Journal of Convex Analysis
Volume 29 (2022), No. 1, 205–220

Some Integral Inequalities for Arithmetically and Geometrically Convex Functions of Two Variables

Vahid Darvish
School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China
vahid.darvish@mail.com

Tahere A. Roushan
Dept. of Mathematics, Faculty of Math. Sciences, University of Mazandaran, Babolsar, Iran
t.roushan@umz.ac.ir

Svetlin G. Georgiev
Sorbonne University, Paris, France
svetlingeorgiev1@gmail.com

Sever S. Dragomir
Mathematics, School of Engineering and Science, Victoria University, Melbourne, Australia
sever.dragomir@vu.edu.au

Received: April 17, 2020
Accepted: March 2, 2021

We introduce arithmetic-geometrically and geometric-geometrically convex functions for two variables. Then, we prove Hermite-Hadamard type inequalities for these functions. As applications, some new inequalities for unitarily invariant norms are established.

Keywords: Hermite-Hadamard inequality, convex function, arithmetic-geometrically convex, unitarily invariant norm.

2010 Mathematics Subject Classification: 47A63, 15A60, 47B05, 47B10, 26D15.

1. Introduction

Let us consider the bi-dimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is called a separately convex function(shortly convex function) on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) := f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) := f(x,v)$ are convex for all $y \in [c,d]$ and $x \in [a,b]$, respectively.

Dragomir proved the following Hermite-Hadamard type inequalities.

Theorem 1.1. [3] *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then we have*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx$$

ISSN 0944-6532 / \$ 2.50

$$
\begin{aligned}
&\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx\right.\\
&\qquad\left. + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right]\\
&\leq \frac{1}{4}\big(f(a,c)+f(a,d)+f(b,c)+f(b,d)\big).
\end{aligned}
$$

Throughout this paper, we introduce geometric-geometrically convex functions for two variables. Then, we prove Hermite-Hadamard type inequalities for these functions. Also, we define arithmetic-geometrically convex functions for two variables. By these functions, we obtain some new inequalities for unitarily invariant norms.

2. Inequalities for geometric-geometrically convex functions of two variables

In this section, we prove Hermite-Hadamard type inequalities for geometric-geometrically convex functions of two variables.

The following definition is for geometrically convex function of one variable.

Definition 2.1. [6]A continuous function $f : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *geometrically convex* (or *multiplicatively convex*) if

$$f(a^\lambda b^{1-\lambda}) \leq f(a)^\lambda f(b)^{1-\lambda} \qquad \text{for } a,b \in I \text{ and } \lambda \in [0,1].$$

In [8, Theorem 3], we proved the following inequalities for geometrically convex functions of one variable

$$
\begin{aligned}
f(\sqrt{ab}) &\leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \leq \exp\left(\frac{1}{\log b - \log a}\int_a^b \frac{\log\circ f(t)}{t}dt\right)\\
&\leq \sqrt{f(\sqrt{ab})}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \leq \sqrt{f(a)f(b)}. \qquad (1)
\end{aligned}
$$

Now, we give the definition of geometric-geometrically convex function of two variables.

Definition 2.2. The function $f : \Delta \subset \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *geometrically* (*geometric-geometrically*) *convex* if

$$f\left(x^\lambda z^{(1-\lambda)}, y^\lambda w^{(1-\lambda)}\right) \leq f(x,y)^\lambda f(z,w)^{(1-\lambda)} \qquad (2)$$

for every $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Below, we will use the following notations:

$$\log(x,y) = (\log x, \log y), \quad x,y > 0,$$

and

$$\exp(x,y) = (\exp x, \exp y), \quad x,y \in \mathbb{R}.$$

Lemma 2.3. *Suppose that $[a,b]\times[c,d]$ is a subinterval of $\mathbb{R}^+\times\mathbb{R}^+$ and that $f:[a,b]\times[c,d]\to(0,\infty)$ is a geometrically convex function. Then*

$$F=\log\circ f\circ\exp:([\log a,\log b]\times[\log c,\log d])\to\mathbb{R}$$

is a convex function.

Proof. Assume that f is geometrically convex. We will show that $F=\log\circ f\circ\exp$ is convex on $[\log a,\log b]\times[\log c,\log d]$, that is,

$$\begin{aligned}&\log\circ f\circ\exp(\lambda\log x+(1-\lambda)\log z,\lambda\log y+(1-\lambda)\log w)\\&\leq\lambda\log\circ f\circ\exp(\log x,\log y)+(1-\lambda)\log\circ f\circ\exp(\log z,\log w)\end{aligned}$$

for all $x,z\in[a,b]$, $y,w\in[c,d]$ and $\lambda\in[0,1]$. We have from the left side of the above inequality

$$\begin{aligned}&\log\circ f\circ\exp(\lambda\log x+(1-\lambda)\log z,\lambda\log y+(1-\lambda)\log w)\\&=\log\circ f\circ\exp\left(\log\left(x^\lambda z^{(1-\lambda)}\right),\log\left(y^\lambda w^{(1-\lambda)}\right)\right)\\&=\log\circ f\left(\left(x^\lambda z^{(1-\lambda)}\right),\left(y^\lambda w^{(1-\lambda)}\right)\right)\\&\leq\log\left(f(x)^\lambda f(z)^{(1-\lambda)},f(y)^\lambda f(w)^{(1-\lambda)}\right)\quad\text{(since } f \text{ is geometrically convex)}\\&=(\lambda\log\circ f(x)+(1-\lambda)\log\circ f(z),\lambda\log\circ f(y)+(1-\lambda)\log\circ f(w))\\&=(\lambda\log\circ f(\exp(\log x))+(1-\lambda)\log\circ f(\exp(\log z)),\lambda\log\circ f(\exp(\log y))\\&+(1-\lambda)\log\circ f(\exp(\log w)))\\&=\lambda\log\circ f\circ\exp(\log x,\log y)+(1-\lambda)\log\circ f\circ\exp(\log z,\log w),\end{aligned}$$

