Journal of Convex Analysis
Volume 29 (2022), No. 1, 165–181

Some Remarks on Orthogonality of Bounded Linear Operators

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Received: September 22, 2020
Accepted: February 16, 2021

We explore the relation between the orthogonality of bounded linear operators in the space of operators and that of elements in the ground space. To be precise, we study if $T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ satisfy $T \perp_B A$, then whether there exists $x \in \mathbb{X}$ such that $Tx \perp_B Ax$ with $\|x\| = 1$, $\|Tx\| = \|T\|$, where $\mathbb{X}, \mathbb{Y}$ are normed linear spaces. In this context, we introduce the notion of Property P_n for a Banach space and illustrate its connection with orthogonality of a bounded linear operator between Banach spaces. We further study Property P_n for various polyhedral Banach spaces.

Keywords: Orthogonality, linear operators, norm attainment, polyhedral Banach spaces.

2010 Mathematics Subject Classification: Primary 46B20; secondary 47L05.

1. Introduction

The purpose of the present article is to continue the study of orthogonality properties of bounded linear operators between Banach spaces, in light of the seminal result obtained by Bhatia and Šemrl [1] regarding orthogonality of linear operators on Euclidean spaces. Let us first establish the relevant notations and the terminologies in this context.

The letters $\mathbb{X}$ and $\mathbb{Y}$ denote Banach spaces. Throughout the article, we work only with real Banach spaces. Let $B_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| \leq 1\}$ and $S_{\mathbb{X}} = \{x \in \mathbb{X} : \|x\| = 1\}$ denote the unit ball and the unit sphere of $\mathbb{X}$ respectively. Let $E_{\mathbb{X}}$ denote the set of all extreme points of $B_{\mathbb{X}}$. For a set $\mathcal{S} \subset \mathbb{X}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$. Let $\mathbb{L}(\mathbb{X}, \mathbb{Y})$ denote the Banach space of all bounded linear operators from $\mathbb{X}$ to $\mathbb{Y}$, endowed with the usual operator norm. We write $\mathbb{L}(\mathbb{X}, \mathbb{Y}) = \mathbb{L}(\mathbb{X})$, if $\mathbb{X} = \mathbb{Y}$. For a bounded linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, let M_T denote the norm attainment set of T, i.e., $M_T = \{x \in S_{\mathbb{X}} : \|Tx\| = \|T\|\}$. The notion of Birkhoff-James orthogonality in a Banach space is well-known and is used extensively in the study of the geometry of Banach spaces. For $x, y \in \mathbb{X}$, x is said to be *orthogonal* to y in the sense of Birkhoff-James [2], written as $x \perp_B y$, if $\|x + \lambda y\| \geq \|x\|$ for all $\lambda \in \mathbb{R}$. Similarly, for

ISSN 0944-6532 / $ 2.50

$T, A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, T is said to be *Birkhoff-James orthogonal* to A, written as $T \perp_B A$, if $\|T + \lambda A\| \geq \|T\|$ for all $\lambda \in \mathbb{R}$. For the n-dimensional Euclidean space $\mathbb{E}^n$, Bhatia and Šemrl [1] proved that for $T, A \in \mathbb{L}(\mathbb{E}^n)$, $T \perp_B A$ if and only if there exists $x \in S_{\mathbb{E}^n}$ such that $\|Tx\| = \|T\|$ and $Tx \perp_B Ax$.

We refer the reader to [4, 5] for another approach in this context. In recent times, various generalizations of this remarkable theorem has been obtained [6, 7] in the setting of Banach spaces. We note that the sufficient part of the above theorem is true whenever the domain space and the co-domain space are normed linear spaces of any dimensions, and not just Euclidean spaces. On the other hand, the necessary part of the said theorem is not true in general Banach spaces, even in the finite-dimensional case [3, 7, 9]. In view of this, the authors [9] defined the Bhatia-Šemrl (BŠ) Property of a bounded linear operator $T \in \mathbb{L}(\mathbb{X})$. It is possible to extend the definition of the BŠ Property in a broader context, by not imposing the restriction that the domain space and the co-domain space must be identical. Therefore, let us first make note of the following definition of the BŠ Property in its full generality.

Definition 1.1. Let $\mathbb{X}, \mathbb{Y}$ be Banach spaces and let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$. We say that T satisfies the *Bhatia-Šemrl* (BŠ) *Property* if for any $A \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, $T \perp_B A$ implies that there exists $x \in M_T$ such that $Tx \perp_B Ax$.

Our aim is to study the BŠ Property of bounded linear operators between Banach spaces, especially when the domain space and the co-domain space are polyhedral. $\mathbb{X}$ is said to be a polyhedral Banach space if $B_{\mathbb{X}}$ has only finitely many extreme points. Equivalently, $\mathbb{X}$ is a polyhedral Banach space if $B_{\mathbb{X}}$ is a polyhedron. In this context, let us mention the following formal definitions:

Definition 1.2. A *polyhedron* P is a non-empty compact subset of $\mathbb{X}$ which is the intersection of finitely many closed half-spaces of $\mathbb{X}$, i.e., $P = \bigcap_{i=1}^{r} M_i$, where M_i are closed half-spaces in $\mathbb{X}$ and $r \in \mathbb{N}$. The *dimension* of the polyhedron P, written as $\dim P$, is defined as the dimension of the subspace generated by the differences $v - w$ of vectors $v, w \in P$.

Definition 1.3. A polyhedron Q is said to be a *face* of the polyhedron P if either $Q = P$ or if we can write $Q = P \cap \delta M$, where M is a closed half-space in $\mathbb{X}$ containing P and δM denotes the boundary of M. If $\dim Q = i$, then Q is called an *i-face* of P. $(n-1)$-faces of P are called *facets* of P and 1-faces of P are called *edges* of P.

Definition 1.4. [8] Let $\mathbb{X}$ be a finite-dimensional polyhedral Banach space. Let F be a facet of the unit ball $B_{\mathbb{X}}$ of $\mathbb{X}$. A functional $f \in S_{\mathbb{X}^*}$ is said to be a *supporting functional corresponding to the facet F of the unit ball $B_{\mathbb{X}}$* if the following two conditions are satisfied:

(1) f attains norm at some point v of F,

(2) $F = (v + \ker f) \cap S_{\mathbb{X}}$.

We also make use of the concept of normal cones in a Banach space in our study.

Definition 1.5. A subset K of $\mathbb{X}$ is said to be a *normal cone* in $\mathbb{X}$ if

(i) $K + K \subset K$, (ii) $\alpha K \subset K$ for all $\alpha \geq 0$, and (iii) $K \cap (-K) = \{\theta\}$.

Normal cones are important in the study of the geometry of Banach spaces, because there is a natural partial ordering $\geq$ associated with a normal cone K. Namely, for any two elements $x, y \in \mathbb{X}$, $x \geq y$ if $x - y \in K$. It is easy to observe that in a two-dimensional Banach space $\mathbb{X}$, any normal cone K is completely determined by the intersection of K with the unit sphere $S_{\mathbb{X}}$. Keeping this in mind, when we say that K is a normal cone in $\mathbb{X}$ determined by v_1, v_2, what we really mean is that

$$K \cap S_{\mathbb{X}} = \left\{ \frac{(1-t)v_1 + tv_2}{\|(1-t)v_1 + tv_2\|} : t \in [0,1] \right\}.$$

Of course, in this case $K = \{\alpha v_1 + \beta v_2 : \alpha, \beta \geq 0\}$.

A complete characterization of linear operators on a two-dimensional real Banach space satisfying the BŠ Property has been obtained in [9] by proving the following theorem:

Theorem 1.6. *A linear operator T on a two-dimensional real Banach space $\mathbb{X}$ satisfies the BŠ Property if and only if the set of unit vectors on which T attains norm is connected in the corresponding projective space $\mathbb{R}P^1 = S_{\mathbb{X}}/\{x \sim -x\}$.*

Equivalently, we can say in the two-dimensional case that $T \in \mathbb{L}(\mathbb{X})$ satisfies the BŠ Property if and only if $M_T = D \cup (-D)$, where D is a closed connected subset of $S_{\mathbb{X}}$. In this context, we refer the readers to the following conjecture [9], which is yet to be proved or disproved, when the dimension of the space is greater than two.

