Journal of Convex Analysis
Volume 29 (2022), No. 1, 257–268

On $(h, g; m)$-Convexity and the Hermite-Hadamard Inequality

Maja Andrić
Faculty of Civil Engineering, Architecture and Geodesy, University of Split, Croatia
maja.andric@gradst.hr

Josip Pečarić
People's Friendship University, 117198 Moscow, Russia
pecaric@element.hr

Received: July 24, 2020
Accepted: May 19, 2021

A new class of $(h, g; m)$-convex functions is presented, together with its properties, thus generalizing several varieties of convexity such as $(h-m)$-convex functions, (s, m)-Godunova-Levin functions or exponentially (s, m)-convex functions. Also, the Hermite-Hadamard inequality for an $(h, g; m)$-convex function is proved and some special results are pointed out.

Keywords: Convex function, Hermite-Hadamard inequality.

2010 Mathematics Subject Classification: 26A51, 26D15.

1. Introduction

Motivated by a large number of different classes of convexity, we present a new convexity that unifies a certain range of them. Starting from the basic definition of a convex function:

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *convex* if the following inequality holds

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1}$$

for all $x, y \in I$ and all $\lambda \in [0, 1]$.

up to a recent convexity [9]:

A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *exponentially* (s, m)*-convex in the* 2^{nd} *sense* if the following inequality holds

$$f(\lambda x + m(1-\lambda) y) \leq \frac{\lambda^s}{e^{\alpha x}} f(x) + \frac{(1-\lambda)^s}{e^{\alpha y}} m f(y) \tag{2}$$

for all $x, y \in I$ and all $\lambda \in [0, 1]$, where $\alpha \in \mathbb{R}$, $s, m \in (0, 1]$.

we noticed that the whole range in-between could be covered if we use on the right-hand side functions h and g in a form

$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda) f(x) g(x) + m\, h(1-\lambda) f(y) g(y).$$

We named this convexity $(h, g; m)$*-convexity*, and in Section 2 we give its definition and certain properties.

ISSN 0944-6532 / $ 2.50

Here are several more varieties of convexity that will be generalized with this:

- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *P-function* if the inequality
$$f(\lambda x + (1-\lambda)y) \leq f(x) + f(y)$$
holds for all $x, y \in I$ and all $\lambda \in [0,1]$.
- A function $f : [0,\infty) \to [0,\infty)$ is called *s-convex in the* 2^{nd} *sense* if the inequality
$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$
holds for all $x, y \in [0,\infty)$ and all $\lambda \in [0,1]$, where $s \in (0,1]$.
- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *Godunova-Levin function* if the inequality
$$f(\lambda x + (1-\lambda)y) \leq \frac{f(x)}{\lambda} + \frac{f(y)}{1-\lambda}$$
holds for all $x, y \in I$ and all $\lambda \in (0,1)$.
- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *h-convex* if the inequality
$$f(\lambda x + (1-\lambda)y) \leq h(\lambda) f(x) + h(1-\lambda) f(y)$$
holds for all $x, y \in I$ and all $\lambda \in (0,1)$, where $h : J \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, $(0,1) \subseteq J$.
- A function $f : [0,b] \to \mathbb{R}$ is called *m-convex* if the inequality
$$f(\lambda x + m(1-\lambda)y) \leq \lambda f(x) + m(1-\lambda) f(y)$$
holds for all $x, y \in [0,b]$ and all $\lambda \in [0,1]$, where $m \in [0,1]$.
- A non-negative function $f : [0,b] \to \mathbb{R}$ is called *$(h-m)$-convex* if the inequality
$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda) f(x) + mh(1-\lambda) f(y)$$
holds for all $x, y \in [0,b]$ and all $\lambda \in (0,1)$, where $h : J \to \mathbb{R}$ is a non-negative function, $h \not\equiv 0$, $(0,1) \subseteq J$ and $m \in [0,1]$.
- A non-negative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *(s,m)-Godunova-Levin function of the* 2^{nd} *kind* if the inequality
$$f(\lambda x + m(1-\lambda)y) \leq \frac{f(x)}{\lambda^s} + \frac{mf(y)}{(1-\lambda)^s}$$
holds for all $x, y \in I$ and all $\lambda \in (0,1)$, where $m \in (0,1]$, $s \in [0,1]$.
- A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *exponential convex* if the inequality
$$f(\lambda x + (1-\lambda)y) \leq \frac{\lambda}{e^{\alpha x}} f(x) + \frac{1-\lambda}{e^{\alpha y}} f(y)$$
holds for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$.
- A function $f : I \subset \mathbb{R} \to \mathbb{R}$ is called *exponentially s-convex in the* 2^{nd} *sense* if the inequality
$$f(\lambda x + (1-\lambda)y) \leq \frac{\lambda^s}{e^{\alpha x}} f(x) + \frac{(1-\lambda)^s}{e^{\alpha y}} f(y)$$
holds for all $x, y \in I$ and all $\lambda \in [0,1]$, where $\alpha \in \mathbb{R}$, $s \in (0,1]$.