for all $x,z\in[a,b]$, $y,w\in[c,d]$ and $\lambda\in[0,1]$. □

Theorem 2.4. *Let f be a geometrically convex function defined on $[a,b]\times[c,d]$ such that $0<a<b$ and $0<c<d$. Then, we have*

$$\begin{aligned}f\left(\sqrt{ab},\sqrt{cd}\right)&\leq\exp\left(\frac{1}{2\log\frac{b}{a}}\int_a^b\frac{\log f(u,\sqrt{cd})}{u}du+\frac{1}{2\log\frac{d}{c}}\int_c^d\frac{\log f(\sqrt{ab},v)}{v}dv\right)\\&\leq\exp\left(\frac{1}{(\log\frac{b}{a})(\log\frac{d}{c})}\int_a^b\int_c^d\frac{\log f(u,v)}{uv}dvdu\right)\\&\leq\exp\left(\frac{1}{4\log\frac{b}{a}}\left[\int_a^b\frac{\log f(u,c)}{u}du+\int_a^b\frac{\log f(u,d)}{u}du\right]\right.\\&\qquad\left.+\frac{1}{4\log\frac{d}{c}}\left[\int_c^d\frac{\log f(a,v)}{v}dv+\int_c^d\frac{\log f(b,v)}{v}dv\right]\right)\\&\leq\sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}.\end{aligned}$$

Proof. Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be geometrically convex. So, by Lemma 2.3

$$F(x,y)=\log\circ f\circ\exp(x,y):[\log a,\log b]\times[\log c,\log d]\to\mathbb{R}$$

is convex.

Then, by Theorem 1.1 we obtain

$$\begin{aligned}
&F\left(\frac{\log a+\log b}{2},\frac{\log c+\log d}{2}\right)\\
&\leq \frac{1}{2}\Bigg[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}F\left(x,\frac{\log c+\log d}{2}\right)dx\\
&\qquad+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}F\left(\frac{\log a+\log b}{2},y\right)dy\Bigg]\\
&\leq \frac{1}{(\log b-\log a)(\log d-\log c)}\int_{\log a}^{\log b}\int_{\log c}^{\log d}F(x,y)dydx
\end{aligned}$$

and

$$\begin{aligned}
&F\left(\frac{\log a+\log b}{2},\frac{\log c+\log d}{2}\right)\\
&\leq \frac{1}{4}\Bigg[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}F(x,\log c)dx+\frac{1}{\log b-\log a}\int_{\log a}^{\log b}F(x,\log d)dx\\
&\qquad+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}F(\log a,y)dy+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}F(\log b,y)dy\Bigg]\\
&\leq \frac{F(\log a,\log c)+F(\log a,\log d)+F(\log b,\log c)+F(\log b,\log d)}{4}.
\end{aligned}$$

By the definition of F, we have

$$\begin{aligned}
&\log\circ f\circ\exp(\log\sqrt{ab},\log\sqrt{cd})\\
&\leq \frac{1}{2}\Bigg[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log\circ f\circ\exp(x,\log\sqrt{cd})dx\\
&\qquad+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log\circ f\circ\exp(\log\sqrt{ab},y)dy\Bigg]\\
&\leq \frac{1}{(\log b-\log a)(\log d-\log c)}\int_{\log a}^{\log b}\int_{\log c}^{\log d}\log\circ f\circ\exp(x,y)dydx,
\end{aligned}$$

whereupon

$$\begin{aligned}
&\log\circ f\circ\exp(\log\sqrt{ab},\log\sqrt{cd})\\
&\leq \frac{1}{4}\Bigg[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log\circ f\circ\exp(x,\log c)dx\\
&\qquad+\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log\circ f\circ\exp(x,\log d)dx\\
&\qquad+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log\circ f\circ\exp(\log a,y)dy\\
&\qquad+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log\circ f\circ\exp(\log b,y)dy\Bigg]
\end{aligned}$$

and
$$\log \circ f \circ \exp(\log \sqrt{ab}, \log \sqrt{cd})$$
$$\leq \frac{1}{4} [\log \circ f \circ \exp(\log a, \log c) + \log \circ f \circ \exp(\log a, \log d)$$
$$+ \log \circ f \circ \exp(\log b, \log c) + \log \circ f \circ \exp(\log b, \log d)] .$$

The above inequalities are equivalent to write

$$\log f(\sqrt{ab}, \sqrt{cd}) \leq \frac{1}{2} \left[\frac{1}{\log b - \log a} \int_{\log a}^{\log b} \log f(\exp(x), \sqrt{cd}) dx \right.$$
$$\left. + \frac{1}{\log d - \log c} \int_{\log c}^{\log d} \log f(\sqrt{ab}, \exp(y)) dy \right]$$
$$\leq \frac{1}{(\log b - \log a)(\log d - \log c)} \int_{\log a}^{\log b} \int_{\log c}^{\log d} \log f(\exp(x), \exp(y)) dy dx$$

and

$$\log f(\sqrt{ab}, \sqrt{cd}) \leq \frac{1}{4} \left[\frac{1}{\log b - \log a} \int_{\log a}^{\log b} \log f(\exp(x), c) dx \right.$$
$$+ \frac{1}{\log b - \log a} \int_{\log a}^{\log b} \log f(\exp(x), d) dx \ + \frac{1}{\log d - \log c} \int_{\log c}^{\log d} \log f(a, \exp(y)) dy$$
$$\left. + \frac{1}{\log d - \log c} \int_{\log c}^{\log d} \log f(b, \exp(y)) dy \right]$$
$$\leq \frac{1}{4} [\log f(a, c) + \log f(a, d) + \log f(b, c) + \log f(b, d)] .$$

