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Some Integral Inequalities for Arithmetically and Geometrically Convex Functions of Two Variables

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We introduce arithmetic-geometrically and geometric-geometrically convex functions for two variables. Then, we prove Hermite-Hadamard type inequalities for these functions. As applications, some new inequalities for unitarily invariant norms are established.

Keywords: Hermite-Hadamard inequality, convex function, arithmetic-geometrically convex, unitarily invariant norm.

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1. Introduction

Let us consider the bi-dimensional interval $\Delta := [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A function $f : \Delta \to \mathbb{R}$ is called a separately convex function(shortly convex function) on the co-ordinates if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) := f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) := f(x,v)$ are convex for all $y \in [c,d]$ and $x \in [a,b]$, respectively.

Dragomir proved the following Hermite-Hadamard type inequalities.

Theorem 1.1. [3] *Suppose that $f : \Delta = [a,b] \times [c,d] \to \mathbb{R}$ is convex on the co-ordinates on Δ. Then we have*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right]$$
$$\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)dydx$$

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$$\leq \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)dx + \frac{1}{b-a}\int_a^b f(x,d)dx + \frac{1}{d-c}\int_c^d f(a,y)dy + \frac{1}{d-c}\int_c^d f(b,y)dy\right]$$
$$\leq \frac{1}{4}\big(f(a,c)+f(a,d)+f(b,c)+f(b,d)\big).$$

Throughout this paper, we introduce geometric-geometrically convex functions for two variables. Then, we prove Hermite-Hadamard type inequalities for these functions. Also, we define arithmetic-geometrically convex functions for two variables. By these functions, we obtain some new inequalities for unitarily invariant norms.

2. Inequalities for geometric-geometrically convex functions of two variables

In this section, we prove Hermite-Hadamard type inequalities for geometric-geometrically convex functions of two variables.

The following definition is for geometrically convex function of one variable.

Definition 2.1. [6]A continuous function $f : I \subset \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *geometrically convex* (or *multiplicatively convex*) if

$$f(a^\lambda b^{1-\lambda}) \leq f(a)^\lambda f(b)^{1-\lambda} \qquad \text{for } a,b \in I \text{ and } \lambda \in [0,1].$$

In [8, Theorem 3], we proved the following inequalities for geometrically convex functions of one variable

$$f(\sqrt{ab}) \leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \leq \exp\left(\frac{1}{\log b-\log a}\int_a^b \frac{\log\circ f(t)}{t}dt\right)$$
$$\leq \sqrt{f(\sqrt{ab})}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \leq \sqrt{f(a)f(b)}. \tag{1}$$

Now, we give the definition of geometric-geometrically convex function of two variables.

Definition 2.2. The function $f : \Delta \subset \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is said to be *geometrically* (*geometric-geometrically*) *convex* if

$$f\left(x^\lambda z^{(1-\lambda)}, y^\lambda w^{(1-\lambda)}\right) \leq f(x,y)^\lambda f(z,w)^{(1-\lambda)} \tag{2}$$

for every $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Below, we will use the following notations:

$$\log(x,y) = (\log x, \log y), \quad x,y>0,$$

and

$$\exp(x,y) = (\exp x, \exp y), \quad x,y \in \mathbb{R}.$$

Lemma 2.3. *Suppose that $[a,b]\times[c,d]$ is a subinterval of $\mathbb{R}^+\times\mathbb{R}^+$ and that $f:[a,b]\times[c,d]\to(0,\infty)$ is a geometrically convex function. Then*

$$F=\log\circ f\circ\exp:([\log a,\log b]\times[\log c,\log d])\to\mathbb{R}$$

is a convex function.

Proof. Assume that f is geometrically convex. We will show that $F=\log\circ f\circ\exp$ is convex on $[\log a,\log b]\times[\log c,\log d]$, that is,

$$\begin{aligned}&\log\circ f\circ\exp(\lambda\log x+(1-\lambda)\log z,\lambda\log y+(1-\lambda)\log w)\\&\leq\lambda\log\circ f\circ\exp(\log x,\log y)+(1-\lambda)\log\circ f\circ\exp(\log z,\log w)\end{aligned}$$

for all $x,z\in[a,b]$, $y,w\in[c,d]$ and $\lambda\in[0,1]$. We have from the left side of the above inequality

$$\begin{aligned}&\log\circ f\circ\exp(\lambda\log x+(1-\lambda)\log z,\lambda\log y+(1-\lambda)\log w)\\&=\log\circ f\circ\exp\left(\log\left(x^\lambda z^{(1-\lambda)}\right),\log\left(y^\lambda w^{(1-\lambda)}\right)\right)\\&=\log\circ f\left(\left(x^\lambda z^{(1-\lambda)}\right),\left(y^\lambda w^{(1-\lambda)}\right)\right)\\&\leq\log\left(f(x)^\lambda f(z)^{(1-\lambda)},f(y)^\lambda f(w)^{(1-\lambda)}\right)\quad\text{(since } f\text{ is geometrically convex)}\\&=(\lambda\log\circ f(x)+(1-\lambda)\log\circ f(z),\lambda\log\circ f(y)+(1-\lambda)\log\circ f(w))\\&=(\lambda\log\circ f(\exp(\log x))+(1-\lambda)\log\circ f(\exp(\log z)),\lambda\log\circ f(\exp(\log y))\\&+(1-\lambda)\log\circ f(\exp(\log w)))\\&=\lambda\log\circ f\circ\exp(\log x,\log y)+(1-\lambda)\log\circ f\circ\exp(\log z,\log w),\end{aligned}$$

for all $x,z\in[a,b]$, $y,w\in[c,d]$ and $\lambda\in[0,1]$. □

Theorem 2.4. *Let f be a geometrically convex function defined on $[a,b]\times[c,d]$ such that $0<a<b$ and $0<c<d$. Then, we have*

