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G-Majorization and Fejér and Hermite-Hadamard Like Inequalities for G-Symmetrized Convex Functions

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For a given finite group G acting on an inner product space V, we introduce G-symmetrized convex functions and G-symmetrized increasing functions. We use them to establish some inequalities of Fejér and Hermite-Hadamard types. In particular, we apply Wright convexity. We interpret the obtained results for the group $G = \mathbb{C}_n$ of sign changes on $V = \mathbb{R}^n$. The case $n = 1$ leads to some recent results by S. S. Dragomir [*Symmetrized convexity and Hermite-Hadamard type inequalities*, J. Math. Inequalities 10/4 (2016) 901–918] and S. Abramovich and L.-E. Persson [*Some new Hermite-Hadamard and Fejer type inequalities without convexity/concavity*, Math. Inequalities Appl. 23/2 (2020) 447–458].

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1. Introduction

The following two results are well-known (see [1, 3], cf. [1, 2, 4, 8, 10, 13, 14, 16, 17, 18]).

Theorem A. [3, p. 137] *If $f : I \to \mathbb{R}$ is a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$, then the following Hermite-Hadamard inequality holds:*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Theorem B. [1] *Let $f : I \to \mathbb{R}$ be a convex function on an interval $I \subset \mathbb{R}$, $a, b \in I$ with $a < b$, and let $\varphi : [a, b] \to \mathbb{R}$ be a non-negative integrable weight on $[a, b]$. Assume that φ is symmetric about $\frac{a+b}{2}$. Then the following Fejér inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b \varphi(x)\,dx \le \int_a^b f(x)\varphi(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b \varphi(x)\,dx. \tag{2}$$

In this paper our aim is to establish generalization of the above two theorems by making use of majorization theory.

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To do so, in the rest of this expository section, we present the notion of majorization on $\mathbb{R}^n$ and its properties. Next, we extend this framework and demonstrate the notion of G-majorization generated by a group G of linear operators acting on a linear space V.

In Section 2 we apply G-majorization techniques to derive some inequalities of Fejér and Hermite-Hadamard types. In Section 3 we deal with Wright convex functions and show their relations to such inequalities. Section 4 is devoted to applications and interpretations of the obtained results.

Throughout $I \subset \mathbb{R}$ is an interval.

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in I^n$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n) \in I^n$. We say that $\mathbf{y}$ is *majorized* by $\mathbf{x}$, written as $\mathbf{y} \prec \mathbf{x}$, if

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i \quad \text{and} \quad \sum_{i=1}^{k} y_{[i]} \leq \sum_{i=1}^{k} x_{[i]} \quad \text{for all } k = 1, 2, \ldots, n \tag{3}$$

(see [9, p. 8], [7]). Hereafter the symbols $x_{[i]}$ and $y_{[i]}$ stand for the ith largest entry of $\mathbf{x}$ and $\mathbf{y}$, respectively, for $i = 1, \ldots, n$.

It is known that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\mathbf{y} \prec \mathbf{x} \quad \text{iff} \quad \mathbf{y} \in \operatorname{conv} \mathbb{P}_n \mathbf{x}, \tag{4}$$

where $\mathbb{P}_n$ is the group of $n \times n$ permutation matrices, $\mathbb{P}_n \mathbf{x} = \{p\mathbf{x} : p \in \mathbb{P}_n\}$, and $\operatorname{conv}(\cdot)$ means the convex hull of $(\cdot)$ (see [9, p. 10]).

A function $F : I^n \to \mathbb{R}$ is said to be *Schur-convex* on I^n (see [9, p. 80]) if

$$\mathbf{y} \prec \mathbf{x} \quad \text{implies} \quad F(\mathbf{y}) \leq F(\mathbf{x}) \quad \text{for } \mathbf{x}, \mathbf{y} \in I^n.$$

In the sequel, unless otherwise stated, $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional real inner product space and G is a closed subgroup of the orthogonal group $O(V)$ acting on V. The group G induces the preorder $\prec_G$ on V, called *G-majorization*, as follows. For $x, y \in V$,

$$y \prec_G x \quad \text{iff} \quad y \in \operatorname{conv} Gx,$$

where $C_G(x) = \operatorname{conv} Gx$ stands for the convex hull of the set $Gx = \{gx : g \in G\}$ [6, 15, 19].

