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Optimal Control of Second Order Sweeping Processes with Discrete and Differential Inclusions

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We discuss the problem of optimal control theory given by second order sweeping processes with discrete and differential inclusions. The main problem is to derive sufficient optimality conditions for second-order sweeping processes with differential inclusions. By using first and second order difference operators in a continuous problem we associate the second order sweeping processes with a discrete-approximate problem. On the basis of the discretization method in the form of Euler-Lagrange inclusions, optimality conditions for discrete approximate inclusions and transversality conditions are obtained. The establishment of Euler-Lagrange type adjoint inclusions is based on the presence of equivalence relations for locally adjoint mappings. To demonstrate the results obtained, a second-order sweeping process with a "linear" differential inclusion is considered.

Keywords: Euler-Lagrange inclusions, adjoint mappings, set-valued map, approximation, second order, sweeping, transversality.

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1. Introduction

A relatively large number of papers dealing with control problems governed by ordinary and partial differential inclusions *(DFIs)* of first order were developed in [2, 3],[5]–[8],[11]–[26], [28]. In the same spirit, we consider in this paper some control problems governed by a second-order discrete *(DSIs)* and *DFIs*; essentially, we consider an optimal control problem with second order sweeping processes. Some results on the existence of solutions of second-order *DFIs* can be found in [1, 4, 9, 10, 14, 18]. Besides, optimal control of second order *DFIs* plays a crucial role in the mathematical theory of optimal processes. We refer the reader to the survey papers [22, 23, 25], and references therein.

As well is known, sweeping process models were introduced by J. J. Moreau in the 1970s to describe dynamical processes arising in elastoplasticity and related mechanical areas; see [29]. Such models were given in the form of discontinuous *DFIs* governed by the normal cone mappings to nicely moving convex sets. It has been well realized in the sweeping process theory that the Cauchy problem for the basic Moreau's sweeping process and its slightly nonconvex extensions admits unique solutions [2]. Thus, this excludes any possible optimization of sweeping *DFIs* and strikingly distinguishes them from the well-developed optimal control theory for their

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Lipschitzian counterparts. On the other hand, the existence and uniqueness results for sweeping trajectories provide a convenient framework for handling simulation and related issues in various applications to mechanics, nonequilibrium processes with rate-independent memory in the mechanics of elastoplastic and thermoelastoplastic materials as well as in ferromagnetism, piezoelectricity or phase transitions, magnetic hysteresis, social-economic modelling, robotics, electronics [3, 4, 5, 6, 7] among more recent publications with the references therein.

The sweeping process with and without perturbation has been extensively studied by several authors in the literature (see [16],[22],[25],[26],[27],[34],[35] and the references therein). In [35] *DFIs* involving the normal cone to a moving set are investigated. A special attention is paid to the sweeping process associated with sets for which no regularity assumption is required. Formally, the perturbed sweeping process is the *DFIs* $x'(t) \in -N_{C(t)}(x(t)) + f(x(t))$, where $C(t)$ is a closed moving set, with normal cone $N_{C(t)}(x)$ at $x \in C(t)$, and the space variable belongs to a Hilbert space. If $C(t)$ is convex, or mildly non-convex (i.e., uniformly prox-regular), and is Lipschitz as a set-valued map depending on the time t, and the perturbation f is Lipschitz, then it is well known that the Cauchy problem admits one and only one Lipschitz solution. The paper [5] is devoted to the study of a perturbed *DFI* governed by a sweeping process in a Hilbert space. The sweeping process is perturbed by the sum of a single-valued map satisfying a Lipschitz condition and a scalarly upper semicontinuous set-valued map.

In [2], the Mayer problem for a controlled Moreau process is considered, the control acts as a perturbation of the dynamics controlled by a normal cone, and the necessary optimality conditions of Pontryagin's Maximum Principle are derived. In the work [11], the class of subsmooth sets was introduced, which strictly contains the class of closed convex sets and the class of prox-regular sets. The main result is related to the study of perturbed differential sweeping process inclusions where the moving set is nonconvex and non prox-regular and depends both on the time and on the state. It is shown the existence of solution under the subsmoothness of the moving set. In the paper [13] in a vector space endowed with a uniformly Gateaux differentiable norm, it is proved that the Moreau envelope enjoys many remarkable differential properties and that its subdifferential can be completely described through a certain approximate proximal mapping. This description shows, in particular, that the Moreau envelope is essentially directionally smooth. New differential properties are obtained for the distance function associated with a closed set.

The paper [30] concerns optimal control problems for a class of sweeping processes governed by discontinuous unbounded *DFIs*, which are described via normal cone mappings to controlled moving sets. Developing the discretized method and using generalized differential tools of variational analysis of the first and second order nondegenerate necessary optimality conditions are obtained for such problems in extended Euler-Lagrange and Hamiltonian forms involving the Hamiltonian maximization. The paper [6] concerns optimal control of discontinuous *DFIs* of the normal cone type governed by a generalized version of the Moreau sweeping process with control functions acting in both nonconvex moving sets and additive perturbations. This is a new class of optimal control problems in comparison with the analogues considered earlier, in which controlled sweeping sets are described by a convex polyhedron. In the work [9], problems of optimal control of the sweeping process are formulated and studied, when control functions enter the moving sweeping set. First, the existence theorem of optimal solutions was established, and then the necessary optimality conditions for this optimal control problem of a new type were obtained.

The paper [23] deals with a Bolza problem of optimal control theory given by second order convex *DFIs* with second order state variable inequality constraints According to the proposed discretization method, problems with discrete-approximate inclusions and inequalities are investigated, necessary and sufficient conditions of optimality including distinctive "transversality" condition are proved in the form of Euler-Lagrange inclusions. The paper [22] is devoted to second-order polyhedral optimization described by ordinary discrete *DFIs*. The stated second-order discrete problem is reduced to the polyhedral minimization problem with polyhedral geometric constraints and necessary and sufficient conditions of optimality are derived in terms of the polyhedral Euler-Lagrange inclusions. Derivation of the sufficient conditions for the second-order polyhedral *DFIs* is based on the discrete-approximation method. The paper [26] studies the Mayer problem with higher order evolution *DFIs* and functional constraints of optimal control theory; to this end first we use an interesting auxiliary problem with second order discrete-time and discrete approximate inclusions. Are proved necessary and sufficient conditions incorporating the Euler-Lagrange inclusion, the Hamiltonian inclusion, the transversality and complementary slackness conditions.

Despite the presence of the above studies for the first-order sweeping processes, second-order sweeping processes have not been studied in the literature; to the best of our knowledge, there are only two papers devoted to such studies [2, 31]. We are trying to fill this gap in the literature in this paper. The problems and results discussed here are new; we consider a special form of the convex optimization problem with second-order sweeping processes with "perturbation". The novelty of our approach is the calculation and equivalence of locally adjoint mappings *(LAMs)* for the discretization problem in order to obtain the necessary and sufficient optimality conditions for convex second-order sweeping processes with *DSI* and *DFI*. In fact, the difficulty in problems with higher order *DFI* is rather to construct the Euler-Lagrange type adjoint inclusions and the suitable transversality conditions. Here we have a twofold goal: to study optimality conditions for second order sweeping processes with *DSIs* of control systems with respect to discrete approximations and to derive sufficient optimality conditions for second order sweeping processes with *DFIs*. We are not familiar with any results in these directions for such systems even in the convex case. It is well known that, in general, many convex optimization applications are still waiting to be discovered and are the basis for several heuristics for solving non-convex problems. There are also theoretical or conceptual advantages of setting the problem as a convex optimization problem. The main property of convex optimization problems is that any locally optimal point is also globally optimal.

The paper conditionally can be divided into four parts and organized as follows, In the first part of the paper, the optimal control problem is studied, in which the dynamics of the system are described by the so-called second-order sweeping processes with DSI. In the second part, for transition to the problem with discrete-approximate sweeping processes the idea of equivalence of $LAMs$ is important. The third part of the paper is devoted to the construction of a discrete-approximate problem for second-order sweeping processes with *DFIs*. In the fourth part of the paper, the optimization of second-order sweeping processes with *DFIs* are considered and sufficient conditions of optimality for sweeping processes with *DFIs* are proved.

Thus, in Section 2 for the reader's convenience from the monograph of Mahmudov [11] and the papers [13]–[16] the necessary notions and results such as LAM properties in finite dimensional Euclidean spaces, Hamiltonian functions, argmaximum sets, cone of tangent directions etc. are given. Then the problems for second order sweeping processes with *DSIs* and *DFIs* are formulated.

In Section 3 the *LAMs* to algebraic sum $F_1 + F_2$ and to product with a real number $\alpha F(\alpha \neq 0)$ of convex set-valued mappings are calculated. Further, in terms of LAM necessary and sufficient conditions of optimality for a sweeping processes with $DSIs$ is proved.

In Section 4, using difference operators of the first and second order with a problem for second order sweeping processes with *DFIs* we associate the second order sweeping processes for a problem with discrete approximation, and establish a connection between discrete and discrete-approximate sweeping *LAMs*. Then we use the equivalence results of *LAMs* connecting the main results of discrete and continuous sweeping processes.

In Section 5, we derive sufficient optimality conditions for second-order sweeping processes with *DFIs*. The formulation of these conditions is based on a formal limit procedure in the optimality conditions of second order sweeping processes with a discrete-approximate problem. Thus, using LAM and the discrete approximation method, in the Euler-Lagrange form, we derive sufficient optimality conditions for second-order sweeping processes with *DFI*. Moreover, Mayer's problem is generalized to the case of Bolza. In fact, using the functional analysis approach in the convex problems, that is, compactness in the corresponding function spaces, uniform convergence and other approaches to functional analysis, we can justify necessity of these sufficient conditions. However, proving the necessary conditions is rather difficult and is a separate subject of discussion and is omitted.