$$\begin{aligned}f\left(\sqrt{ab},\sqrt{cd}\right)&\leq\exp\left(\frac{1}{2\log\frac{b}{a}}\int_a^b\frac{\log f(u,\sqrt{cd})}{u}du+\frac{1}{2\log\frac{d}{c}}\int_c^d\frac{\log f(\sqrt{ab},v)}{v}dv\right)\\&\leq\exp\left(\frac{1}{(\log\frac{b}{a})(\log\frac{d}{c})}\int_a^b\int_c^d\frac{\log f(u,v)}{uv}dvdu\right)\\&\leq\exp\left(\frac{1}{4\log\frac{b}{a}}\left[\int_a^b\frac{\log f(u,c)}{u}du+\int_a^b\frac{\log f(u,d)}{u}du\right]\right.\\&\qquad\left.+\frac{1}{4\log\frac{d}{c}}\left[\int_c^d\frac{\log f(a,v)}{v}dv+\int_c^d\frac{\log f(b,v)}{v}dv\right]\right)\\&\leq\sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}.\end{aligned}$$

Proof. Let $f:[a,b]\times[c,d]\to\mathbb{R}$ be geometrically convex. So, by Lemma 2.3

$$F(x,y)=\log\circ f\circ\exp(x,y):[\log a,\log b]\times[\log c,\log d]\to\mathbb{R}$$

is convex.

Then, by Theorem 1.1 we obtain

$$\begin{aligned}
&F\left(\frac{\log a+\log b}{2},\frac{\log c+\log d}{2}\right)\\
&\leq \frac{1}{2}\left[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}F\left(x,\frac{\log c+\log d}{2}\right)dx\right.\\
&\qquad\left.+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}F\left(\frac{\log a+\log b}{2},y\right)dy\right]\\
&\leq \frac{1}{(\log b-\log a)(\log d-\log c)}\int_{\log a}^{\log b}\int_{\log c}^{\log d}F(x,y)dydx
\end{aligned}$$

and

$$\begin{aligned}
&F\left(\frac{\log a+\log b}{2},\frac{\log c+\log d}{2}\right)\\
&\leq \frac{1}{4}\left[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}F(x,\log c)dx+\frac{1}{\log b-\log a}\int_{\log a}^{\log b}F(x,\log d)dx\right.\\
&\qquad\left.+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}F(\log a,y)dy+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}F(\log b,y)dy\right]\\
&\leq \frac{F(\log a,\log c)+F(\log a,\log d)+F(\log b,\log c)+F(\log b,\log d)}{4}.
\end{aligned}$$

By the definition of F, we have

$$\begin{aligned}
&\log\circ f\circ\exp(\log\sqrt{ab},\log\sqrt{cd})\\
&\leq \frac{1}{2}\left[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log\circ f\circ\exp(x,\log\sqrt{cd})dx\right.\\
&\qquad\left.+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log\circ f\circ\exp(\log\sqrt{ab},y)dy\right]\\
&\leq \frac{1}{(\log b-\log a)(\log d-\log c)}\int_{\log a}^{\log b}\int_{\log c}^{\log d}\log\circ f\circ\exp(x,y)dydx,
\end{aligned}$$

whereupon

$$\begin{aligned}
&\log\circ f\circ\exp(\log\sqrt{ab},\log\sqrt{cd})\\
&\leq \frac{1}{4}\left[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log\circ f\circ\exp(x,\log c)dx\right.\\
&\qquad+\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log\circ f\circ\exp(x,\log d)dx\\
&\qquad+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log\circ f\circ\exp(\log a,y)dy\\
&\qquad\left.+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log\circ f\circ\exp(\log b,y)dy\right]
\end{aligned}$$

and
$$\begin{aligned}&\log\circ f\circ\exp(\log\sqrt{ab},\log\sqrt{cd})\\&\leq\frac{1}{4}\left[\log\circ f\circ\exp(\log a,\log c)+\log\circ f\circ\exp(\log a,\log d)\right.\\&\qquad\left.+\log\circ f\circ\exp(\log b,\log c)+\log\circ f\circ\exp(\log b,\log d)\right].\end{aligned}$$

The above inequalities are equivalent to write

$$\begin{aligned}\log f(\sqrt{ab},\sqrt{cd})&\leq\frac{1}{2}\left[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log f(\exp(x),\sqrt{cd})dx\right.\\&\qquad\left.+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log f(\sqrt{ab},\exp(y))dy\right]\\&\leq\frac{1}{(\log b-\log a)(\log d-\log c)}\int_{\log a}^{\log b}\int_{\log c}^{\log d}\log f(\exp(x),\exp(y))dydx\end{aligned}$$

and

$$\begin{aligned}\log f(\sqrt{ab},\sqrt{cd})&\leq\frac{1}{4}\left[\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log f(\exp(x),c)dx\right.\\&+\frac{1}{\log b-\log a}\int_{\log a}^{\log b}\log f(\exp(x),d)dx\ +\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log f(a,\exp(y))dy\\&\left.+\frac{1}{\log d-\log c}\int_{\log c}^{\log d}\log f(b,\exp(y))dy\right]\\&\leq\frac{1}{4}\left[\log f(a,c)+\log f(a,d)+\log f(b,c)+\log f(b,d)\right].\end{aligned}$$

By change of variable, $u=\exp(x)$ and $v=\exp(y)$, we get

$$\begin{aligned}&\log f(\sqrt{ab},\sqrt{cd})\\&\leq\frac{1}{2}\left[\frac{1}{\log b-\log a}\int_a^b\frac{\log f(u,\sqrt{cd})}{u}du\ +\frac{1}{\log d-\log c}\int_c^d\frac{\log f(\sqrt{ab},v)}{v}dv\right]\\&\leq\frac{1}{(\log b-\log a)(\log d-\log c)}\int_a^b\int_c^d\frac{\log f(u,v)}{uv}dvdu\\&\leq\frac{1}{4}\left[\frac{1}{\log b-\log a}\left(\int_a^b\frac{\log f(u,c)}{u}du+\int_a^b\frac{\log f(u,d)}{u}du\right)\right.\\&\qquad\left.+\frac{1}{\log d-\log c}\left(\int_c^d\frac{\log f(a,v)}{v}dv+\int_c^d\frac{\log f(b,v)}{v}dv\right)\right]\\&\leq\log\sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}.\end{aligned}$$

Since $\exp(x)$ is increasing, we have the desired result. □

3. Inequalities for arithmetic-geometrically convex functions of two variables

In this section, by the results obtained in Section 2, we establish Hermite-Hadamard type inequalities for arithmetic-geometrically convex (AG-convex) functions for two variables.