We introduce the subspace

$$M_G(V) = \{v \in V : gv = v \text{ for all } g \in G\}.$$

It is known that $M_G(V)$ is the set of minimal points in $\prec_G$ [6, 19].

For a given finite group $G = \{g_1, \ldots, g_m\}$, we define

$$\bar{x} = \frac{1}{m} \sum_{i=1}^{m} g_i x \quad \text{for } x \in V.$$

It is clear that $\bar{x} \in C_G(x) \cap M_G(V)$ for each $x \in V$.

Let C be a G-invariant convex subset of V. A function $f : C \to \mathbb{R}$ is said to be *G-increasing* on C (see [5]) if for $x, y \in C$,

$$y \prec_G x \quad \text{implies} \quad f(y) \le f(x).$$

It follows that if f is convex on C and f is G-invariant on C, then f is G-increasing on C.

2. G-majorization approach to Fejér and Hermite-Hadamard like inequalities

Throughout this section we assume that

(I) V is a finite dimensional inner product space,

(II) G is a closed subgroup of the orthogonal group $O(V)$ acting on V,

(III) for a vector $a \in V$, the symbol $C_G(a)$ stands for the G-invariant convex set $\operatorname{conv} Ga$,

(IV) μ is a G-invariant measure on the σ-field of Borel subsets of V,

(V) there exist all integrals under consideration.

For a vector $a \in V$ and a function $f : C_G(a) \to \mathbb{R}$, we introduce

$$\hat{f}(x) = \frac{1}{m} \sum_{i=1}^{m} f(g_i x) \quad \text{for } x \in C_G(a), \tag{5}$$

whenever $G = \{g_1, \ldots, g_m\}$ is a finite group.

Lemma 2.1. *Let $a \in V$. Let $f : C_G(a) \to \mathbb{R}$ be an integrable function and let $\varphi : C_G(a) \to \mathbb{R}$ be an integrable G-invariant function.*

(i) *Then* $$\int_{C_G(a)} f(x)\varphi(x)\, d\mu(x) = \int_{C_G(a)} f(gx)\varphi(x)\, d\mu(x) \quad \text{for all } g \in G. \tag{6}$$

(ii) *If $G = \{g_1, \ldots, g_m\}$ is finite, then*

$$\int_{C_G(a)} \hat{f}(x)\varphi(x)\, d\mu(x) = \int_{C_G(a)} f(x)\varphi(x)\, d\mu(x). \tag{7}$$

Proof. (i) The assertion follows, because φ, $C_G(a)$ and μ are G-invariant.

(ii) By (5) and (6) we have

$$\int_{C_G(a)} \hat{f}(x)\varphi(x)\, d\mu(x) = \int_{C_G(a)} \frac{1}{m} \sum_{i=1}^{m} f(g_i x)\varphi(x)\, d\mu(x)$$

$$= \frac{1}{m} \sum_{i=1}^{m} \int_{C_G(a)} f(g_i x)\varphi(x)\, d\mu(x) = \frac{1}{m} \sum_{i=1}^{m} \int_{C_G(a)} f(x)\varphi(x)\, d\mu(x)$$

$$= \int_{C_G(a)} f(x)\varphi(x)\, d\mu(x). \qquad \square$$

We say that a function $f : C \to \mathbb{R}$ is *G-symmetrized convex* on a G-invariant convex set $C \subset V$, if the function

$$\hat{f}(x) = \frac{f(g_1 x) + \ldots + f(g_m x)}{m}, \quad x \in C, \tag{8}$$

is convex on C, where $G = \{g_1, \ldots, g_m\}$ is a finite group

We say that a function $f : C \to \mathbb{R}$ is *G-symmetrized increasing monotone* on a G-invariant convex set $C \subset V$, if function (8) is G-increasing on C.