More detailed information may be found in [1]-[11]. Also, recall that a real valued function f on the interval I is said to be *starshaped* if

$$f(\lambda x) \leq \lambda f(x)$$

whenever $x \in I, \lambda x \in I$ and $\lambda \in [0, 1]$.

Section 3 is dedicated to the famous Hermite-Hadamard inequality which gives us an estimate of the (integral) mean value of a continuous convex function.

Theorem 1.1. (The Hermite-Hadamard inequality) *Let $f : [a, b] \to \mathbb{R}$ be a continuous convex function. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Of course, equality holds in either side only for affine functions. We prove the Hermite-Hadamard inequality for an $(h, g; m)$-convex function and point out some special results. Also, we improve certain known inequalities.

By $L_p[a, b]$, $1 \leq p < \infty$, the space of all Lebesgue measurable functions f for which $|f^p|$ is Lebesgue integrable on $[a, b]$ is denoted.

2. Definition and basic results

Definition 2.1. Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$ and g be a positive function on $I \subseteq \mathbb{R}$. Also, let $m \in (0, 1]$. A function $f : I \to \mathbb{R}$ is said to be $(h, g; m)$-*convex* if it is nonnegative and if

$$f(\lambda x + m(1-\lambda)y) \leq h(\lambda)f(x)g(x) + m\,h(1-\lambda)f(y)g(y) \tag{3}$$

holds for all $x, y \in I$ and all $\lambda \in (0, 1)$.
If (3) holds in the reversed sense, then f is said to be $(h, g; m)$-*concave.*

Remark 2.2. For a different choices of functions h, g and parameter m in (3), we can get corresponding convexity from *Introduction.* E.g., if we set $h(\lambda) = \lambda^s$, $s \in (0, 1]$, $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, then $(h, g; m)$-convexity reduces to exponentially (s, m)-convexity in the 2^{nd} sense (2).

Lemma 2.3. *If $f : I \to [0, \infty)$ is an $(h, g; m)$-convex function such that $f(0) = 0$, $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then f is a starshaped.*

Proof. Let f be an $(h, g; m)$-convex function. Then we have

$$\begin{aligned} f(\lambda x) &= f(\lambda x + m(1-\lambda)0) \\ &\leq h(\lambda)f(x)g(x) + m\,h(1-\lambda)f(0)g(0) \\ &\leq \lambda f(x). \end{aligned}$$

Therefore, f is a starshaped. □

Remark 2.4. Let g be a positive function such that $g(x) \geq 1$. If f is a nonnegative $(h-m)$-convex function on $[0,\infty)$, then we have

$$\begin{aligned} f(\lambda x + m(1-\lambda)y) &\leq h(\lambda)f(x) + m\,h(1-\lambda)f(y) \\ &\leq h(\lambda)f(x)g(x) + m\,h(1-\lambda)f(y)g(y). \end{aligned}$$

Hence, f is an $(h, g; m)$-convex function.

If in addition $h(\lambda) \geq \lambda$, then for nonnegative m-convex functions f on $[0,\infty)$ we have

$$\begin{aligned} f(\lambda x + m(1-\lambda)y) &\leq \lambda f(x) + m\,(1-\lambda)f(y) \\ &\leq h(\lambda)f(x) + m\,h(1-\lambda)f(y) \\ &\leq h(\lambda)f(x)g(x) + m\,h(1-\lambda)f(y)g(y), \end{aligned}$$

that is, f is an $(h, g; m)$-convex function. An example of a function that satisfies $h(\lambda) \geq \lambda$ is $h(\lambda) = \lambda^k$, where $k \leq 1$ and $\lambda > 0$.