By change of variable, $u = \exp(x)$ and $v = \exp(y)$, we get

$$\log f(\sqrt{ab}, \sqrt{cd})$$
$$\leq \frac{1}{2} \left[\frac{1}{\log b - \log a} \int_a^b \frac{\log f(u, \sqrt{cd})}{u} du \ + \frac{1}{\log d - \log c} \int_c^d \frac{\log f(\sqrt{ab}, v)}{v} dv \right]$$
$$\leq \frac{1}{(\log b - \log a)(\log d - \log c)} \int_a^b \int_c^d \frac{\log f(u, v)}{uv} dv du$$
$$\leq \frac{1}{4} \left[\frac{1}{\log b - \log a} \left(\int_a^b \frac{\log f(u, c)}{u} du + \int_a^b \frac{\log f(u, d)}{u} du \right) \right.$$
$$\left. + \frac{1}{\log d - \log c} \left(\int_c^d \frac{\log f(a, v)}{v} dv + \int_c^d \frac{\log f(b, v)}{v} dv \right) \right]$$
$$\leq \log \sqrt[4]{f(a, c) f(a, d) f(b, c) f(b, d)}.$$

Since $\exp(x)$ is increasing, we have the desired result. □

3. Inequalities for arithmetic-geometrically convex functions of two variables

In this section, by the results obtained in Section 2, we establish Hermite-Hadamard type inequalities for arithmetic-geometrically convex (AG-convex) functions for two variables.

In [9] we proved the following theorem for one variable.

Theorem 3.1. *Let f be an AG-convex function defined on $[a,b]$. Then, we have*

$$f\left(\frac{a+b}{2}\right) \leq \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)} \leq \exp\left(\frac{1}{b-a}\int_a^b \log\circ f(u)du\right)$$
$$\leq \sqrt{f\left(\frac{a+b}{2}\right)}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \leq \sqrt{f(a)f(b)}, \quad \textit{where } u=\log t. \tag{3}$$

Here, we present the definition of AG-convex function for two variables.

Definition 3.2. The function $f:\Delta\subset\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ is said to be AG-convex if

$$f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w) \leq f(x,y)^{\lambda} f(z,w)^{(1-\lambda)} \tag{4}$$

for every $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Remark 3.3. The condition (4) can be written as

$$f\circ\log(\exp(\lambda x)\exp((1-\lambda)z), \exp(\lambda y)\exp((1-\lambda)w))$$
$$\leq (f\circ\log(\exp(x),\exp(y)))^{\lambda}(f\circ\log(\exp(z).\exp(w)))^{1-\lambda}, \tag{5}$$

Then we can see $f:\Delta\to\mathbb{R}^+$ is AG-convex on Δ if and only if $f\circ\log$ is geometrically convex on $\exp(\Delta):=\{(\exp(x),\exp(y)),(x,y)\in\Delta\}$.

Observe that if $\Delta=[a,b]\times[c,d]$, then $\exp(\Delta)=[\exp(a),\exp(b)]\times[\exp(c),\exp(d)]$.

Theorem 3.4. *Let f be an AG-convex function defined on $[a,b]\times[c,d]$. Then, we have*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \exp\left(\frac{1}{2(b-a)}\int_a^b \log\circ f\left(s,\frac{c+d}{2}\right)ds + \frac{1}{2(d-c)}\int_c^d \log\circ f\left(\frac{a+b}{2},t\right)dt\right)$$
$$\leq \exp\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \log\circ f(s,t)dsdt\right)$$
$$\leq \exp\left(\frac{1}{4(b-a)}\left[\int_a^b \log\circ f(s,c)ds+\int_a^b \log\circ f(s,d)ds\right]\right.$$
$$\left.+\frac{1}{4(d-c)}\left[\int_c^d \log\circ f(a,t)dt+\int_c^d \log\circ f(b,t)dt\right]\right)$$
$$\leq \sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}.$$

Proof. Let $f:[a,b]\to\mathbb{R}^+$ be a AG-convex function. Then, by Remark 3.3, the function $f\circ\log$ is geometrically convex on $[\exp(a),\exp(b)]\times[\exp(c),\exp(d)]$.

Applying the inequalities of Theorem 2.4, we obtain

$$
\begin{aligned}
& f\circ\log\left(\sqrt{e^a e^b},\sqrt{e^c e^d}\right)\\
&\le \exp\left(\frac{1}{2\log\frac{e^b}{e^a}}\int_{e^a}^{e^b}\frac{\log\circ f\circ\log(u,\sqrt{e^c e^d})}{u}du\ +\frac{1}{2\log\frac{e^d}{e^c}}\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(\sqrt{e^a e^b},v)}{v}dv\right)\\
&\le \exp\left(\frac{1}{(\log\frac{e^b}{e^a})(\log\frac{e^d}{e^c})}\int_{e^a}^{e^b}\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(u,v)}{uv}dudv\right)\\
&\le \exp\left(\frac{1}{4\log\frac{e^b}{e^a}}\left[\int_{e^a}^{e^b}\frac{\log\circ f\circ\log(u,e^c)}{u}du+\int_{e^a}^{e^b}\frac{\log\circ f\circ\log(u,e^d)}{u}du\right]\right.\\
&\quad\left.+\frac{1}{4\log\frac{e^d}{e^c}}\left[\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(e^a,v)}{v}dv+\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(e^b,v)}{v}dv\right]\right)\\
&\le \sqrt[4]{f\circ\log(e^a,e^c)f\circ\log(e^c,e^d)f\circ\log(e^b,e^c)f\circ\log(e^b,e^d)}.
\end{aligned}
$$

Hence, using Theorem 1.1, we can write

$$
\begin{aligned}
& f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\le \exp\left(\frac{1}{2(b-a)}\int_{e^a}^{e^b}\frac{\log\circ f(\log u,\frac{c+d}{2})}{u}du\ +\frac{1}{2(c-d)}\int_{e^c}^{e^d}\frac{\log\circ f(\frac{a+b}{2},\log v)}{v}dv\right)\\
&\le \exp\left(\frac{1}{(b-a)(d-c)}\int_{e^a}^{e^b}\int_{e^c}^{e^d}\frac{\log\circ f(\log u,\log v)}{uv}dudv\right)\\
&\le \exp\left(\frac{1}{4(b-a)}\left[\int_{e^a}^{e^b}\frac{\log\circ f(\log u,c)}{u}du+\int_{e^a}^{e^b}\frac{\log\circ f(\log u,d)}{u}du\right]\right.\\
&\qquad\left.+\frac{1}{4(d-c)}\left[\int_{e^c}^{e^d}\frac{\log\circ f(a,\log v)}{v}dv+\int_{e^c}^{e^d}\frac{\log\circ f(b,\log v)}{v}dv\right]\right)\\
&\le \sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}.
\end{aligned}
$$

By change of variable $s=\log u$ and $t=\log v$, we have the desired result. □

4. Some unitarily invariant norm inequalities

In this section, we prove some unitarily invariant norm inequalities for operators.