Some inequalities of Fejér and Hermite-Hadamard type have been studied in the context of symmetrized convexity on a real interval (see [2, 4]). We develop this idea to G-symmetrized increasing monotonicity on some subsets of an inner product space V endowed with an action of a finite group $G \subset O(V)$.

Theorem 2.2. (Fejér type inequality) *Let $G = \{g_1, \ldots, g_m\}$ be a finite group and $a \in V$. Let $f : C_G(a) \to \mathbb{R}$ be an integrable function and $\varphi : C_G(a) \to \mathbb{R}$ be an integrable nonnegative G-invariant function.*

(i) *If f is G-symmetrized increasing monotone on $C_G(a)$, then*

$$f(\bar{a}) \int\limits_{C_G(a)} \varphi(x)\, d\mu(x) \le \int\limits_{C_G(a)} f(x)\varphi(x)\, d\mu(x)$$

$$\le \frac{f(g_1 a) + \ldots + f(g_m a)}{m} \int\limits_{C_G(a)} \varphi(x)\, d\mu(x), \tag{9}$$

where $\bar{a} = \frac{1}{m} \sum\limits_{i=1}^{m} g_i a$.

(ii) *If f is G-symmetrized convex on $C_G(a)$, then (9) holds.*

(iii) *If f is convex on $C_G(a)$ then* (9) *holds.*

Proof. Fix any $x \in C_G(a)$. Then $x \prec_G a$. Simultaneously, $\bar{a} \prec_G x$.

(i) By the G-increasity of $\hat{f}$ on $C_G(a)$, we see that $\hat{f}(\bar{a}) \le \hat{f}(x) \le \hat{f}(a)$, that is,

$$\frac{f(g_1 \bar{a}) + \ldots + f(g_m \bar{a})}{m} \le \frac{f(g_1 x) + \ldots + f(g_m x)}{m} \le \frac{f(g_1 a) + \ldots + f(g_m a)}{m}. \tag{10}$$

However $\bar{a} \in M_G(V)$, so $g\bar{a} = \bar{a}$ for all $g \in G$. For this reason (10) takes the form

$$f(\bar{a}) \le \hat{f}(x) \le \frac{f(g_1 a) + \ldots + f(g_m a)}{m}.$$

In consequence, by $\varphi(x) \ge 0$, we obtain

$$f(\bar{a})\varphi(x) \le \hat{f}(x)\varphi(x) \le \frac{f(g_1 a) + \ldots + f(g_m a)}{m}\varphi(x) \quad \text{for all } x \in C_G(a). \tag{11}$$

Hence,

$$\int\limits_{C_G(a)} f(\bar{a})\varphi(x)\, d\mu(x) \le \int\limits_{C_G(a)} \hat{f}(x)\varphi(x)\, d\mu(x) \le \int\limits_{C_G(a)} \frac{f(g_1 a) + \ldots + f(g_m a)}{m}\, \varphi(x)\, d\mu(x),$$

which implies

$$f(\bar{a}) \int\limits_{C_G(a)} \varphi(x)\, d\mu(x) \leq \int\limits_{C_G(a)} \hat{f}(x)\varphi(x)\, d\mu(x)$$

$$\leq \frac{f(g_1 a) + \ldots + f(g_m a)}{m} \int\limits_{C_G(a)} \varphi(x)\, d\mu(x). \tag{12}$$

Now, by employing inequality (7) in Lemma 2.1, item (ii), we conclude that inequality (9) holds valid, as required.

(ii) If f is G-symmetrized convex on $C_G(a)$, then $\hat{f}$ is convex on $C_G(a)$. It is easily seen that $\hat{f}$ is G-invariant. Consequently, $\hat{f}$ is G-increasing on $C_G(a)$. In other words, f is G-symmetrized increasing monotone on $C_G(a)$. So, by the proved part (i) of Theorem 2.2, inequality (9) is fulfilled for f.

