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Approximate Calculation of the Chebyshev Center for a Convex Compact Set in $\mathbb{R}^n$

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We present an approximate algorithm for calculation of the Chebyshev center for a convex compact subset from $\mathbb{R}^n$ which is given via its supporting function. We reduce the problem to the solution of a linear programming problem and estimate the error between an approximate and the exact solutions in terms of the step of a grid.

Keywords: Chebyshev center, stability of minimization problem, Hausdorff distance, linear programming, supporting function.

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1. Introduction and main notations

Let $\mathbb{R}^n$ be an n-dimensional Euclidean space with the inner product $(\cdot,\cdot)$, the Euclidean norm $\|\cdot\|$ and $B_r(a) = \{x \in \mathbb{R}^n \mid \|x-a\| \le r\}$ for $a \in \mathbb{R}^n$, $r > 0$. If $M \subset \mathbb{R}^n$ is a symmetric with respect to zero compact subset with nonempty interior, then we can consider a particular norm $n_M(x) = \mu(x, M) = \inf\{t > 0 \mid \frac{x}{t} \in M\}$ for all $x \in \mathbb{R}^n$ as Minkowski functional of M. We shall define the particular ball of radius r with center $a \in \mathbb{R}^n$: $rM + a = \{x \in \mathbb{R}^n \mid n_M(x-a) \le r\}$. Denote by $\operatorname{int} A$ and ∂A the interior and the boundary of the set A.

Let $A \subset \mathbb{R}^n$ be a convex compact subset. The *Chebyshev center* of the set A is the set of solutions $C_M(A)$ of the problem

$$R_M(A) = \min_{x\in\mathbb{R}^n} \max_{a\in A} n_M(x-a), \tag{1}$$

i.e. for any $x \in C_M(A)$ we have $\max_{a\in A} n_M(x-a) = R_M(A)$. Obviously $C_M(A) \ne \emptyset$ due to compactness arguments. So, $C_M(A)$ is the set of centers of balls $R_M(A)M + x$ of the smallest radius $R_M(A)$ each of which contains A. Note that if a compact set A is not convex, then $C_M(A) = C_M(\operatorname{co} A)$. Further we shall assume that the set A is convex.

In general, $C_M(A)$ is not necessarily a singleton and does not belong to the set A. But in the case of Euclidean norm $n_M(x) = \|x\|$ the set $C(A)$, the solution of the problem

$$R(A) = \min_{x\in\mathbb{R}^n} \max_{a\in A} \|x-a\|, \tag{2}$$

is singleton and $C(A) \in A$ [6, Theorem 1].

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The concept of Chebyshev center is an important tool in different applications [4, 5, 7, 10, 11], for example in some identification problems. It is known, that the problem of finding the Chebyshev center (2) is NP-hard and there are only few situation (in certain cases on the plane, for special sets: finite sets, intersection of two ellipsoids) when the Chebyshev center can be found numerically efficiently, see details in [10].

Recall that the *Minkowski-Pontryagin difference* [9] for subsets $P, Q \subset \mathbb{R}^n$ is

$$P \overset{*}{-} Q = \{x \in \mathbb{R}^n \mid x + Q \subset P\} = \bigcap_{x\in Q}(P - x),$$

and the *Hausdorff distance* is

$$h(P,Q) = \max\left\{\max_{x\in P}\varrho_Q(x),\ \max_{x\in Q}\varrho_P(x)\right\}.$$

Here $\varrho_Q(x) = \inf\limits_{y\in Q}\|x-y\|$ is the distance function.

Then we can rewrite $C_M(A)$ as follows

$$\forall x \in C_M(A) \quad A \subset R_M(A)\cdot M + x,$$

$$\forall x \in C_M(A) \quad -x \in R_M(A)\cdot M \overset{*}{-} A,$$

i.e.

$$-C_M(A) = (R_M(A)\cdot M) \overset{*}{-} A = \bigcap_{a\in A}(R_M(A)\cdot M - a). \tag{3}$$

We plan to find an approximate solution for problem (2). The idea is to approximate the unit ball $B_1(0)$ on a grid $\mathbb{G}_1 = \{p_i\}_{i=1}^I$ of unit vectors and to solve problem (1) with the "ball" $M = \{x \in \mathbb{R}^n \mid (p_i, x) \le 1,\ i = 1,\dots,I\}$. Note that approximation M is not necessary a symmetric body, thus M is a quasiball. But this fact does not matter for future speculations.