Conjecture 1.7. *A linear operator T on an n-dimensional real Banach space $\mathbb{X}$ satisfies the BŠ Property if and only if the set of unit vectors on which T attains norm is connected in the corresponding projective space $\mathbb{R}P^{n-1} = S_{\mathbb{X}}/\{x \sim -x\}$.*

However, the sufficient part of the conjecture was proved in [7]. Indeed, if $\mathbb{X}$ is a finite-dimensional real Banach space and $T \in \mathbb{L}(\mathbb{X})$ is such that $M_T = D \cup (-D)$, where D is a closed connected subset of $S_{\mathbb{X}}$, then T satisfies the BŠ Property. We are interested in exploring the converse to the above result. To be precise, if $\dim \mathbb{X} > 2$ and $M_T \neq D \cup (-D)$, where D is a closed connected subset of $S_{\mathbb{X}}$, then whether $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ satisfies the BŠ Property, where $\mathbb{Y}$ is any Banach space. With this motivation in mind, we introduce the following definition for a Banach space $\mathbb{X}$, which plays a significant role in our study.

Definition 1.8. Let $\mathbb{X}$ be a Banach space. Given $n \in \mathbb{N}$, we say that $\mathbb{X}$ has *Property P_n* if for every choice of n vectors $x_1, x_2, \ldots, x_n \in S_{\mathbb{X}}$ we have $\bigcup_{i=1}^{n} x_i^{\perp} \subsetneqq \mathbb{X}$.

It is clear from the above definition that if $\mathbb{X}$ has Property P_n then $\mathbb{X}$ has Property P_m for all $m \in \mathbb{N}$, with $m \leq n$. We illustrate the connection between Property P_n for a Banach space and bounded linear operators not satisfying the BŠ Property. We further explore Property P_n for different polyhedral Banach spaces.

2. Main results

We begin this section with the observation that Theorem 2.3 of [9] holds true even if the co-domain space is any Banach space with dimension at least two. Indeed, the said theorem can be stated in the following more general form, by using essentially the same arguments presented in the proof of the original result.

Theorem 2.1. *Let $\mathbb{X}$ be a two-dimensional Banach space and let $\mathbb{Y}$ be a Banach space of dimension greater than or equal to two. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that M_T has more than two components. Then T does not satisfy the BŠ Property.*

As a corollary to Theorem 2.1, we can provide an elementary condition on M_T so that T does not satisfy the BŠ Property, when $\mathbb{X}$ is a two-dimensional Banach space.

Corollary 2.2. *Let $\mathbb{X}$ be a two-dimensional Banach space and let $\mathbb{Y}$ be a Banach space of dimension greater than or equal to two. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that there exist $x, y \in M_T$ with $x \neq \pm y$ and $\frac{x+y}{\|x+y\|}, \frac{x-y}{\|x-y\|} \notin M_T$. Then T does not satisfy the BŠ Property.*

Proof. We claim that M_T has more than two components.
Let $u_1 = \frac{x+y}{\|x+y\|}$ and $u_2 = \frac{x-y}{\|x-y\|}$. Consider the following subsets of $S_{\mathbb{X}}$:

$$S_1 = \left\{ \frac{(1-t)u_1 + tu_2}{\|(1-t)u_1 + tu_2\|} : t \in (0,1) \right\}, \quad S_2 = \left\{ \frac{(1-t)u_1 + t(-u_2)}{\|(1-t)u_1 - tu_2\|} : t \in (0,1) \right\},$$

$$S_3 = -S_1 \text{ and } S_4 = -S_2.$$

Then clearly S_i, $i = 1, 2, 3, 4$, are connected subsets of $S_{\mathbb{X}}$ and by the construction of S_i we have $x \in S_1$, $y \in S_2$, $-x \in S_3$ and $-y \in S_4$. Also, $S_i \cap S_j = \phi$ for all $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$.
As $S_{\mathbb{X}} \setminus \{\pm\frac{x+y}{\|x+y\|}, \pm\frac{x-y}{\|x-y\|}\} = \bigcup_{i=1}^{4} S_i$ we conclude $M_T \subseteq \bigcup_{i=1}^{4} S_i$.
Also for each disjoint connected set S_i, $S_i \cap M_T \neq \phi$. Therefore, M_T must have more than two components. Hence using Theorem 2.1, we conclude that T does not satisfy the BŠ Property. □

The following example illustrates the applicability of Theorem 2.1 in studying the BŠ Property of a bounded linear operator between Banach spaces.

Example 2.3. Let us consider a bounded linear operator $T : \ell_1^2 \to \ell_\infty^3$, defined by $T(x, y) = (x, y, \frac{x+y}{2})$ for all $(x, y) \in \ell_1^2$. Then it is easy to see that $\|T\| = 1$ and $M_T = \{\pm(1, 0), \pm(0, 1)\}$. Therefore by using Theorem 2.1, we conclude that T does not satisfy the BŠ Property.

If $\mathbb{X}$ is a two-dimensional Banach space, then from Theorem 2.1 of [7] and Theorem 2.3 of [9], it follows that $T \in \mathbb{L}(\mathbb{X})$ satisfies the BŠ Property if and only if we have $M_T = D \cup (-D)$, where D is a connected subset of $S_{\mathbb{X}}$. If $\dim \mathbb{X} \geq 3$ and M_T has more than two components then it is not known whether T will satisfy the BŠ Property. Our next result gives some insight in that direction, under certain assumptions on the co-domain space $\mathbb{Y}$ and the norm attainment set M_T, for a bounded linear operator $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$.

Theorem 2.4. *Let $\mathbb{X}$ be an n-dimensional Banach space, where $n \geq 3$ and let $\mathbb{Y}$ be any Banach space. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$, with $|M_T| \geq 4$, be such that the following conditions are satisfied:*

(a) *There exists a basis $\{x_1, x_2, x_3, \ldots, x_n\}$ of $\mathbb{X}$ such that $x_1, x_2 \in M_T$.*

(b) *There exist scalars* $\alpha_3, \alpha_4, \dots, \alpha_n$ *and* $\beta_3, \beta_4, \dots, \beta_n$ *such that we have for each* $w = c_{1w}x_1 + c_{2w}x_2 + \ldots + c_{nw}x_n \in M_T$*:*

$c_{1w} + c_{2w} + \alpha_3 c_{3w} + \ldots + \alpha_n c_{nw} \neq 0$ *and* $c_{1w} - c_{2w} + \beta_3 c_{3w} + \ldots + \beta_n c_{nw} \neq 0.$

Then at least one of the following is true:

(i) $\bigcup_{x \in M_T} (Tx)^{\perp} = \mathbb{Y}$.

(ii) T *does not satisfy the BŠ Property.*

Proof. Assuming that (i) is not true, we show that T does not satisfy the BŠ Property. Under this assumption, $\bigcup_{x \in M_T} (Tx)^{\perp} \subsetneq \mathbb{Y}$. Take $z \in \mathbb{Y} \setminus \bigcup_{x \in M_T} (Tx)^{\perp}$. We note that it follows from Proposition 2.1 of [6] that for each $i = 1, 2$, either $z \in (Tx_i)^+$ or $z \in (Tx_i)^-$.

Case 1: Let $z \in (Tx_1)^+ \cap (Tx_2)^+$ or $z \in (Tx_1)^- \cap (Tx_2)^-$. We define $A : \mathbb{X} \to \mathbb{Y}$ by

$$Ax_1 = z, \ Ax_2 = -z \ \text{ and } \ Ax_i = \beta_i z \ \text{ for } \ i = 3, 4, \dots, n.$$

If $z \in (Tx_1)^+ \cap (Tx_2)^+$, then $Ax_1 \in (Tx_1)^+$ and $Ax_2 \in (Tx_2)^-$. On the other hand, if $z \in (Tx_1)^- \cap (Tx_2)^-$, then $Ax_1 \in (Tx_1)^-$ and $Ax_2 \in (Tx_2)^+$. Therefore, using Theorem 2.2 of [6], we have $T \perp_B A$, in both cases. We claim that $Tu \not\perp_B Au$ for any $u \in M_T$. Let $u = c_{1u}x_1 + c_{2u}x_2 + \ldots + c_{nu}x_n \in M_T$.
Then,

$$Au = (c_{1u} - c_{2u} + c_{3u}\beta_3 + \ldots + c_{nu}\beta_n)z = \gamma z, \text{ (say)}.$$

As $\gamma \neq 0$ and $Tu \not\perp_B z$, so we conclude that $Tu \not\perp_B Au$. Thus T does not satisfy the BŠ Property.

Case 2: Let $z \in (Tx_1)^+ \cap (Tx_2)^-$ or $z \in (Tx_1)^- \cap (Tx_2)^+$. Let us define $A : \mathbb{X} \to \mathbb{Y}$ by $Ax_1 = z$, $Ax_2 = z$ and $Ax_i = \alpha_i z$ for $i = 3, 4, \dots, n$. Proceeding in a similar manner, we can conclude that $T \perp_B A$ but there exists no $u \in M_T$ such that $Tu \perp_B Au$. Therefore T does not satisfy the BŠ Property. This completes the proof of the theorem. □

The above theorem indirectly hints at the importance of the notion of Property P_n for a Banach space in the study of the BŠ Property of a bounded linear operator. We state this formally in the following corollary.