(iii) If f is convex on $C_G(a)$, then $\hat{f}$ is also convex on $C_G(a)$. That is, f is G-symmetrized convex on $C_G(a)$. On account of the proved part (ii) of Theorem 2.2, inequality (9) is satisfied for f. □

Theorem 2.3. (Hermite-Hadamard type inequality) *Let $G = \{g_1, \ldots, g_m\}$ be a finite group and $a \in V$. Let $f : C_G(a) \to \mathbb{R}$ be an integrable function.*

(i) *If f is G-symmetrized increasing monotone on $C_G(a)$, then*

$$f(\bar{a})\mu(C_G(a)) \leq \int\limits_{C_G(a)} f(x)\, d\mu(x) \leq \frac{f(g_1 a) + \ldots + f(g_m a)}{m}\, \mu(C_G(a)). \tag{13}$$

(ii) *If f is G-symmetrized convex on $C_G(a)$, then* (13) *holds.*

(iii) *If f is convex on $C_G(a)$ then* (13) *holds.*

Proof. It is enough to utilize Theorem 2.2 for the constant function $\varphi(x) = 1$ for $x \in C_G(a)$. □

Remark 2.4. In the case f is G-invariant, the fraction

$$\frac{f(g_1 a) + \ldots + f(g_m a)}{m}$$

in inequalities (9) and (13) takes the value $f(a)$.

Remark 2.5. Some multivariate inequalities similar to (9) and (13) for convex functions can be found in [11].

3. A relationship with Wright-convexity

Let $V^m = V \times \cdots \times V$ (m times) be the mth Cartesian power of the space V. Elements of V^m are m-tuples $(x_1, \ldots, x_m)$ with $x_i \in V$, $i = 1, \ldots, m$.

We define the action of the permutation group $P = \mathbb{P}_m$ on V^m via

$$p \cdot \mathbf{y} = (y_1, \ldots, y_m)p_\pi = (y_{i_1}, \ldots, y_{i_m}) \in V^m,$$

where $\mathbf{y} = (y_1, \ldots, y_m) \in V^m$ and $p = p_\pi \in P$ is the $m \times m$ permutation matrix corresponding to a permutation π of the index set $\{1, \ldots, m\}$ such that $(\pi(1), \ldots, \pi(m)) = (i_1, \ldots, i_m)$.

Following [12], we define the *P-majorization* $\prec_P$ on the real linear space V^m, as follows. For $\mathbf{x} = (x_1, \ldots, x_m) \in V^m$ and $\mathbf{y} = (y_1, \ldots, y_m) \in V^m$,

$$\mathbf{x} \prec_P \mathbf{y} \quad \text{iff} \quad \mathbf{x} \in \operatorname{conv}\{\mathbf{y}p : p \in P\}.$$

Equivalently,
$$\mathbf{x} \prec_P \mathbf{y} \quad \text{iff} \quad \mathbf{x} = \mathbf{y}S$$

for some $m \times m$ doubly stochastic matrix S.

We are interested in functions $f : C_G(a) \to \mathbb{R}$ such that the map

$$F(x_1, \ldots, x_m) = f(x_1) + \ldots + f(x_m) \quad \text{for } x_i \in C_G(a),\ i = 1, \ldots, m,$$

is P-increasing on $(C_G(a))^m$.

That is, we shall be focused on functions such that if $\mathbf{x} = (x_1, \ldots, x_m) \in (C_G(a))^m$ and $\mathbf{y} = (y_1, \ldots, y_m) \in (C_G(a))^m$, then

$$\mathbf{x} \prec_P \mathbf{y} \quad \text{implies} \quad f(x_1) + \ldots + f(x_m) \le f(y_1) + \ldots + f(y_m). \tag{14}$$

If $m = 2$ then the only doubly stochastic 2×2 matrices are of the form $\begin{pmatrix} t & 1-t \\ 1-t & t \end{pmatrix}$ for $t \in [0, 1]$. In this situation, condition (14) can be rewritten as

$$f(ty_1 + (1-t)y_2) + f((1-t)y_1 + ty_2) \le f(y_1) + f(y_2) \quad \text{for } t \in [0, 1]. \tag{15}$$

Functions having property (15) are said to be *Wright-convex*.