2. Problem statement and preliminaries

For the convenience of the reader the basic notions can be found in [20]. Let $\mathbb{R}^n$ be the standard n-dimensional Euclidean space, $\langle x, y\rangle$ be the usual inner product of elements $x, y \in \mathbb{R}^n$ and (x, y) be a pair of x, y. A set-valued mapping $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is *convex* if its graph $gphF = \{(x, y, z) : z \in F(x, y)\}$ is a convex subset of $\mathbb{R}^{3n}$. It is *convex-valued* if $F(x, y)$ is a convex set for each $x, y \in domF = \{(x, y) : F(x, y) \neq \varnothing\}$.

As usual the *Hamiltonian function* and *argmaximum set* for a set-valued mapping F is defined as follows

$$H_F(x, y, z^*) = \sup_z \left\{\langle z, z^*\rangle : z \in F(x, y)\right\}, z^* \in \mathbb{R}^n,$$

$$F(x, y; z^*) = \left\{z \in F(x, y) : \langle z, z^*\rangle = H_F(x, y, z^*)\right\}$$

respectively. The *interior* and the *relative interior* of the set $A \subset \mathbb{R}^{3n}$ are denoted by $intA$ and riA, respectively, where we put $H_F(x, y, z^*) = -\infty$ if $F(x, y) = \varnothing$. Obviously, for a convex set-valued mapping F the Hamiltonian function $H_F(\cdot, \cdot, z^*)$ is concave.

Definition 2.1. Let A be a convex set. Then the convex cone $K_A(w)$, $w = (x, y, z)$ is called the *cone of tangent directions* at a point $w \in A$ to the set A if from $\bar{w} = (\bar{x}, \bar{y}, \bar{z}) \in K_A(w)$ it follows that is a tangent vector at $w \in A$.
Then the set-valued mapping F at a point $(x, y, z) \in gphF$ satisfies

$$K_{gphF}(x, y, z) = cone\big[gphF - (x, y, z)\big]$$
$$= \Big\{(\bar{x}, \bar{y}, \bar{z}) : \bar{x} = \alpha(\tilde{x} - x),\ \bar{y} = \alpha(\tilde{y} - y),\ \bar{z} = \alpha(\tilde{z} - z),\ \alpha > 0\Big\}, \forall\ (\tilde{x}, \tilde{y}, \tilde{z}) \in gphF.$$

Definition 2.2. Let $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ be a convex set-valued mapping. Then a set-valued mapping $F^*(\cdot, w) : \mathbb{R}^n \rightrightarrows \mathbb{R}^{2n}$ defined by

$$F^*(z^*; w) := \Big\{(x^*, y^*) : (x^*, y^*, -z^*) \in K^*_{gphF}(w)\Big\}$$

is called a *LAM* to F at a point $(x, y, z) \in gphF$, where $K^*_{gphF}(x, y, z)$ is the dual to the cone of tangent vectors $K_{gphF}(x, y, z)$. Basically, we will use a more suitable equivalent definition of *LAM* for $(x, y, z) \in gphF$, $z \in F_A(x, y, z^*)$:

$$F^*(z^*; (x, y, z)) := \left\{(x^*, y^*) : \begin{array}{c} H_F(\tilde{x}, \tilde{y}, z^*) - H_F(x, y, z^*) \\ \le \langle x^*, \tilde{x} - x\rangle + \langle y^*, \tilde{y} - y\rangle, \end{array} \forall(\tilde{x}, \tilde{y}) \in \mathbb{R}^{2n}\right\}.$$

We state here the following widely used Theorem 2.1 of [20]:

Lemma 2.3. *Let $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ be a convex set-valued mapping. Then one has*

$$F^*\big(z^*; (x, y, z)\big) = \begin{cases} \partial_{(x,y)} H_F(x, y, z^*), & z \in F_A(x, y; z^*), \\ \varnothing\ , & z \notin F_A(x, y; z^*) \end{cases}$$

where $\partial_{(x,y)} H_F(x, y, z^*) := -\partial_{(x,y)}\big[-H_F(x, y, z^*)\big]$ is the usual subdifferential of $H_F(\cdot, \cdot, z^*)$ at the point (x, y).

For the further presentation we present Proposition 4.5 of [24].

Proposition 2.4. *Let $\Omega(\cdot, \cdot)$ be a proper convex function defined by the relation $\Omega(x, y) \equiv \Phi\Big(x, \frac{y-x}{h}\Big)$. Then the following inclusions are equivalent:*

(1) $\quad \Big(\bar{x}^*,\ \bar{y}^*\Big) \in \partial_{(x,y)}\Omega(\tilde{x}, \tilde{y}),\ (\tilde{x}, \tilde{y}) \in dom\Omega,$

(2) $\quad \Big(\bar{x}^* + \bar{y}^*,\ h\bar{y}^*\Big) \in \partial\Phi\Big(\tilde{x},\ \dfrac{\tilde{y} - \tilde{x}}{h}\Big).$

The optimal control of second-order sweeping processes with $DSIs$ is the subject of research in Section 3, which deals with the following second-order discrete model labelled as (SWD):

$$(SWD) \qquad \begin{cases} \text{minimize } \Phi(x_{T-1}, x_T), & (1) \\ x_{t+2} \in -N_{C(x_t)}(x_{t+1}) + F(x_t, x_{t+1}, t)\ , t = 0, \dots, T-2 & (2) \\ x_0 = \bar{\omega}_0,\ x_1 = \bar{\omega}_1 \in C(\bar{\omega}_0), & (3) \end{cases}$$

where $x_t \in \mathbb{R}^n$, $\Phi(\cdot,\cdot) : \mathbb{R}^{2n} \to \mathbb{R}^1$ are real-valued functions, $F(\cdot, t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, C is a convex mapping defined as $t \to C(t) \equiv C(x_t)$, $t = 0, \dots, T$, $C(t) \subset \mathbb{R}^n$; T is a fixed natural number, and $\bar{\omega}_0, \bar{\omega}_1$ are fixed vectors, $N_C(x)$ is the usual normal cone: $N_C(x) := \{u \in \mathbb{R}^n : \langle u, y - x\rangle \le 0, y \in C\}$ if $x \in C$. In fact, N_C is cone-valued mapping. If C is convex, the normal cone $N_C(x)$ is a closed convex cone containing the origin and $N_C(x) = \{0\}$ for $x \in intC$. By convention, we put $N_C(x) := \varnothing$ for $x \notin C$. Clearly,

$$dom(-N_C + F(\cdot, t)) = dom(-N_C) \cap domF(\cdot, t) = gphC \cap domF(\cdot, t)$$

where $gphC = \{(x, y) : y \in C(x)\}$.

Condition (3) is a discrete analogous of Cauchy initial conditions for second-order sweeping processes with *DFIs*. A sequence $\{x_t\}_{t=0}^T = \{x_t : t = 0, 1, \dots, T\}$ is called a *feasible trajectory* for the stated problem (1)–(3).

We will investigate the convex problem *(SWD)*, where $F(\cdot, t)$ is a convex set-valued mapping, $C(t)$ is convex closed mapping and Φ is a proper convex function, that is, it does not assume the value $-\infty$ and is not identically equal to $+\infty$.

Definition 2.5. Let us denote $G(x, y, t) = -N_{C(x)}(y) + F(x, y, t)$. Then a convex problem (SWD) satisfies the *non-degeneracy condition*, if for some feasible solution $\{x_t\}_{t=0}^T$ we have either (i) or (ii):

(i) $\big(x_t, x_{t+1}, x_{t+2}\big) \in int[gphG(\cdot, t)]$, $t = 0, \dots, T-2$ (with the possible exception of one fixed t) and $\Phi(\cdot,\cdot)$ is continuous at (x_{T-1}, x_T),

(ii) $\big(x_t, x_{t+1}, x_{t+2}\big) \in ri[gphG(\cdot, t)], (x_{T-1}, x_T) \in ri[dom\Phi]$.

We recall that the convex cones $K_1, K_2, \dots, K_m$ are called separable, if there exist vectors $x_i^* \in K_i^* (i = 1, \dots, m)$, not all zero such that $x_1^* + \dots + x_m^* = 0$. Besides, by Theorem 1.12 [20, p.20] either $\big(K_1 \cap \dots \cap K_m\big)^* = K_1^* + \dots + K_m^*$ or the cones $K_1, K_2, \dots, K_m$ are not separable.

The non-degeneracy condition provides that if $\{\tilde{x}_t\}_{t=0}^T$ is the optimal trajectory in the problem *(SWD)*, then the cones of tangent directions $K_{gphG(\cdot,t)}(\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2})$ are not separable. It follows that the requirement of Theorem 3.2 in [20] regarding to necessary conditions for an extremum in convex programming problem is satisfied.

Section 5 is devoted to the convex problem for second-order sweeping processes with *DFIs* labelled as (SWC):

$$(SWC) \qquad \begin{cases} \text{minimize } \Phi(x(T), x'(T)), & (4) \\ x''(t) \in -N_{C(x(t))}(x'(t)) + F(x(t), x'(t), t),\ \text{a.e. } t \in [0, T], & (5) \\ x(0) = \omega_0,\ x'(0) = \omega_1 \in C(\omega_0), & (6) \end{cases}$$

where $F(\cdot, t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$, is a convex set-valued mapping, Φ is continuous and convex function $\Phi(\cdot,\cdot) : \mathbb{R}^{2n} \to \mathbb{R}^1, \omega_0, \omega_1$ are fixed vectors, $N_C(x)$ as usual normal cone, where C is closed convex mapping, T is a positive real number. The problem is to find an arc $\tilde{x}(\cdot)$ of the Cauchy problem *(SWC)* for sweeping processes with *DFIs* satisfying the second-order *DFIs* almost everywhere (*a.e.*) on $[0,T]$ and the initial conditions (6) that minimize the Mayer functional $\Phi(x(T), x'(T))$. To do this, we first derive the necessary and sufficient optimality conditions for the discrete-approximate problem *(SWDA)*, and then, by going over to the formal limit, establish sufficient optimality conditions for the problem *(SWC)* described by *DFI*. The arc $x(\cdot)$ and its first-order derivative are absolutely continuous function on a time interval $[0,T]$ and $x''(\cdot) \in L_1^n([0,T])$. Clearly, such class of functions is a Banach space, endowed with the different equivalent norms.