We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$ stand for C^*-algebra of all bounded linear operators and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV|||=|||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $B(H)$. For $1\le p<\infty$, the Schatten p-norm of a compact operator A is defined by $\|A\|_p=(\mathrm{Tr}\,|A|^p)^{1/p}$, where Tr is the usual trace functional. These norms are special examples of the more general class of the Schatten p-norms which are unitarily invariant [2].

As the first application of AG-convex functions of one variable for unitarily invariant norms, we need the following lemma. Before we state our lemma let us mention that the norm version of the Cauchy-Schwarz inequality ([2], Problem IV.5.7, pp. 108) for $A, B \in M_n(\mathbb{C})$ is as follows

$$|||AB^*|||^2 \leq |||A^*A||||||B^*B|||. \tag{6}$$

Lemma 4.1. *Let $A, B, X \in B(H)$ be such that A and B are positive operators. Then $\varphi(t) = |||AB^tA|||$ is AG-convex for $t \in [0, \infty)$.*

Proof. Since φ is continuous, it is enough to prove the mid-point AG-convexity, i.e.,

$$\varphi\left(\frac{s+t}{2}\right) \leq \varphi(s)^{\frac{1}{2}}\varphi(t)^{\frac{1}{2}},$$

for $s, t \in [0, \infty)$. In fact, we have

$$\begin{aligned}
\varphi\left(\frac{s+t}{2}\right) &= |||AB^{\frac{s+t}{2}}A||| = |||(AB^{\frac{s}{2}})(B^{\frac{t}{2}}A)||| \\
&\leq |||AB^{\frac{s}{2}}B^{\frac{s}{2}}A|||^{\frac{1}{2}}|||AB^{\frac{t}{2}}B^{\frac{t}{2}}A|||^{\frac{1}{2}} \quad \text{the Cauchy-Schwarz inequality} \\
&= \varphi(AB^sA)^{\frac{1}{2}}\varphi(AB^tA)^{\frac{1}{2}} = \varphi(s)^{\frac{1}{2}}\varphi(t)^{\frac{1}{2}}.
\end{aligned}$$

So, $\varphi(t) = |||AB^tA|||$ is an AG-convex function. $\square$

Now we give the following inequalities for weighted geometric mean

$$A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^\nu A^{\frac{1}{2}}$$

for two positive operators and all $0 \leq \nu \leq 1$. In the case of $\nu = \frac{1}{2}$, we denote $A\sharp_{\frac{1}{2}}B$ by $A\sharp B$.

We apply inequalities (3) to the function $\varphi(t) = |||AB^tA|||$ on the interval $[\nu, 1-\nu]$ when $\nu \in [0, \frac{1}{2})$ and on the interval $[1-\nu, \nu]$ when $\nu \in (\frac{1}{2}, 1]$. We obtain the following result.

Theorem 4.2. *Let $A, B, X \in B(H)$ be such that A, B are positive operators and $\nu \in [0, 1]$. Then, we have*

$$\begin{aligned}
|||A\sharp B||| &\leq |||A\sharp_{\frac{1+2\nu}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3-2\nu}{4}}B|||^{\frac{1}{2}} \leq \exp\left(\frac{1}{1-2\nu}\int_\nu^{1-\nu}\log|||A\sharp_u B|||\,du\right) \\
&\leq |||A\sharp B|||^{\frac{1}{2}}|||A\sharp_\nu B|||^{\frac{1}{4}}|||A\sharp_{1-\nu}B|||^{\frac{1}{4}} \leq |||A\sharp_\nu B|||^{\frac{1}{2}}|||A\sharp_{1-\nu}B|||^{\frac{1}{2}}.
\end{aligned} \tag{7}$$

Proof. For $\nu \in [0, \frac{1}{2})$, by inequalities (3), we have

$$\begin{aligned}
|||AB^{\frac{1}{2}}A||| &\leq |||AB^{\frac{1+2\nu}{4}}A|||^{\frac{1}{2}}|||AB^{\frac{3-2\nu}{4}}A|||^{\frac{1}{2}} \\
&\leq \exp\left(\frac{1}{1-2\nu}\int_\nu^{1-\nu}\log|||AB^uA|||du\right) \\
&\leq |||AB^{\frac{1}{2}}A|||^{\frac{1}{2}}|||AB^\nu A|||^{\frac{1}{4}}|||AB^{1-\nu}A|||^{\frac{1}{4}} \leq |||AB^\nu A|||^{\frac{1}{2}}|||AB^{1-\nu}A|||^{\frac{1}{2}}.
\end{aligned}$$

We replace A and B by $A^{\frac{1}{2}}$ and $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$, respectively, and we obtain the following inequalities

$$\begin{aligned}
&|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}|||\\
&\leq |||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1+2\nu}{4}}A^{\frac{1}{2}}|||^{\frac{1}{2}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{3-2\nu}{4}}A^{\frac{1}{2}}|||^{\frac{1}{2}}\\
&\leq \exp\left(\frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\log|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{u}A^{\frac{1}{2}}|||du\right)\\
&\leq |||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}|||^{\frac{1}{2}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\nu}A^{\frac{1}{2}}|||^{\frac{1}{4}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{1-\nu}A^{\frac{1}{2}}|||^{\frac{1}{4}}\\
&\leq |||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\nu}A^{\frac{1}{2}}|||^{\frac{1}{2}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{1-\nu}A^{\frac{1}{2}}|||^{\frac{1}{2}}.
\end{aligned}$$