Ng [12, Theorem 3] proved that statements (15) and (14) are equivalent for any positive integer $m \ge 2$ (provided that $V = \mathbb{R}^n$). He also showed that f is Wright-convex if and only if f has the form $f = f_1 + f_2$ for some convex f_1 and additive f_2 (for details, see [12, Theorem 3]).

For this reason, we shall call a function $f : C_G(a) \to \mathbb{R}$ *Wright convex* on $C_G(a)$, if condition (14) is satisfied.

Lemma 3.1. *Let $G = \{g_1, \ldots, g_m\}$ be a finite group and $x, y \in V$.*

(i) *If $x \prec_G y$ then* $$(g_1 x, \ldots, g_m x) \prec_P (g_1 y, \ldots, g_m y), \tag{16}$$ *where $P = \mathbb{P}_m$ is the permutation group on V^m.*

(ii) *Let $f : C_G(y) \to \mathbb{R}$ be a Wright convex function on $C_G(y)$. Let $x, y \in C_G(a)$. If $x \prec_G y$ then*
$$f(g_1 x) + \ldots + f(g_m x) \le f(g_1 y) + \ldots + f(g_m y). \tag{17}$$
In consequence, f is G-symmetrized increasing monotone on $C_G(a)$.

Proof. (i) Since $x \prec_G y$, we get $x = \sum\limits_{i=1}^{m} t_i h_i y$ for some $h_i \in G$ and $t_i \ge 0$, $i = 1, \ldots, m$, with $\sum\limits_{i=1}^{m} t_i = 1$.

Hence,
$$g_j x = \sum_{i=1}^{m} t_i g_j h_i y \quad \text{for } j = 1, \dots, m.$$

Thus we arrive at

$$(g_1 x, \dots, g_m x) = \left(\sum_{i=1}^{m} t_i g_1 h_i y, \dots, \sum_{i=1}^{m} t_i g_m h_i y \right) = \sum_{i=1}^{m} t_i \left(g_1 h_i y, \dots, g_m h_i y \right).$$

For each $i = 1, \dots, m$, there exists a permutation π_i of the index set $\{1, \dots, m\}$ such that $g_j h_i = g_{\pi_i(j)}$ for all $j = 1, \dots, m$. Therefore,

$$(g_1 x, \dots, g_m x) = \sum_{i=1}^{m} t_i \left(g_{\pi_i(1)} y, \dots, g_{\pi_i(m)} y \right) = \sum_{i=1}^{m} t_i p_i \left(g_1 y, \dots, g_m y \right)$$

for some permutation $p_i \in P$. This proves (16), as claimed.

(ii) Since f is Wright convex on $C_G(a)$, the mapping $F : (C_G(a))^m \to \mathbb{R}$ given by

$$F(x_1, \dots, x_m) = f(x_1) + \dots + f(x_m) \quad \text{for } (x_1, \dots, x_m) \in (C_G(a))^m$$

is P-increasing on $(C_G(a))^m$.

Let $x, y \in C_G(a)$ be such that $x \prec_G y$. By the proved part (i) of Lemma 3.1,

$$(g_1 x, \dots, g_m x) \prec_P (g_1 y, \dots, g_m y). \tag{18}$$

Therefore we infer that

$$F(g_1 x, \dots, g_m x) \le F(g_1 y, \dots, g_m y), \tag{19}$$

which yields (17), as wanted. Hence f is G-symmetrized increasing monotone on $C_G(a)$. □

Theorem 3.2. *Let $G = \{g_1, \dots, g_m\}$ be a finite group and $a \in V$. Let $f : C_G(a) \to \mathbb{R}$ be an integrable function and $\varphi : C_G(a) \to \mathbb{R}$ be an integrable nonnegative G-invariant function.*

If f is Wright convex on $C_G(a)$, then the Fejér type inequality (9) *and the Hermite-Hadamard type inequality* (13) *are satisfied.*