3. Optimality conditions for sweeping processes with DSIs

In this section we consider the convex problem (1)–(3). The algebraic sum of set-valued mappings $-N_{C(x_t)}(x_{t+1})$ and $F(x_t, x_{t+1}, t)$ is denoted by G:

$$G(x_t, x_{t+1}, t) \equiv -N_{C(x_t)}(x_{t+1}) + F(x_t, x_{t+1}, t).$$

Proposition 3.1. *Let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a convex closed set-valued mapping. Then the cone of tangent directions $K_{C(\tilde{x}_t)}(\tilde{x}_{t+1}), \tilde{x}_{t+1} \in C(\tilde{x}_t)$ is closed.*

Proof. By the definition of cone of tangent directions, we have

$$K_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) = \{\bar{x}_{t+1} : \bar{x}_{t+1} = \alpha(x_{t+1} - \tilde{x}_{t+1}),\ x_{t+1} \in C(\tilde{x}_t),\ \alpha > 0\}.$$

Since C is closed-valued, for arbitrary convergent sequences $\{x_{t+1}^n\}$, $x_{t+1}^n \to x_{t+1}^0$, as $n \to \infty$, $x_{t+1}^n \in C(\tilde{x}_t)$ it follows $x_{t+1}^0 \in C(\tilde{x}_t)$. Then since $x_{t+1}^0 - \tilde{x}_{t+1} \in C(\tilde{x}_t)$ the limit of $\bar{x}_{t+1}^n = \alpha(x_{t+1}^n - \tilde{x}_{t+1})$ designated by $\bar{x}_{t+1}^0 = \alpha(x_{t+1}^0 - \tilde{x}_{t+1})$ belongs to $K_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$, i.e., $\bar{x}_{t+1}^0 \in K_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$ and $K_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$ is closed. □

Proposition 3.2. *Let $C(\tilde{x}_t)$ be a convex set. Then*

$$-N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) = K^*_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$$

and the Hamiltonian function $H_{-N_{C(\tilde{x}_t)}}(\tilde{x}_{t+1}, \cdot)$ is the indicator function of the cone $K_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$, i.e.

$$H_{-N_{C(\tilde{x}_t)}}(\tilde{x}_{t+1}, -x^*_{t+2}) = \begin{cases} 0, & \text{if} \quad x^*_{t+2} \in K_{C(\tilde{x}_t)}(\tilde{x}_{t+1}), \\ +\infty, & \text{if} \quad x^*_{t+2} \notin K_{C(\tilde{x}_t)}(\tilde{x}_{t+1}), \end{cases}$$

where $K^*_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$ is the dual to the cone of tangent vectors $K_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$.

Proof. In fact, by the definition of a normal cone at a point $\tilde{x}_{t+1} \in C(\tilde{x}_t)$

$$\begin{aligned} N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) &= \{\bar{u}_{t+2} : \langle \bar{u}_{t+2}, x_{t+1} - \tilde{x}_{t+1} \rangle \le 0,\ x_{t+1} \in C(\tilde{x}_t)\} \\ &= \{\bar{u}_{t+2} : \langle -\bar{u}_{t+2}, x_{t+1} - \tilde{x}_{t+1} \rangle \ge 0,\ x_{t+1} \in C(\tilde{x}_t)\} \\ &= \{u_{t+2} : \langle u_{t+2}, x_{t+1} - \tilde{x}_{t+1} \rangle \ge 0,\ x_{t+1} - \tilde{x}_{t+1} \in K_{C(\tilde{x}_t)}(\tilde{x}_{t+1})\} \\ &= -K^*_{C(\tilde{x}_t)}(\tilde{x}_{t+1}). \end{aligned}$$

We note that, if $\tilde{x}_{t+1} \in int\, C(\tilde{x}_t)$, then we have $K_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) = \mathbb{R}^n$ and consequently, $K^*_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) = -N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) = \{0\}$.

Furthermore,
$$\begin{aligned} H_{-N_{C(\tilde{x}_t)}}(\tilde{x}_{t+1}, -x^*_{t+2}) &= H_{N_{C(\tilde{x}_t)}}(\tilde{x}_{t+1}, x^*_{t+2}) \\ &= \sup_{u_{t+2}} \left\{ \langle u_{t+2}, -x^*_{t+2} \rangle : u_{t+2} \in -N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) \right\} \\ &= \sup_{u_{t+2}} \left\{ \langle u_{t+2}, -x^*_{t+2} \rangle : u_{t+2} \in K^*_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) \right\} \\ &= \begin{cases} 0, & \text{if} \quad x^*_{t+2} \in K^{**}_{C(\tilde{x}_t)}(\tilde{x}_{t+1}), \\ +\infty, & \text{otherwise}, \end{cases} \end{aligned}$$

where $K^{**}_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) = \bar{K}_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$ and $\bar{K}_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) = K_{C(\tilde{x}_t)}(\tilde{x}_{t+1})$ because of Proposition 3.2. □

Proposition 3.3. *Let $F_i : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n, i = 1, 2,$ be convex set-valued mappings and assume that the functions $H_{F_i}(\cdot, z^*)$ are continuous except possibly one at a point $(\bar{x}, \bar{y}) \in domF_1 \cap domF_2$.*
Then for the LAM of the sum of the set-valued mappings

$$G(x, y) = (F_1 + F_2)(x, y) \equiv F_1(x, y) + F_2(x, y), \ (x, y) \in domF_1 \cap domF_2$$

we have $G^*(z^*; (x, y, z)) = F_1^*(z^*; (x, y, z_1)) + F_2^*(z^*; (x, y, z_2)),$ *where* $z = z_1 + z_2,$ $z_i \in F_i(x, y),\ i = 1, 2.$

Proof. In fact, the sum of convex set-valued mappings is convex. Furthermore,

$$H_G(x, y, z^*) = H_{F_1}(x, y, z^*) + H_{F_2}(x, y, z^*), \ (x, y) \in domF_1 \cap domF_2$$

by the hypothesis of the proposition we derive from Theorem 1.29 [20]

$$\partial_{(x,y)} H_G(x, y, z^*) = \partial_{(x,y)} H_{F_1}(x, y, z^*) + \partial_{(x,y)} H_{F_2}(x, y, z^*). \tag{7}$$

Besides,
$$\begin{aligned} G_A(x, y, z^*) &= \left\{ z \in G(x, y) : \langle z, z^* \rangle = H_G(x, y, z^*) \right\} \\ &= \left\{ z_1 + z_2, z_i \in F_i(x, y) : \langle z_1 + z_2, z^* \rangle \right. \\ &\quad \left. = H_{F_1}(x, y, z^*) + H_{F_2}(x, y, z^*),\ i = 1, 2 \right\} \\ &= F_{1A}(x, y, z^*) + F_{2A}(x, y, z^*) \end{aligned} \tag{8}$$

Now, we remind that by Lemma 2.3 of Section 2 for we have

$$G^*(z^*; (x, y, z)) = \partial_{(x,y)} H_G(x, y, z^*). \tag{9}$$

Then taking into account (8), (9) in (7) we have the needed result. □

Proposition 3.4. *Let $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ be convex set-valued mapping and assume $G(x, y) = \alpha F(x, y)$, where $\alpha \neq 0$ is any real number. Then*

$$G^*(z^*; (x, y, \alpha z)) = |\alpha| F^*(z^* sign\, \alpha; (x, y, z)).$$

Proof. It is elementary to verify that for a convex mapping F the set-valued mapping $G(x, y) = \alpha F(x, y)$ is convex. Then it is not hard to see that

$$H_G(x, y, z^*) = \sup_z \left\{ \langle z, z^* \rangle : z \in G(x, y) \right\}$$
$$= \sup_z \left\{ \langle z, z^* \rangle : z \in \alpha F(x, y) \right\} = \sup_{z_1} \left\{ \langle \alpha z_1, z^* \rangle : z_1 \in F(x, y) \right\}$$
$$= |\alpha| \sup_{z_1} \left\{ \langle z_1, sign\, \alpha z^* \rangle : z_1 \in F(x, y) \right\} = |\alpha| H_F(x, y, sign\, \alpha z^*).$$

Therefore $\partial_{(x,y)}H_G(x,y,z^*) = |\alpha|\partial_{(x,y)}H_F(x,y,sign\,\alpha z^*)$. Now using Lemma 2.3 we obtain the required result. □

Theorem 3.5. *Let $F(\cdot,t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ be convex set-valued mappings and let $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^1 \cup \{+\infty\}$ be a proper convex function. Moreover, let $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be convex closed set-valued mapping. Then, in order for $\{\tilde{x}_y\}_{t=0}^T$ to be an optimal trajectory of the problem (SVD), it is necessary that there exists a number $\lambda \in \{0,1\}$ and vectors $x_t^*, x_T^*, \eta_t^*, t = 0,\dots,T-1$ not all equal to zero, satisfying the following Euler-Lagrange inclusions and the transversality condition:*

$$\begin{aligned}
&(1) \qquad \left(x_t^* - \eta_t^*, \eta_{t+1}^*\right) \in F^*\left(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{v}_{t+2}), t\right),\ t = 0,\dots,T-2,\\
&\qquad\quad \tilde{u}_{t+2} \in -N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}),\ \tilde{v}_{t+2} \in F(\tilde{x}_t, \tilde{x}_{t+1}, t),\ \tilde{x}_{t+2} = \tilde{u}_{t+2} + \tilde{v}_{t+2},\\
&\qquad\quad H_{-N_{C(\tilde{x}_t)}}(\tilde{x}_{t+1}, x_{t+2}^*) = -\langle x_{t+2}^*, \tilde{u}_{t+2}\rangle,\\
&(2) \qquad \left(\eta_{T-1}^* - x_{T-1}^*, -x_T^*\right) \in \lambda\partial_{(x,y)}\Phi(\tilde{x}_{T-1}, \tilde{x}_T).
\end{aligned}$$

Moreover, if the non-degeneracy condition is satisfied, then $\lambda = 1$ and these conditions are sufficient for optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$.