Corollary 2.5. *Let* $\mathbb{X}$ *be an* n*-dimensional Banach space, where* $n \geq 3$ *and let* $\mathbb{Y}$ *be a Banach space such that* $\mathbb{Y}$ *has Property* P_m*, for some* $m \in \mathbb{N}$*. Let* $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ *be such that the following conditions are satisfied:*

(a) $|M_T| \geq 4$ *and* $|T(M_T)| \leq 2m$*,*

(b) *There exists a basis* $\{x_1, x_2, x_3, \dots, x_n\}$ *of* $\mathbb{X}$ *such that* $x_1, x_2 \in M_T$*,*

(c) *There exist scalars* $\alpha_3, \alpha_4, \dots, \alpha_n$ *and* $\beta_3, \beta_4, \dots, \beta_n$ *such that we have for each* $w = c_{1w}x_1 + c_{2w}x_2 + \ldots + c_{nw}x_n \in M_T$*:*
$c_{1w} + c_{2w} + \alpha_3 c_{3w} + \ldots + \alpha_n c_{nw} \neq 0$ *and* $c_{1w} - c_{2w} + \beta_3 c_{3w} + \ldots + \beta_n c_{nw} \neq 0.$

Then T *does not satisfy the BŠ Property.*

Proof. As $T(M_T)$ contains at most $2m$ elements which are pairwise scalar multiples of one another and the co-domain space $\mathbb{Y}$ has Property P_m, we must have $\bigcup_{x \in M_T} (Tx)^{\perp} \subsetneq \mathbb{Y}$. Then from Theorem 2.4, we conclude that T does not satisfy the BŠ Property. □

We now give an example to illustrate the applicability of Corollary 2.5 in studying the BŠ Property of a bounded linear operator between Banach spaces. Here we would like to mention that every smooth Banach space of dimension at least 2, has Property P_n, for each $n \in \mathbb{N}$.

Example 2.6. We consider a bounded linear operator $T : \ell_\infty^3 \to \ell_2^3$, defined by $Tx = \frac{x}{\sqrt{3}}$ for all $x \in \ell_\infty^3$. Then it is easy to see that $\|T\| = 1$,

$$M_T = \{\pm(1,1,1),\ \pm(-1,1,1),\ \pm(-1,-1,1),\ \pm(1,-1,1)\} \text{ and}$$

$$T(M_T) = \{\pm(\tfrac{1}{\sqrt{3}},\ \tfrac{1}{\sqrt{3}},\ \tfrac{1}{\sqrt{3}}),\ \pm(\tfrac{-1}{\sqrt{3}},\ \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}),\ \pm(\tfrac{-1}{\sqrt{3}}, \tfrac{-1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}),\ \pm(\tfrac{1}{\sqrt{3}}, \tfrac{-1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}})\}.$$

Consider $x_1 = (1,1,1)$, $x_2 = (-1,1,1)$, $x_3 = (-1,-1,1)$ and $x_4 = (1,-1,1)$. Clearly $\{x_1, x_2, x_3\}$ forms a basis of ℓ_∞^3. If we choose $\alpha = \beta = 1$, then condition (c) of Corollary 2.5 is satisfied. Also, ℓ_2^3 has Property P_n, for any $n \in \mathbb{N}$, as ℓ_2^3 is a smooth space. Therefore, by using Corollary 2.5, we conclude that T does not satisfy the BŠ Property. $\square$

It is worth mentioning that Theorem 2.2 of [9] holds true when the domain space is any finite-dimensional Banach space and the co-domain space is any smooth Banach space of dimension at least two. Indeed, the said theorem can be stated in the following more general form, by using the same arguments presented in the proof of the original result.

Theorem 2.7. *Let $\mathbb{X}$ be a finite-dimensional Banach space and $\mathbb{Y}$ be a smooth Banach space of dimension greater than or equal to two. Let $T \in \mathbb{L}(\mathbb{X}, \mathbb{Y})$ be such that M_T is a countable set with more than two points. Then T does not satisfy the BŠ Property.*

In the remaining part of this article, we focus on Property P_n for polyhedral Banach spaces. Our first observation reveals that given any polyhedral Banach space $\mathbb{X}$, there exists a natural number n_0 such that $\mathbb{X}$ does not have Property P_n, for any $n \geq n_0$.

Theorem 2.8. *Let $\mathbb{X}$ be a finite-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ has exactly $2n$ extreme points. Then $\mathbb{X}$ does not have Property P_n.*

Proof. Let us denote the extreme points of $B_{\mathbb{X}}$ by $\pm u_1, \pm u_2, \ldots, \pm u_n$. We claim that $\bigcup_{i=1}^n u_i^\perp = \mathbb{X}$. Let $y \in \mathbb{X}$ be arbitrary. Given $z \in \mathbb{X}$ there exists a scalar $a \in \mathbb{R}$ such that $ay + z \perp_B y$, by Theorem 2.3 of [2]. Take $x = \frac{ay+z}{\|ay+z\|}$, then $x \perp_B y$. If x is an extreme point of $B_{\mathbb{X}}$, then we have nothing more to show. Now, suppose that x is not an extreme point of $B_{\mathbb{X}}$. As $x \perp_B y$, by using Theorem 2.1 of [2], there exists a linear functional $f \in S_{\mathbb{X}^*}$ such that $f(x) = \|x\| = 1$ and $f(y) = 0$. Since f attains norm, it is easy to see that there exists an extreme point u_i of $B_{\mathbb{X}}$ such that $|f(u_i)| = \|f\| = 1$. Therefore, by Theorem 2.1 of [2], we have $u_i \perp_B y$. Thus $\bigcup_{i=1}^n u_i^\perp = \mathbb{X}$. This completes the proof of the theorem. $\square$

Our next theorem shows that we have a definitive answer for two-dimensional polyhedral Banach spaces, regarding Property P_n. To prove the theorem, we need the following lemma:

Lemma 2.9. *Let $\mathbb{X}$ be a two-dimensional polyhedral Banach space. Then for any $x \in E_{\mathbb{X}}$, there exists a normal cone K of $\mathbb{X}$ such that $x^{\perp} = K \cup (-K)$. In addition, if the normal cone K is determined by $v_1, v_2 \in S_{\mathbb{X}}$, then*

$$\{(1-t)v_1 + tv_2 : t \in (0,1)\} \cap y^{\perp} = \phi \quad \textit{for each} \quad y \in E_{\mathbb{X}} \setminus \{\pm x\}.$$

Proof. Let $g \in S_{\mathbb{X}^*}$ and $g(x) = \|x\| = 1$, i.e., g is a supporting functional of $B_{\mathbb{X}}$ at x. Let f_1 and f_2 be the two supporting functionals corresponding to the two edges of $S_{\mathbb{X}}$ meeting at x. Now, $x + \ker g$ is a supporting line to $B_{\mathbb{X}}$ at x, that lies entirely within the cone formed by the straight lines $x + \ker f_1$ and $x + \ker f_2$. For $i = 1, 2$, let

$$f_i^+ = \{z \in \mathbb{X} : f_i(z) \geq 0\} \text{ and } f_i^- = \{z \in \mathbb{X} : f_i(z) \leq 0\}.$$

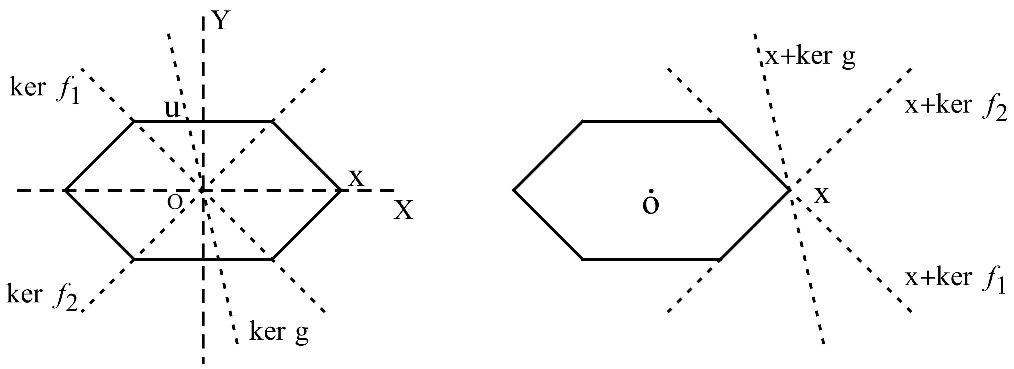

Figure 2.1: Illustration for Lemma 2.9

We note that each f_i^+ (f_i^-) is a closed half-space in $\mathbb{X}$. Let $u \in \ker g$ be arbitrary. The situation is illustrated in Figure 2.1. It follows immediately that either of the following must be true:

(i) $u \in f_1^+$ and $u \in f_2^-$, (ii) $u \in f_1^-$ and $u \in f_2^+$.