Equivalently, we can write

$$\begin{aligned}
&|||A\sharp B||| \leq |||A\sharp_{\frac{1+2\nu}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3-2\nu}{4}}B|||^{\frac{1}{2}}\\
&\leq \exp\left(\frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\log|||A\sharp_u B|||\ du\right)\\
&\leq |||A\sharp_{\frac{1}{2}}B|||^{\frac{1}{2}}|||A\sharp_{\nu}B|||^{\frac{1}{4}}|||A\sharp_{1-\nu}B|||^{\frac{1}{4}} \leq |||A\sharp_{\nu}B|||^{\frac{1}{2}}|||A\sharp_{1-\nu}B|||^{\frac{1}{2}}. \qquad (8)
\end{aligned}$$

On the other hand, for $\nu \in (\frac{1}{2}, 1]$, by inequalities (3), we have

$$\begin{aligned}
|||A\sharp B||| &\leq |||A\sharp_{\frac{1+2\nu}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3-2\nu}{4}}B|||^{\frac{1}{2}}\\
&\leq \exp\left(\frac{1}{2\nu-1}\int_{1-\nu}^{\nu}\log|||A\sharp_u B|||\ du\right)\\
&\leq |||A\sharp_{\frac{1}{2}}B|||^{\frac{1}{2}}|||A\sharp_{\nu}B|||^{\frac{1}{4}}|||A\sharp_{1-\nu}B|||^{\frac{1}{4}} \leq |||A\sharp_{\nu}B|||^{\frac{1}{2}}|||A\sharp_{1-\nu}B|||^{\frac{1}{2}}. \qquad (9)
\end{aligned}$$

In the last inequalities we used

$$\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\log|||A\sharp_u B|||\ du\right) = |||A\sharp B|||.$$

This completes the proof. □

In the special case $\nu = 0$, by the above result, one can obtain the following corollary.

Corollary 4.3. *Let $A, B, X \in B(H)$ be such that A, B are positive operators. Then we have*

$$\begin{aligned}
|||A\sharp B||| &\leq |||A\sharp_{\frac{1}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3}{4}}B|||^{\frac{1}{2}} \leq \exp\left(\int_0^1\log|||A\sharp_u B|||\ du\right)\\
&\leq |||A\sharp B|||^{\frac{1}{2}}|||A|||^{\frac{1}{4}}|||B|||^{\frac{1}{4}} \leq |||A|||^{\frac{1}{2}}|||B|||^{\frac{1}{2}} \qquad (10)\\
&\leq \frac{1}{2}\big(|||A||| + |||B|||\big). \qquad (11)
\end{aligned}$$

From now on, we give some applications of AG-convex functions of two variables for unitarily invariant norms.

Lemma 4.4. [7] *Let $A, B, X \in B(H)$ such that A and B are positive. Then*

$$f(p,q) = |||A^p X B^q|||$$

is AG-convex for $t \in [0,1]$ and $p, q \in [0,\infty)$.

Applying inequalities of Theorem 3.4 to the function $f(p,q) = |||A^p X B^q|||$ on the interval $[\nu, 1-\nu] \times [\nu, 1-\nu]$ when $0 \le \nu < \frac{1}{2}$ and on the interval $[1-\nu, \nu] \times [1-\nu, \nu]$ when $\frac{1}{2} < \nu \le 1$, we obtain the following theorem.

Theorem 4.5. *Let $A, B, X \in B(H)$ such that A and B are positive. Then*

$$\begin{aligned} |||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| &\le \exp\left(\frac{1}{2-4\nu} \int_\nu^{1-\nu} \log |||A^s X B^{\frac{1}{2}}||| ds \right. \\ &\qquad \left. + \frac{1}{2-4\nu} \int_\nu^{1-\nu} \log |||A^{\frac{1}{2}} X B^t||| dt \right) \\ &\le \exp\left(\frac{1}{(1-2\nu)^2} \int_\nu^{1-\nu} \int_\nu^{1-\nu} \log |||A^s X B^t||| ds dt \right) \end{aligned}$$

and

$$\begin{aligned} |||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| &\le \exp\left(\frac{1}{4-8\nu} \left[\int_\nu^{1-\nu} \log |||A^s X B^\nu||| ds + \int_\nu^{1-\nu} \log |||A^s X B^{1-\nu}||| ds \right] \right. \\ &\qquad \left. + \frac{1}{4-8\nu} \left[\int_\nu^{1-\nu} \log |||A^\nu X B^t||| dt + \int_\nu^{1-\nu} \log |||A^{1-\nu} X B^t||| dt \right] \right) \\ &\le \sqrt[4]{|||A^\nu X B^\nu||| |||A^\nu X B^{1-\nu}||| |||A^{1-\nu} X B^\nu||| |||A^{1-\nu} X B^{1-\nu}|||}. \end{aligned}$$

Proof. For $0 \le \nu < \frac{1}{2}$, by applying the inequalities of Theorem 3.4, we get

$$\begin{aligned} f\left(\frac{1}{2}, \frac{1}{2}\right) &\le \exp\left(\frac{1}{2(1-2\nu)} \int_\nu^{1-\nu} \log \circ f\left(s, \frac{1}{2}\right) ds \right. \\ &\qquad \left. + \frac{1}{2(1-2\nu)} \int_\nu^{1-\nu} \log \circ f\left(\frac{1}{2}, t\right) dt \right) \\ &\le \exp\left(\frac{1}{(1-2\nu)^2} \int_\nu^{1-\nu} \int_\nu^{1-\nu} \log \circ f(s,t) ds dt \right) \end{aligned}$$

and from here,

$$\begin{aligned} f\left(\frac{1}{2}, \frac{1}{2}\right) &\le \exp\left(\frac{1}{4(1-2\nu)} \left[\int_\nu^{1-\nu} \log \circ f(s,\nu) ds + \int_\nu^{1-\nu} \log \circ f(s, 1-\nu) ds \right] \right. \\ &\qquad \left. + \frac{1}{4(1-2\nu)} \left[\int_\nu^{1-\nu} \log \circ f(\nu, t) dt + \int_\nu^{1-\nu} \log \circ f(1-\nu, t) dt \right] \right) \\ &\le \sqrt[4]{f(\nu,\nu) f(\nu, 1-\nu) f(1-\nu, \nu) f(1-\nu, 1-\nu)}. \end{aligned}$$