We shall assume that we know the supporting function of the set A, i.e. for any $p \in \mathbb{R}^n$ we know

$$f(p) = \max_{a\in A}(p,a).$$

This is quite natural for a number of applications. In particular, if

$$A = \{x \in \mathbb{R}^n \mid (q_j, x) \le \mu_j,\ j = 1,\dots,N\},\ \|q_j\| = 1,\ \mu_j \in \mathbb{R},$$

then we can find $f(p)$ for any $\|p\| = 1$ as the solution of the next linear programming problem

$$f(p) = \inf\left\{\sum_{j=1}^N \lambda_j\mu_j \;\middle|\; \sum_{j=1}^N \lambda_j q_j = p,\quad \lambda_j \ge 0,\ j = 1,\dots,N\right\}.$$

Further we shall consider an external approximation of the set

$$A = \{x \in \mathbb{R}^n \mid (p,x) \le f(p),\ \|p\| = 1\}$$

on a grid $\mathbb{G}_2$ of the form

$$\hat{A} = \{x \in \mathbb{R}^n \mid (q_j, x) \le f(q_j),\ \{q_j\}_{j=1}^N = \mathbb{G}_2\}.$$

We shall assume that $\mathbb{G}_2 = \mathbb{G}_1$, but the last is not necessary.

Considering an approximation M of the ball $B_1(0)$ and an approximation of the set A on a grid $\mathbb{G}_1$ we obtain the next problem

$$R \to \min, \quad R + (p_i, x) \geq f(p_i), \ i = 1, \dots, I. \tag{4}$$

Any solution of problem (4) is a pair $(x, R) \in \mathbb{R}^n \times \mathbb{R}$ which gives the center and the smallest radius of a ball containing $\hat{A}$:

$$\hat{A} \subset R \cdot M + x.$$

Our purpose is to estimate the difference $\|x - C(A)\|$, where $C(A)$ is the solution (2), via the Hausdorff distance $h(A, \hat{A})$ and $h(M, B_1(0))$ and finally via step of the grid and geometrical properties of the set A.

There are few possibilities to estimate the error. We want to recall one result about stability of minimization problems [8, Theorem 1.19.3].

Proposition 1.1. *Let $A_1, A_2 \subset \mathbb{R}^n$ be compact convex subsets, $A = \mathrm{co}\,(A_1 \cup A_2)$. We consider two minimization problems $\min_{x\in A_i} f_i(x)$, $i = 1, 2$, with the following properties:*

(1) *$f_1 : A \to \mathbb{R}$ is Lipschitz continuous with constant $L > 0$,*

(2) *$f_2 : A \to \mathbb{R}$ is continuous and strongly convex with constant $\kappa > 0$,*

(3) *$\exists \varepsilon > 0$ such that $\forall x \in A$ we have $|f_1(x) - f_2(x)| \leq \varepsilon$.*

Suppose that $u_1 \in A_1$ is some solution of the first problem, $u_2 \in A_2$ is the solution of the second problem and $f_1(u_1) \leq f_1(u_2)$. Then

$$\|u_1 - u_2\| \leq \sqrt{\frac{2Lh(A_1, A_2) + 4\varepsilon}{\kappa}} + h(A_1, A_2).$$

Proof. Put $h = h(A_1, A_2)$. Fix a solution u_1. There exists $x_2 \in A_2$: $\|u_1 - x_2\| \leq h$.