Proof. Let f be Wright convex on $C_G(a)$. In light of Lemma 3.1(ii), f is G-symmetrized increasing monotone on $C_G(a)$. By making use of Theorem 2.2(i), and of Theorem 2.3(i), we conclude that inequalities (9) and (13) are fulfilled. □

Remark 3.3. In the special case $V = \mathbb{R}$ and $G = \{\mathrm{id}, -\mathrm{id}\}$ (see the discussion after the proof of Theorem 4.2), a related result to Theorem 3.2 for Wright convex functions is that of Olbryś [18, Theorem 11].

4. Applications for Lebesque measure on $\mathbb{R}^n$

In this section we interpret Theorem 2.2 in the situation when

$$V = \mathbb{R}^n, \quad G = \mathbb{C}_n, \quad m = 2^n, \tag{20}$$

where $\mathbb{C}_n$ is the *coordinate sign changes group* consisting of all $n \times n$ orthogonal diagonal matrices, that is,

$$G = \left\{ g = \begin{pmatrix} \pm 1 & 0 & \dots & 0 \\ 0 & \pm 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \pm 1 \end{pmatrix} : \quad \text{for all possible choices of } \pm \text{ signs.} \right\}$$

The group $\mathbb{C}_n$ acts on $\mathbb{R}^n$ via

$$gx = (\pm x_1, \dots, \pm x_n) \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n \text{ and } g \in \mathbb{C}_n,$$

with all possible changes of $\pm$ signs.

Then for any $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we have

$$\begin{aligned} Ga &= \{(\pm a_1, \dots, \pm a_n) : \text{for all choices of } \pm \text{ signs}\}, \\ C_G(a) &= \operatorname{conv} Ga = \Delta(a) = \text{parallelepiped in } \mathbb{R}^n \text{ with vertices} \\ &\qquad \text{at the points } (\pm a_1, \dots, \pm a_n) \text{ for all choices of } \pm \text{ signs}, \\ \Delta(a) &= \{b = (b_1, \dots, b_n) \in \mathbb{R}^n : |b_i| \le |a_i|, i = 1, \dots, n\}, \\ \bar{a} &= 0 \in \mathbb{R}^n, \quad x \prec_G a \quad \text{iff} \quad |x_i| \le |a_i| \quad \text{for} \quad i = 1, \dots, n, \\ \mu &= \mu_n \quad \text{is the Lebesque measure on Borel sets of } \mathbb{R}^n, \\ \mu(C_G(a)) &= 2^n |a_1 \cdots a_n|. \end{aligned}$$

In what follows, for $a = (a_1, \dots, a_n) \in \mathbb{R}$,

(a) the expression "φ is $\mathbb{C}_n$-invariant on $\Delta(a)$" means that φ is even on $\Delta(a)$ in each variable,

(b) the expression "f is $\mathbb{C}_n$-symmetrized increasing monotone on $\Delta(a)$" means that the function

$$\hat{f}(x) = \frac{f(g_1 x) + \ldots + f(g_{2^n} x)}{2^n}$$

$$= \frac{f((\pm x_1, \dots, \pm x_n)) + \ldots + f((\pm x_1, \dots, \pm x_n))}{2^n}, \quad |x_i| \le |a_i|,\ i = 1, \dots, n,$$

(with all possible choices of $\pm$ signs), is $\mathbb{C}_n$-increasing on $\Delta(a)$ in the sense that

$$|x_i| \le |y_i|, \quad i = 1, \dots, n,$$

implies

$$\frac{f((\pm x_1, \dots, \pm x_n)) + \ldots + f((\pm x_1, \dots, \pm x_n))}{2^n} \le \frac{f((\pm y_1, \dots, \pm y_n)) + \ldots + f((\pm y_1, \dots, \pm y_n))}{2^n},$$

(c) the expression "f is $\mathbb{C}_n$-symmetrized convex on $\Delta(a)$" means that the function

$$\hat{f}(x) = \frac{f(g_1 x) + \ldots + f(g_{2^n} x)}{2^n} \quad \text{for } x \in \Delta(a),$$

is convex on $\Delta(a)$.