In [9] we proved the following theorem for one variable.

Theorem 3.1. *Let f be an AG-convex function defined on $[a,b]$. Then, we have*

$$f\left(\frac{a+b}{2}\right) \leq \sqrt{f\left(\frac{3a+b}{4}\right) f\left(\frac{a+3b}{4}\right)} \leq \exp\left(\frac{1}{b-a}\int_a^b \log\circ f(u)du\right)$$
$$\leq \sqrt{f\left(\frac{a+b}{2}\right)}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \leq \sqrt{f(a)f(b)}, \quad \text{where } u=\log t. \tag{3}$$

Here, we present the definition of AG-convex function for two variables.

Definition 3.2. The function $f:\Delta\subset\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ is said to be AG-convex if

$$f(\lambda x+(1-\lambda)z, \lambda y+(1-\lambda)w) \leq f(x,y)^{\lambda} f(z,w)^{(1-\lambda)} \tag{4}$$

for every $(x,y),(z,w)\in\Delta$ and $\lambda\in[0,1]$.

Remark 3.3. The condition (4) can be written as

$$f\circ\log(\exp(\lambda x)\exp((1-\lambda)z), \exp(\lambda y)\exp((1-\lambda)w))$$
$$\leq (f\circ\log(\exp(x),\exp(y)))^{\lambda}(f\circ\log(\exp(z).\exp(w)))^{1-\lambda}, \tag{5}$$

Then we can see $f:\Delta\to\mathbb{R}^+$ is AG-convex on Δ if and only if $f\circ\log$ is geometrically convex on $\exp(\Delta):=\{(\exp(x),\exp(y)),(x,y)\in\Delta\}$.

Observe that if $\Delta=[a,b]\times[c,d]$, then $\exp(\Delta)=[\exp(a),\exp(b)]\times[\exp(c),\exp(d)]$.

Theorem 3.4. *Let f be an AG-convex function defined on $[a,b]\times[c,d]$. Then, we have*

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)$$
$$\leq \exp\left(\frac{1}{2(b-a)}\int_a^b \log\circ f\left(s,\frac{c+d}{2}\right)ds \ + \frac{1}{2(d-c)}\int_c^d \log\circ f\left(\frac{a+b}{2},t\right)dt\right)$$
$$\leq \exp\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \log\circ f(s,t)dsdt\right)$$
$$\leq \exp\left(\frac{1}{4(b-a)}\left[\int_a^b \log\circ f(s,c)ds+\int_a^b \log\circ f(s,d)ds\right]\right.$$
$$\left. + \frac{1}{4(d-c)}\left[\int_c^d \log\circ f(a,t)dt+\int_c^d \log\circ f(b,t)dt\right]\right)$$
$$\leq \sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}.$$

Proof. Let $f:[a,b]\to\mathbb{R}^+$ be a AG-convex function. Then, by Remark 3.3, the function $f\circ\log$ is geometrically convex on $[\exp(a),\exp(b)]\times[\exp(c),\exp(d)]$.

Applying the inequalities of Theorem 2.4, we obtain

$$
\begin{aligned}
&f\circ\log\left(\sqrt{e^a e^b},\sqrt{e^c e^d}\right)\\
&\leq \exp\left(\frac{1}{2\log\frac{e^b}{e^a}}\int_{e^a}^{e^b}\frac{\log\circ f\circ\log(u,\sqrt{e^c e^d})}{u}du \;+\frac{1}{2\log\frac{e^d}{e^c}}\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(\sqrt{e^a e^b},v)}{v}dv\right)\\
&\leq \exp\left(\frac{1}{(\log\frac{e^b}{e^a})(\log\frac{e^d}{e^c})}\int_{e^a}^{e^b}\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(u,v)}{uv}dudv\right)\\
&\leq \exp\left(\frac{1}{4\log\frac{e^b}{e^a}}\left[\int_{e^a}^{e^b}\frac{\log\circ f\circ\log(u,e^c)}{u}du+\int_{e^a}^{e^b}\frac{\log\circ f\circ\log(u,e^d)}{u}du\right]\right.\\
&\quad\left.+\frac{1}{4\log\frac{e^d}{e^c}}\left[\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(e^a,v)}{v}dv+\int_{e^c}^{e^d}\frac{\log\circ f\circ\log(e^b,v)}{v}dv\right]\right)\\
&\leq \sqrt[4]{f\circ\log(e^a,e^c)f\circ\log(e^c,e^d)f\circ\log(e^b,e^c)f\circ\log(e^b,e^d)}.
\end{aligned}
$$