Taking $K = f_1^+ \cap f_2^-$, it is easy to see that $-K = f_1^- \cap f_2^+$. Thus for any $u \in \mathbb{X}$, with $x \perp_B u$, we have $u \in K \cup (-K)$. Therefore, $x^{\perp} = \{w \in \mathbb{X} : x \perp_B w\} = K \cup (-K)$. This completes the proof of the first part of the lemma.

Next, suppose K is determined by $v_1, v_2 \in S_{\mathbb{X}}$. From the construction of K it is clear that $v_1 \in \ker f_1 \cap S_{\mathbb{X}}$ and $v_2 \in \ker f_2 \cap S_{\mathbb{X}}$. Let $V = \{(1-t)v_1 + tv_2 : t \in (0,1)\}$. We show that $V \cap y^{\perp} = \phi$, for each $y \in E_{\mathbb{X}} \setminus \{\pm x\}$. If possible, suppose that $V \cap y^{\perp} \neq \phi$, for some $y \in E_{\mathbb{X}} \setminus \{\pm x\}$. Then there exists $v = (1-t)v_1 + tv_2 \in V$ such that $v \in y^{\perp}$.

Since $x \perp_B v$, there exists $f \in S_{\mathbb{X}^*}$ such that $f(x) = \|x\| = 1$ and $f(v) = 0$. On the other hand, since $y \perp_B v$, there exists $h \in S_{\mathbb{X}^*}$ such that $h(y) = \|y\| = 1$ and $h(v) = 0$. Since $\mathbb{X}$ is two-dimensional, it is easy to deduce that $f = \pm h$. Therefore, either x, y are adjacent vertices and $f = h$ is an extreme supporting functional corresponding to the facet $L[x, y] = \{(1-t)x + ty : t \in [0,1]\}$, or, $x, -y$ are adjacent vertices and $f = -h$ is an extreme supporting functional corresponding to the facet $L[x, -y] = \{(1-t)x + t(-y) : t \in [0,1]\}$. As f_1 and f_2 are two supporting functionals corresponding to the two edges of $S_{\mathbb{X}}$ meeting at x, f is equal to either f_1 or f_2.

Therefore, v is equal to either v_1 or v_2, which is a contradiction to our assumption that $v \in V$. This completes the proof of the lemma. □

We now prove the desired theorem.

Theorem 2.10. *Let $\mathbb{X}$ be a two-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ has exactly $2n$ extreme points for some $n \in \mathbb{N}$. Then $\mathbb{X}$ has Property P_{n-1}.*

Proof. If $x \in S_{\mathbb{X}}$ is a non-extreme point of $B_{\mathbb{X}}$, then there exist $x_1, x_2 \in S_{\mathbb{X}}$ such that $x = (1-t)x_1 + tx_2$ for some $t \in (0,1)$ and x_1, x_2 are extreme points of $B_{\mathbb{X}}$. It is easy to observe that $x^{\perp} \subsetneqq x_1^{\perp}$ and $x^{\perp} \subsetneqq x_2^{\perp}$. Therefore, without loss of generality we may consider any $(n-1)$ extreme points of $B_{\mathbb{X}}$, instead of any $(n-1)$ points in $S_{\mathbb{X}}$, to prove that $\mathbb{X}$ has Property P_{n-1}.

Let $E_{\mathbb{X}} = \{\pm x_1, \pm x_2, \ldots, \pm x_n\}$. Then from Lemma 2.9, for each $i \in \{1, 2, \ldots, n\}$, there exists normal cone K_i of $\mathbb{X}$ such that $x_i^{\perp} = K_i \cup (-K_i)$. If the normal cone K_i is determined by $v_{i1}, v_{i2} \in S_{\mathbb{X}}$, then

$$\{(1-t)v_{i1} + tv_{i2} : t \in (0,1)\} \cap \bigcup_{\substack{j=1 \\ j \neq i}}^{n} x_j^{\perp} = \phi.$$

Since x_i is any extreme point of $B_{\mathbb{X}}$, we conclude that the union of the Birkhoff-James orthogonal sets of any $(n-1)$ extreme points of $B_{\mathbb{X}}$ must be a proper subset of $\mathbb{X}$. However, this is clearly equivalent to the fact that $\mathbb{X}$ has Property P_{n-1}. This establishes the theorem. □

As an application of Corollary 2.5, we next give an example of a bounded linear operator T between a three-dimensional polyhedral Banach space $\mathbb{X}$ and a two-dimensional polyhedral Banach space $\mathbb{Y}$ such that T does not satisfy the BŠ Property.

Example 2.11. Let $\mathbb{X} = \ell_{\infty}^3$ and let $\mathbb{Y}$ be a two-dimensional real polyhedral Banach space such that $S_{\mathbb{Y}}$ is regular decagon with vertices $(\cos \frac{j\pi}{5}, \sin \frac{j\pi}{5})$, $j \in \{0, 1, 2, \ldots, 9\}$. Consider the linear operator $T : \mathbb{X} \to \mathbb{Y}$, defined by

$$T(x, y, z) = \Big(\frac{x+y}{2} + \frac{(y-x)\cos\frac{2\pi}{5}}{2},\ \frac{(y-x)\sin\frac{2\pi}{5}}{2} \Big).$$

It is easy to check that $\|T\| = 1$, $M_T = \{\pm(1,1,z),\ \pm(-1,1,z) : z \in [-1,1]\}$ and $T(M_T) = \{\pm(1,0),\ \pm(\cos\frac{2\pi}{5},\ \sin\frac{2\pi}{5})\}$. Consider $x_1 = (1,1,1), x_2 = (-1,1,1)$ and $x_3 = (-1,-1,1)$. Clearly $\{x_1, x_2, x_3\}$ forms a basis of $\mathbb{X}$. If we choose $\alpha = -10$ and $\beta = \frac{-3}{2}$, then condition (c) of Corollary 2.5 is satisfied. From Theorem 2.10, we know that $\mathbb{Y}$ has Property P_4. Therefore, by using Corollary 2.5, we conclude that T does not satisfy the BŠ Property. □

For any two Banach spaces $\mathbb{X}, \mathbb{Y}$, it is easy to see that $\mathbb{X} \times \mathbb{Y}$, equipped with the norm $\|(x,y)\| = \max\{\|x\|, \|y\|\}$ for all $(x,y) \in \mathbb{X} \times \mathbb{Y}$, is a Banach space. Let us denote this space by $\mathbb{X} \oplus_{\infty} \mathbb{Y}$. Similarly, $\mathbb{X} \times \mathbb{Y}$, equipped with the norm $\|(x,y)\| = \|x\| + \|y\|$ for all $(x,y) \in \mathbb{X} \times \mathbb{Y}$, is a Banach space which is denoted by $\mathbb{X} \oplus_1 \mathbb{Y}$. In the following theorems, we study Property P_n for Banach spaces $\mathbb{X} \oplus_{\infty} \mathbb{Y}$ and $\mathbb{X} \oplus_1 \mathbb{Y}$, where $\mathbb{X}$ is a polyhedral Banach space.

Theorem 2.12. *Let $\mathbb{X}$ be a polyhedral Banach space such that $\mathbb{X}$ does not have Property P_n, for some $n \in \mathbb{N}$. Then $\mathbb{X} \oplus_{\infty} \mathbb{Y}$ does not have Property P_n, for any Banach space $\mathbb{Y}$. Moreover, if $\mathbb{Y}$ is a polyhedral Banach space such that $\mathbb{Y}$ does not have Property P_m, for some $m \in \mathbb{N}$, then $\mathbb{X} \oplus_{\infty} \mathbb{Y}$ does not have Property P_r, where $r = \min\{m, n\}$.*

Proof. As $\mathbb{X}$ does not have Property P_n, there exist $x_1, x_2, \ldots, x_n \in S_{\mathbb{X}}$ such that $\bigcup\limits_{i=1}^{n} x_i^{\perp} = \mathbb{X}$. Now we claim that $\bigcup\limits_{i=1}^{n} (x_i, 0)^{\perp} = \mathbb{X} \oplus_{\infty} \mathbb{Y}$.