Proof. By Theorem 2.2 [22] for optimality of the trajectory $\{\tilde{x}_t\}_{t=0}^T$ it is necessary that there is a number $\lambda \in \{0,1\}$ and vectors $x_t^*, x_T^*, \eta_t^*, t = 0,\dots,T-1$ at the same time, not all equal to zero, satisfying discrete Euler-Lagrange inclusions

$$\left(x_t^* - \eta_t^*, \eta_{t+1}^*\right) \in G^*\left(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2}), t\right),\ t = 0,\dots,T-2 \tag{10}$$

and transversality condition

$$\left(\eta_{T-1}^* - x_{T-1}^*, -x_T^*\right) \in \lambda\partial_{(x,y)}\Phi(\tilde{x}_{T-1}, \tilde{x}_T), \tag{11}$$

where $\lambda = 1$ is guaranteed that these conditions are sufficient for optimality.
Let $M(x_t, x_{t+1}) \equiv -N_{C(x_t)}(x_{t+1})$. Then according to Proposition 3.3 the *LAM* G^* corresponding to the algebraic sum $G(x_t, x_{t+1}) = M(x_t, x_{t+1}) + F(x_t, x_{t+1}, t)$ is

$$\begin{gathered}
G^*\left(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{x}_{t+2})\right) = M^*\left(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, -\tilde{u}_{t+2})\right) + F^*\left(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, \tilde{v}_{t+2}), t\right)\\
\tilde{x}_{t+2} = \tilde{u}_{t+2} + \tilde{v}_{t+2},\ \tilde{u}_{t+2} \in -N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}),\ \tilde{v}_{t+2} \in F(\tilde{x}_t, \tilde{x}_{t+1}, t). \qquad (12)
\end{gathered}$$

In turn, by Proposition 3.4 in the case $\alpha = -1$ we have

$$\begin{aligned}
M^*\Big(x_{t+2}^*; (\tilde{x}_t, \tilde{x}_{t+1}, -\tilde{u}_{t+2})\Big) &= N_{C(\tilde{x}_t)}^*(-x_{t+2}^*; \tilde{x}_{t+1}, \tilde{u}_{t+2}),\\
\partial_{(x,y)}H_M(x_t, x_{t+1}, x_{t+2}^*) &= \partial_{(x,y)}H_{-N_{C(x_t)}}(x_{t+1}, -x_{t+2}^*),
\end{aligned}$$

where on the Lemma 2.3 we can write

$$\begin{aligned}
&N_{C(\tilde{x}_t)}^*(-x_{t+2}^*; \tilde{x}_{t+1}, \tilde{u}_{t+2})\\
&= \begin{cases} \partial_{(x,y)}H_{-N_{C(\tilde{x}_t)}}(x_{t+1}, -x_{t+2}^*), & \text{if} \quad \tilde{u}_{t+2} \in N_{C(\tilde{x}_t)A}(\tilde{x}_{t+1}; -x_{t+2}^*),\\ \varnothing\ , & \text{if} \quad \tilde{u}_{t+2} \notin N_{C(\tilde{x}_t)A}(\tilde{x}_{t+1}; -x_{t+2}^*), \end{cases}\\
&N_{C(\tilde{x}_t)A}(\tilde{x}_{t+1}; -x_{t+2}^*)\\
&= \Big\{u_{t+2} \in N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) : -\langle x_{t+2}^*, \tilde{u}_{t+2}\rangle = H_{-N_{C(\tilde{x}_t)}}(\tilde{x}_{t+1}, -x_{t+2}^*)\Big\}.
\end{aligned}$$

But by Proposition 3.2 for $(x_t, x_{t+1}) \in domH_{-N_{C(\cdot)}}(\cdot, x^*_{t+2})$

$$\partial_{(x,y)} H_{-N_{C(\tilde{x}_t)}}(x_{t+1}, -x^*_{t+2}) = \{(0,0)\},\ x^*_{t+2} \in -K_{C(\tilde{x}_t)}(x_{t+1}),$$

where $x^*_{t+2} \in -K_{C(\tilde{x}_t)}(x_{t+1})$ is equivalent to $H_{C(x_t)}(x^*_{t+2}) = \langle x^*_{t+2}, x_{t+1}\rangle$. Then taking into account (10)–(12) we obtain the desired result. □

Remark 3.6. Suppose that for all $x \in domC$ we have $C(x) \equiv \mathbb{R}^n$. Then obviously, $N_{C(\tilde{x}_t)}(\tilde{x}_{t+1}) \equiv \{0\}$. Consequently, $\tilde{u}_{t+2} = 0$ in Theorem 3.5 and so $\tilde{x}_{t+2} = \tilde{v}_{t+2}$. In turn it follows that the condition $H_{-N_{C(\tilde{x}_t)}}(\tilde{x}_{t+1}, -x^*_{t+2}) = -\langle x^*_{t+2}, \tilde{u}_{t+2}\rangle$ is superfluous. Thus, by substituting $\tilde{v}_{t+2} = \tilde{x}_{t+2}$ in the adjoint Euler-Lagrange inclusion (1) we obtain the necessary and sufficient condition of optimality for *DSIs*.

4. Necessary and sufficient conditions of optimality for discrete-approximate sweeping problems

Let h be a step on the t-axis and let $x(t) \equiv x_h(t)$ be a grid functions on a uniform grid on $[0,T]$. We introduce the following first and second order difference operators (forward and backward difference approximations) $\Delta_+ x(t) \equiv \Delta x(t)$ and $\Delta_- x(t)$, $t = 0, \dots, T-2h$

$$\Delta x(t) = \frac{1}{h}[x(t+h) - x(t)],\quad \Delta_- x(t) = \frac{1}{h}[x(t) - x(t-h),],$$
$$\Delta^2 x(t) = \frac{1}{h}[\Delta x(t+h) - \Delta x(t)]$$

and associate with the problem *(SWC)* the following second order discrete-approximate evolution discrete-approximate problem:

$$\begin{cases} \text{minimize } \Phi(x(T), \Delta_- x(T)), \\ \Delta^2 x(t) \in -N_{C(x(t))}(\Delta x(t)) + F(x(t), \Delta x(t), t),\ t = 0, \dots, T-2h, \\ x(0) = \omega_0,\ \Delta x(0) = \omega_1 \in C(\omega_0). \end{cases} \tag{13}$$

Here $\Delta^2 x(t)$ can be represented as $\Delta^2 x(t) = \Delta^2 u(t) + \Delta^2 v(t)$, where

$$\Delta^2 u(t) \in -N_{C(x(t))}(\Delta x(t));\ \Delta^2 v(t) \in F(x(t), \Delta x(t), t).$$

We reduce the problem (13) to a problem of the form *(SWD)*. To do this, we introduce the following auxiliary set-valued mappings and function

$$P(x,y,t) = 2y - x + h^2 F\Big(x, \frac{y-x}{h}, t\Big),\ D(x) = x + hC(x),$$
$$\Omega(x(T-h), x(t)) = \Phi(x(T), \Delta_- x(T)). \tag{14}$$

Proposition 4.1. *The following equality holds:*

$$N_{D(\tilde{x}(t))}(\tilde{x}(t+h)) = N_{C(\tilde{x}(t))}(\Delta\tilde{x}(t)).$$

Proof. In fact

$$N_{D(\tilde{x}(t))}(\tilde{x}(t+h)) = \{\bar{u}(t+2h) : \langle \bar{u}(t+2h), x(t+h) - \tilde{x}(t+h)\rangle \le 0,$$

$$x(t+h) \in D(\tilde{x}(t))\}$$

$$= \Big\{\bar{u}(t+2h) : \Big\langle h\bar{u}(t+2h), \frac{x(t+h)-\tilde{x}(t)}{h} - \frac{\tilde{x}(t+h)-\tilde{x}(t)}{h}\Big\rangle \le 0, \frac{\tilde{x}(t+h)-\tilde{x}(t)}{h} \in C(\tilde{x}(t))\Big\}$$

$$= \Big\{\frac{1}{h}\bar{v}(t+2h) : \Big\langle \bar{v}(t+2h), \frac{x(t+h)-\tilde{x}(t)}{h} - \Delta\tilde{x}(t)\Big\rangle \le 0,\ \Delta\tilde{x}(t) \in C(\tilde{x}(t))\Big\}$$

$$= \frac{1}{h}N_{C(\tilde{x}(t))}(\Delta\tilde{x}(t)) = N_{C(\tilde{x}(t))}(\Delta\tilde{x}(t)). \qquad \square$$

Then using first and second order difference operators $\Delta x(t)$ and $\Delta^2 x(t)$ we can rewrite the problem (13) as follows:

$$\textit{(SWDA)} \qquad \begin{cases} \text{minimize } \ \Omega(x(T-h), x(T)), \\ x(t+2h) \in -N_{D(x(t))}(x(t+h)) + P(x(t), x(t+h), t), \\ x(0) = \omega_0,\ x(h) = \omega_0 + h\omega_1 \in C(\omega_0),\ t = 0, \dots, T-2h. \end{cases}$$

Here it is taken into account here that since $N_{C(x(t))}(\Delta x(t))$ is a normal cone $h^2 N_{C(x(t))}(\Delta x(t)) = N_{C(x(t))}(\Delta x(t))$.