Similarly, if $g(x) \leq 1$, then all nonnegative $(h-m)$-concave functions are $(h, g; m)$-concave functions on $[0,\infty)$. Also, if $g(x) \leq 1$ and $h(\lambda) \leq \lambda$, then all nonnegative m-concave functions are $(h, g; m)$-concave functions on $[0,\infty)$.

Proposition 2.5. *Let h_1, h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1, h_2 \not\equiv 0$, such that $h_2(\lambda) \leq h_1(\lambda), \quad \lambda \in (0,1)$.*

Let g be a positive function on $I \subseteq \mathbb{R}$ and $m \in (0,1]$.

If $f : I \to [0,\infty)$ is an $(h_2, g; m)$-convex function, then f is an $(h_1, g; m)$-convex.

If $f : I \to [0,\infty)$ is an $(h_1, g; m)$-concave function, then f is an $(h_2, g; m)$-concave.

Proof. Let f be an $(h_2, g; m)$-convex function. Then we have

$$\begin{aligned} f(\lambda x + m(1-\lambda)y) &\leq h_2(\lambda)f(x)g(x) + m\,h_2(1-\lambda)f(y)g(y) \\ &\leq h_1(\lambda)f(x)g(x) + m\,h_1(1-\lambda)f(y)g(y). \end{aligned}$$

Hence, f is an $(h_1, g; m)$-convex.

If f is an $(h_1, g; m)$-concave function, then analogously follows that f is an $(h_2, g; m)$-concave. $\square$

Proposition 2.6. *Let h be a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and g be a positive function on $I \subseteq \mathbb{R}$. Also, let $m \in (0,1]$ and $\alpha > 0$.*

If $f_1, f_2 : I \to [0,\infty)$ are $(h, g; m)$-convex function, then $f_1 + f_2$ and αf_1 are $(h, g; m)$-convex.

If $f_1, f_2 : I \to [0,\infty)$ are $(h, g; m)$-concave function, then $f_1 + f_2$ and αf_1 are $(h, g; m)$-concave.

Proof. Let f_1, f_2 be $(h, g; m)$-convex functions and $\alpha > 0$. Then we have

$$f_1(\lambda x + m(1-\lambda)y) \leq h(\lambda)f_1(x)g(x) + m\,h(1-\lambda)f_1(y)g(y) \tag{4}$$

and

$$f_2(\lambda x + m(1-\lambda)y) \leq h(\lambda)f_2(x)g(x) + m\,h(1-\lambda)f_2(y)g(y). \tag{5}$$

Adding (4) and (5) we get

$$\begin{aligned}
&[f_1+f_2](\lambda x+m(1-\lambda)y)\\
&\quad\le h(\lambda)[f_1+f_2](x)g(x)+m\,h(1-\lambda)[f_1+f_2](y)g(y)
\end{aligned}$$

and

$$\begin{aligned}
&[\alpha f_1](\lambda x+m(1-\lambda)y)\\
&\quad\le \alpha h(\lambda)f_1(x)g(x)+\alpha m\,h(1-\lambda)f_1(y)g(y)\\
&\quad= h(\lambda)[\alpha f_1](x)g(x)+m\,h(1-\lambda)[\alpha f_1](y)g(y).
\end{aligned}$$

We conclude that f_1+f_2 and αf_1 are $(h_1,g;m)$-convex.

If $f_1,f_2:I\to[0,\infty)$ are $(h,g;m)$-concave function, then analogously follows that f_1+f_2 and αf_1 are $(h,g;m)$-concave. □

Proposition 2.7. *Let h be a nonnegative function on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h\not\equiv 0$ and g be a positive increasing function on $I\subseteq\mathbb{R}$. Also, let $0<n<m\le 1$. If $f:I\to[0,\infty)$ is an $(h,g;m)$-convex function such that $f(0)=0$, $g(x)\le 1$ and $h(\lambda)\le\lambda$, then f is an $(h,g;n)$-convex.*