Hence, by Lemma 4.4, we obtain

$$\begin{aligned}
|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \exp\left(\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^{s}XB^{\frac{1}{2}}|||ds\right.\\
&\quad\left.+\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^{\frac{1}{2}}XB^{t}|||dt\right)\\
&\le \exp\left(\frac{1}{(1-2\nu)^2}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||A^{s}XB^{t}|||dsdt\right)
\end{aligned}$$

and

$$\begin{aligned}
|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \exp\left(\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^{s}XB^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||A^{s}XB^{1-\nu}|||ds\right]\right.\\
&\quad\left.+\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^{\nu}XB^{t}|||dt+\int_{\nu}^{1-\nu}\log|||A^{1-\nu}XB^{t}|||dt\right]\right)\\
&\le \sqrt[4]{|||A^{\nu}XB^{\nu}||||||A^{\nu}XB^{1-\nu}||||||A^{1-\nu}XB^{\nu}||||||A^{1-\nu}XB^{1-\nu}|||}.
\end{aligned}$$

For $\frac{1}{2}<\nu\le 1$, by making use of Theorem 3.4, we have

$$\begin{aligned}
f\left(\frac{1}{2},\frac{1}{2}\right) &\le \exp\left(\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log\circ f\left(s,\frac{1}{2}\right)ds\right.\\
&\quad\left.+\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log\circ f\left(\frac{1}{2},t\right)dt\right)\\
&\le \exp\left(\frac{1}{(2\nu-1)^2}\int_{1-\nu}^{\nu}\int_{1-\nu}^{\nu}\log\circ f(s,t)dsdt\right)
\end{aligned}$$

and

$$\begin{aligned}
f\left(\frac{1}{2},\frac{1}{2}\right) &\le \exp\left(\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log\circ f(s,1-\nu)ds+\int_{1-\nu}^{\nu}\log\circ f(s,\nu)ds\right]\right.\\
&\quad\left.+\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log\circ f(1-\nu,t)dt+\int_{1-\nu}^{\nu}\log\circ f(\nu,t)dt\right]\right)\\
&\le \sqrt[4]{f(1-\nu,1-\nu)f(1-\nu,\nu)f(\nu,1-\nu)f(\nu,\nu)}.
\end{aligned}$$

By Lemma 4.4, we have

$$\begin{aligned}
|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\le \exp\left(\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log|||A^{s}XB^{\frac{1}{2}}|||ds\right.\\
&\quad\left.+\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log|||A^{\frac{1}{2}}XB^{t}|||dt\right)\\
&\le \exp\left(\frac{1}{(2\nu-1)^2}\int_{1-\nu}^{\nu}\int_{1-\nu}^{\nu}\log|||A^{s}XB^{t}|||dsdt\right)
\end{aligned}$$

and

$$\begin{aligned}
|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \exp\left(\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log|||A^sXB^{1-\nu}|||ds+\int_{1-\nu}^{\nu}\log|||A^sXB^{\nu}|||ds\right]\right.\\
&\quad\left.+\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log|||A^{1-\nu}XB^t|||dt+\int_{1-\nu}^{\nu}\log|||A^{\nu}XB^t|||dt\right]\right)\\
&\leq \sqrt[4]{|||A^{1-\nu}XB^{1-\nu}||||||A^{1-\nu}XB^{\nu}||||||A^{\nu}XB^{1-\nu}||||||A^{\nu}XB^{\nu}|||}.
\end{aligned}$$

Note that in the last inequalities we used

$$\begin{aligned}
&\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^sXB^{\frac{1}{2}}|||ds+\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^{\frac{1}{2}}XB^t|||dt\right)\\
&=\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{(1-2\nu)^2}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||A^sXB^t|||dsdt\right)\\
&=\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^sXB^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||A^sXB^{1-\nu}|||ds\right]\right.\\
&\quad\left.+\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^{\nu}XB^t|||dt+\int_{\nu}^{1-\nu}\log|||A^{1-\nu}XB^t|||dt\right]\right)\\
&=|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.
\end{aligned}$$

This completes the proof. □

Corollary 4.6. *Let $A, B, X \in B(H)$. Then*

$$\begin{aligned}
|||AXB^*|||^4 &\leq \exp\left(\frac{2}{1-2\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^{\frac{1}{2}}|||ds\right.\\
&\qquad\left.+\frac{2}{1-2\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{\frac{1}{2}}X(B^*B)^t|||dt\right)\\
&\leq \exp\left(\frac{4}{(1-2\nu)^2}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^t|||dsdt\right)\\
&\leq \exp\left(\frac{1}{1-2\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^{1-\nu}|||ds\right]\right.\\
&\quad\left.+\frac{1}{1-2\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^{\nu}X(B^*B)^t|||dt+\int_{\nu}^{1-\nu}\log|||(A^*A)^{1-\nu}X(B^*B)^t|||dt\right]\right)\\
&\leq |||(A^*A)^{\nu}X(B^*B)^{\nu}||||||(A^*A)^{\nu}X(B^*B)^{1-\nu}|||\\
&\quad |||(A^*A)^{1-\nu}X(B^*B)^{\nu}||||||(A^*A)^{1-\nu}X(B^*B)^{1-\nu}|||.
\end{aligned}$$

