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Ohlin Type Theorem and its Applications for Strongly Convex Set-Valued Maps

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A version of the Ohlin theorem for strongly convex set-valued maps is presented. As an application counterparts of the Jensen, converse Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps are proved in a simple unified way.

Keywords: Ohlin's lemma, strongly convex set-valued maps, Jensen inequality, converse Jensen inequality, Hermite-Hadamard inequalities.

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1. Introduction

Over fifty years ago J. Ohlin [10] proved the following result on convex stochastic ordering ($\mathbb{E}[X]$ denotes the expectation of the random variable X):

Lemma 1.1. [10] *Let X_1, X_2 be two real valued random variables such that $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If there exists a number $t_0 \in \mathbb{R}$ such that the distribution functions F_{X_1}, F_{X_2} satisfy*

$$F_{X_1}(t) \leq F_{X_2}(t) \quad for\ t < t_0 \quad and \quad F_{X_1}(t) \geq F_{X_2}(t) \quad for\ t > t_0, \tag{1}$$

then $\mathbb{E}[f(X_1)] \leq \mathbb{E}[f(X_2)]$, *for every convex function $f : \mathbb{R} \to \mathbb{R}$.*

This Ohlin's lemma plays a fundamental role in the theory of optimal reinsurance structures. However, it is also a very useful tool in the theory of functional inequalities. In [15], T. Rajba used the Ohlin lemma to get a very simple proof of the Hermite-Hadamard inequalities, as well as to obtain some new inequalities related to convex functions (see also [7, 11, 16, 18] for other results of this type). Recently a version of Ohlin's lemma for strongly convex functions was proved in [7] and for convex set-valued maps in [8].

In this note we prove a counterpart of the Ohlin lemma for strongly convex set-valued maps. As an application of this result we present a simple and unified way to obtain Jensen, converse Jensen and Hermite-Hadamard type inclusions for strongly convex set-valued maps.

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2. Preliminaries

In the whole paper we assume that $(Y, \|\cdot\|)$ is a separable Banach space, B is the closed unit ball in Y and $I \subset \mathbb{R}$ is an open interval. By $n(Y)$ we will denote the family of all nonempty subsets of Y and by $cl(Y)$ the family of all closed nonempty subsets of Y. Let $(\Omega, \mathcal{A}, P)$ be a probability space with a nonatomic measure P. For a given set-valued map $G : \Omega \to n(Y)$ the integral $\int_\Omega G(\omega)dP$ is understood in the sense of Aumann, i.e. it is the set of integrals of all integrable (in the sense of Bochner) selections of the map G (cf. [1], [2]). A set-valued map $G : \Omega \to n(Y)$ is called *integrable bounded* if there exists a nonnegative integrable function $k : \Omega \to \mathbb{R}$ such that $G(\omega) \subset k(\omega)B$, for all $\omega \in \Omega$. In this case every measurable selection of G is integrable and, consequently, the Aumann integral of G is nonempty whenever G is measurable (i.e. the inverse image $\{\omega \in \Omega : G(\omega) \cap U \neq \emptyset\} \in \mathcal{A}$ for every open set $U \subset Y$).

The following properties of the Aumann integral will be needed in our investigations (the assumption that Y is separable is needed here):

Lemma 2.1. [1, Theorems 8.6.3, 8.6.4]
Let $G : \Omega \to cl(Y)$ be a measurable set-valued map.

(a) *The closure of the integral of G is convex and*

$$\overline{\int_\Omega G(\omega)dP} = \overline{conv}\Big(\int_\Omega G(\omega)dP\Big).$$

(b) *If Y is finite dimensional then the integral of G is convex. In particular, if $Y = \mathbb{R}$ and G is of the form $G(\omega) = [g_1(\omega), g_2(\omega)]$, $\omega \in \Omega$, where $g_1, g_2 : \Omega \to \mathbb{R}$ are integrable and $g_1 \leq g_2$, then*

$$\int_\Omega G(\omega)dP = \Big[\int_\Omega g_1(\omega)dP, \int_\Omega g_2(\omega)dP\Big].$$

(c) *If G is integrable bounded, then*

$$\overline{\int_\Omega G(\omega)dP} = \int_\Omega \overline{conv} G(\omega)dP.$$

(d) *If G is integrable bounded with convex values and the space Y is reflexive, then the integral of G is closed.*