Hence, using Theorem 1.1, we can write

$$
\begin{aligned}
&f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\\
&\leq \exp\left(\frac{1}{2(b-a)}\int_{e^a}^{e^b}\frac{\log\circ f(\log u,\frac{c+d}{2})}{u}du \;+\frac{1}{2(c-d)}\int_{e^c}^{e^d}\frac{\log\circ f(\frac{a+b}{2},\log v)}{v}dv\right)\\
&\leq \exp\left(\frac{1}{(b-a)(d-c)}\int_{e^a}^{e^b}\int_{e^c}^{e^d}\frac{\log\circ f(\log u,\log v)}{uv}dudv\right)\\
&\leq \exp\left(\frac{1}{4(b-a)}\left[\int_{e^a}^{e^b}\frac{\log\circ f(\log u,c)}{u}du+\int_{e^a}^{e^b}\frac{\log\circ f(\log u,d)}{u}du\right]\right.\\
&\qquad\left.+\frac{1}{4(d-c)}\left[\int_{e^c}^{e^d}\frac{\log\circ f(a,\log v)}{v}dv+\int_{e^c}^{e^d}\frac{\log\circ f(b,\log v)}{v}dv\right]\right)\\
&\leq \sqrt[4]{f(a,c)f(a,d)f(b,c)f(b,d)}.
\end{aligned}
$$

By change of variable $s=\log u$ and $t=\log v$, we have the desired result. □

4. Some unitarily invariant norm inequalities

In this section, we prove some unitarily invariant norm inequalities for operators.

We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on an ideal in $B(H)$ stand for C^*-algebra of all bounded linear operators and it will be implicitly understood that when we talk of $|||T|||$, then the operator T belongs to the norm ideal associated with $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV|||=|||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $B(H)$. For $1\leq p<\infty$, the Schatten p-norm of a compact operator A is defined by $\|A\|_p=(\mathrm{Tr}\,|A|^p)^{1/p}$, where Tr is the usual trace functional. These norms are special examples of the more general class of the Schatten p-norms which are unitarily invariant [2].

As the first application of AG-convex functions of one variable for unitarily invariant norms, we need the following lemma. Before we state our lemma let us mention that the norm version of the Cauchy-Schwarz inequality ([2], Problem IV.5.7, pp. 108) for $A, B \in M_n(\mathbb{C})$ is as follows

$$|||AB^*|||^2 \leq |||A^*A||||||B^*B|||. \tag{6}$$

Lemma 4.1. *Let $A, B, X \in B(H)$ be such that A and B are positive operators. Then $\varphi(t) = |||AB^tA|||$ is AG-convex for $t \in [0, \infty)$.*

Proof. Since φ is continuous, it is enough to prove the mid-point AG-convexity, i.e.,

$$\varphi\left(\frac{s+t}{2}\right) \leq \varphi(s)^{\frac{1}{2}}\varphi(t)^{\frac{1}{2}},$$

for $s, t \in [0, \infty)$. In fact, we have

$$\begin{aligned}\varphi\left(\frac{s+t}{2}\right) &= |||AB^{\frac{s+t}{2}}A||| = |||(AB^{\frac{s}{2}})(B^{\frac{t}{2}}A)||| \\ &\leq |||AB^{\frac{s}{2}}B^{\frac{s}{2}}A|||^{\frac{1}{2}}|||AB^{\frac{t}{2}}B^{\frac{t}{2}}A|||^{\frac{1}{2}} \quad \text{the Cauchy-Schwarz inequality} \\ &= \varphi(AB^sA)^{\frac{1}{2}}\varphi(AB^tA)^{\frac{1}{2}} = \varphi(s)^{\frac{1}{2}}\varphi(t)^{\frac{1}{2}}.\end{aligned}$$

So, $\varphi(t) = |||AB^tA|||$ is an AG-convex function. □

Now we give the following inequalities for weighted geometric mean

$$A\sharp_\nu B = A^{\frac{1}{2}}\left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^\nu A^{\frac{1}{2}}$$

for two positive operators and all $0 \leq \nu \leq 1$. In the case of $\nu = \frac{1}{2}$, we denote $A\sharp_{\frac{1}{2}}B$ by $A\sharp B$.

We apply inequalities (3) to the function $\varphi(t) = |||AB^tA|||$ on the interval $[\nu, 1-\nu]$ when $\nu \in [0, \frac{1}{2})$ and on the interval $[1-\nu, \nu]$ when $\nu \in (\frac{1}{2}, 1]$. We obtain the following result.

Theorem 4.2. *Let $A, B, X \in B(H)$ be such that A, B are positive operators and $\nu \in [0, 1]$. Then, we have*

$$\begin{aligned}|||A\sharp B||| &\leq |||A\sharp_{\frac{1+2\nu}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3-2\nu}{4}}B|||^{\frac{1}{2}} \leq \exp\left(\frac{1}{1-2\nu}\int_\nu^{1-\nu}\log|||A\sharp_u B|||\ du\right) \\ &\leq |||A\sharp B|||^{\frac{1}{2}}|||A\sharp_\nu B|||^{\frac{1}{4}}|||A\sharp_{1-\nu}B|||^{\frac{1}{4}} \leq |||A\sharp_\nu B|||^{\frac{1}{2}}|||A\sharp_{1-\nu}B|||^{\frac{1}{2}}. \end{aligned}\tag{7}$$

Proof. For $\nu \in [0, \frac{1}{2})$, by inequalities (3), we have

$$\begin{aligned}|||AB^{\frac{1}{2}}A||| &\leq |||AB^{\frac{1+2\nu}{4}}A|||^{\frac{1}{2}}|||AB^{\frac{3-2\nu}{4}}A|||^{\frac{1}{2}} \\ &\leq \exp\left(\frac{1}{1-2\nu}\int_\nu^{1-\nu}\log|||AB^uA|||du\right) \\ &\leq |||AB^{\frac{1}{2}}A|||^{\frac{1}{2}}|||AB^\nu A|||^{\frac{1}{4}}|||AB^{1-\nu}A|||^{\frac{1}{4}} \leq |||AB^\nu A|||^{\frac{1}{2}}|||AB^{1-\nu}A|||^{\frac{1}{2}}.\end{aligned}$$