Proof. From the inequality (3.4) of [9], we have

$$|||AXB^*|||^2 \leq |||(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}}||||||(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}}||| \tag{12}$$

for all $A, B \in B(H)$. Now, applying Theorem 4.5, we get the desired result. □

Remark 4.7. Let $X = I$ in the above theorem. Then for two operators A and B in $B(H)$, we have

$$
\begin{aligned}
&|||(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}}||| \qquad (13)\\
&\leq \exp\left(\frac{1}{2-4\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{\frac{1}{2}}|||ds\right.\\
&\qquad\qquad \left.+\frac{1}{2-4\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{\frac{1}{2}}(B^*B)^{t}|||dt\right)\\
&\leq \exp\left(\frac{1}{(1-2\nu)^2}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{t}|||dsdt\right)\\
&\leq \exp\left(\frac{1}{4-8\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{1-\nu}|||ds\right]\right.\\
&\quad \left.+\frac{1}{4-8\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^{\nu}(B^*B)^{t}|||dt+\int_{\nu}^{1-\nu}\log|||(A^*A)^{1-\nu}(B^*B)^{t}|||dt\right]\right)\\
&\leq \sqrt[4]{|||(A^*A)^{\nu}(B^*B)^{\nu}|||\,|||(A^*A)^{\nu}(B^*B)^{1-\nu}|||}\\
&\qquad \times\sqrt[4]{|||(A^*A)^{1-\nu}(B^*B)^{\nu}|||\,|||(A^*A)^{1-\nu}(B^*B)^{1-\nu}|||}. \qquad (14)
\end{aligned}
$$

From [5], we have

$$|||(A^*A)^{\nu}(B^*B)^{\nu}||| \leq |||A^*AB^*B|||^{\nu} \qquad (15)$$

and from [1] we have

$$|||(A^*A)^{\nu}(B^*B)^{1-\nu}||| \leq |||\nu A^*A+(1-\nu)B^*B||| \qquad (16)$$

for all A and B in $B(H)$ and $\nu \in [0,1]$. Now, applying (14), (15) and (16), we obtain

$$
\begin{aligned}
&\sqrt[4]{|||(A^*A)^{\nu}(B^*B)^{\nu}|||\,|||(A^*A)^{\nu}(B^*B)^{1-\nu}|||}\\
&\qquad \times\sqrt[4]{|||(A^*A)^{1-\nu}(B^*B)^{\nu}|||\,|||(A^*A)^{1-\nu}(B^*B)^{1-\nu}|||}\\
&\leq \sqrt[4]{|||A^*AB^*B|||\,|||\nu A^*A+(1-\nu)B^*B|||\,|||(1-\nu)A^*A+\nu B^*B|||}. \qquad (17)
\end{aligned}
$$

On the other hand, from the inequality (12), for $X = I$, we have

$$|||AB^*|||^2 \leq |||(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}}|||\,|||(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}}||| \qquad (18)$$

for all $A, B \in B(H)$.

By putting (17) and (18) into (14), we have

$$
\begin{aligned}
|||AB^*|||^4 \leq \exp\Bigg(& \frac{2}{1-2\nu}\int_\nu^{1-\nu} \log |||(A^*A)^s(B^*B)^{\frac{1}{2}}|||ds \\
& + \frac{2}{1-2\nu}\int_\nu^{1-\nu} \log |||(A^*A)^{\frac{1}{2}}(B^*B)^t|||dt\Bigg) \\
\leq \exp\Bigg(& \frac{4}{(1-2\nu)^2}\int_\nu^{1-\nu}\int_\nu^{1-\nu} \log |||(A^*A)^s(B^*B)^t|||dsdt\Bigg) \\
\leq \exp\Bigg(& \frac{1}{1-2\nu}\left[\int_\nu^{1-\nu} \log |||(A^*A)^s(B^*B)^\nu|||ds + \int_\nu^{1-\nu} \log |||(A^*A)^s(B^*B)^{1-\nu}|||ds\right] \\
& + \frac{1}{1-2\nu}\left[\int_\nu^{1-\nu} \log |||(A^*A)^\nu(B^*B)^t|||dt + \int_\nu^{1-\nu} \log |||(A^*A)^{1-\nu}(B^*B)^t|||dt\right]\Bigg) \\
\leq\ & |||(A^*A)^\nu(B^*B)^\nu||| |||(A^*A)^\nu(B^*B)^{1-\nu}||| \\
& \times |||(A^*A)^{1-\nu}(B^*B)^\nu||| |||(A^*A)^{1-\nu}(B^*B)^{1-\nu}||| \\
\leq\ & |||A^*AB^*B||| |||\nu A^*A+(1-\nu)B^*B||| |||(1-\nu)A^*A+\nu B^*B|||. \qquad (19)
\end{aligned}
$$

Applying (19) for $\nu = 0$, we obtain the following inequalities.

Corollary 4.8. *Let $A, B \in B(H)$. Then*

$$
\begin{aligned}
& |||AB^*|||^2 \\
& \leq \exp\left(\int_0^1 \log |||(A^*A)^s(B^*B)^{\frac{1}{2}}|||ds + \int_0^1 \log |||(A^*A)^{\frac{1}{2}}(B^*B)^t|||dt\right) \\
& \leq \exp\left(2\int_0^1\int_0^1 \log |||(A^*A)^s(B^*B)^t|||dsdt\right) \\
& \leq \exp\Bigg(\frac{1}{2}\int_0^1 \log |||(A^*A)^s|||ds + \frac{1}{2}\int_0^1 \log |||(A^*A)^sB^*B|||ds \\
& \qquad + \frac{1}{2}\int_0^1 \log |||(B^*B)^t|||dt + \frac{1}{2}\int_0^1 \log |||A^*A(B^*B)^t|||dt\Bigg) \\
& \leq \sqrt{|||A^*A||| |||B^*B||| |||A^*AB^*B|||} \\
& \leq |||A^*A||| |||B^*B|||.
\end{aligned}
$$

We should note here that the above corollary is a refinement for the Cauchy-Schwarz inequality.