3. Ohlin type theorem

Let $I \subset \mathbb{R}$ be an interval and c be a positive number. A function $f : I \to \mathbb{R}$ is called *strongly convex with modulus c* (cf. [13]) if

$$f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2 \tag{2}$$

for all $x_1, x_2 \in I$ and $t \in [0, 1]$. f is called strongly concave with modulus c if $-f$ is strongly convex with modulus c. Many properties and applications of strongly convex functions can be found in the literature (see, for instance, [6, 12, 13] and the references therein). Recently the following counterpart of Ohlin's lemma was proved in [7] (as usual, $\mathbb{D}^2[X]$ denotes the variance of X):

Lemma 3.1. [7] *Let $X_1, X_2 : \Omega \to I$ be square integrable random variables such that $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If there exists a $t_0 \in \mathbb{R}$ such that condition* (1) *holds, then*

$$\mathbb{E}[f(X_1)] - c\mathbb{D}^2[X_1] \le \mathbb{E}[f(X_2)] - c\mathbb{D}^2[X_2] \tag{3}$$

for every function $f : I \to \mathbb{R}$ strongly convex with modulus c.

In [3] Huang extended the definition (2) of strongly convex function to set-valued maps. He used such maps to investigate error bounds for some inclusion problems with set constraints. Some further properties of strongly convex set-valued maps can be found in [4, 5, 9].

Let B be a closed unit ball in Y and c be a positive constant. A set-valued map $G : I \to n(Y)$ is called *strongly convex with modulus c* if

$$tG(x_1) + (1-t)G(x_2) + ct(1-t)(x_1 - x_2)^2 B \subset G(tx_1 + (1-t)x_2) \tag{4}$$

for all $x_1, x_2 \in I$ and $t \in [0,1]$. Clearly, the usual notion of convex set-valued maps corresponds to relation (4) with $c = 0$.

The following lemma characterizes strongly convex integrable bounded set-valued maps with values in $cl(\mathbb{R})$.

Lemma 3.2. ([9, Lem. 2.1]) *An integrable bounded set-valued map $G : I \to cl(\mathbb{R})$ is strongly convex with modulus $c > 0$ if and only if it is of the form*

$$G(x) = [g_1(x), g_2(x)], \; x \in I,$$

where $g_1 : I \to \mathbb{R}$ is strongly convex with modulus $c > 0$ and $g_2 : I \to \mathbb{R}$ is strongly concave with modulus $c > 0$.

We will need also the following two lemmas.

Lemma 3.3. [5, Thm 12] *If a set-valued map $G : I \to cl(Y)$ is strongly convex with modulus c, then the set-valued map F defined by $F(x) = G(x) + cx^2 B$, $x \in I$, is convex.*

Lemma 3.4. [8, Thm 4] *Let $(\Omega, \mathcal{A}, P)$ be a probability space with a nonatomic measure P and $X_1, X_2 : \Omega \to I$ be integrable random variables with $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If there exists a $t_0 \in \mathbb{R}$ such that* (1) *holds then*

$$\int_\Omega G\big(X_2(\omega)\big)dP \subset \int_\Omega G\big(X_1(\omega)\big)dP. \tag{5}$$

for every convex integrable bounded set-valued map $G : I \to cl(Y)$.

Let us recall also the classical Rådström cancelation law:

Lemma 3.5. [14] *Let A, B, C be subsets of Y such that $A + C \subset B + C$. If B is closed convex and C is bounded and nonempty, then $A \subset B$.*

The main result of this paper is the following counterpart the Ohlin lemma for strongly convex set-valued maps.

Theorem 3.6. *Let $(Y, \|\cdot\|)$ be a separable Banach space and $(\Omega, \mathcal{A}, P)$ be a probability space with a nonatomic measure P. Assume that $I \subset \mathbb{R}$ is an open interval and $X_1, X_2 : \Omega \to I$ are square integrable random variables such that $\mathbb{E}[X_1] = \mathbb{E}[X_2]$. If for some $t_0 \in \mathbb{R}$ condition* (1) *holds, then*

$$\int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)B \subset \int_\Omega G\big(X_1(\omega)\big)dP. \tag{6}$$

for every integrable bounded set-valued map $G : I \to cl(Y)$ strongly convex with modulus c.

We will present two different proofs of this theorem. The first one is direct and is similar to the proof of Lemma 3.4 given in [8]. The second one is based on the representation of strongly convex set-valued maps proved in [5]. However, in the second proof we need the additional assumption that the space Y is reflexive.