We replace A and B by $A^{\frac{1}{2}}$ and $A^{\frac{-1}{2}}BA^{\frac{-1}{2}}$, respectively, and we obtain the following inequalities

$$\begin{aligned}
&|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}||| \\
&\leq |||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1+2\nu}{4}}A^{\frac{1}{2}}|||^{\frac{1}{2}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{3-2\nu}{4}}A^{\frac{1}{2}}|||^{\frac{1}{2}} \\
&\leq \exp\left(\frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\log|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{u}A^{\frac{1}{2}}|||du\right) \\
&\leq |||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}|||^{\frac{1}{2}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\nu}A^{\frac{1}{2}}|||^{\frac{1}{4}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{1-\nu}A^{\frac{1}{2}}|||^{\frac{1}{4}} \\
&\leq |||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\nu}A^{\frac{1}{2}}|||^{\frac{1}{2}}|||A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{1-\nu}A^{\frac{1}{2}}|||^{\frac{1}{2}}.
\end{aligned}$$

Equivalently, we can write

$$\begin{aligned}
&|||A\sharp B||| \leq |||A\sharp_{\frac{1+2\nu}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3-2\nu}{4}}B|||^{\frac{1}{2}} \\
&\leq \exp\left(\frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\log|||A\sharp_u B|||\ du\right) \\
&\leq |||A\sharp_{\frac{1}{2}}B|||^{\frac{1}{2}}|||A\sharp_{\nu}B|||^{\frac{1}{4}}|||A\sharp_{1-\nu}B|||^{\frac{1}{4}} \leq |||A\sharp_{\nu}B|||^{\frac{1}{2}}|||A\sharp_{1-\nu}B|||^{\frac{1}{2}}. \qquad (8)
\end{aligned}$$

On the other hand, for $\nu \in (\frac{1}{2}, 1]$, by inequalities (3), we have

$$\begin{aligned}
|||A\sharp B||| &\leq |||A\sharp_{\frac{1+2\nu}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3-2\nu}{4}}B|||^{\frac{1}{2}} \\
&\leq \exp\left(\frac{1}{2\nu-1}\int_{1-\nu}^{\nu}\log|||A\sharp_u B|||\ du\right) \\
&\leq |||A\sharp_{\frac{1}{2}}B|||^{\frac{1}{2}}|||A\sharp_{\nu}B|||^{\frac{1}{4}}|||A\sharp_{1-\nu}B|||^{\frac{1}{4}} \leq |||A\sharp_{\nu}B|||^{\frac{1}{2}}|||A\sharp_{1-\nu}B|||^{\frac{1}{2}}. \qquad (9)
\end{aligned}$$

In the last inequalities we used

$$\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\log|||A\sharp_u B|||\ du\right) = |||A\sharp B|||.$$

This completes the proof. □

In the special case $\nu = 0$, by the above result, one can obtain the following corollary.

Corollary 4.3. *Let $A, B, X \in B(H)$ be such that A, B are positive operators. Then we have*

$$\begin{aligned}
|||A\sharp B||| &\leq |||A\sharp_{\frac{1}{4}}B|||^{\frac{1}{2}}|||A\sharp_{\frac{3}{4}}B|||^{\frac{1}{2}} \leq \exp\left(\int_0^1\log|||A\sharp_u B|||\ du\right) \\
&\leq |||A\sharp B|||^{\frac{1}{2}}|||A|||^{\frac{1}{4}}|||B|||^{\frac{1}{4}} \leq |||A|||^{\frac{1}{2}}|||B|||^{\frac{1}{2}} \qquad (10) \\
&\leq \frac{1}{2}\big(|||A||| + |||B|||\big). \qquad (11)
\end{aligned}$$

From now on, we give some applications of AG-convex functions of two variables for unitarily invariant norms.

Lemma 4.4. [7] *Let $A, B, X \in B(H)$ such that A and B are positive. Then*

$$f(p,q) = |||A^p X B^q|||$$

is AG-convex for $t \in [0,1]$ and $p, q \in [0,\infty)$.

Applying inequalities of Theorem 3.4 to the function $f(p,q) = |||A^p X B^q|||$ on the interval $[\nu, 1-\nu] \times [\nu, 1-\nu]$ when $0 \leq \nu < \frac{1}{2}$ and on the interval $[1-\nu, \nu] \times [1-\nu, \nu]$ when $\frac{1}{2} < \nu \leq 1$, we obtain the following theorem.

Theorem 4.5. *Let $A, B, X \in B(H)$ such that A and B are positive. Then*

$$\begin{aligned}|||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| &\leq \exp\left(\frac{1}{2-4\nu}\int_\nu^{1-\nu} \log |||A^s X B^{\frac{1}{2}}|||ds \right.\\ &\qquad \left. + \frac{1}{2-4\nu}\int_\nu^{1-\nu} \log |||A^{\frac{1}{2}} X B^t|||dt\right)\\ &\leq \exp\left(\frac{1}{(1-2\nu)^2}\int_\nu^{1-\nu}\int_\nu^{1-\nu} \log |||A^s X B^t|||dsdt\right)\end{aligned}$$

and

$$\begin{aligned}|||A^{\frac{1}{2}} X B^{\frac{1}{2}}||| &\leq \exp\left(\frac{1}{4-8\nu}\left[\int_\nu^{1-\nu} \log |||A^s X B^\nu|||ds + \int_\nu^{1-\nu} \log |||A^s X B^{1-\nu}|||ds\right]\right.\\ &\qquad \left. + \frac{1}{4-8\nu}\left[\int_\nu^{1-\nu} \log |||A^\nu X B^t|||dt + \int_\nu^{1-\nu} \log |||A^{1-\nu} X B^t|||dt\right]\right)\\ &\leq \sqrt[4]{|||A^\nu X B^\nu||||||A^\nu X B^{1-\nu}||||||A^{1-\nu} X B^\nu||||||A^{1-\nu} X B^{1-\nu}|||}.\end{aligned}$$