We first show that $x_i^{\perp} \times \mathbb{Y} \subseteq (x_i, 0)^{\perp}$, for each $i \in \{1, 2, \ldots, n\}$. Let $(x, y) \in x_i^{\perp} \times \mathbb{Y}$. Then for any scalar λ,

$$\begin{aligned}\|(x_i, 0) + \lambda(x, y)\| &= \|(x_i + \lambda x, \lambda y)\| = \max\{\|x_i + \lambda x\|, \|\lambda y\|\} \\ &\geq \|x_i + \lambda x\| \geq \|x_i\| = \|(x_i, 0)\|,\end{aligned}$$

as $x \in x_i^{\perp}$. Therefore, $(x, y) \in (x_i, 0)^{\perp}$. Hence $x_i^{\perp} \times \mathbb{Y} \subseteq (x_i, 0)^{\perp}$. Therefore we have

$$\bigcup_{i=1}^{n} (x_i, 0)^{\perp} \supseteq \bigcup_{i=1}^{n} (x_i^{\perp} \times \mathbb{Y}) = \mathbb{X} \oplus_{\infty} \mathbb{Y}.$$

Hence $\bigcup_{i=1}^{n} (x_i, 0)^{\perp} = \mathbb{X} \oplus_{\infty} \mathbb{Y}$. Further if $\mathbb{Y}$ does not have Property P_m, then as before we can show that $\mathbb{X} \oplus_{\infty} \mathbb{Y}$ does not possess Property P_m. Thus $\mathbb{X} \oplus_{\infty} \mathbb{Y}$ does not have Property P_r, where $r = \min\{m, n\}$. This completes the proof of the theorem. □

Corollary 2.13. *Consider* $\mathbb{X} = \ell_{\infty}^n$ *for any* $n \in \mathbb{N}$, $n \geq 2$. *Then* $\mathbb{X}$ *does not have Property* P_2.

Proof. From Theorem 2.8, it follows that ℓ_{∞}^2 does not have Property P_2. Also, we know that $\ell_{\infty}^3 = \ell_{\infty}^2 \oplus_{\infty} \mathbb{R}$. Therefore, by using Theorem 2.12, we conclude that ℓ_{∞}^3 does not have Property P_2. Continuation of this argument proves that ℓ_{∞}^n, for any n $(\geq 2) \in \mathbb{N}$ does not have Property P_2. □

Applying similar arguments, the proofs of the following results are now apparent:

Theorem 2.14. *Let* $\mathbb{X}$ *be a polyhedral Banach space such that* $\mathbb{X}$ *does not have Property* P_n, *for some* $n \in \mathbb{N}$. *Then* $\mathbb{X} \oplus_1 \mathbb{Y}$ *does not have Property* P_n, *for any Banach space* $\mathbb{Y}$. *Moreover, if* $\mathbb{Y}$ *is a polyhedral Banach space such that* $\mathbb{Y}$ *does not have Property* P_m, *for some* $m \in \mathbb{N}$, *then* $\mathbb{X} \oplus_1 \mathbb{Y}$ *does not have Property* P_r, *where* $r = \min\{m, n\}$.

Corollary 2.15. *Consider* $\mathbb{X} = \ell_1^n$ *for any* $n \in \mathbb{N}$, $n \geq 2$. *Then* $\mathbb{X}$ *does not have Property* P_2.

Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ is a prism with vertices $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1)$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, $n \geq 2$. Then it is trivial to see that $\mathbb{X} = \mathbb{Y} \oplus_{\infty} \mathbb{R}$, where $\mathbb{Y}$ is a two-dimensional polyhedral Banach space so that the extreme points of $B_{\mathbb{Y}}$ are given by $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n})$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, $n \geq 2$. Therefore, by using Theorem 2.8 and Theorem 2.12, we can conclude that $\mathbb{X}$ does not have Property P_n. However, in the next theorem we show that $\mathbb{X}$ does not have Property P_2, for any $n \geq 2$.

Theorem 2.16. *Let* $\mathbb{X}$ *be a three-dimensional polyhedral Banach space such that* $B_{\mathbb{X}}$ *is a prism with vertices* $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1)$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, $n \geq 2$. *Then* $\mathbb{X}$ *does not have Property* P_2.

Proof. Let the vertices of $B_{\mathbb{X}}$ be $v_{\pm(j+1)}$, $j \in \{0, 1, 2, \ldots, 2n-1\}$, where $v_{\pm(j+1)} = (\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1)$. The unit sphere $S_{\mathbb{X}}$ is shown in Figure 2.2.

A simple computation reveals the explicit expression for the norm function on $\mathbb{X}$. Given any $(x, y, z) \in \mathbb{X}$, we have,

$$\|(x,y,z)\| = \max_{0\le j\le 2n-1}\left\{\frac{|\cos\frac{(2j+1)\pi}{2n}||x|}{\cos\frac{\pi}{2n}} + \frac{|\sin\frac{(2j+1)\pi}{2n}||y|}{\cos\frac{\pi}{2n}}, |z|\right\}.$$

We claim that $v_1^{\perp} \cup v_{n+1}^{\perp} = \mathbb{X}$. Let $(x, y, z) \in \mathbb{X}$ be such that $x \ge 0, z \le 0$ and y is arbitrary.

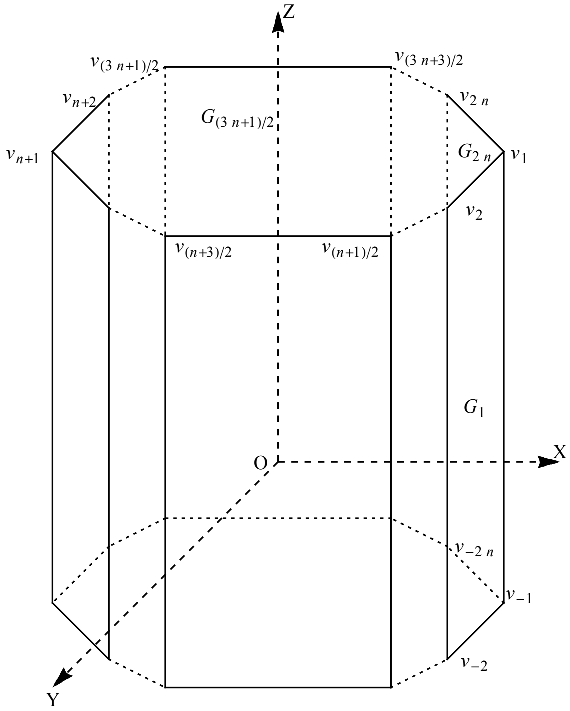

Figure 2.2: Illustration for Theorem 2.16

Now, for any scalar $\lambda \ge 0$, we have

$$\begin{aligned}\|(1,0,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, \lambda y, 1+\lambda z)\|\\ &= \max_{0\le j\le 2n-1}\left\{\frac{|\cos\frac{(2j+1)\pi}{2n}||1+\lambda x|}{\cos\frac{\pi}{2n}} + \frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}},\ |1+\lambda z|\right\}\\ &\ge |1+\lambda x| + \tan\frac{\pi}{2n}|\lambda y| \ge 1 = \|(1,0,1)\|.\end{aligned}$$

Also, for any scalar $\lambda \le 0$,

$$\begin{aligned}\|(1,0,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, \lambda y, 1+\lambda z)\|\\ &= \max_{0\le j\le 2n-1}\left\{\frac{|\cos\frac{(2j+1)\pi}{2n}||1+\lambda x|}{\cos\frac{\pi}{2n}} + \frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}},\ |1+\lambda z|\right\}\\ &\ge |1+\lambda z| \ge 1 = \|(1,0,1)\|.\end{aligned}$$

Therefore, $(1,0,1)\perp_B(x,y,z)$, for all $x \ge 0, z \le 0$ and for any y. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(1,0,1)\perp_B(x,y,z)$, for all $x \le 0$, $z \ge 0$ and for any y.