Now by Theorem 3.5 for optimality of the trajectory $\{\tilde{x}(t)\} \equiv \{\tilde{x}(t) : t = 0, h, \dots, T\}$, in the problem $(SWDA)$ it is necessary that there exist a pair of vectors $\{x^*(t), \mu^*(t)\}$ and a number $\lambda \in \{0, 1\}$, not all zero, such that

$$\begin{gathered} \big(x^*(t) - \mu^*(t), \mu^*(t+h)\big) \in P^*\big(x^*(t+2h);\ (\tilde{x}(t), \tilde{x}(t+h), \tilde{v}(t+2h)), t\big), \\ \tilde{u}(t+2h) \in 2\tilde{u}(t+h) - \tilde{u}(t) - h^2 N_{D(\tilde{x}(t))}(\tilde{x}(t+h)), \\ \tilde{v}(t+2h) \in 2\tilde{v}(t+h) - \tilde{v}(t) + h^2 F(\tilde{x}(t), \tilde{x}(t+h), t), \qquad (15) \\ \Delta^2\tilde{x}(t) = \Delta^2\tilde{u}(t) + \Delta^2\tilde{v}(t),\ t = 0, \dots, T-2h, \\ H_{-N_{D(\tilde{x}(t))}}(\tilde{x}(t+h), x^*(t+2h)) = -\langle x^*(t+2h), \Delta^2\tilde{u}(t)\rangle, \\ \big(\mu^*(T-h) - x^*(T-h), -x^*(T)\big) \in \lambda\partial_{(x,y)}\Omega(\tilde{x}(T-h), \tilde{x}(T)). \qquad (16) \end{gathered}$$

In the above relationship (15) we should express the *LAM* P^* in terms of *LAM* F^*. Before all let us prove two lemmas crucial in what follows.

Proposition 4.2. *The following equality is valid*

$$H_{-N_{D(\tilde{x}(t))}}(\tilde{x}(t+h), x^*(t+2h)) = H_{-N_{C(\tilde{x}(t))}}(\Delta\tilde{u}(t), x^*(t+2h).$$

Proof. Indeed, using Proposition 4.1 we can write

$$\begin{aligned} & H_{-N_{D(\tilde{x}(t))}}(\tilde{x}(t+h), x^*(t+2h)) \\ & = \sup\big\{\langle \Delta u(t), x^*(t+2h)\rangle : \Delta u(t) \in -N_{D(\tilde{x}(t))}(\tilde{x}(t+h))\big\} \\ & = \sup\big\{\langle \Delta u(t), x^*(t+2h)\rangle : \Delta u(t) \in -N_{C(\tilde{x}(t))}(\tilde{x}(t+h))\big\} \\ & = H_{-N_{C(\tilde{x}(t))}}(\Delta\tilde{x}(t), x^*(t+2h)). \qquad \square \end{aligned}$$

Lemma 4.3. *Let F and C be convex set-valued mappings. Then, between the Hamiltonian functions H_P, H_F and H_D, H_C, there are the following connections*

(1) $$H_P\big(x, y, z^*\big) = \langle 2y - xz^*\rangle + h^2 H_F\Big(x, \frac{y-x}{h}, z^*\Big),$$

(2) $$H_{D(x)}(y^*) = \langle x, y^*\rangle + hH_{C(x)}(y^*).$$

Proof. Indeed, by definition of the Hamiltonian function and set-valued mapping P (see(14)) we can write

$$\begin{aligned}
H_P\big(x, y, z^*\big) &= \sup\big\{\langle z, z^*\rangle : z \in P(x, y)\big\}\\
&= \langle 2y - x, z^*\rangle + h^2 \sup\Big\{\langle z_1, z^*\rangle : z_1 \in F\Big(x, \frac{y-x}{h}, z^*\Big)\Big\}\\
&= \langle 2y - x, z^*\rangle + h^2 H_F\Big(x, \frac{y-x}{h}, z^*\Big).
\end{aligned}$$

Furthermore

$$\begin{aligned}
H_D(y^*) &= \sup\big\{\langle y, y^*\rangle : y \in D(x)\big\} = \sup\big\{\langle x + hy_1, y^*\rangle : y_1 \in C(x)\big\}\\
&= \langle x, y^*\rangle + \sup\big\{\langle hy_1, y^*\rangle : y_1 \in C(x)\big\} = \langle x, y^*\rangle + hH_C(y^*). \qquad \square
\end{aligned}$$

The following Lemma 4.4 and Proposition 4.5 play a crucial role in the construction of LAM to the original discrete approximate problem (14) or (15).

Lemma 4.4. *Let $F : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ be a convex set-valued mapping and $P : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ be defined as $P(x, y) = 2y - x + h^2F(x,(y - x)/h)$. Then the following inclusions of LAMs are equivalent*

(1) $$(x^*, y^*) \in P^*(z^*; (x, y, z)),\ v \in P_A(x, y, z^*),$$

(2) $$\Big(\frac{x^* + y^* - z^*}{h^2}, \frac{y^* - 2z^*}{h}\Big) \in F^*\Big(z^*; \Big(x, \frac{y-x}{h}, \frac{z - 2y^* + x}{h^2}\Big)\Big),$$
$$\frac{z - 2y + x}{h^2} \in F_A\Big(x, \frac{y-x}{h}; z^*\Big),\ z^* \in \mathbb{R}^n.$$

Proof. Let $K_{gphP}(x, y, z), (x, y, z) \in gphP$ be the cone of tangent directions. It is easy to verify that, the inclusions

$$(\bar{x}, \bar{y}, \bar{z}) \in K_{gphP}(x, y, z) \tag{17}$$

and
$$\Big(\bar{x}, \frac{\bar{y}, -\bar{x}}{h}, \frac{\bar{z} - 2\bar{y} + \bar{x}}{h^2}\Big) \in K_{gphF}\Big(x, \frac{y-x}{h}, \frac{z - 2y + x}{h^2}\Big) \tag{18}$$

are equivalent. For example, we prove that (18) implies (17). The converse is proved by analogy. In fact, satisfaction of (18) means that for $\alpha > 0$

$$\begin{aligned}
\bar{x} = \alpha(\tilde{x} - x),\quad \frac{\bar{y} - \bar{x}}{h} &= \alpha\Big(\frac{\bar{y}, -\bar{x}}{h}, \frac{y - x}{h}\Big),\\
\frac{\bar{z} - 2\bar{y} + \bar{x}}{h^2} &= \alpha\Big(\frac{\tilde{z} - 2\tilde{y} + \tilde{x}}{h^2} - \frac{z - 2y + x}{h^2}\Big),
\end{aligned}$$

and for all $(\tilde{x}, \tilde{y}, \tilde{z})$ such that $\frac{\tilde{z}-2\tilde{y}+\tilde{x}}{h^2} \in F\Big(\tilde{x}, \frac{\tilde{y}-\tilde{z}}{h}\Big)$ or equivalently, $\tilde{z} \in P(\tilde{x}, \tilde{y})$.

Simplifying the latter relations, we have $\bar{x} = \alpha(\tilde{x}-x)$, $\bar{y}-\bar{x} = \alpha(\tilde{y}-\tilde{x})-\alpha(y-x)$, $\bar{z}-2\bar{y}+\bar{x} = \alpha(\tilde{z}-z)-2\alpha(\tilde{y}-y)+\alpha(\tilde{x}-x)$.

Now, substituting $\bar{x}$ into the second relation, we have $\bar{y} = \alpha(\tilde{y}-y)$, and then substituting both $\bar{x} = \alpha(\tilde{x}-x)$ and $\bar{y} = \alpha(\tilde{y}-y)$ into the third relation, we derive $\bar{z} = \alpha(\tilde{z}-z)$, which means that $(\bar{x},\bar{y},\bar{z}) \in K_{gphP}(x,y,z)$.

Suppose now that $(x^*,y^*) \in P^*(z^*;(x,y,z)),\ z \in P_A(x,y,z^*)$ or equivalently

$$\langle \bar{x}, x^* \rangle + \langle \bar{y}, y^* \rangle - \langle \bar{z}, z^* \rangle \geq 0;\ (\bar{x},\bar{y},\bar{z}) \in K_P(x,y,z). \tag{19}$$

We rewrite this inequality in the form

$$\langle \bar{x}, a \rangle + \Big\langle \frac{\bar{y}-\bar{x}}{h}, b \Big\rangle - \Big\langle \frac{\bar{z}-2\bar{y}+\bar{x}}{h^2}, z^* \Big\rangle \geq 0,$$

for which (18) is satisfied. Here the vectors a and b should be appropriately defined. It is not hard to see that the latter inequality is equivalent to the inequality

$$\langle \bar{x}, h^2 a - hb - z^* \rangle + \langle \bar{y}, ha + 2z^* \rangle - \langle \bar{z}, z^* \rangle \geq 0.$$

Then comparing it with (19), we find that $x^* = h^2 a - hb - z^*$, $y^* = ha + 2z^*$, whence

$$a = \frac{x^*+y^*-z^*}{h^2}, \quad b = \frac{y^*-2z^*}{h}.$$

Then from the equivalence of (17) and (18) we have

$$\Big(\frac{x^*+y^*-z^*}{h^2}, \frac{y^*-2z^*}{h}\Big) \in F^*\Big(z^*; \Big(x, \frac{y-x}{h}, \frac{z-2y+x}{h^2}\Big)\Big).$$

Moreover, it is easy to see that $P^*(z^*;(x,y,z)) \neq \varnothing$ if $z \in P_A(x,y,z^*)$ and

$$F^*\Big(z^*; \Big(x, \frac{y-x}{h}, \frac{z-2y+x}{h^2}\Big)\Big) \neq \varnothing \ \text{ if } \ \frac{z-2y+x}{h^2} \in F_A\Big(x, \frac{y-x}{h}; z^*\Big),$$

respectively. Hence, the theorem is proved. □

Proposition 4.5. *If $P(x,y) = 2y - x + h^2 F(x)$, where $F; \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping independent of y, then the following statements for the LAMs, provided that $y^* = 2z^*$ are equivalent:*

(1) $(x^*,y^*) \in P^*(z^*;(x,y,z)),\ z \in P_A(x,y,z^*)$,

(2) $(x^*+z^*)/h^2 \in F^*\big(z^*\big(x,(z-2y+x)/h^2\big)\big),\ (z-2y+x)/h^2 \in F_A(x;z^*),\ z^* \in \mathbb{R}^n$.