Proof. For $0 \leq \nu < \frac{1}{2}$, by applying the inequalities of Theorem 3.4, we get

$$\begin{aligned}f\left(\frac{1}{2},\frac{1}{2}\right) &\leq \exp\left(\frac{1}{2(1-2\nu)}\int_\nu^{1-\nu} \log\circ f\left(s,\frac{1}{2}\right) ds \right.\\ &\qquad \left. + \frac{1}{2(1-2\nu)}\int_\nu^{1-\nu} \log\circ f\left(\frac{1}{2},t\right) dt\right)\\ &\leq \exp\left(\frac{1}{(1-2\nu)^2}\int_\nu^{1-\nu}\int_\nu^{1-\nu} \log\circ f(s,t) dsdt\right)\end{aligned}$$

and from here,

$$\begin{aligned}f\left(\frac{1}{2},\frac{1}{2}\right) &\leq \exp\left(\frac{1}{4(1-2\nu)}\left[\int_\nu^{1-\nu} \log\circ f(s,\nu) ds + \int_\nu^{1-\nu} \log\circ f(s,1-\nu) ds\right]\right.\\ &\qquad \left. + \frac{1}{4(1-2\nu)}\left[\int_\nu^{1-\nu} \log\circ f(\nu,t) dt + \int_\nu^{1-\nu} \log\circ f(1-\nu,t) dt\right]\right)\\ &\leq \sqrt[4]{f(\nu,\nu) f(\nu,1-\nu) f(1-\nu,\nu) f(1-\nu,1-\nu)}.\end{aligned}$$

Hence, by Lemma 4.4, we obtain

$$\begin{aligned}|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \exp\left(\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^{s}XB^{\frac{1}{2}}|||ds\right.\\ &\quad\left.+\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^{\frac{1}{2}}XB^{t}|||dt\right)\\ &\leq \exp\left(\frac{1}{(1-2\nu)^{2}}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||A^{s}XB^{t}|||dsdt\right)\end{aligned}$$

and

$$\begin{aligned}|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \exp\left(\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^{s}XB^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||A^{s}XB^{1-\nu}|||ds\right]\right.\\ &\quad\left.+\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^{\nu}XB^{t}|||dt+\int_{\nu}^{1-\nu}\log|||A^{1-\nu}XB^{t}|||dt\right]\right)\\ &\leq \sqrt[4]{|||A^{\nu}XB^{\nu}||||||A^{\nu}XB^{1-\nu}||||||A^{1-\nu}XB^{\nu}||||||A^{1-\nu}XB^{1-\nu}|||}.\end{aligned}$$

For $\frac{1}{2}<\nu\leq 1$, by making use of Theorem 3.4, we have

$$\begin{aligned}f\left(\frac{1}{2},\frac{1}{2}\right) &\leq \exp\left(\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log\circ f\left(s,\frac{1}{2}\right)ds\right.\\ &\quad\left.+\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log\circ f\left(\frac{1}{2},t\right)dt\right)\\ &\leq \exp\left(\frac{1}{(2\nu-1)^{2}}\int_{1-\nu}^{\nu}\int_{1-\nu}^{\nu}\log\circ f(s,t)dsdt\right)\end{aligned}$$

and

$$\begin{aligned}f\left(\frac{1}{2},\frac{1}{2}\right) &\leq \exp\left(\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log\circ f(s,1-\nu)ds+\int_{1-\nu}^{\nu}\log\circ f(s,\nu)ds\right]\right.\\ &\quad\left.+\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log\circ f(1-\nu,t)dt+\int_{1-\nu}^{\nu}\log\circ f(\nu,t)dt\right]\right)\\ &\leq \sqrt[4]{f(1-\nu,1-\nu)f(1-\nu,\nu)f(\nu,1-\nu)f(\nu,\nu)}.\end{aligned}$$

By Lemma 4.4, we have

$$\begin{aligned}|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| &\leq \exp\left(\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log|||A^{s}XB^{\frac{1}{2}}|||ds\right.\\ &\quad\left.+\frac{1}{2(2\nu-1)}\int_{1-\nu}^{\nu}\log|||A^{\frac{1}{2}}XB^{t}|||dt\right)\\ &\leq \exp\left(\frac{1}{(2\nu-1)^{2}}\int_{1-\nu}^{\nu}\int_{1-\nu}^{\nu}\log|||A^{s}XB^{t}|||dsdt\right)\end{aligned}$$

and

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}||| \leq \exp\left(\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log|||A^sXB^{1-\nu}|||ds+\int_{1-\nu}^{\nu}\log|||A^sXB^{\nu}|||ds\right]\right.$$
$$\left.+\frac{1}{4(2\nu-1)}\left[\int_{1-\nu}^{\nu}\log|||A^{1-\nu}XB^{t}|||dt+\int_{1-\nu}^{\nu}\log|||A^{\nu}XB^{t}|||dt\right]\right)$$
$$\leq \sqrt[4]{|||A^{1-\nu}XB^{1-\nu}||||||A^{1-\nu}XB^{\nu}||||||A^{\nu}XB^{1-\nu}||||||A^{\nu}XB^{\nu}|||}.$$

Note that in the last inequalities we used

$$\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^sXB^{\frac{1}{2}}|||ds+\frac{1}{2(1-2\nu)}\int_{\nu}^{1-\nu}\log|||A^{\frac{1}{2}}XB^{t}|||dt\right)$$
$$=\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{(1-2\nu)^2}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||A^sXB^{t}|||dsdt\right)$$
$$=\lim_{\nu\to\frac{1}{2}}\exp\left(\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^sXB^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||A^sXB^{1-\nu}|||ds\right]\right.$$
$$\left.+\frac{1}{4(1-2\nu)}\left[\int_{\nu}^{1-\nu}\log|||A^{\nu}XB^{t}|||dt+\int_{\nu}^{1-\nu}\log|||A^{1-\nu}XB^{t}|||dt\right]\right)$$
$$=|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|||.$$