Let $(x, y, z) \in \mathbb{X}$ be such that $x \geq 0, z \geq 0$. Now for any $\lambda \geq 0$,

$$\begin{aligned}\|(-1,0,1)+\lambda(x,y,z)\| &= \|(-1+\lambda x, \lambda y, 1+\lambda z)\| \\ &= \max_{0\leq j\leq 2n-1}\left\{\frac{|\cos\frac{(2j+1)\pi}{2n}||-1+\lambda x|}{\cos\frac{\pi}{2n}}+\frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}},\ |1+\lambda z|\right\} \\ &\geq |1+\lambda z| \geq 1 = \|(-1,0,1)\|.\end{aligned}$$

Also, for any $\lambda \leq 0$,

$$\begin{aligned}\|(-1,0,1)+\lambda(x,y,z)\| &= \|(-1+\lambda x, \lambda y, 1+\lambda z)\| \\ &= \max_{0\leq j\leq 2n-1}\left\{\frac{|\cos\frac{(2j+1)\pi}{2n}||-1+\lambda x|}{\cos\frac{\pi}{2n}}+\frac{|\sin\frac{(2j+1)\pi}{2n}||\lambda y|}{\cos\frac{\pi}{2n}},\ |1+\lambda z|\right\} \\ &\geq |-1+\lambda x| + \tan\frac{\pi}{2n}|\lambda y| \geq 1 = \|(-1,0,1)\|.\end{aligned}$$

Therefore, $(-1,0,1)\perp_B(x,y,z)$, for all $x \geq 0, z \geq 0$ and for any y. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(-1,0,1)\perp_B(x,y,z)$, for all $x \leq 0, z \leq 0$ and for any y. Hence for any $(x,y,z) \in \mathbb{X}$, either $(x,y,z) \in v_1^{\perp}$ or $(x,y,z) \in v_{n+1}^{\perp}$, i.e., $v_1^{\perp} \cup v_{n+1}^{\perp} = \mathbb{X}$. Therefore, $\mathbb{X}$ does not have Property P_2.

This completes the proof of the theorem. □

Remark 2.17. Theorem 2.16 shows that the space $\mathbb{X} \oplus_\infty \mathbb{Y}$ may not have Property P_n even if both spaces have Property P_n.

Our previous results point to the fact that there is a large family of three-dimensional polyhedral Banach spaces not having Property P_2. The concerned spaces have been constructed by taking ℓ_∞ sum of two-dimensional polyhedral Banach spaces with $\mathbb{R}$. In the next theorem, we give another such example of a three-dimensional Polyhedral Banach space, not having Property P_2, which cannot be constructed by taking ℓ_∞ sum of lower dimensional Banach space.

Theorem 2.18. *Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ is a polyhedron obtained by gluing two pyramids at the opposite base faces of a right prism having square base, with vertices $\pm(1,1,1), \pm(-1,1,1), \pm(-1,-1,1), \pm(1,-1,1), \pm(0,0,2)$. Then $\mathbb{X}$ does not have Property P_2.*

Proof. Let the vertices of $B_{\mathbb{X}}$ be $v_{\pm j}$, $j \in \{1,2,3,4\}$ and $w_{\pm 1}$, where $v_{\pm 1} = (1,1,\pm 1)$, $v_{\pm 2} = (-1,1,\pm 1)$, $v_{\pm 3} = (-1,-1,\pm 1)$, $v_{\pm 4} = (1,-1,\pm 1)$ and $w_{\pm 1} = (0,0,\pm 2)$. The unit sphere $S_{\mathbb{X}}$ is shown in Figure 2.3.

Given any $(x,y,z) \in \mathbb{X}$, the expression for the norm function on $\mathbb{X}$ turns out to be the following:

$$\|(x,y,z)\| = \max\left\{|x|, |y|, \frac{|x|}{2}+\frac{|z|}{2}, \frac{|y|}{2}+\frac{|z|}{2}\right\}.$$

We claim that, $v_1^\perp \cup v_4^\perp = \mathbb{X}$. Let $(x, y, z) \in \mathbb{X}$ be such that $x \geq 0, y \geq 0$.

Now, for any $\lambda \geq 0$, we have

$$\begin{aligned}
\|(1,-1,1) + \lambda(x,y,z)\| &= \|(1+\lambda x, -1+\lambda y, 1+\lambda z)\| \\
&= \max\left\{|1+\lambda x|,\ |-1+\lambda y|,\ \frac{|1+\lambda x|}{2} + \frac{|1+\lambda z|}{2},\ \frac{|-1+\lambda y|}{2} + \frac{|1+\lambda z|}{2}\right\} \\
&\geq |1+\lambda x| \geq 1 = \|(1,-1,1)\|.
\end{aligned}$$

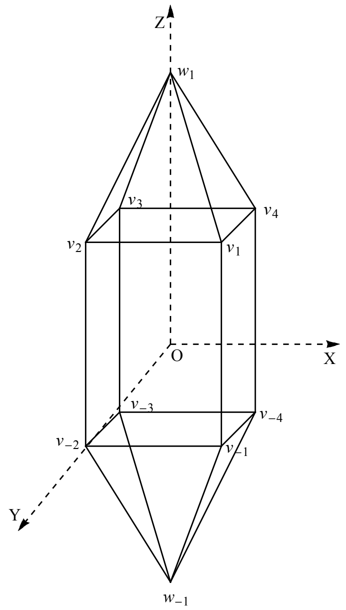

Figure 2.3: Illustration for Theorem 2.18

Also, for any $\lambda \leq 0$,

$$\begin{aligned}
\|(1,-1,1) + \lambda(x,y,z)\| &= \|(1+\lambda x, -1+\lambda y, 1+\lambda z)\| \\
&= \max\left\{|1+\lambda x|,\ |-1+\lambda y|,\ \frac{|1+\lambda x|}{2} + \frac{|1+\lambda z|}{2},\ \frac{|-1+\lambda y|}{2} + \frac{|1+\lambda z|}{2}\right\} \\
&\geq |-1+\lambda y| \geq 1 = \|(1,-1,1)\|.
\end{aligned}$$

Therefore, $(1,-1,1) \perp_B (x,y,z)$, for all $x \geq 0, y \geq 0$ and for any z. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(1,-1,1) \perp_B (x,y,z)$, for all $x \leq 0$, $y \leq 0$ and for any z.

Let $(x, y, z) \in \mathbb{X}$ be such that $x \geq 0, y \leq 0$. For any $\lambda \geq 0$,

$$\begin{aligned}
\|(1,1,1) + \lambda(x,y,z)\| &= \|(1+\lambda x, 1+\lambda y, 1+\lambda z)\| \\
&= \max\left\{|1+\lambda x|,\ |1+\lambda y|,\ \frac{|1+\lambda x|}{2} + \frac{|1+\lambda z|}{2},\ \frac{|1+\lambda y|}{2} + \frac{|1+\lambda z|}{2}\right\} \\
&\geq |1+\lambda x| \geq 1 = \|(1,1,1)\|.
\end{aligned}$$

Also, for any $\lambda \leq 0$,

$$\begin{aligned}\|(1,1,1)+\lambda(x,y,z)\| &= \|(1+\lambda x, 1+\lambda y, 1+\lambda z)\| \\ &= \max\left\{|1+\lambda x|,\ |1+\lambda y|,\ \frac{|1+\lambda x|}{2}+\frac{|1+\lambda z|}{2},\ \frac{|1+\lambda y|}{2}+\frac{|1+\lambda z|}{2}\right\} \\ &\geq |1+\lambda y| \geq 1 = \|(1,1,1)\|.\end{aligned}$$

Therefore, $(1,1,1)\perp_B(x,y,z)$, for all $x \geq 0$, $y \leq 0$ and for any z. From the homogeneity property of Birkhoff-James orthogonality, it follows that $(1,1,1)\perp_B(x,y,z)$, for all $x \leq 0$, $y \geq 0$ and for any z.

Hence for any $(x,y,z) \in \mathbb{X}$, either $(x,y,z) \in v_1^{\perp}$ or $(x,y,z) \in v_4^{\perp}$, i.e., $v_1^{\perp} \cup v_4^{\perp} = \mathbb{X}$. Therefore, $\mathbb{X}$ does not have Property P_2. This completes the proof. □

We next give an example of a three-dimensional polyhedral Banach space which has Property P_2 but does not have Property P_3.