Proof. (1) The proof is divided into two steps. First, assume that $Y = \mathbb{R}$. Then, by Lemma 3.2, G is of the form $G(x) = [g_1(x), g_2(x)]$, $x \in I$, where $g_1 : I \to \mathbb{R}$ is strongly convex with modulus c and $g_2 : I \to \mathbb{R}$ is strongly concave with modulus c. By Lemma 3.1 we have

$$\mathbb{E}[g_1(X_1)] + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big) \le \mathbb{E}[g_1(X_2)]$$

and

$$\mathbb{E}[g_2(X_1)] \ge \mathbb{E}[g_2(X_2)] + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big).$$

Hence, using Lemma 2.1(b) we get

$$\begin{aligned}
\int_\Omega G\big(X_1(\omega)\big)dP &= \big[\mathbb{E}[g_1(X_1)], \mathbb{E}[g_2(X_1)]\big] \\
&\supset \big[\mathbb{E}[g_1(X_2)] - c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big), \mathbb{E}[g_2(X_2)] + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)\big] \\
&= \int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)[-1, 1]
\end{aligned}$$

which proves (6) with $B = [-1, 1] \subset \mathbb{R}$.

Now, assume that Y is an arbitrary separable Banach space. Take a nonzero continuous linear functional $y^* \in Y^*$ and consider the set-valued map $x \mapsto \overline{y^*(G(x))}$, $x \in I$. This set-valued map is strongly convex with modulus $c\|y^*\|$ and has closed values in $\mathbb{R}$. Therefore, by the previous step,

$$\int_\Omega \overline{y^*\Big(G\big(X_2(\omega)\big)\Big)}dP + c\|y^*\|\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)[-1, 1] \subset \int_\Omega \overline{y^*\Big(G\big(X_1(\omega)\big)\Big)}dP. \tag{7}$$

Fix arbitrary $b \in B$ and $z \in \int_\Omega G\big(X_2(\omega)\big)dP$. There exists an integrable selection $g \circ X_2$ of $G \circ X_2$ such that $z = \int_\Omega g\big(X_2(\omega)\big)dP$. Using (7) we obtain

$$\begin{aligned}
y^*\Big(z + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)b\Big) &= y^*\Big(\int_\Omega g\big(X_2(\omega)\big)dP\Big) + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)y^*(b) \\
&\in \int_\Omega \overline{y^*\Big(G\big(X_2(\omega)\big)\Big)}dP + c\|y^*\|\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)[-1, 1] \subset \int_\Omega \overline{y^*\Big(G\big(X_1(\omega)\big)\Big)}dP.
\end{aligned} \tag{8}$$

Since G is integrable bounded and the values $y^*\Big(G\big(X_1(\omega)\big)\Big)$ are convex , by Lemma 2.1(c) we have

$$\int_\Omega \overline{y^*\Big(G\big(X_1(\omega)\big)\Big)}dP = \int_\Omega \overline{conv}\ y^*\Big(G\big(X_1(\omega)\big)\Big)dP = \overline{\int_\Omega y^*\Big(G\big(X_1(\omega)\big)dP\Big)} \tag{9}$$
$$= \overline{y^*\Big(\int_\Omega G\big(X_1(\omega)\big)dP\Big)} \subset \overline{y^*\Big(\overline{\int_\Omega G\big(X_1(\omega)\big)dP}\Big)}.$$

From (8) and (9) we get

$$y^*\Big(z + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)b\Big) \in \overline{y^*\Big(\overline{\int_\Omega G\big(X_1(\omega)\big)dP}\Big)}.$$

Since this condition holds for arbitrary $y^* \in Y^*$ and, by Lemma 2.1(a) the set $\overline{\int_\Omega G\big(X_1(\omega)\big)dP}$ is convex and closed, by the separation theorem (see [17], Corollary 2.5.11) we obtain

$$z + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)b \in \overline{\int_\Omega G\big(X_1(\omega)\big)dP},$$

and hence, using once more Lemma 2.1(c),

$$z + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)b \in \int_\Omega G\big(X_1(\omega)\big)dP.$$

This shows that

$$\int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)B \subset \int_\Omega G\big(X_1(\omega)\big)dP,$$

and finishes the first part of the proof.