Proof. By Lemma 4.3 $H_P(x,y,z^*) = \langle 2y-x, z^* \rangle + h^2 H_F(x,z^*)$. Then it is easy to calculate that

$$\partial_{(x,y)} H_P(x,y,z^*) = \{-z^*\} \times \{2z^*\} + h^2 \partial_x H_F(x,z^*) \times \{0\}$$

or

$$\partial_{(x,y)} H_P(x,y,z^*) = \{-z^* + h^2 \partial_x H_F(x,z^*)\} \times \{2z^*\},$$

where by convention it is understood that $\partial_{(x,y)} H_P(x,y,z^*) = -\partial_{(x,y)}\big[-H_P(x,y,z^*)\big]$.

Then we obtain from Lemma 2.3(Section 2) and the latter relation

$$P^*(z^*;(x,y,z)) = \{-z^* + h^2F^*(z^*;(x,z))\} \times \{2z^*\}, \tag{20}$$

where $\qquad z \in P_A(x,y,z^*),\ (z-2y+x)/h^2 \in F_A(x;z^*),\ z^* \in \mathbb{R}^n.$

Moreover, if $(x^*,y^*) \in P^*(z^*;(x,y,z))$, then from (20) under the condition $y^* = 2z^*$ we derive that $x^* = -z^* + h^2F^*(z^*;(x,z))$. The proof is completed. $\square$

Lemma 4.6. *The adjoint inclusion of Euler-Lagrange type*

$$\big(x^*(t)-\mu^*(t),\mu^*(t+h)\big) \in P^*\big(x^*(t+2h);(\tilde{x}(t),\tilde{x}(t+h),\tilde{v}(t+2h),t)$$

has the form

$$\big(\Delta^2x^*(t)+\Delta\eta^*(t),\eta^*(t+h)\big) \in F^*\big(x^*(t+2h);(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{v}(t)),t\big),$$

$t = 2h\ldots,T-2h$, *where* $\eta^*(t) = (\mu^*(t)-2x^*(t^h))/h$.

Proof. By Lemma 4.4 the *LAM* inclusion

$$\big(x^*(t)-\mu^*(t),\mu^*(t+h)\big) \in P^*\big(x^*(t+2h);(\tilde{x}(t),\tilde{x}(t+h),\tilde{v}(t+2h)),t\big)$$

is equivalent to

$$\Big(\frac{x^*(t)-\mu^*(t)+\mu^*(t+h)-x^*(t+2h)}{h^2},\frac{\mu^*(t+h)-2x^*(t+2h)}{h}\Big)$$
$$\in F^*\big(x^*(t+2h);(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{v}(t)),t\big),\ t=2h,3h,\ldots,T-2h;$$

Then since $\mu^*(t+h) = h\eta^*(t+h)+2x^*(t+2h)$ it follows that

$$\frac{x^*(t)-\mu^*(t)+\mu^*(t+h)-x^*(t+2h)}{h^2}$$
$$= \frac{x^*(t)-h\eta^*(t)-2x^*(t+h)+h\eta^*(t+h)+x^*(t+2h)}{h^2} = \Delta^2x^*(t)+\Delta\eta^*(t). \qquad \square$$

Theorem 4.7. *Suppose that $F(\cdot,t):\mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ is a convex set-valued mapping, and $\Phi:\mathbb{R}^{2n} \to \mathbb{R}^1 \cup \{+\infty\}$ is a proper convex function. Besides, suppose that $C:\mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a closed set-valued mapping. Then for the optimality of the trajectory $\{\tilde{x}(t)\}$ in the discrete approximate problem $(SWDA)$ it is necessary that there exist a number $\lambda \in \{0,1\}$ and a pair of vectors $\{x^*(t),\mu^*(t)\}$ not all equal to zero, satisfying the approximate Euler-Lagrange inclusions and transversality condition:*

(i) $\quad \big(\Delta^2x^*(t)+\Delta\eta^*(t),\eta^*(t+h)\big) \in F^*\big(x^*(t+2h);(\tilde{x}(t),\Delta\tilde{x}(t),\Delta^2\tilde{v}(t)),t\big)$,
$\quad t=2h,\ldots,T-2h;\ \Delta^2\tilde{u}(t) \in -N_{C(\tilde{x}(t))}(\Delta\tilde{x}(t));\ \Delta^2\tilde{v}(t) \in F(\tilde{x}(t),\Delta\tilde{x}(t),t)$,

(ii) $\quad \Delta^2\tilde{x}(t) = \Delta^2\tilde{u}(t)+\Delta^2\tilde{v}(t), t=0,\ldots,T-2h$,
$\quad H_{-N_{C(\tilde{x}(t))}}\big(\Delta\tilde{u}(t),x^*(t+2h)\big) = -\langle x^*(t+2h),\Delta^2\tilde{u}(t)\rangle$

(iii) $\quad \big(\eta^*(T-h)+\Delta_-x^*(T),-x^*(T)\big) \in \lambda\partial_{(x,y)}\Phi(\tilde{x}(T-h),\Delta_-\tilde{x}(T))$.

In addition, under the non-degeneracy condition, these conditions are also sufficient for optimality of $\{\tilde{x}(t)\}$.

Proof. It is clear that, taking into account the positive homogeneity of the Euler-Lagrange inclusion in the first argument, condition (i) is the result of Lemma 4.6. Let us prove the remaining statements. The condition

$$H_{-N_{D(\tilde{x}(t))}}(\tilde{x}(t+h), x^*(t+2h)) = -\langle x^*(t+2h), \Delta^2\tilde{u}(t)\rangle$$

by Proposition 4.2 means that

$$H_{-N_{D(\tilde{x}(t))}}(\Delta\tilde{u}(t), x^*(t+2h)) = -\langle x^*(t+2h), \Delta^2\tilde{u}(t)\rangle.$$

Furthermore, in view of the relation $\mu^*(t+h) = h\eta^*(t+h)+2x^*(t+2h)$ the inclusion (16) is equivalent to

$$\big(h\eta^*(T-h) + x^*(T) - x^*(T-h), -x^*(T)\big) \in \lambda\partial_{(x,y)}\Omega(\tilde{x}(T-h), \tilde{x}(T)).$$

Then by Proposition 2.4 of Section 2

$$\big(h\eta^*(T-h)+x^*(T)-x^*(T-h), -hx^*(T)\big) \in \lambda\partial_{(x,y)}\Phi\Big(\tilde{x}(T-h), \frac{\tilde{x}(T)-\tilde{x}(T-h)}{h}\Big),$$

whence $\big(\eta^*(T-h) + \Delta_{-}x^*(T) - x^*(T)\big) \in \lambda\partial_{(x,y)}\Phi\big(\tilde{x}(T-h), \Delta\tilde{x}(T)\big).$ □

Similarly, using Proposition 4.5, we have

Theorem 4.8. *Assume that in problem* (SWC) $F(x, x', t) \equiv F(x, t)$*, i.e. the mapping* $F(\cdot, t)$ *is independent of the* x'*. Then the conjugate second-order Euler-Lagrange DSI* (i) *and the discrete transversality condition* (iii) *of Theorem* 4.7 *have the form*

(1) $$\Delta^2 x^*(t) \in F^*\big(x^*(t+2h);\ (\tilde{x}(t), \Delta^2\tilde{v}(t)), t\big),\ t = 0, \dots, T-2h;$$

(2) $$\big(\Delta_{-}x^*(T), -x^*(T)\big) \in \lambda\partial_{(x,y)}\Phi\big(\tilde{x}(T-h), \Delta_{-}\tilde{x}(T)\big),$$

respectively.

Proof. By condition (2) of Proposition 4.5 we deduce from the inclusion

$$\big(x^*(t) - \mu^*(t), \mu^*(t+h)\big) \in Q^*\big(x^*(t+2h); (\tilde{x}(t), \tilde{x}(t+h), \tilde{v}(t+2h)), t\big)$$

that we have

$$\frac{x^*(t) - \mu^*(t) + x^*(t+2h)}{h^2} \in F^*\big(x^*(t+2h); (x(t), \Delta^2 v(t)), t\big).$$

Note that the condition $y^* = 2z^*$ of Proposition 4.5 means that $\mu^*(t) = 2x^*(t+h)$. Then by substituting $\mu^*(t) = 2x^*(t+h)$ into the last inclusion we have the adjoint second order Euler-Lagrange *DSIs* (1) of the theorem.