Consider the function g: $g(x) = f_2(x)$ for all $x \in A_2$ and $g(x) = +\infty$ otherwise. This function is strongly convex with constant κ and thus

$$f_2(x_2) - f_2(u_2) = g(x_2) - g(u_2) \geq \frac{\kappa}{2}\|x_2 - u_2\|^2.$$

By (3) we get

$$\frac{\kappa}{2}\|x_2 - u_2\|^2 \leq f_2(x_2) - f_2(u_2) \leq f_1(x_2) - f_1(u_2) + 2\varepsilon$$

and from (1) and inequality $f_1(u_1) \leq f_1(u_2)$ we obtain

$$\frac{\kappa}{2}\|x_2 - u_2\|^2 \leq f_1(x_2) - f_1(u_1) + 2\varepsilon \leq Lh + 2\varepsilon, \quad \|x_2 - u_2\| \leq \sqrt{\frac{2Lh + 4\varepsilon}{\kappa}}.$$

Hence $\|u_1 - u_2\| \leq \|u_1 - x_2\| + \|x_2 - u_2\| \leq h + \sqrt{\frac{2Lh+4\varepsilon}{\kappa}}$. □

Taking $A_2 = A$, $A_1 = \hat{A}$, $f_2(x) = \max_{a\in A}\|x - a\|^2$, $f_1(x) = \max_{a\in\hat{A}} n_M^2(x - a)$ one can estimate constants κ (obviously $\kappa = 2$ for f_2), L, ε, etc. and obtain Holder dependence between $C(A)$ and an approximate Chebyshev center (which belongs to $\hat{A}$ in this situation) via $h(M, B_1(0))$, $h(A, \hat{A})$.

2. Approximation on a grid

We specify the definition of a grid and recall the results about the error of approximation of a set in the Hausdorff metric.

Definition 2.1. ([1],[2],[8]) Let $\Delta \in (0, \frac{1}{2})$. The set $\mathbb{G} = \{p_i\}_{i=1}^{I}$, $\|p_i\| = 1$ for all i, is called a *grid* with the step Δ if for any unit vector $p \in \mathbb{R}^n$ the exists $I_p \subset \{1, \ldots, I\}$ and $\alpha_i > 0$, $i \in I_p$, such that

$$p = \sum_{i \in I_p} \alpha_i p_i, \quad \|p_i - p_j\| < \Delta \ \forall i, j \in I_p.$$

Proposition 2.2. ([1, Remark 1]) *Let $\mathbb{G}$ be a grid with the step $\Delta \in (0, \frac{1}{2})$ and $M = \{x \in \mathbb{R}^n \mid (p_i, x) \le 1,\ i = 1, \ldots, I\}$. Then $B_1(0) \subset M$ and*

$$h(B_1(0), M) \le \frac{\Delta^2}{1 - \frac{1}{2}\Delta^2}. \tag{5}$$

Analogously we have for any $R > 0$: $h(B_R(0), RM) \le \dfrac{R\Delta^2}{1 - \frac{1}{2}\Delta^2}$. (6)

Proposition 2.3. [2], [1, Definition 3, Remark 1] *Let $\mathbb{G}$ be a grid with the step $\Delta \in (0, \frac{1}{2})$ and the supporting function of the convex compact set A is uniformly continuous subdifferentiable with modulus $w(\cdot)$,*

$$\hat{A} = \{x \in \mathbb{R}^n \mid (p_i, x) \le s(p_i, A),\ i = 1, \ldots, I\}.$$

Then
$$h(A, \hat{A}) \le \frac{w(\Delta)}{1 - \frac{1}{2}\Delta^2}. \tag{7}$$

In particular, if $A = \bigcap_{x \in X} B_R(x)$ then $w(\Delta) = R\Delta^2$, if A is a polyhedron then $w(\Delta) = \Delta \cdot \operatorname{diam} A$.

3. The distance between $C(A)$ and $C_M(\hat{A})$

Lemma 3.1. *Suppose that $A \subset \mathbb{R}^n$ is a convex compact set, $R(A)$ is the solution of problem* (2) *and R_M is the solution of problem* (4) *on a grid $\mathbb{G}_1$ with the step $\Delta \in (0, \frac{1}{2})$. Then*

$$R_M \le R(A) + h(A, \hat{A}) \qquad \text{and} \qquad C(A) + (R(A) + \varepsilon)B_1(0) \supset \hat{A}, \tag{8}$$

where $\varepsilon = h(A, \hat{A}) + \dfrac{(R(A) + h(A, \hat{A}))\Delta^2}{1 - \frac{1}{2}\Delta^2}$.