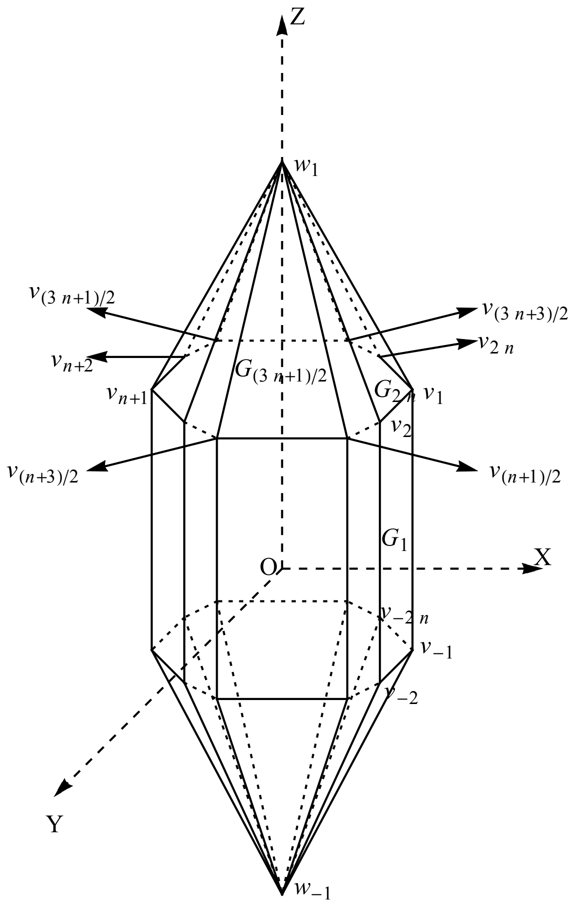

Figure 2.4: Illustration for Theorem 2.19

Theorem 2.19. *Let $\mathbb{X}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{X}}$ is a polyhedron with vertices $(\cos\frac{j\pi}{n}, \sin\frac{j\pi}{n}, \pm 1), (0,0,\pm 2)$, $j \in \{0,1,2,\ldots,2n-1\}$, $n \geq 3$. Then $\mathbb{X}$ has Property P_2 but $\mathbb{X}$ does not have Property P_3.*

Proof. We prove the theorem by assuming that n is an odd integer. Similar calculations hold true, when n is an even integer.

Let the vertices of $B_{\mathbb{X}}$ be $v_{\pm(j+1)}$, $j \in \{0,1,2,\ldots,2n-1\}$ and $w_{\pm 1}$, where

$$v_{\pm(j+1)} = (\cos\tfrac{j\pi}{n}, \sin\tfrac{j\pi}{n}, \pm 1) \quad \text{and} \quad w_{\pm 1} = (0,0,\pm 2).$$

The unit sphere $S_{\mathbb{X}}$ is shown in Figure 2.4. Let G_{j+1} denote the facet of $B_{\mathbb{X}}$ containing $v_{j+1}, v_{-j-1}, v_{j+2}, v_{-j-2}$, where $v_{2n+1} = v_1$ and $v_{-2n-1} = v_{-1}$. Let $F_{\pm(j+1)}$ denote the facet of $B_{\mathbb{X}}$ containing $v_{\pm(j+1)}, v_{\pm(j+2)}, w_{\pm 1}$. For each $j \in \{0,1,2,\ldots,2n-1\}$, let g_{j+1} and $f_{\pm(j+1)}$ be the supporting functionals corresponding to the facets G_{j+1} and $F_{\pm(j+1)}$, respectively, i.e.,

$$(v_{j+1} + \ker g_{j+1}) \cap S_{\mathbb{X}} = G_{j+1} \quad \text{and} \quad (v_{\pm(j+1)} + \ker f_{\pm(j+1)}) \cap S_{\mathbb{X}} = F_{\pm(j+1)}.$$

For every $v_{j+1} \in S_{\mathbb{X}}$, $j = 0,1,2,\ldots,2n-1$, there are four adjacent facets G_j, G_{j+1}, F_j, F_{j+1}. Here we assume that $G_0 = G_{2n}$ and $F_0 = F_{2n}$. Therefore by Lemma 2.1 of [8], extreme supporting functionals corresponding to the vertices v_{j+1}, $j = 0,1,2,\ldots,2n-1$, are g_j, g_{j+1}, f_j, f_{j+1}. Here also we assume that $g_0 = g_{2n}$ and $f_0 = f_{2n}$. Now consider the following subsets of $S_{\mathbb{X}^*}$:

$$H_{j+1} = \left\{ h \in S_{\mathbb{X}^*} \ : \begin{array}{l} h = \lambda_1 g_j + \lambda_2 g_{j+1} + \lambda_3 f_j + \lambda_4 f_{j+1}, \lambda_k \geq 0 \\ \forall\ k \in \{1,2,3,4\} \text{ and } \sum_{k=1}^{4} \lambda_k = 1 \end{array} \right\},$$

for each $j = 0,1,2,\ldots,2n-1$. Then by using Theorem 2.1 of [2] and Lemma 2.1 of [8], we conclude that $v_{j+1}^{\perp} = \bigcup_{h \in H_{j+1}} \ker h$, for each $j = 0,1,2,\ldots,2n-1$.

Again, for w_1, there are $2n$ adjacent facets F_{j+1}, $j = 0,1,2,\ldots,2n-1$. Therefore, as before, extreme supporting functionals corresponding to the vertex w_1 are f_{j+1}, $j = 0,1,2,\ldots,2n-1$. Now, consider the following subset of $S_{\mathbb{X}^*}$:

$$H = \left\{ h \in S_{\mathbb{X}^*} \ : \begin{array}{l} h = \sum_{k=0}^{2n-1} \lambda_{k+1} f_{k+1}, \lambda_{k+1} \geq 0 \\ \forall\ k \in \{0,1,2,\ldots,2n-1\} \text{ and } \sum_{k=0}^{2n-1} \lambda_{k+1} = 1 \end{array} \right\}.$$

From this, we conclude that $w_1^{\perp} = \bigcup_{h \in H} \ker h$. Now, for any $(x,y,z) \in \mathbb{X}$, we have

$$\|(x,y,z)\| = \max_{0 \leq j \leq \frac{n-1}{2}} \left\{ \frac{\cos\frac{(2j+1)\pi}{2n}|x|}{\cos\frac{\pi}{2n}} + \frac{\sin\frac{(2j+1)\pi}{2n}|y|}{\cos\frac{\pi}{2n}}, \frac{\cos\frac{(2j+1)\pi}{2n}|x|}{2\cos\frac{\pi}{2n}} + \frac{\sin\frac{(2j+1)\pi}{2n}|y|}{2\cos\frac{\pi}{2n}} + \frac{|z|}{2} \right\}.$$

Using the above expression of the norm function, we can compute the Birkhoff-James orthogonal sets of the extreme points of $B_{\mathbb{X}}$. For the extreme points of $B_{\mathbb{X}}$, lying above the plane $z = 0$, we have,

$$\begin{aligned} v_1^{\perp} = (1,0,1)^{\perp} = \pm\Big[&\{(x,y,z) \in \mathbb{X} : x \geq 0, y \geq 0, z \geq 0, x - \tan(\tfrac{\pi}{2n})y \leq 0\} \\ &\cup \{(x,y,z) \in \mathbb{X} : x \leq 0,\ y \geq 0,\ z \geq 0, \tfrac{x}{2} + \tfrac{\tan(\frac{\pi}{2n})y}{2} + \tfrac{z}{2} \geq 0\} \\ &\cup \{(x,y,z) \in \mathbb{X} : x \leq 0,\ y \leq 0,\ z \geq 0, \tfrac{x}{2} - \tfrac{\tan(\frac{\pi}{2n})y}{2} + \tfrac{z}{2} \geq 0\} \\ &\cup \{(x,y,z) \in \mathbb{X} : x \geq 0,\ y \leq 0,\ z \geq 0, x + \tan(\tfrac{\pi}{2n})y \leq 0\}\Big]. \end{aligned}$$

Now, for each $j \in \{1,2,\ldots,\frac{n-1}{2}\}$, we have

$$\begin{aligned} v_{(j+1)}^{\perp} = (\cos\tfrac{j\pi}{n}, \sin\tfrac{j\pi}{n}, 1)^{\perp} = \pm\Big[&\{(x,y,z) \in \mathbb{X} : x \leq 0,\ y \geq 0,\ z \geq 0, \\ &\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0,\ \frac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \leq 0\} \\ &\cup \{(x,y,z) \in \mathbb{X} : x \leq 0,\ y \leq 0,\ z \geq 0,\ \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \geq 0\} \end{aligned}$$

$$\cup \big\{(x,y,z) \in \mathbb{X} : x \le 0,\ y \le 0,\ z \ge 0,\ \tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \le 0,\ z \ge 0,\ \tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0,$$
$$\tfrac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \le 0\big\}\Big].$$

Again, for each $j \in \{1,2,\dots,\frac{n-1}{2}\}$, we have

$$v^{\perp}_{(j+\frac{n+1}{2})} = (-\cos\tfrac{j\pi}{n}, \sin\tfrac{j\pi}{n}, 1)^{\perp} = \pm\Big[\big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \ge 0,\ z \ge 0,$$
$$-\tfrac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0,\ -\tfrac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \le 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \le 0, y \le 0, z \ge 0,\ -\tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0,$$
$$-\tfrac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \le 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \le 0,\ z \ge 0,\ -\tfrac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \le 0,\ z \ge 0,\ -\tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0\big\}\Big].$$