Now, since we have $\mu^*(t) = h\eta^*(t)+2x^*(t+h)$ (see Lemma 4.6) and $\mu^*(t) = 2x^*(t+h)$ it follows that $\eta^*(t) = 0$, i.e., $\eta^*(T-h) = 0$ in the condition (iii) of Theorem 4.7. The proof is completed. □

5. Sufficient conditions of optimality for second order sweeping processes (SWC)

Setting $\lambda = 1$ and by passing to the formal limit in the conditions of Lemma 4.3 as $h \to 0$, we establish the so-called second-order Euler-Lagrange *DFIs*:

$$\text{(a)} \qquad \Big(\frac{d^2 x^*(t)}{dt^2} + \frac{d\eta^*(t)}{dt}, \eta^*(t)\Big) \in F^*\big(x^*(t); (\tilde{x}(t), \tilde{x}'(t), \tilde{v}''(t)), t\big) \text{ a.e. } t \in [0,T],$$

$$\tilde{u}''(t) \in -N_{C(\tilde{x}(t))}(\tilde{x}(t));\ \tilde{v}''(t) \in F(\tilde{x}(t), \tilde{x}'(t), t);\ \tilde{x}''(t) = \tilde{u}''(t) + \tilde{v}''(t);$$

$$\text{(b)} \qquad H_{-N_{C(\tilde{x}(t))}}(\tilde{x}'(t), -x^*(t)) = -\langle \tilde{u}''(t), x^*(t)\rangle,\ t \in [0,T]$$

and the transversality condition

$$\text{(c)} \qquad \Big(\eta^*(T) + {x^*}'(T), -x^*(T)\Big) \in \partial_{(x,y)}\Phi(\tilde{x}(T), \tilde{x}'(T)).$$

We assume that the function $x^*(t), t \in [0,T]$, is absolutely continuous together with the first order derivatives for which ${x^*}''(\cdot) \in L_1^n([0,T])$. Besides, $v^*(t), t \in [0,T]$ is absolutely continuous and ${\eta^*}'(\cdot) \in L_1^n([0,T])$.

In addition, the following condition ensures that the *LAM* F^* is not empty at a given point:

$$\text{(d)} \qquad \frac{d^2\tilde{v}(t)}{dt^2} \in F_A\big(\tilde{x}(t), \tilde{x}'(t); x^*(t), t\big), \text{a.e. } t \in [0,T].$$

It turns out that the following statement is true.

Theorem 5.1. *Suppose that $\Phi : \mathbb{R}^{2n} \to \mathbb{R}^1 \cup \{+\infty\}$ is a continuous proper convex function, and that $F(\cdot,t) : \mathbb{R}^{2n} \rightrightarrows \mathbb{R}^n$ and $C : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are convex set-valued mappings. Then for the optimality of the arc $\tilde{x}(\cdot)$ in the problem (SWC) it is sufficient that there exists a pair of absolutely continuous function $\{x^*(t), \eta^*(t)\}, t \in [0,T]$ satisfying a.e. the Euler-Lagrange type adjoint inclusion* (a), (d) *and transversality condition* (c).

Proof. Since $F^*\big(z^*, (x,y,z), t\big) = \partial_{(x,y)} H_F(x,y,z^*), z \in F_A(x,y;z^*,t)$ by using the Moreau-Rockafellar theorem [12, 20, 28] and the fact that from condition (a) we obtain the second order adjoint *DFI*

$$\Big(\frac{d^2 x^*(t)}{dt^2} + \frac{d\eta^*(t)}{dt}, \eta^*(t)\Big) \in \partial_{(x,y)}\Big[H_F\big(\tilde{x}(t), \tilde{x}'(t), x^*(t)\big)\Big],\ t \in [0,T].$$

On the definition of sudifferential set of the Hamiltonian function H_F we rewrite the last relation in the form:

$$\begin{aligned} &H_F\big(x(t), x'(t), x^*(t)\big) - H_F\big(\tilde{x}(t), \tilde{x}'(t), x^*(t)\big) \\ &\le \Big\langle \frac{d^2 x^*(t)}{dt^2} + \frac{d\eta^*(t)}{dt}, x(t) - \tilde{x}(t)\Big\rangle + \langle \eta^*(t), x'(t) - \tilde{x}'(t)\rangle. \end{aligned} \tag{21}$$

In turn, by using the definition of the Hamiltonian function, (21) can be converted to the relation

$$\Big\langle \frac{d^2 v(t)}{dt^2}, x^*(t)\Big\rangle - \Big\langle \frac{d^2\tilde{v}(t)}{dt^2}, x^*(t)\Big\rangle \le \Big\langle \frac{d^2 x^*(t)}{dt^2}, x(t) - \tilde{x}(t)\Big\rangle + \frac{d}{dt}\langle \eta^*(t), x(t) - \tilde{x}(t)\rangle.$$

Since $\tilde{v}''(t) = \tilde{u}''(t) - \tilde{x}''(t)$, let us rewrite the latter inequality as follows

$$\Big\langle \frac{d^2(x(t)-u(t))}{dt^2}, x^*(t) \Big\rangle - \Big\langle \frac{d^2(\tilde{x}(t)-\tilde{u}(t)}{dt^2}, x^*(t) \Big\rangle$$
$$- \Big\langle \frac{d^2 x^*(t)}{dt^2}, x(t)-\tilde{x}(t) \Big\rangle - \frac{d}{dt}\langle \eta^*(t), x(t)-\tilde{x}(t)\rangle \le 0$$

or

$$\Big\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t) \Big\rangle - \Big\langle \frac{d^2\tilde{x}(t)}{dt^2}, x(t)-\tilde{x}(t) \Big\rangle$$
$$- \Big\langle \frac{d^2(u(t)-\tilde{u}(t))}{dt^2}, x^*(t) \Big\rangle - \frac{d}{dt}\langle \eta^*(t), x(t)-\tilde{x}(t)\rangle \le 0 \qquad (22)$$

Now, in view of condition (b)

$$\Big\langle \frac{d^2 u(t)}{dt^2}, -x^*(t) \Big\rangle \le -\Big\langle \frac{d^2\tilde{u}(t)}{dt^2}, x^*(t) \Big\rangle,\ u''(t) \in -N_{C(\tilde{x}(t))}(\tilde{x}'(t))$$

or

$$- \Big\langle \frac{d^2(u(t)-\tilde{u}(t))}{dt^2}, x^*(t) \Big\rangle,\ u''(t) \le 0 \text{ for all } u''(t) \in -N_{C(\tilde{x}(t))}(\tilde{x}'(t))$$

and so the inequality (22) can be rewritten as

$$\Big\langle \frac{d^2(x(t)-\tilde{x}(t))}{dt^2}, x^*(t) \Big\rangle - \Big\langle \frac{d^2 x^*(t)}{dt^2}, x(t)-\tilde{x}(t) \Big\rangle - \frac{d}{dt}\langle \eta^*(t), x(t)-\tilde{x}(t)\rangle \le 0. \quad (23)$$

Integrating (23) over the interval $[0,T]$ and taking into account that $x(\cdot), \tilde{x}(\cdot)$ are feasible ($x(0)=\tilde{x}(0)=\omega_0$) we can write

$$\int_0^T \Big[\big\langle x''(t)-\tilde{x}''(t), x^*(t) \big\rangle - \big\langle x^{*''}(t), x(t)-\tilde{x}(t) \big\rangle \Big] dt$$
$$+ \big\langle \eta^*(0), x(0)-\tilde{x}(0) \big\rangle - \big\langle \eta^*(T), x(T)-\tilde{x}(T) \big\rangle$$
$$= \int_0^T \Big[\big\langle x''(t)-\tilde{x}''(t), x^*(t) \big\rangle - \big\langle x^{*''}(t), x(t)-\tilde{x}(t) \big\rangle \Big] dt$$
$$- \big\langle \eta^*(T), x(T)-\tilde{x}(T) \big\rangle \le 0. \qquad (24)$$

Now, transform the expression in the square parentheses of (24) as follows

$$\langle x''(t)-\tilde{x}''(t), x^*(t)\rangle - \big\langle x^{*''}(t), x(t)-\tilde{x}(t) \big\rangle$$
$$= \frac{d}{dt}\Big\langle \frac{d(x(t)-\tilde{x}(t))}{dt}, x^*(t) \Big\rangle - \frac{d}{dt}\Big\langle \frac{dx^*(t)}{dt}, x(t)-\tilde{x}(t) \Big\rangle.$$

Then we can compute the integral on the left hand side of (24) as follows

$$\int_0^T \Big[\big\langle x''(t)-\tilde{x}''(t), x^*(t) \big\rangle - \big\langle x^{*''}(t), x(t)-\tilde{x}(t) \big\rangle \Big] dt$$
$$= \Big\langle \frac{d(x(T)-\tilde{x}(T))}{dt}, x^*(T) \Big\rangle - \Big\langle \frac{d(x(0)-\tilde{x}(0))}{dt}, x^*(0) \Big\rangle$$
$$- \Big\langle \frac{dx^*(T)}{dt}, x(T)-\tilde{x}(T) \Big\rangle + \Big\langle \frac{dx^*(0)}{dt}, x(0)-\tilde{x}(0) \Big\rangle. \qquad (25)$$

Again since $x(0) = \tilde{x}(0) = \omega_0, x'(0) = \tilde{x}'(0) = \omega_1$ we have

$$\Big\langle \frac{d(x(0) - \tilde{x}(0))}{dt}, x^*(0) \Big\rangle = \Big\langle \frac{dx^*(0)}{dt}, x(0) - \tilde{x}(0) \Big\rangle = 0$$

and so (25) can be rewritten as follows

$$\int_0^T \Big[\langle x''(t) - \tilde{x}''(t), x^*(t) \rangle - \langle x^{*''}(t), x(t) - \tilde{x}(t) \rangle \Big] dt$$
$$= \Big\langle \frac{d(x(T) - \tilde{x}(T))}{dt}, x^*(T) \Big\rangle - \Big\langle \frac{dx^*(T)}{dt}, x(T) - \tilde{x}(T) \Big\rangle.$$

Thus (24) and (25) imply

$$\Big\langle \frac{dx^*(T)}{dt} + \psi^*(T), x(T) - \tilde{x}(T) \Big\rangle + \Big\langle \frac{d(x(T) - \tilde{x}(T))}{dt}, -x^*(T) \Big\rangle \geq 0. \tag{26}$$

Recalling that by condition (c) for all feasible arcs $x(t)$, $t \in [0, T]$:

$$\Phi\big(x(T), x'(T)\big) - \Phi\big(\tilde{x}(T), \tilde{x}'(T)\big)$$
$$\geq \Big\langle \eta^*(T) + \frac{dx^*(T)}{dt}, x(T) - \tilde{x}(T) \Big\rangle - \Big\langle x^*(T), \frac{d(x(T) - \tilde{x}(T))}{dt} \Big\rangle \geq 0. \tag{27}$$

Then from the inequalities (26),(27) we obtain that

$$\Phi\big(x(T), x'(T)\big) \geq \Phi\big(\tilde{x}(T), \tilde{x}'(T)\big) \text{ for all } x(t), t \in [0, T],$$

i.e. $\tilde{x}(t)$, $t \in [0, T]$ is optimal. □

Below we prove that if the mapping $F(\cdot, t)$ depends only on the first argument, then in the second-order Euler-Lagrange *DFIs* there is no auxiliary variable $\eta^*(t)$.