Proof. Put $h = h(A, \hat{A})$. By the inclusion $A \subset C(A) + B_{R(A)}(0)$ we have

$$\hat{A} \subset A + hB_1(0) \subset C(A) + (R(A) + h)B_1(0) \subset C(A) + (R(A) + h)M.$$

Thus $R_M \le R(A) + h$. From Proposition 2.2, formula (6) with $R = R(A) + h$, we get $(R(A) + h)M \subset (R(A) + \varepsilon)B_1(0)$ and we obtain the right inclusion in (8). $\square$

By formula (3) and Lemma 3.1 we obtain

$$-C_M(\hat{A}) = R_M M \overset{*}{—} \hat{A} \subset B_{R(A)+\varepsilon}(0) \overset{*}{—} A = \bigcap_{a\in A} B_{R(A)+\varepsilon}(-a), \tag{9}$$

where ε is from Lemma 3.1, simultaneously with the equality

$$B_{R(A)}(0) \overset{*}{—} A = -C(A).$$

Theorem 3.2. *Assume that conditions of Lemma 3.1 are fulfilled. Then*

$$h(C_M(\hat{A}), C(A)) \le \sqrt{2R(A)\varepsilon + \varepsilon^2}, \qquad \varepsilon = h(A,\hat{A}) + \frac{(R(A) + h(A,\hat{A}))\Delta^2}{1 - \frac{1}{2}\Delta^2}.$$

Proof. Without loss of generality put $C(A) = 0$; put $R = R(A)$. Geometrically by (9) $C_M(\hat{A})$ is contained in the collection of shifts $v \in \mathbb{R}^n$ such that $-v + A \subset B_{R+\varepsilon}(0)$ or $v - A \subset B_{R+\varepsilon}(0)$, moreover $0 = B_R(0) \overset{*}{—} A$.

By the Jung theorem [3, Theorem 11.5.8] there exists a symplex $S = \mathrm{co}\,\{s_i\}_{i=1}^k$, $2 \le k \le n+1$, with $S \subset A$ and $0 = B_R(0) \overset{*}{—} S$. The latter means that there exist nonnegative numbers $\{\lambda_i\}_{i=1}^k$, $\sum_{i=1}^k \lambda_i = 1$ and $\sum_{i=1}^k \lambda_i s_i = 0 = C(A)$.

Fix $v \in C_M(\hat{A})$. Then $\sum_{i=1}^k \lambda_i (v, s_i) = 0$ and there exists i, $1 \le i \le k$, with $(v, -s_i) \ge 0$. Hence $v - s_i \in v - A$ and, on the other hand,

$$v - s_i \in B_{R+\varepsilon}(0) \cap \{x \in \mathbb{R}^n \mid (x, -s_i) \ge \|s_i\|^2\} = \Sigma.$$

The set Σ is a segment and any farthest from $-s_i$ point $a \in \Sigma$ is a point of the intersection $a \in \partial B_{R+\varepsilon}(0) \cap \{x \in \mathbb{R}^n \mid (x, -s_i) = \|s_i\|^2\}$. From the Pythagoras theorem we conclude

$$\|v\| \le \|a - (-s_i)\| = \sqrt{\|a\|^2 - \|s_i\|^2} = \sqrt{(R+\varepsilon)^2 - R^2} = \sqrt{2R\varepsilon + \varepsilon^2}. \qquad \square$$

Theorem 3.3. *Assume that the conditions of Lemma* 3.1 *hold. Let* $D_A = A \cap \partial B_{R(A)}(C(A))$ *and assume that there exists* $\delta > 0$ *with* $B_\delta(C(A)) \subset \mathrm{co}\, D_A$.

Then $h(C_M(\hat{A}), C(A)) \le \varepsilon \dfrac{R(A)}{\delta}$, *where* $\varepsilon = h(A,\hat{A}) + \dfrac{(R(A) + h(A,\hat{A}))\Delta^2}{1 - \frac{1}{2}\Delta^2}$.