This completes the proof. □

Corollary 4.6. *Let $A, B, X \in B(H)$. Then*

$$|||AXB^*|||^4 \leq \exp\left(\frac{2}{1-2\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^{\frac{1}{2}}|||ds\right.$$
$$\left.+\frac{2}{1-2\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{\frac{1}{2}}X(B^*B)^{t}|||dt\right)$$
$$\leq \exp\left(\frac{4}{(1-2\nu)^2}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^{t}|||dsdt\right)$$
$$\leq \exp\left(\frac{1}{1-2\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||(A^*A)^sX(B^*B)^{1-\nu}|||ds\right]\right.$$
$$\left.+\frac{1}{1-2\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^{\nu}X(B^*B)^{t}|||dt+\int_{\nu}^{1-\nu}\log|||(A^*A)^{1-\nu}X(B^*B)^{t}|||dt\right]\right)$$
$$\leq |||(A^*A)^{\nu}X(B^*B)^{\nu}||||||(A^*A)^{\nu}X(B^*B)^{1-\nu}|||$$
$$|||(A^*A)^{1-\nu}X(B^*B)^{\nu}||||||(A^*A)^{1-\nu}X(B^*B)^{1-\nu}|||.$$

Proof. From the inequality (3.4) of [9], we have

$$|||AXB^*|||^2 \leq |||(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}}||||||(A^*A)^{\frac{1}{2}}X(B^*B)^{\frac{1}{2}}||| \tag{12}$$

for all $A, B \in B(H)$. Now, applying Theorem 4.5, we get the desired result. □

Remark 4.7. Let $X = I$ in the above theorem. Then for two operators A and B in $B(H)$, we have

$$
\begin{aligned}
&|||(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}}||| \qquad (13)\\
&\leq \exp\left(\frac{1}{2-4\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{\frac{1}{2}}|||ds\right.\\
&\qquad\left.+\frac{1}{2-4\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{\frac{1}{2}}(B^*B)^{t}|||dt\right)\\
&\leq \exp\left(\frac{1}{(1-2\nu)^2}\int_{\nu}^{1-\nu}\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{t}|||dsdt\right)\\
&\leq \exp\left(\frac{1}{4-8\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{\nu}|||ds+\int_{\nu}^{1-\nu}\log|||(A^*A)^{s}(B^*B)^{1-\nu}|||ds\right]\right.\\
&\quad\left.+\frac{1}{4-8\nu}\left[\int_{\nu}^{1-\nu}\log|||(A^*A)^{\nu}(B^*B)^{t}|||dt+\int_{\nu}^{1-\nu}\log|||(A^*A)^{1-\nu}(B^*B)^{t}|||dt\right]\right)\\
&\leq \sqrt[4]{|||(A^*A)^{\nu}(B^*B)^{\nu}||||||(A^*A)^{\nu}(B^*B)^{1-\nu}|||}\\
&\qquad\times\sqrt[4]{|||(A^*A)^{1-\nu}(B^*B)^{\nu}||||||(A^*A)^{1-\nu}(B^*B)^{1-\nu}|||}. \qquad (14)
\end{aligned}
$$

From [5], we have

$$|||(A^*A)^{\nu}(B^*B)^{\nu}||| \leq |||A^*AB^*B|||^{\nu} \qquad (15)$$

and from [1] we have

$$|||(A^*A)^{\nu}(B^*B)^{1-\nu}||| \leq |||\nu A^*A+(1-\nu)B^*B||| \qquad (16)$$

for all A and B in $B(H)$ and $\nu\in[0,1]$. Now, applying (14), (15) and (16), we obtain

$$
\begin{aligned}
&\sqrt[4]{|||(A^*A)^{\nu}(B^*B)^{\nu}||||||(A^*A)^{\nu}(B^*B)^{1-\nu}|||}\\
&\qquad\times\sqrt[4]{|||(A^*A)^{1-\nu}(B^*B)^{\nu}||||||(A^*A)^{1-\nu}(B^*B)^{1-\nu}|||}\\
&\leq \sqrt[4]{|||A^*AB^*B||||||\nu A^*A+(1-\nu)B^*B||||||(1-\nu)A^*A+\nu B^*B|||}. \qquad (17)
\end{aligned}
$$

On the other hand, from the inequality (12), for $X = I$, we have

$$|||AB^*|||^2 \leq |||(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}}||||||(A^*A)^{\frac{1}{2}}(B^*B)^{\frac{1}{2}}||| \qquad (18)$$

for all $A, B\in B(H)$.

By putting (17) and (18) into (14), we have

$$
\begin{aligned}
|||AB^*|||^4 &\leq \exp\left(\frac{2}{1-2\nu}\int_\nu^{1-\nu}\log|||(A^*A)^s(B^*B)^{\frac{1}{2}}|||ds\right.\\
&\qquad\left.+\frac{2}{1-2\nu}\int_\nu^{1-\nu}\log|||(A^*A)^{\frac{1}{2}}(B^*B)^t|||dt\right)\\
&\leq \exp\left(\frac{4}{(1-2\nu)^2}\int_\nu^{1-\nu}\int_\nu^{1-\nu}\log|||(A^*A)^s(B^*B)^t|||dsdt\right)\\
&\leq \exp\left(\frac{1}{1-2\nu}\left[\int_\nu^{1-\nu}\log|||(A^*A)^s(B^*B)^\nu|||ds+\int_\nu^{1-\nu}\log|||(A^*A)^s(B^*B)^{1-\nu}|||ds\right]\right.\\
&\quad\left.+\frac{1}{1-2\nu}\left[\int_\nu^{1-\nu}\log|||(A^*A)^\nu(B^*B)^t|||dt+\int_\nu^{1-\nu}\log|||(A^*A)^{1-\nu}(B^*B)^t|||dt\right]\right)\\
&\leq |||(A^*A)^\nu(B^*B)^\nu||||||(A^*A)^\nu(B^*B)^{1-\nu}|||\\
&\qquad\times|||(A^*A)^{1-\nu}(B^*B)^\nu||||||(A^*A)^{1-\nu}(B^*B)^{1-\nu}|||\\
&\leq |||A^*AB^*B||||||\nu A^*A+(1-\nu)B^*B||||||(1-\nu)A^*A+\nu B^*B|||. \qquad (19)
\end{aligned}
$$

Applying (19) for $\nu=0$, we obtain the following inequalities.