Proof. Let f be an $(h,g;m)$-convex function. From $f(0)=0$, $g(x)\le 1$ and $h(\lambda)\le\lambda$ by Lemma 2.3 follows $f(\lambda x)\le\lambda f(x)$. Considering also that g is an increasing function, we obtain

$$\begin{aligned}
f(\lambda x+n(1-\lambda)y) &= f(\lambda x+m(1-\lambda)(\tfrac{n}{m}y))\\
&\le h(\lambda)f(x)g(x)+m\,h(1-\lambda)f(\tfrac{n}{m}y)g(\tfrac{n}{m}y)\\
&\le h(\lambda)f(x)g(x)+m\,h(1-\lambda)\frac{n}{m}f(y)g(y),
\end{aligned}$$

which proves that f is an $(h,g;n)$-convex. □

Proposition 2.8. *Let h_1,h_2 be nonnegative functions on $J\subseteq\mathbb{R}$, $(0,1)\subseteq J$, $h_1,h_2\not\equiv 0$ and let*

$$h(t)=\max\{h_1(t),h_2(t)\},\quad t\in J.$$

Let g_1,g_2 be positive functions on $I\subseteq\mathbb{R}$ and let $m_1,m_2\in(0,1]$. For $i=1,2$, let $f_i:I\to[0,\infty)$ be an $(h_i,g_i;m_i)$-convex function. If the functions $f_1\,g_1$ and $f_2\,g_2$ are monotonic in the same sense, i.e.

$$[f_1(x)g_1(x)-f_1(y)g_1(y)]\,[f_2(x)g_2(x)-f_2(y)g_2(y)]\ge 0,\quad x,y\in I,$$

and if $c>0$ such that

$$h(\lambda)+mh(1-\lambda)\le c,\quad \lambda\in(0,1),$$

where $m=\max\{m_1,m_2\}$, then $f_1\,f_2$ is a $(ch,g_1g_2;m)$-convex function.

Proof. Let $f_i:I\to[0,\infty)$ be $(h_i,g_i;m_i)$-convex function, $i=1,2$. From the hypotheses on the functions, for $x,y\in I$ we have

$$\begin{aligned}
&f_1(x)g_1(x)f_2(x)g_2(x)+f_1(y)g_1(y)f_2(y)g_2(y)\\
&\quad\ge\ f_1(x)g_1(x)f_2(y)g_2(y)+f_1(y)g_1(y)f_2(x)g_2(x).
\end{aligned}$$

Let α and $\beta > 0$ be positive numbers such that $\alpha + \beta = 1$. Then we have

$$
\begin{aligned}
& f_1 f_2(\alpha x + \beta y) \\
&\quad \le \ [h_1(\alpha) f_1(x) g_1(x) + m_1 h_1(\beta) f_1(y) g_1(y)] \\
&\qquad \times [h_2(\alpha) f_2(x) g_2(x) + m_2 h_2(\beta) f_2(y) g_2(y)] \\
&\quad \le \ [h(\alpha) f_1(x) g_1(x) + m h(\beta) f_1(y) g_1(y)] \\
&\qquad \times [h(\alpha) f_2(x) g_2(x) + m h(\beta) f_2(y) g_2(y)] \\
&\quad = \ h^2(\alpha) f_1(x) g_1(x) f_2(x) g_2(x) + m h(\alpha) h(\beta) f_1(x) g_1(x) f_2(y) g_2(y) \\
&\qquad + m h(\alpha) h(\beta) f_1(y) g_1(y) f_2(x) g_2(x) + m^2 h^2(\beta) f_1(y) g_1(y) f_2(y) g_2(y),
\end{aligned}
$$

hence

$$
\begin{aligned}
& f_1 f_2(\alpha x + \beta y) \\
&\quad \le \ h^2(\alpha) f_1(x) g_1(x) f_2(x) g_2(x) + m h(\alpha) h(\beta) f_1(x) g_1(x) f_2(x) g_2(x) \\
&\qquad + m h(\alpha) h(\beta) f_1(y) g_1(y) f_2(y) g_2(y) + m^2 h^2(\beta) f_1(y) g_1(y) f_2(y) g_2(y) \\
&\quad = \ [h(\alpha) + m\, h(\beta)] \\
&\qquad \times [h(\alpha) f_1(x) f_2(x) g_1(x) g_2(x) + m\, h(\beta) f_1(y) f_2(y) g_1(y) g_2(y)] \\
&\quad \le \ c\, h(\alpha) f_1(x) f_2(x) g_1(x) g_2(x) + m\, c\, h(\beta) f_1(y) f_2(y) g_1(y) g_2(y).
\end{aligned}
$$

This proves that $f_1\, f_2$ is a $(ch, g_1 g_2; m)$-convex. □

Analogously we obtain the following proposition.