Applying inequalities of Theorem 3.4 to the function

$$f(p, q) = |||A^pXB^q|||$$

on the interval $[0, \nu] \times [0, \nu]$ when $0 < \nu \leq \frac{1}{2}$ and on the interval $[\nu, 1] \times [\nu, 1]$ when $\frac{1}{2} \leq \nu < 1$, we obtain the following theorem by a similar proof of Theorem 4.5.

Theorem 4.9. *Let $A, B, X \in B(H)$ be such that A and B are positive. Then*

(1) *For $\nu \in (0, \frac{1}{2}]$, we have*

$$\begin{aligned}
|||A^{\frac{\nu}{2}} X B^{\frac{\nu}{2}}||| &\leq \exp\left(\frac{1}{2\nu} \int_0^\nu \log |||A^s X B^{\frac{\nu}{2}}||| ds \ + \frac{1}{2\nu} \int_0^\nu \log |||A^{\frac{\nu}{2}} X B^t||| dt \right) \\
&\leq \exp\left(\frac{1}{\nu^2} \int_0^\nu \int_0^\nu \log |||A^s X B^t||| ds dt \right) \\
&\leq \exp\left(\frac{1}{4\nu} \left[\int_0^\nu \log |||A^s X||| ds + \int_0^\nu \log |||A^s X B^\nu||| ds \right] \right. \\
&\qquad \left. + \frac{1}{4\nu} \left[\int_0^\nu \log |||X B^t||| dt + \int_0^\nu \log |||A^\nu X B^t||| dt \right] \right) \\
&\leq \sqrt[4]{|||X||| |||XB^\nu||| |||A^\nu X||| |||A^\nu X B^\nu|||}.
\end{aligned}$$

(2) *Also, for $\nu \in [\frac{1}{2}, 1)$, we have*

$$\begin{aligned}
|||A^{\frac{\nu+1}{2}} X B^{\frac{\nu+1}{2}}||| &\leq \exp\left(\frac{1}{2(1-\nu)} \int_\nu^1 \log |||A^s X B^{\frac{\nu+1}{2}}||| ds \right. \\
&\qquad \left. + \frac{1}{2(1-\nu)} \int_\nu^1 \log |||A^{\frac{\nu+1}{2}} X B^t||| dt \right) \\
&\leq \exp\left(\frac{1}{(1-\nu)^2} \int_\nu^1 \int_\nu^1 \log |||A^s X B^t||| ds dt \right) \\
&\leq \exp\left(\frac{1}{4(1-\nu)} \left[\int_\nu^1 \log |||A^s X B^\nu||| ds + \int_\nu^1 \log |||A^s X B||| ds \right] \right. \\
&\qquad \left. + \frac{1}{4(1-\nu)} \left[\int_\nu^1 \log |||A^\nu X B^t||| dt + \int_\nu^1 \log |||A X B^t||| dt \right] \right) \\
&\leq \sqrt[4]{|||A^\nu X B^\nu||| |||A^\nu X B||| |||A X B^\nu||| |||A X B|||}.
\end{aligned}$$

By considering the above theorem on $[0, 1] \times [0, 1]$ and the inequality (12) for all $A, B, X \in B(H)$, we have the following result.

Corollary 4.10. *Let $A, B, X \in B(H)$. Then*

$$\begin{aligned}
|||AXB^*|||^4 &\leq \exp\left(2 \int_0^1 \log |||(A^*A)^s X (B^*B)^{\frac{1}{2}}||| ds \right. \\
&\qquad \left. + 2 \int_0^1 \log |||(A^*A)^{\frac{1}{2}} X (B^*B)^t||| dt \right) \\
&\leq \exp\left(4 \int_0^1 \int_0^1 \log |||(A^*A)^s X (B^*B)^t||| ds dt \right) \\
&\leq \exp\left(\int_0^1 \log |||(A^*A)^s X||| ds + \int_0^1 \log |||(A^*A)^s X (B^*B)||| ds \right. \\
&\qquad \left. + \int_0^1 \log |||X (B^*B)^t||| dt + \int_0^1 \log |||(A^*A) X (B^*B)^t||| dt \right) \\
&\leq |||X||| |||(A^*A)X||| |||X(B^*B)||| |||(A^*A)X(B^*B)|||.
\end{aligned}$$

Acknowledgments. The first author is supported by the Talented Young Scientist Program of Ministry of Science and Technology of China (Iran-19-001).

References

[1] T. Ando: *Matrix Young inequalities*, Oper. Theory Adv. Appl. 75 (1995) 33–38.

[2] R. Bhatia: *Matrix Analysis*, Graduate Texts in Mathematics 169, Springer, New York (1997).

[3] S. S. Dragomir: *On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane*, Taiwanese J. Math. 4 (2001) 775–788.

[4] S. S. Dragomir, C. E. M. Pearce: *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University (2000), online: http://rgmia.vu.edu.au/monographs.

[5] F. Kittaneh: *Norm inequalities for fractional powers of positive operators*, Lett. Math. Phys. 27 (1993) 279–285.

[6] C. P. Niculescu: *Convexity according to the geometric mean*, Math. Inequal. Appl. 3/2 (2000) 155–167.

[7] M. Sababheh: *Interpolated inequalities for unitarily invariant norms*, Linear Algebra Appl. 475 (2015) 240–250.

[8] A. Taghavi, V. Darvish, H. M. Nazari, S. S. Dragomir: *Hermite-Hadamard type inequalities for operator geometrically convex functions*, Monatshefte Math. 181 (2016) 187–203.

[9] A. Taghavi, T. Azimi Roushan, V. Darvish: *Some refinements for the arithmetic-geometric mean and Cauchy-Schwarz matrix norm interpolating inequalities*, Bull. Iranian Math. Soc. 44/4 (2018) 927–936.