Corollary 4.8. *Let $A,B\in B(H)$. Then*

$$
\begin{aligned}
&|||AB^*|||^2\\
&\leq \exp\left(\int_0^1\log|||(A^*A)^s(B^*B)^{\frac{1}{2}}|||ds+\int_0^1\log|||(A^*A)^{\frac{1}{2}}(B^*B)^t|||dt\right)\\
&\leq \exp\left(2\int_0^1\int_0^1\log|||(A^*A)^s(B^*B)^t|||dsdt\right)\\
&\leq \exp\left(\frac{1}{2}\int_0^1\log|||(A^*A)^s|||ds+\frac{1}{2}\int_0^1\log|||(A^*A)^sB^*B|||ds\right.\\
&\qquad\left.+\frac{1}{2}\int_0^1\log|||(B^*B)^t|||dt+\frac{1}{2}\int_0^1\log|||A^*A(B^*B)^t|||dt\right)\\
&\leq \sqrt{|||A^*A||||||B^*B||||||A^*AB^*B|||}\\
&\leq |||A^*A||||||B^*B|||.
\end{aligned}
$$

We should note here that the above corollary is a refinement for the Cauchy-Schwarz inequality.

Applying inequalities of Theorem 3.4 to the function

$$f(p,q)=|||A^pXB^q|||$$

on the interval $[0,\nu]\times[0,\nu]$ when $0<\nu\leq\frac{1}{2}$ and on the interval $[\nu,1]\times[\nu,1]$ when $\frac{1}{2}\leq\nu<1$, we obtain the following theorem by a similar proof of Theorem 4.5.

Theorem 4.9. *Let $A, B, X \in B(H)$ be such that A and B are positive. Then*

(1) *For $\nu \in (0, \frac{1}{2}]$, we have*

$$\begin{aligned}
|||A^{\frac{\nu}{2}} X B^{\frac{\nu}{2}}||| &\leq \exp\left(\frac{1}{2\nu}\int_0^\nu \log|||A^s X B^{\frac{\nu}{2}}|||ds \ + \frac{1}{2\nu}\int_0^\nu \log|||A^{\frac{\nu}{2}} X B^t|||dt\right)\\
&\leq \exp\left(\frac{1}{\nu^2}\int_0^\nu\int_0^\nu \log|||A^s X B^t|||dsdt\right)\\
&\leq \exp\left(\frac{1}{4\nu}\left[\int_0^\nu \log|||A^s X|||ds + \int_0^\nu \log|||A^s X B^\nu|||ds\right]\right.\\
&\qquad \left. + \frac{1}{4\nu}\left[\int_0^\nu \log|||X B^t|||dt + \int_0^\nu \log|||A^\nu X B^t|||dt\right]\right)\\
&\leq \sqrt[4]{|||X||||||XB^\nu||||||A^\nu X||||||A^\nu X B^\nu|||}.
\end{aligned}$$

(2) *Also, for $\nu \in [\frac{1}{2}, 1)$, we have*

$$\begin{aligned}
|||A^{\frac{\nu+1}{2}} X B^{\frac{\nu+1}{2}}||| &\leq \exp\left(\frac{1}{2(1-\nu)}\int_\nu^1 \log|||A^s X B^{\frac{\nu+1}{2}}|||ds\right.\\
&\qquad \left. + \frac{1}{2(1-\nu)}\int_\nu^1 \log|||A^{\frac{\nu+1}{2}} X B^t|||dt\right)\\
&\leq \exp\left(\frac{1}{(1-\nu)^2}\int_\nu^1\int_\nu^1 \log|||A^s X B^t|||dsdt\right)\\
&\leq \exp\left(\frac{1}{4(1-\nu)}\left[\int_\nu^1 \log|||A^s X B^\nu|||ds + \int_\nu^1 \log|||A^s X B|||ds\right]\right.\\
&\qquad \left. + \frac{1}{4(1-\nu)}\left[\int_\nu^1 \log|||A^\nu X B^t|||dt + \int_\nu^1 \log|||AXB^t|||dt\right]\right)\\
&\leq \sqrt[4]{|||A^\nu X B^\nu||||||A^\nu XB||||||AXB^\nu||||||AXB|||}.
\end{aligned}$$

By considering the above theorem on $[0,1]\times[0,1]$ and the inequality (12) for all $A, B, X \in B(H)$, we have the following result.

Corollary 4.10. *Let $A, B, X \in B(H)$. Then*

$$\begin{aligned}
|||AXB^*|||^4 &\leq \exp\left(2\int_0^1 \log|||(A^*A)^s X (B^*B)^{\frac{1}{2}}|||ds\right.\\
&\qquad \left. + 2\int_0^1 \log|||(A^*A)^{\frac{1}{2}} X (B^*B)^t|||dt\right)\\
&\leq \exp\left(4\int_0^1\int_0^1 \log|||(A^*A)^s X (B^*B)^t|||dsdt\right)\\
&\leq \exp\left(\int_0^1 \log|||(A^*A)^s X|||ds + \int_0^1 \log|||(A^*A)^s X (B^*B)|||ds\right.\\
&\qquad \left. + \int_0^1 \log|||X (B^*B)^t|||dt + \int_0^1 \log|||(A^*A) X (B^*B)^t|||dt\right)\\
&\leq |||X||||||(A^*A)X||||||X(B^*B)||||||(A^*A)X(B^*B)|||.
\end{aligned}$$

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