Proposition 2.9. *Let h_1, h_2 be nonnegative functions on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h_1, h_2 \not\equiv 0$ and let*

$$h(t) = \min\{h_1(t), h_2(t)\}, \quad t \in J.$$

Let g_1, g_2 be positive functions on $I \subseteq \mathbb{R}$ and let $m_1, m_2 \in (0,1]$. For $i = 1, 2$, let $f_i : I \to [0, \infty)$ be an $(h_i, g_i; m_i)$-concave function. If the functions $f_1\, g_1$ and $f_2\, g_2$ are monotonic in the opposite sense, i.e.

$$[f_1(x) g_1(x) - f_1(y) g_1(y)]\, [f_2(x) g_2(x) - f_2(y) g_2(y)] \le 0, \quad x, y \in I,$$

and if $c > 0$ such that

$$h(\lambda) + m h(1 - \lambda) \ge c, \quad \lambda \in (0,1),$$

where $m = \min\{m_1, m_2\}$, then $f_1\, f_2$ is a $(ch, g_1 g_2; m)$-concave function.

3. The Hermite-Hadamard inequality for an $(h, g; m)$-convex function

In this section we prove the Hermite-Hadamard inequality for an $(h, g; m)$-convex function and we point out some special results. Also, several known inequalities are improved.

Theorem 3.1. *Let f be a nonnegative $(h, g; m)$-convex function on $[0, \infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0, 1) \subseteq J$, $h \not\equiv 0$, g is a positive function on $[0, \infty)$ and $m \in (0, 1]$. If $f, g, h \in L_1[a, b]$, where $0 \leq a < b < \infty$, then following inequalities hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(x)g(x) + mf\left(\frac{x}{m}\right)g\left(\frac{x}{m}\right)\right]dx \\ &\leq \frac{h\left(\frac{1}{2}\right)f(a)g(a)}{b-a}\int_a^b h\left(\frac{b-x}{b-a}\right)g(x)\,dx \\ &\quad + \frac{mh\left(\frac{1}{2}\right)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{b-a}\int_a^b h\left(\frac{x-a}{b-a}\right)g(x)\,dx \\ &\quad + \frac{mh\left(\frac{1}{2}\right)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{b-a}\int_a^b h\left(\frac{b-x}{b-a}\right)g\left(\frac{x}{m}\right)dx \\ &\quad + \frac{m^2h\left(\frac{1}{2}\right)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{b-a}\int_a^b h\left(\frac{x-a}{b-a}\right)g\left(\frac{x}{m}\right)dx. \end{aligned} \tag{6}$$

Proof. Let f be an $(h, g; m)$-convex function. Then for $\lambda = \frac{1}{2}$ we have

$$f\left(\frac{x+my}{2}\right) \leq h\left(\tfrac{1}{2}\right)[f(x)g(x) + mf(y)g(y)]$$

for all $x, y \in I$. Choosing $y \equiv \frac{y}{m}$ we get

$$f\left(\frac{x+y}{2}\right) \leq h\left(\tfrac{1}{2}\right)\left[f(x)g(x) + mf\left(\frac{y}{m}\right)g\left(\frac{y}{m}\right)\right] \tag{7}$$

Let $x = \lambda a + (1-\lambda)b$ and $y = (1-\lambda)a + \lambda b$. Then

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \leq\ & h\left(\tfrac{1}{2}\right)\Big[f(\lambda a + (1-\lambda)b)\,g(\lambda a + (1-\lambda)b) \\ & +mf\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)g\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)\Big]. \end{aligned}$$

In the following step we will need to integrate the above over $\lambda \in [0, 1]$. From

$$\int_0^1 f(\lambda a + (1-\lambda)b)\,g(\lambda a + (1-\lambda)b)\,d\lambda = \frac{1}{b-a}\int_a^b f(u)g(u)\,du$$

and

$$\int_0^1 f\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)g\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)d\lambda = \frac{1}{b-a}\int_a^b f\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du$$

we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(u)g(u) + mf\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)\right]du. \tag{8}$$

By the $(h, g; m)$-convexity of f we have

$$f(\lambda a + (1-\lambda)b) \leq h(\lambda)f(a)g(a) + mh(1-\lambda)f\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right).$$

Multiplying the above inequality by $g(\lambda a + (1-\lambda)b)$ and integrating over $\lambda \in [0,1]$ we obtain