(2) By Lemma 3.3 the set-valued map F defined by $F(x) = G(x) + cx^2B,\ x \in I$, is convex. Therefore, from Lemma 3.4, we have

$$\int_\Omega F\big(X_2(\omega)\big)dP \subset \int_\Omega F\big(X_1(\omega)\big)dP, \quad \text{or}$$
$$\int_\Omega G\big(X_2(\omega)\big)dP + \int_\Omega cX_2^2(\omega)BdP \subset \int_\Omega G\big(X_1(\omega)\big)dP + \int_\Omega cX_1^2(\omega)BdP. \tag{10}$$

Note that $\int_\Omega cX_i^2(\omega)BdP = c\mathbb{E}[X_i^2]B,\ i = 1, 2$. Note also that from Lemma 1.1 applied for $f(x) = x^2$, we have $\mathbb{E}[X_1^2] \le \mathbb{E}[X_2^2]$ and hence, by the convexity of B,

$$\mathbb{E}[X_2^2]B = \big(\mathbb{E}[X_2^2] - \mathbb{E}[X_1^2]\big)B + \mathbb{E}[X_1^2]B.$$

Consequently, we can rewrite (10) in the form

$$\int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{E}[X_2^2] - \mathbb{E}[X_1^2]\big)B + c\mathbb{E}[X_1^2]B \subset \int_\Omega G\big(X_1(\omega)\big)dP + c\mathbb{E}[X_1^2]B.$$

Since the set $c\mathbb{E}[X_1^2]B$ is bounded and, by Lemma 2.1(a) and Lemma 2.1(d), the set $\int_\Omega G\big(X_1(\omega)\big)dP$ is convex and closed (here we need the reflexivity of Y), using the Rådström cancelation law (Lemma 3.5) we get

$$\int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{E}[X_2^2] - \mathbb{E}[X_1^2]\big)B \subset \int_\Omega G\big(X_1(\omega)\big)dP.$$

Since $\mathbb{E}[X_1] = \mathbb{E}[X_2]$, the last condition is equivalent to

$$\int_\Omega G\big(X_2(\omega)\big)dP + c\big(\mathbb{D}^2[X_2] - \mathbb{D}^2[X_1]\big)B \subset \int_\Omega G\big(X_1(\omega)\big)dP,$$

and the proof is finished. □

4. Consequences and applications

In this section we give examples of possible applications of the above Ohlin type theorem. The results presented here are mostly known, but they are proved in a new, unified and short way.

We start with a counterpart of the classical Jensen inequality.

Theorem 4.1. (cf. [9]) *Let $(\Omega, \mathcal{A}, P)$ be a probability space with a nonatomic measure P. If $G : I \to cl(Y)$ is strongly convex with modulus c and integrable bounded, then*

$$\int_\Omega G\big(X(\omega)\big)dP + c\mathbb{D}^2[X]B \subset G\Big(\int_\Omega X(\omega)dP\Big), \tag{11}$$

for every square integrable random variable $X : \Omega \to I$.

Proof. Take a random variable $X_1 : \Omega \to I$ with the distribution $\mu_{X_1} = \delta_{\mathbb{E}[X]}$. Then the distribution functions F_X, F_{X_1} satisfy condition (1), $\mathbb{E}[X] = \mathbb{E}[X_1]$ and $\mathbb{D}^2[X_1] = 0$. Moreover, $\int_\Omega G\big(X_1(\omega)\big)dP = G(\mathbb{E}[X])$. Therefore, by Theorem 3.6,

$$\int_\Omega G\big(X(\omega)\big)dP + c\mathbb{D}^2[X]B \subset \int_\Omega G\big(X_1(\omega)\big)dP = G(\mathbb{E}[X]) = G\Big(\int_\Omega X(\omega)dP\Big)$$

which finishes the proof. □

Now assume that the random variable X in the above theorem has the distribution $\mu_X = p_1\delta_{x_1} + \cdots + p_n\delta_{x_n}$, where $x_1, \ldots, x_n \in I$ and $p_1, \ldots, p_n > 0$ are such that $p_1 + \cdots + p_n = 1$. Then

$$\bar{x} := \mathbb{E}[X]) = \sum_{i=1}^n p_i x_i, \qquad \mathbb{D}^2[X] = \sum_{i=1}^n p_i(x_i - \bar{x})^2$$

and
$$\int_\Omega G\big(X(\omega)\big)dP = \sum_{i=1}^n p_i G(x_i).$$

Therefore, as a consequence of Theorem 4.1, we obtain a counterpart of the discrete Jensen inequality.

Corollary 4.2. (cf. [9]) *If a set-valued map $G : I \to cl(Y)$ is strongly convex with modulus c and integrable bounded, then*

$$\sum_{i=1}^n p_i G(x_i) + c\sum_{i=1}^n p_i(x_i - m)^2 B \subset G\Big(\sum_{i=1}^n p_i x_i\Big),$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in I$ and $p_1, \ldots, p_n > 0$ with $p_1 + \cdots + p_n = 1$.