With these specifications, Theorem 2.2 on Fejér type inequalities, takes the following form.

Theorem 4.1. *Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Let $f : \Delta(a) \to \mathbb{R}$ be an integrable function and $\varphi : \Delta(a) \to \mathbb{R}$ be an integrable nonnegative $\mathbb{C}_n$-invariant function.*

(i) *If f is $\mathbb{C}_n$-symmetrized increasing monotone on $\Delta(a)$, then*

$$f(0) \int\limits_{\Delta(a)} \varphi(x)\, d\mu_n(x) \le \int\limits_{\Delta(a)} f(x)\varphi(x)\, d\mu_n(x)$$

$$\le \frac{f((\pm a_1, \dots, \pm a_n)) + \dots + f((\pm a_1, \dots, \pm a_n))}{2^n} \int\limits_{\Delta(a)} \varphi(x)\, d\mu_n(x) \tag{21}$$

with all choices of $\pm$ signs.

(ii) *If f is $\mathbb{C}_n$-symmetrized convex on $\Delta(a)$, then (21) holds.*

(iii) *If f is convex on $\Delta(a)$ then (21) holds.*

Proof. Apply Theorem 2.2 for the specifications given before Theorem 4.1 with remarks (a), (b), (c). □

A variant of Theorem 4.1 for Hermite-Hadamard type inequalities is as follows.

Theorem 4.2. *Let $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ with $a_1 \neq 0, \dots, a_n \neq 0$. Let $f : \Delta(a) \to \mathbb{R}$ be an integrable function.*

(i) *If f is $\mathbb{C}_n$-symmetrized increasing monotone on $\Delta(a)$, then*

$$f(0) \le \frac{1}{2^n |a_1 \cdots a_n|} \int\limits_{\Delta(a)} f(x)\, d\mu_n(x)$$

$$\le \frac{f((\pm a_1, \dots, \pm a_n)) + \dots + f((\pm a_1, \dots, \pm a_n))}{2^n} \tag{22}$$

with all choices of $\pm$ signs.

(ii) *If f is $\mathbb{C}_n$-symmetrized convex on $\Delta(a)$, then (22) holds.*

(iii) *If f is convex on $\Delta(a)$ then (22) holds.*

Proof. We make use of Theorem 4.1 for $\varphi \equiv 1$ on $\Delta(a)$ and of the identity $\mu_n(\Delta(a)) = 2^n |a_1 \cdots a_n|$. □

A special case of the above situation is for $n = 1$. We now assume that

$$V = \mathbb{R}, \quad G = \mathbb{C}_1 = \{\mathrm{id}, -\mathrm{id}\}, \quad m = 2, \tag{23}$$

where id denotes the identity map on $\mathbb{R}$.

Then for any $a = a_1 \in \mathbb{R}$ and $x \in \mathbb{R}$, we have

$$Ga = \{a, -a\}, \quad C_G(a) = \operatorname{conv} Ga = \Delta(a) = \text{the closed interval } [-|a|, |a|] \text{ in } \mathbb{R},$$

$$\bar{a} = 0 \in \mathbb{R}, \quad x \prec a \quad \text{iff} \quad |x| \le |a|,$$

$$\mu = dx \ \text{ is the Lebesque measure on Borel sets of } \mathbb{R}, \ \ \mu(C_G(a)) = 2|a|.$$

In the forthcoming corollaries, for $0 < a \in \mathbb{R}$,

(a) the expression "φ is $\mathbb{C}_1$-invariant on $[-a, a]$" means that φ is even on $[-a, a]$,

(b) the expression "f is $\mathbb{C}_1$-symmetrized increasing monotone on $[-a, a]$" means that the function

$$\hat{f}(x) = \frac{f(x) + f(-x)}{2}, \quad x \in [-a, a],$$

is $\mathbb{C}_1$-increasing on $[-a, a]$ in the sense that

$$|x| \le |y| \quad \text{implies} \quad \frac{f(x) + f(-x)}{2} \le \frac{f(y) + f(-y)}{2}.$$