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f(u)\,g(u)du \\
&\leq f(a)g(a)\int_0^1 h(\lambda)g(\lambda a + (1-\lambda)b)\,d\lambda \\
&\quad + mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)\int_0^1 h(1-\lambda)g(\lambda a + (1-\lambda)b)\,d\lambda \\
&= \frac{f(a)g(a)}{b-a}\int_a^b h\left(\frac{b-u}{b-a}\right)g(u)\,du \\
&\quad + \frac{mf\left(\frac{b}{m}\right)g\left(\frac{b}{m}\right)}{b-a}\int_a^b h\left(\frac{u-a}{b-a}\right)g(u)\,du. \qquad (9)
\end{aligned}$$

Again, by the $(h, g; m)$-convexity of f we have

$$f\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right) \leq h(1-\lambda)f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) + mh(\lambda)f\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)$$

and if we multiply the above inequality by $g\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)$ and integrate over $\lambda \in [0,1]$ we get

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b f\left(\frac{u}{m}\right)g\left(\frac{u}{m}\right)du \\
&\leq f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1 h(1-\lambda)g\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)d\lambda \\
&\quad + mf\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)\int_0^1 h(\lambda)g\left((1-\lambda)\frac{a}{m} + \lambda\frac{b}{m}\right)d\lambda \\
&= \frac{f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)}{b-a}\int_a^b h\left(\frac{b-u}{b-a}\right)g\left(\frac{u}{m}\right)du \\
&\quad + \frac{mf\left(\frac{b}{m^2}\right)g\left(\frac{b}{m^2}\right)}{b-a}\int_a^b h\left(\frac{u-a}{b-a}\right)g\left(\frac{u}{m}\right)du. \qquad (10)
\end{aligned}$$

Now from (8), (9) and (10) we obtain (6). □

In the following we state several corollaries, using special functions for h and/or g. We start with the first special case: if $g \equiv 1$, then we have the Hermite-Hadamard inequality for an $(h-m)$-convex function.

Corollary 3.2. *Let f be a nonnegative $(h-m)$-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$ and $m \in (0,1]$. If $f, h \in L_1[a,b]$, where $0 \leq a < b < \infty$, then following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(x) + mf\left(\frac{x}{m}\right)\right] dx$$

$$\leq h\left(\tfrac{1}{2}\right)\int_0^1 h(x)\,dx \times \left[f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right]. \tag{11}$$

Proof. We use

$$\int_a^b h\left(\frac{b-x}{b-a}\right) dx = \int_a^b h\left(\frac{x-a}{b-a}\right) dx = (b-a)\int_0^1 h(u)\,du. \qquad \square$$

Remark 3.3. In [8, Theorem 9] the authors gave the following Hermite-Hadamard type inequality for an $(h-m)$-convex function

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b \left[f(x) + mf\left(\frac{x}{m}\right)\right] dx \\ &\leq h\left(\tfrac{1}{2}\right)\left[f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right]. \end{aligned} \tag{12}$$

For all functions h such that $\int_0^1 h(x)\,dx \leq 1$, our result (11) improves (12).

If $g \equiv 1$ and $m = 1$, then we have the Hermite-Hadamard inequality for an h-convex function ([11]):

Corollary 3.4. *Let f be a nonnegative h-convex function on $[0,\infty)$ where h is a nonnegative function on $J \subseteq \mathbb{R}$, $(0,1) \subseteq J$, $h \not\equiv 0$. If $f, h \in L_1[a,b]$, where $0 \leq a < b < \infty$, then following inequalities hold*

$$\begin{aligned} \frac{1}{2} f\left(\frac{a+b}{2}\right) &\leq \frac{h\left(\frac{1}{2}\right)}{b-a}\int_a^b f(x)\,dx \\ &\leq h\left(\tfrac{1}{2}\right)[f(a)+f(b)]\int_0^1 h(x)\,dx. \end{aligned} \tag{13}$$

For h being identity and $g \equiv 1$, the Hermite-Hadamard type inequality for a m-convex function holds ([3]):

Corollary 3.5. *Let f be a nonnegative m-convex function on $[0,\infty)$ with $m \in (0,1]$. If $f \in L_1[a,b]$, where $0 \leq a < b < \infty$, then following inequalities hold*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2(b-a)}\int_a^b \left[f(x) + mf\left(\frac{x}{m}\right)\right] dx \\ &\leq \frac{1}{4}\left[f(a) + mf\left(\frac{b}{m}\right) + mf\left(\frac{a}{m}\right) + m^2 f\left(\frac{b}{m^2}\right)\right]. \end{aligned}$$

Of course, if $h(x) = x$, $g \equiv 1$ and $m = 1$, then we have the Hermite-Hadamard inequality given in Theorem 1.1.