We have also the following converse Jensen inclusion for strongly convex set-valued maps.

Theorem 4.3. (cf. [4]) *Let $m, M \in I$, $m < M$. If $G : I \to cl(Y)$ is strongly convex with modulus c and integrable bounded, then*

$$\frac{M-\bar{x}}{M-m}G(m)+\frac{\bar{x}-m}{M-m}G(M)+c\sum_{i=1}^{n}p_i(M-x_i)(x_i-m)B\subset\sum_{i=1}^{n}p_iG(x_i), \quad (12)$$

for all $x_1,\dots,x_n \in [m, M]$, $p_1,\dots,p_n > 0$ with the property $p_1+\cdots+p_n = 1$, and $\bar{x} = p_1x_1+\cdots+p_nx_n$.

Proof. Assume that $(\Omega, \mathcal{A}, P)$ is a probability space with a nonatomic measure P and take random variables $X_1, X_2 : \Omega \to I$ with the distributions

$$\mu_{X_1}=\sum_{i=1}^{n}p_i\delta_{x_i} \quad\text{and}\quad \mu_{X_2}=\frac{M-\bar{x}}{M-m}\delta_m+\frac{\bar{x}-m}{M-m}\delta_M.$$

Then the distribution functions F_{X_1}, F_{X_2} satisfy condition (1) and

$$\mathbb{E}[X_1]=\sum_{i=1}^{n}p_ix_i=\bar{x}=\frac{M-\bar{x}}{M-m}m+\frac{\bar{x}-m}{M-m}M=\mathbb{E}[X_2],$$

$$\begin{aligned}\mathbb{D}^2[X_2]-\mathbb{D}^2[X_1]=\mathbb{E}[X_2^2]-\mathbb{E}[X_1^2]&=\frac{M-\bar{x}}{M-m}m^2+\frac{\bar{x}-m}{M-m}M^2-\sum_{i=1}^{n}p_ix_i^2\\&=\sum_{i=1}^{n}p_i(M-x_i)(x_i-m).\end{aligned}$$

We have also
$$\int_{\Omega}G\big(X_1(\omega)\big)dP=\sum_{i=1}^{n}p_iG(x_i),$$
and
$$\int_{\Omega}G\big(X_2(\omega)\big)dP=\frac{M-\bar{x}}{M-m}G(m)+\frac{\bar{x}-m}{M-m}G(M).$$

Therefore, by Theorem 3.6, we obtain (12). □

The next result is a counterpart of the classical Hermite-Hadamard inequality for strongly convex set-valued maps.

Theorem 4.4. (cf. [9]) *If a set-valued map $G : I \to cl(Y)$ is strongly convex with modulus c and integrable bounded, then we have for all $a, b \in I$, $a < b$,*

$$\frac{1}{b-a}\int_a^b G(x)dx+\frac{c}{12}(a-b)^2B\subset G\left(\frac{a+b}{2}\right) \quad (13)$$

and

$$\frac{G(a)+G(b)}{2}+\frac{c}{6}(a-b)^2B\subset\frac{1}{b-a}\int_a^b G(x)dx. \quad (14)$$

Proof. Assume that $(\Omega, \mathcal{A}, P)$ is a probability space with a nonatomic measure P. Let $X_1, X_2 : \Omega \to I$ be random variables with the distributions $\mu_{X_1} = \delta_{(a+b)/2}$, $\mu_{X_2} = \frac{1}{2}(\delta_a + \delta_b)$ and let $X_3 : \Omega \to I$ has the uniform distribution on $[a, b]$. Then the pairs X_1, X_3 and X_3, X_2 satisfy the assumptions of Theorem 3.6.

We have

$$\int_\Omega G\big(X_1(\omega)\big)dP = G\Big(\frac{a+b}{2}\Big), \quad \int_\Omega G\big(X_2(\omega)\big)dP = \frac{G(a)+G(b)}{2},$$

and

$$\int_\Omega G\big(X_3(\omega)\big)dP = \frac{1}{b-a}\int_a^b G(x)dx.$$

Moreover,

$$\mathbb{D}^2[X_1] = 0, \quad \mathbb{D}^2[X_2] = \frac{(b-a)^2}{4},$$

and

$$\mathbb{D}^2[X_3] = \frac{1}{b-a}\int_a^b x^2 dx - \Big(\frac{1}{b-a}\int_a^b x dx\Big)^2 = \frac{(b-a)^2}{12}.$$

Hence, using Theorem 3.6, we obtain (13) and (14). □

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