Now,

$$v^{\perp}_{(n+1)} = (-1,0,1)^{\perp} = \pm\Big[\big\{(x,y,z) \in \mathbb{X} : x \ge 0, y \ge 0, z \ge 0, -\tfrac{x}{2} + \tfrac{\tan(\frac{\pi}{2n})y}{2} + \tfrac{z}{2} \ge 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \le 0,\ y \ge 0,\ z \ge 0,\ -x - \tan(\tfrac{\pi}{2n})y \le 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \le 0,\ y \le 0,\ z \ge 0,\ -x + \tan(\tfrac{\pi}{2n})y \le 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \le 0,\ z \ge 0,\ -\tfrac{x}{2} - \tfrac{\tan(\frac{\pi}{2n})y}{2} + \tfrac{z}{2} \ge 0\big\}\Big].$$

Again, for each $j \in \{1,2,\dots,\frac{n-1}{2}\}$, we have

$$v^{\perp}_{(j+n+1)} = (-\cos\tfrac{j\pi}{n}, -\sin\tfrac{j\pi}{n}, 1)^{\perp} = \pm\Big[\big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \ge 0,\ z \ge 0,$$
$$-\tfrac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \ge 0,\ z \ge 0,\ -\tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \le 0,\ y \ge 0, z\ \ge 0,\ -\tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0,$$
$$-\tfrac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \le 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \le 0,\ z \ge 0,\ -\tfrac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0,$$
$$-\tfrac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \le 0\big\}\Big].$$

Also, for each $j \in \{1,2,\dots,\frac{n-1}{2}\}$, we have

$$v^{\perp}_{(j+\frac{3n+1}{2})} = (\cos\tfrac{j\pi}{n}, -\sin\tfrac{j\pi}{n}, 1)^{\perp} = \pm\Big[\big\{(x,y,z) \in \mathbb{X} : x \ge 0,\ y \ge 0,\ z \ge 0,$$
$$\tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0, \tfrac{\cos\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j+1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \le 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \le 0,\ y \ge 0,\ z \ge 0,\ \tfrac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{X} : x \le 0,\ y \ge 0,\ z \ge 0,\ \tfrac{\cos\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \tfrac{\sin\frac{(2j-1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \ge 0\big\}$$

$$\cup \{(x,y,z) \in \mathbb{X} : x \leq 0,\ y \leq 0,\ z \geq 0,\ \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \tfrac{z}{2} \geq 0,$$
$$\frac{\cos\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}x - \frac{\sin\frac{(2j-1)\pi}{2n}}{\cos\frac{\pi}{2n}}y \leq 0\}\Big].$$

Now, $w_1^\perp = (0,0,2)^\perp = \pm\Big[A \cup B \cup C \cup D\Big]$, where

$$A = \bigcup_{j=0}^{\frac{n-1}{2}} A_j, \quad B = \bigcup_{j=0}^{\frac{n-1}{2}} B_j, \quad C = \bigcup_{j=0}^{\frac{n-1}{2}} C_j, \quad D = \bigcup_{j=0}^{\frac{n-1}{2}} D_j,$$

$$A_j = \{(x,y,z) \in \mathbb{X} : x \geq 0, y \geq 0, z \geq 0, -\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \leq 0\},$$

$$B_j = \{(x,y,z) \in \mathbb{X} : x \leq 0,\ y \geq 0,\ z \geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x - \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \leq 0\},$$

$$C_j = \{(x,y,z) \in \mathbb{X} : x \leq 0,\ y \leq 0,\ z \geq 0, \frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \leq 0\},$$

$$D_j = \{(x,y,z) \in \mathbb{X} : x \geq 0,\ y \leq 0,\ z \geq 0, -\frac{\cos\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}x + \frac{\sin\frac{(2j+1)\pi}{2n}}{2\cos\frac{\pi}{2n}}y + \frac{z}{2} \leq 0\}.$$

From the above expressions of the Birkhoff-James orthogonal sets of extreme points of $B_{\mathbb{X}}$, it follows that, for any two extreme points $u, v \in S_{\mathbb{X}}$, $u^\perp \cup v^\perp \subsetneqq \mathbb{X}$. Therefore $\mathbb{X}$ has Property P_2.

Again, by using the same expressions, we can show that $(1,0,1)^\perp \cup (-1,0,1)^\perp \cup (0,0,2)^\perp = \mathbb{X}$. Therefore, $\mathbb{X}$ does not have Property P_3. This completes the proof of the theorem. □

Finally, we give an example of a linear operator T between a three-dimensional polyhedral Banach space and a three-dimensional polyhedral Banach space having Property P_2, such that T does not satisfy the BŠ Property.

Example 2.20. Let $\mathbb{X} = \ell_\infty^3$ and let $\mathbb{Y}$ be a three-dimensional polyhedral Banach space such that $B_{\mathbb{Y}}$ is a polyhedron with vertices $(1,0,\pm1)$, $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \pm1)$, $(0,1,\pm1)$, $(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \pm1)$, $(-1,0,\pm1)$, $(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \pm1)$, $(0,-1,\pm1)$, $(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \pm1)$, $(0,0,\pm2)$.

Consider a bounded linear operator $T : \mathbb{X} \to \mathbb{Y}$, defined by

$$T(x,y,z) = \Big(\frac{x+y}{2}, \frac{y-x}{2}, y\Big).$$

Then it is easy to check that $\|T\| = 1$, $M_T = \{\pm(1,1,z),\ \pm(-1,1,z) : z \in [-1,1]\}$ and $T(M_T) = \{\pm(1,0,1),\ \pm(0,1,1)\}$. Consider $x_1 = (1,1,1)$, $x_2 = (-1,1,1)$ and $x_3 = (-1,-1,1)$. Clearly $\{x_1, x_2, x_3\}$ forms a basis of $\mathbb{X}$. If we choose $\alpha = -10$ and $\beta = \frac{-3}{2}$, then condition (c) of Corollary 2.5 is satisfied. From Theorem 2.19, we know that $\mathbb{Y}$ has Property P_2. Therefore, by using Corollary 2.5, we conclude that T does not satisfy the BŠ Property. □

In view of the methods employed to study the BŠ Property of linear operators and the results obtained in the present article, it is perhaps appropriate to end it with the following remark:

Remark 2.21. We have illustrated the important role played by Property P_n in determining the BŠ Property of linear operators. Indeed, using this concept, we have extended the previously obtained results in [9]. It is worth mentioning in this connection that Example 2.6, Example 2.11 and Example 2.20 provided in this article are beyond the scope of Proposition 2.1 of [9]. We note that Property P_n is essentially a structural concept, associated especially with polyhedral Banach spaces. Therefore, it might be interesting to further study various polyhedral Banach spaces in light of the newly introduced concept of Property P_n.

Acknowledgements. A. Ray would like to thank DST, Govt. of India, for the financial support in the form of a doctoral fellowship. D. Sain would like to thank UGC, Govt. of India for D. S. Kothari post-doctoral fellowship and is indebted to Professor Gadadhar Misra for his motivating inspirations. The research of K. Paul is partially supported by RUSA 2.0, JU.

References

[1] R. Bhatia, P. Šemrl: *Orthogonality of matrices and some distance problems*, Linear Algebra Appl. 287/1-3 (1999) 77–85.

[2] R. C. James: *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. 61 (1947) 265–292.

[3] C. K. Li, H. Schneider: *Orthogonality of matrices*, Linear Algebra Appl. 347 (2002) 115–122.

[4] K. Paul: *Translatable radii of an operator in the direction of another operator*, Sci. Math. 2/1 (1999) 119–122.

[5] K. Paul, Sk. M. Hossein, K. C. Das: *Orthogonality on $B(H, H)$ and minimal-norm Operator*, J. Analysis Appl. 6/3 (2008) 169–178.

[6] D. Sain: *Birkhoff-James orthogonality of linear operators on finite dimensional Banach spaces*, J. Math. Analysis Appl. 447 (2017) 860–866.

[7] D. Sain, K. Paul: *Operator norm attainment and inner product spaces*, Linear Algebra Appl. 439/8 (2013) 2448–2452.

[8] D. Sain, K. Paul, P. Bhunia, S. Bag: *On the numerical index of polyhedral Banach spaces*, Linear Algebra Appl. 577 (2019) 121–133.

[9] D. Sain, K. Paul, S. Hait: *Operator norm attainment and Birkhoff-James orthogonality*, Linear Algebra Appl. 476 (2015) 85–97.