(For instance, it is sufficient that f is even on $[-a, a]$ and f is (usually) increasing on $[0, a]$.) This says that $\hat{f}$ is *symmetrically increasing* on $[-a, a]$ according to terminology of Abramovich and Persson [2, Definition 1],

(c) the expression "f is $\mathbb{C}_1$-symmetrized convex on $[-a, a]$" means that the function

$$\hat{f}(x) = \frac{f(x) + f(-x)}{2}, \quad x \in [-a, a],$$

is convex on $[-a, a]$ in the usual sense.

In fact, this is the standard notion of *symmetrized convexity* [4, Definition 1].

Corollary 4.3. *Let $0 < a \in \mathbb{R}$. Let $f : [-a, a] \to \mathbb{R}$ be an integrable function and $\varphi : [-a, a] \to \mathbb{R}$ be an integrable nonnegative $\mathbb{C}_1$-invariant function.*

(i) *If f is $\mathbb{C}_1$-symmetrized increasing monotone on $[-a, a]$, then*

$$f(0) \int_{-a}^{a} \varphi(x)\, dx \le \int_{-a}^{a} f(x)\varphi(x)\, dx \le \frac{f(a) + f(-a)}{2} \int_{-a}^{a} \varphi(x)\, dx. \tag{24}$$

(ii) *If f is $\mathbb{C}_1$-symmetrized convex on $[-a, a]$, then* (24) *holds.*

(iii) *If f is convex on $[-a, a]$ then* (24) *holds.*

Proof. Use Theorem 4.1 for $n = 1$. □

Remark 4.4. In Corollary 4.3, part (iii) amounts to the classical Fejér inequality on $[-a, a]$ (see Theorem B in Section 1). Part (ii) is Dragomir's result [4, Corollary 1]. More general is part (i). It is related to a result of Abramovich and Persson [2, Theorem 1].

Corollary 4.5. *Let $0 < a \in \mathbb{R}$. Let $f : [-a, a] \to \mathbb{R}$ be an integrable function.*

(i) *If f is $\mathbb{C}_1$-symmetrized increasing monotone on $[-a, a]$, then*

$$f(0) \le \frac{1}{2a} \int_{-a}^{a} f(x)\, dx \le \frac{f(a) + f(-a)}{2}. \tag{25}$$

(ii) *If f is $\mathbb{C}_1$-symmetrized convex on $[-a, a]$, then* (25) *holds.*

(iii) *If f is convex on $[-a, a]$ then* (25) *holds.*

Proof. It is enough to apply Corollary 4.3 for $\varphi \equiv 1$ on $[-a, a]$. $\square$

Remark 4.6. Corollary 4.5(iii) states the classical Hermite-Hadamard inequality on $[-a, a]$ (see Theorem A in Section 1). Part (ii) of Corollary 4.5 is a result due to Dragomir [4, Theorem 1]. And (i) generalizes (ii) (cf. [2, Theorem 1] by Abramovich and Persson).

We finish with some concluding remarks. The results of the present paper concern integral inequalities on sets of type $C_G(a)$ for $a \in V$, where V is an inner product space, $\dim V < \infty$, with the action of a finite group $G \subset O(V)$.

These sets, in the situation described by (23), correspond to usual intervals of the form $[-\alpha, \alpha]$ with center at 0 for $0 < \alpha \in \mathbb{R}$.

It is also of interest to have inequalities similar to (9) and (13) on the sets $b + C_G(a)$ for $a, b \in V$ (that, under (23), correspond to usual intervals $[\beta, \alpha]$ with center $\frac{\alpha+\beta}{2}$ for $\beta < \alpha \in \mathbb{R}$; see (1) and (2)).

To this end one can employ (9) and (13) and integrate by substitution using suitable translation with an additional assumption that the measure μ is translations-invariant (e.g., Lebesque measure μ_n). The details are left for the interested reader.

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