An interesting Hermite-Hadamard type inequality follows if h is identity.

Corollary 3.6. *Suppose that the assumptions of Theorem* 3.1 *hold and let* $h(x) = x$. *Then we have*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)} \int_a^b \left[f(x)g(x) + mf\left(\frac{x}{m}\right) g\left(\frac{x}{m}\right)\right] dx \tag{14}$$
$$\leq \frac{f(a)g(a)}{2(b-a)^2} \int_a^b (b-x)g(x)\, dx + \frac{mf\left(\frac{b}{m}\right) g\left(\frac{b}{m}\right)}{2(b-a)^2} \int_a^b (x-a)g(x)\, dx$$
$$+ \frac{mf\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right)}{2(b-a)^2} \int_a^b (b-x) g\left(\frac{x}{m}\right) dx + \frac{m^2 f\left(\frac{b}{m^2}\right) g\left(\frac{b}{m^2}\right)}{2(b-a)^2} \int_a^b (x-a) g\left(\frac{x}{m}\right) dx.$$

Next we discuss $h(\lambda) = \lambda^s$, $s \in (0, 1]$ and a special case of the positive function $g(x) = e^{-\alpha x}$, $\alpha \in \mathbb{R}$, to obtain the following new Hermite-Hadamard inequality for the exponentially (s, m)-convex function in the 2^{nd} sense.

Corollary 3.7. *Let f be a nonnegative exponentially (s, m)-convex function in the 2^{nd} sense on $[0, \infty)$ where $s, m \in (0, 1]$. If $f \in L_1[a, b]$, where $0 \leq a < b < \infty$, then following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^s(b-a)} \int_a^b \left[f(x)e^{-\alpha x} + mf\left(\frac{x}{m}\right) e^{-\frac{\alpha x}{m}}\right] dx \tag{15}$$
$$\leq \frac{f(a)e^{-\alpha a}}{2^s(b-a)^{s+1}} \int_a^b (b-x)^s e^{-\alpha x}\, dx + \frac{mf\left(\frac{b}{m}\right) e^{-\frac{\alpha b}{m}}}{2^s(b-a)^{s+1}} \int_a^b (x-a)^s e^{-\alpha x}\, dx$$
$$+ \frac{mf\left(\frac{a}{m}\right) e^{-\frac{\alpha a}{m}}}{2^s(b-a)^{s+1}} \int_a^b (b-x)^s e^{-\frac{\alpha x}{m}} dx + \frac{m^2 f\left(\frac{b}{m^2}\right) e^{-\frac{\alpha b}{m^2}}}{2^s(b-a)^{s+1}} \int_a^b (x-a)^s e^{-\frac{\alpha x}{m}} dx.$$

The last result is for an exponentially convex function: a special case of Theorem 3.1 when $h(\lambda) = \lambda$, $g(x) = e^{-\alpha x}$ and $m = 1$.

Corollary 3.8. *Let f be a nonnegative exponentially convex function, $\alpha \neq 0$. If $f \in L_1[a, b]$, where $0 \leq a < b < \infty$, then following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)e^{-\alpha x}\, dx$$
$$\leq \frac{e^{-2\alpha a} f(a) - e^{-2\alpha b} f(b)}{\alpha(b-a)} + \left[e^{-\alpha b} - e^{-\alpha a}\right] \frac{e^{-\alpha a} f(a) - e^{-\alpha b} f(b)}{\alpha^2 (b-a)^2}. \tag{16}$$

Proof. The second part of (16) follows from

$$\frac{e^{-\alpha a} f(a)}{(b-a)^2} \int_a^b (b-x)e^{-\alpha x}\, dx + \frac{e^{-\alpha b} f(b)}{(b-a)^2} \int_a^b (x-a)e^{-\alpha x}\, dx$$
$$= \frac{e^{-2\alpha a} f(a) - e^{-2\alpha b} f(b)}{\alpha(b-a)} + \left[e^{-\alpha b} - e^{-\alpha a}\right] \frac{e^{-\alpha a} f(a) - e^{-\alpha b} f(b)}{\alpha^2 (b-a)^2}.$$

