rangle^2} - |\langle x, y\rangle|\right)^2} \leq \frac{64}{9(1 - \|x\|^2)},$$

for any $y \in \mathsf{S}_H$ with $\frac{\sqrt{1-\|x\|^2}}{4\sqrt{5}} \leq |\langle x, y\rangle| \leq \frac{\sqrt{1-\|x\|^2}}{2\sqrt{2}}$. Hence,

$$|DP(x)y| \leq \max\left\{\frac{40}{9(1 - \|x\|^2)}, \frac{64}{9(1 - \|x\|^2)}, 24\|x\|\right\} = 24\|x\|,$$

for any $y \in \mathsf{S}_H$. Assume now that $\|x\| > \frac{\sqrt{3}}{3}$. In this last case, the function $\frac{24|\langle x,y\rangle|}{(1-\|x\|^2+\langle x,y\rangle^2)^{3/2}}$ is stritcly decreasing in $|\langle x, y\rangle|$ on $\left(\frac{\sqrt{1-\|x\|^2}}{\sqrt{2}}, \|x\|\right]$. Hence, if $y \in \mathsf{S}_H$ is such that $\frac{\sqrt{1-\|x\|^2}}{\sqrt{2}} < |\langle x, y\rangle|$, then

$$|DP(x)y| \leq \frac{24|\langle x, y\rangle|}{(1 - \|x\|^2 + \langle x, y\rangle^2)^{3/2}} \leq \frac{16}{\sqrt{3}(1 - \|x\|^2)}.$$

Hence, using the same arguments as above,

$$|DP(x)y| \leq \max\left\{\frac{40}{9(1 - \|x\|^2)}, \frac{64}{9(1 - \|x\|^2)}, \frac{16}{\sqrt{3}(1 - \|x\|^2)}\right\} = \frac{16}{\sqrt{3}(1 - \|x\|^2)},$$

for any $y \in \mathsf{S}_H$.

Problem 2.9. (i) *Does there exist a real Hilbert space of dimension* > 1 *such that* $\mathcal{B}_{3,2}(x, H) > \mathcal{B}_{3,2}(\|x\|)$ *for every* $x \in \mathsf{B}_H$ *with* $\frac{\sqrt{3}}{3} < \|x\| < 1$*?*

(ii) *Let* $\overline{\mathcal{B}_{3,2}}(x)\colon \left(\frac{\sqrt{3}}{3}, 1\right) \to \mathbb{R}$ *be defined by* $\overline{\mathcal{B}_{3,2}}(x) := \frac{16}{\sqrt{3}(1-x^2)}$.

Does there exist a sequence $\{H_k\}_{k\in\mathbb{N}}$ *of real Hilbert spaces such that the sequence of functions*

$$\left\{\mathcal{B}_{3,2}(x, H_k) \restriction \left\{x \in \mathsf{B}_{H_k}\colon \frac{\sqrt{3}}{3} \leq \|x\| < 1\right\}\right\}_{k\in\mathbb{N}}$$

converges pointwise to $\overline{\mathcal{B}_{3,2}}$*?*

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