Proof. Without loss of generality suppose that $C(A) = 0$; put $R = R(A)$.

Fix $\tau \in (0,1)$. Let $D_\tau = \{d_i\}_{i=1}^N \subset D_A$ be such a finite set that

$$h\,(\mathrm{co}\, D_\tau, \mathrm{co}\, D_A) < (1-\tau)\delta.$$

Then $B_{\tau\delta}(0) \subset \mathrm{co}\, D_\tau$.

Let $v \ne 0$ be a shift: $-v + D_\tau \subset -v + A \subset B_{R+\varepsilon}(0)$.

Suppose that $-v \in \mathrm{cone}\,\{d_{i_1}, \dots, d_{i_k}\}$ and moreover $\mathrm{co}\,\{d_{i_1}, \dots, d_{i_k}\}$ is an edge of $\mathrm{co}\, D_\tau$.

Define L as the affine hull of the points $\{d_{i_j}\}_{j=1}^k$. We have $h = \varrho_L(0) \geq \tau\delta$. Define $e = P_L 0$.

The angle φ_0 between d_{i_j} (for any $1 \leq j \leq k$) and e fulfills

$$\cos\varphi_0 = \frac{h}{R} \geq \frac{\tau\delta}{R}.$$

We claim that there exists j, $1 \leq j \leq k$, with $\angle(d_{i_j}, -v) \leq \varphi_0$. Indeed, put $v_1 = (-L) \cap \{tv \mid t \geq 0\}$. We have $-v_1 \in \operatorname{co}\{d_{i_1}, \ldots, d_{i_k}\} = \mathcal{S}$. The point e is the center of the circumscribed ball for $\operatorname{co}\{d_{i_1}, \ldots, d_{i_k}\}$. Thus by the inclusion

$$\mathcal{S} \subset \bigcup_{j=1}^{k} \left(\mathcal{S} \cap B_{\|d_{i_j}-e\|}(d_{i_j})\right)$$

there exists a number j, $1 \leq j \leq k$, with $-v_1 \in \mathcal{S} \cap B_{\|d_{i_j}-e\|}(d_{i_j})$ and hence $\|d_{i_j} - (-v_1)\| \leq \|d_{i_j} - e\|$. Denote $\|d_{i_j}\| = a$, $\|v_1\| = b$, $\|e\| = b_1$, $\|d_{i_j} - (-v_1)\| = c$, $\|d_{i_j} - e\| = c_1$. Note that $b_1 \leq b$, $c \leq c_1$ and $a^2 = b_1^2 + c_1^2$. Then we conclude from the Cosine theorem

$$\cos\varphi_0 = \frac{2b_1 b}{2ab} \leq \frac{b_1^2 + b^2}{2ab} = \frac{b_1^2 + c_1^2 + b^2 - c_1^2}{2ab} = \frac{a^2 + b^2 - c_1^2}{2ab}$$
$$\leq \frac{a^2 + b^2 - c^2}{2ab} = \cos\angle(-v_1, d_{i_j}).$$

Consider a cone K with the vertex $-d_{i_j}$, the axis $\{-td_{i_j} \mid t \geq 1\}$ and the angle between the axis and a generatrix equals φ_0.