Corollary 5.2. *Suppose that in the problem* (SWC) $F(x, x', t) \equiv F(x, t)$ *and that the conditions of Theorem* 4.7 *are satisfied. Then the second order Euler-Lagrange DFI and the transversality condition of Theorem* 4.7 *consist of the following*

(i) $$\frac{d^2x^*(t)}{dt^2} \in F^*\big(x^*(t); (\tilde{x}(t), \tilde{v}''(t)), t\big) \text{ a.e. } t \in [0, T],$$
$$\tilde{v}''(t) \in F(\tilde{x}(t), t);\ \tilde{u}''(t) \in -N_{C(\tilde{x}(t))}(\tilde{x}(t));\ \tilde{x}''(t) = \tilde{u}''(t) + \tilde{v}''(t);$$

(ii) $$\frac{d^2\tilde{v}(t)}{dt^2} \in F_A\big(\tilde{x}(t); x^*(t), t\big) \text{ a.e. } t \in [0, T]$$

and the transversality condition

(iii) $$\big(x^{*'}(T), -x^*(T)\big) \in \partial_{(x,y)}\Phi(\tilde{x}(T), \tilde{x}'(T)).$$

Proof. Indeed, by passing to the formal limit in the discrete Euler-Lagrange type inclusion $\Delta^2 x^*(t) \in F^*\big(x^*(t + 2h); (\tilde{x}(t), \Delta^2\tilde{v}(t)), t\big)$, $t = 0, \ldots, T - 2h$, of Theorem 4.8 we have the condition (i) of the theorem. The rest of the theorem is simply a repetition of Theorem 4.7. □

Corollary 5.3. *In addition to the assumptions of Theorem* 5.1 *let* $F(\cdot,t)$ *be a closed set-valued mapping. Then the conditions* (a), (c) *of Theorem* 5.1 *can be rewritten in terms of the Hamiltonian function in symmetric form as follows*

$$\Big(\frac{d^2x^*(t)}{dt^2}+\frac{d\eta^*(t)}{dt},\eta^*(t)\Big)\in\partial_{(x,y)}H_F\big(\tilde{x}(t),\tilde{x}'(t),x^*(t)\big),$$
$$\frac{d^2\tilde{x}(t)}{dt^2}\in\partial_{z^*}H_F\big(\tilde{x}(t),\tilde{x}'(t);x^*(t)\big),\ a.e.\ t\in[0,T].$$

Proof. By Lemma 2.3 $F^*\big(z^*,(x,y,z),t\big)=\partial_{(x,y)}H_F(x,y,z^*)$. On the other hand it is elementary to verify that $F_A(x,y,z^*,t)=\partial_{z^*}H_F(x,y,z^*)$. □

Note that the technique of proof of Theorem 5.1 can easily be extended to the case of the Bolza problem:

$$(SWB)\qquad\begin{cases}\text{minimize}\ \ J[x(\cdot)]=\displaystyle\int_0^T f(x(t),t)dt+\varphi(x(T)),\\ x''(t)\in -N_{C(x(t))}(x'(t))+F(x(t),x'(t),t),\ \text{a.e. } t=[0,T],\\ x(0)=\omega_0,\ x'(0)=\omega_1,\end{cases}$$

where $f(\cdot,t)$ and φ are proper convex functions.
The conditions of Theorem 5.1 in this case consist of the following:

(1) $\Big(\dfrac{d^2x^*(t)}{dt^2}+\dfrac{d\eta^*(t)}{dt},\eta^*(t)\Big)\in F^*\big(x^*(t);(\tilde{x}(t),\tilde{x}'(t),\tilde{v}''(t)),t\big)-\partial f(\tilde{x}(t),t)\times\{0\}$,
a.e. $t\in[0,T]$;

(2) $\tilde{v}''(t)\in F(\tilde{x}(t),\tilde{x}'(t),t);\ \tilde{u}''(t)\in -N_{C(\tilde{x}(t))}(\tilde{x}'(t));\tilde{x}''(t)=\tilde{u}''(t)+\tilde{v}''(t);$
$H_{-N_{C(\tilde{x}(t))}}(\tilde{x}'(t),-x^*(t))=-\langle\tilde{u}''(t),-x^*(t)\rangle,\ t\in[0,T],$
$\dfrac{d^2\tilde{v}(t)}{dt^2}\in F_A\big(\tilde{x}(t),\tilde{x}'(t),x^*(t),t\big),$ a.e. $t\in[0,T]$

and the transversality condition

(3) $\eta^*(T)+x^{*'}(T)\in\partial\varphi\big(\tilde{x}(T)\big),\ x^*(T)=0.$

Theorem 5.4. *Suppose that* $f(\cdot,t),\varphi:\mathbb{R}^n\to\mathbb{R}^1\cup\{+\infty\}$ *are continuous proper convex functions and* $F:\mathbb{R}^{2n}\rightrightarrows\mathbb{R}^n$ *and* $C:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ *are convex set-valued mappings. Then for the optimality of the arc* $\tilde{x}(\cdot)$ *in the Bolza problem* (SWB) *it is sufficient that there exists a pair of absolutely continuous function* $\{x^*(t),\eta^*(t)\}$, $t\in[0,1]$ *satisfying the conditions* (1)–(3).

Proof. The proof of the theorem is a slight modification of the proof of Theorem 5.1, therefore we omit it. □

Suppose now that we have the so-called linear sweeping problem for second order *DFIs*:

$$(SWL)\qquad\begin{cases}\text{minimize}\ \ J[x(\cdot)]=\displaystyle\int_0^T d(x(t),t)dt+\varphi(x(T)),\\ x''(t)\in -N_{C(x(t))}(x'(t))+F(x(t),x'(t)),\ \text{a.e. } t\in[0,T],\\ x(0)=\omega_0,\ x'(0)=\omega_1,\ F(x,y)=A_1x+A_2y+BU,\end{cases}$$

where $f(\cdot,t)$ and φ are continuously differentiable function in $x, A_i, i=1,2$ and B are $n\times n$ and $n\times r$ matrices, respectively, U is a convex compact in $\mathbb{R}^r$. The problem is to find a controlling parameter $\tilde{w}(t)\in U$ such that the arc $\tilde{x}(t)$ corresponding to it minimizes $J[x(\cdot)]$.

Theorem 5.5. *The arc $\tilde{x}(\cdot)$ corresponding to the controlling parameter $\tilde{w}(t)$ minimizes $J[x(\cdot)]$ in the problem (SWL) if the conditions* (1), (2) *of Theorem* 5.4 *are satisfied, where an absolutely continuous function $x^*(\cdot)$ satisfies the second order adjoint differential equation, the transversality and the Weierstrass-Pontryagin conditions:*

$$\frac{d^2x^*(t)}{dt^2} = -A_2^*\frac{dx^*(t)}{dt} + A_1^*x^*(t) - f'(\tilde{x}(t),t) \;\; a.e.\ t\in[0,T],$$
$$x^{*'}(T) = \varphi'(\tilde{x}(T)),\ x^*(T)=0,\ a.e.\ t\in[0,T],$$
$$\langle B\tilde{w}(t), x^*(t)\rangle = \max_{w\in U}\langle Bw(t), x^*(t)\rangle.$$

Proof. In this problem, we rely on Theorem 5.1. Further

$$H_F(x,y,z^*) = \sup_z\big\{\langle z,z^*\rangle : z\in F(x,y)\big\} = \langle x, A_1^*z^*\rangle + \langle y, A_2^*z^*\rangle + \max_{w\in U}\langle Bw(t), z^*\rangle,$$

where A^* is the transposed matrix of A.

Then by Lemma 2.3 one has $F^*\big(z^*;(\tilde{x},\tilde{y},\tilde{z})\big) = \{A_1^*z^*, A_2^*z^*\}$ if $-B^*z^*\in K_U^*(\tilde{w})$, where $\tilde{z} = A_1\tilde{x} + A_2\tilde{y} + B\tilde{w}, \tilde{w}\in U$. Thus, using Theorem 5.4, we obtain the following result

$$\frac{d^2x^*(t)}{dt^2} + \frac{d\eta^*(t)}{dt} = A_1^*x^*(t), \quad \eta^*(t) = A_2^*x^*(t). \tag{28}$$

Differentiating the second equation of (28) and substituting it into the first equation we easily derive the second order adjoint differential equation. The remaining conditions of the theorem follow from condition (3) of Theorem 5.4 and the formula $-B^*z^*\in K_U^*(\tilde{w})$. Notice that here the conditions $\eta^*(T) = A_2^*x^*(T)$ and $x^*(T)=0$ entail $\eta^*(T)=0$. □

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