□

Remark 3.9. In [1, Theorem 1] authors gave the following Hermite-Hadamard type inequality for exponentially convex function

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)e^{-\alpha x}\,dx \leq \frac{e^{-\alpha a}f(a)+e^{-\alpha b}f(b)}{2}.$$

Actually, this results holds only for $\alpha > 0$ (not for all $\lambda \in \mathbb{R}$, as the authors stated). In that case, our result (16) is an improvement of the above inequality since

$$\begin{aligned}
&\frac{e^{-2\alpha a}f(a)-e^{-2\alpha b}f(b)}{\alpha(b-a)} + \left[e^{-\alpha b}-e^{-\alpha a}\right]\frac{e^{-\alpha a}f(a)-e^{-\alpha b}f(b)}{\alpha^2(b-a)^2} \\
= &\ \frac{e^{-\alpha a}f(a)}{(b-a)^2}\int_a^b (b-x)e^{-\alpha x}\,dx + \frac{e^{-\alpha b}f(b)}{(b-a)^2}\int_a^b (x-a)e^{-\alpha x}\,dx \\
\leq &\ \frac{e^{-\alpha a}f(a)}{(b-a)^2}\int_a^b (b-x)\,dx + \frac{e^{-\alpha b}f(b)}{(b-a)^2}\int_a^b (x-a)\,dx \\
= &\ \frac{e^{-\alpha a}f(a)+e^{-\alpha b}f(b)}{2}.
\end{aligned}$$

We use the basic property of the exponential function: $e^{-\alpha x} \leq 1$ for $x \in [0, \infty)$ and $\alpha > 0$.

At the end of this paper, we emphasize that all other Hermite-Hadamard inequalities for different types of convexity, which follow from Theorem 3.1, can be treated analogously.

Acknowledgements. This research is partially supported through project number KK.01.1.1.02.0027, a project co-financed by the Croatian Government and the European Union through the European Regional Development Fund, the Competitiveness and Cohesion Operational Programme and the Ministry of Education and Science of the Russian Federation (Agreement No. 02.a03.21.0008).

References

[1] M. U. Awan, M. A. Noor, K. I. Noor: *Hermite-Hadamard inequalities for exponentially convex functions*, Appl. Math. Inf. Sci. 2/2 (2018) 405–409.

[2] W. W. Breckner: *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, Publ. Inst. Math. 23 (1978) 13–20.

[3] S. S. Dragomir: *On some new inequalities of Hermite-Hadamard type for m-convex functions*, Tamkang J. Math. 33/1 (2002) 45–56.

[4] S. S. Dragomir, J. Pečarić, L. E. Persson: *Some inequalities of Hadamard type*, Soochow J. Math. 21 (1995) 335–341.

[5] E. K. Godunova, V. I. Levin: *Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye i nekotorye drugie vidy funkcii*, Vyčislitel. Mat. i. Mat. Fiz. Meždvuzov. Sb. Nauč. Trudov, MGPI, Moskva (1985) 138–142.

[6] N. Mehreen, M. Anwar: *Hermite–Hadamard type inequalities for exponentially p-convex functions and exponentially s-convex functions in the second sense with applications*, J. Inequalities Appl. (2019), art. no. 92.

[7] M. A. Noor, K. I. Noor, M. U. Awan: *Fractional Ostrowski inequalities for (s,m)-Godunova-Levin functions*, Facta Univ. Ser. Math. Inform. 30 (2015) 489–499.

[8] M. E. Özdemir, A. O. Akdemir, E. Set: *On $(h-m)$-convexity and Hadamard-type inequalities*, Transylv. J. Math. Mech. 8/1 (2016) 51–58.

[9] X. Qiang, G. Farid, J. Pečarić, S. B. Akbar: *Generalized fractional integral inequalities for exponentially (s,m)-convex functions*, J. Inequalities Appl. (2020), art. no. 70.

[10] G. H. Toader: *Some generalizations of the convexity*, Proc. Colloq. Approx. Optim. Cluj-Napoca (1984) 329–338.

[11] S. Varošanec: *On h-convexity*, J. Math. Analysis Appl. 326 (2007) 303–311.