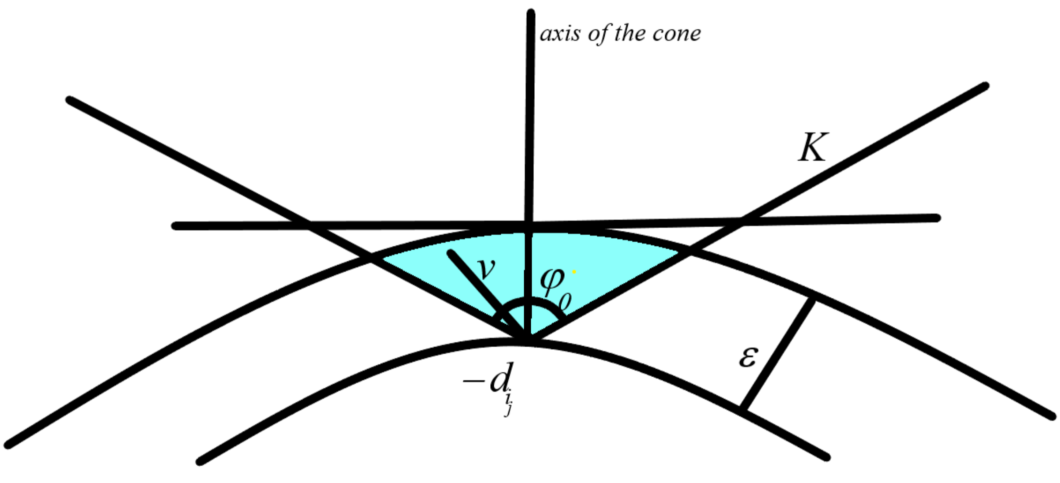

Figure 3.1: Intersection of the layer and the cone K.

Considering the intersection of the layer $B_{R+\varepsilon}(0) \backslash B_R(0)$, see Figure 3.1, with the cone K we get

$$\frac{\varepsilon}{\|v\|} \geq \cos\varphi_0 \geq \frac{\tau\delta}{R}.$$

Hence for any vector v with $v - A \subset B_{R+\varepsilon}(0)$ we obtain that $\|v\| \leq \varepsilon\frac{R}{\tau\delta}$. Thus $h(C_M(\hat{A}), C(A)) \leq \varepsilon\frac{R(A)}{\tau\delta}$ for all $\tau \in (0,1)$. Passing $\tau \to 1-0$ we obtain

$$h(C_M(\hat{A}), C(A)) \leq \varepsilon\frac{R(A)}{\delta}. \qquad \square$$

Remark 3.4. Theorem 3.3 gives a sufficient condition. Consider now in $\mathbb{R}^2$ the set $A = \operatorname{co}\{(0,-1),(0,1)\}$ and a grid $\mathbb{G} = \{(\cos\frac{2\pi m}{N}, \sin\frac{2\pi m}{N})\}_{m=0}^{N-1}$ which contains the vectors $\pm(1,0)$, but does not contain the vectors $\pm(0,1)$. Then $\hat{A} = A$ and $C_M(A) = C(A) = (0,0)$.

Example 3.5. In $\mathbb{R}^2$ consider $A = \operatorname{co}\{(0,-1),(0,1)\}$ and a grid

$$\mathbb{G} = \{(\cos\tfrac{2\pi m}{N}, \sin\tfrac{2\pi m}{N})\}_{m=0}^{N-1}$$

which contains the vectors $\pm(0,1)$ and $\pm(1,0)$. Then $\hat{A} = A$, the length of a side of the regular N-polygon M equals $2\tan\frac{\pi}{N}$ and $\Delta = 2\sin\frac{\pi}{N}$. It is easy to see that $C(A) = (0,0)$, $C_M(A) \supset [(-\frac{1}{2}\Delta, 0), (\frac{1}{2}\Delta, 0)]$. Thus $h(C(A), C_M(A)) \geq \frac{1}{2}\Delta$. We see that Theorem 3.2 is realized in the present example, but not Theorem 3.3.

Suppose that under the conditions of Theorem 3.3 $A \subset \mathbb{R}^n$ is a polyhedron and the grid $\mathbb{G}_1$ with the step Δ is such that $\hat{A} = A$. Then the number I of elements of the grid is $I \asymp O\left(\frac{1}{\Delta^{n-1}}\right)$, $\Delta \to 0$. The error for approximate Chebyshev center is

$$h(C_M(\hat{A}), C(A)) \leq \frac{R}{\delta}\frac{R(A)\Delta^2}{1-\frac{1}{2}\Delta^2} \asymp \Delta^2, \ \Delta \to 0.$$

In other words, after $O(N)$ operations we get an estimate

$$h(C_M(\hat{A}), C(A)) \asymp \frac{1}{N^{2/(n-1)}}.$$

In the same situation under the conditions of Theorem 3.2 after $O(N)$ operations we get an estimate

$$h(C_M(\hat{A}), C(A)) \asymp \frac{1}{N^{1/(n-1)}}.$$

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