Journal of Convex Analysis
Volume 29 (2022), No. 1, 291–320

Existence of Solutions for a Class of Nonlocal Problems Driven by an Anisotropic Operator via Sub-Supersolutions

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Received: June 23, 2020
Accepted: May 27, 2021

We consider some anisotropic problems involving nonlocal terms. New results are provided involving sub-supersolutions for the anisotropic spaces setting. Then some applications are considered.

Keywords: Fixed point, nonlocal problems, anisotropic operator.

2010 Mathematics Subject Classification: Primary 35J60; secondary 35Q53.

1. Introduction

In this manuscript we consider the existence of positive solutions for the following nonlocal anisotropic problems

$$\begin{cases} -\mathcal{H}(x,\|u\|_{L^r})\Delta_{\overrightarrow{p}}u = \|u\|^{\alpha}_{L^t} + \|u\|^{\gamma}_{L^s}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_1}$$

and

$$\begin{cases} -\mathcal{H}(x,\|v\|_{L^{r_1}})\Delta_{\overrightarrow{p}}u = \|v\|^{\alpha_1}_{L^{t_1}} + \|v\|^{\gamma_1}_{L^{s_1}}, & \text{in } \Omega, \\ -\mathcal{H}(x,\|u\|_{L^{r_2}})\Delta_{\overrightarrow{q}}v = \|u\|^{\alpha_2}_{L^{t_2}} + \|u\|^{\gamma_2}_{L^{s_2}}, & \text{in } \Omega, \\ u, v = 0, & \text{on } \partial\Omega, \end{cases} \tag{P_2}$$

where $N \geq 2$, $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 boundary,$-\Delta_{\overrightarrow{p}}$ denotes the anisotropic operator

$$-\Delta_{\overrightarrow{p}}u := -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left[\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right] \tag{1}$$

and p_i, $i = 1, ..., N$ are real numbers with $2 \leq p_1 \leq p_2 \leq \cdots \leq p_N < +\infty$, $\alpha_0 := \alpha$ and $\gamma_0 := \gamma$ are non-negative constants, $\mathcal{H} : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying some hypotheses and $\|\cdot\|_{L^t}$ is the usual norm of $L^t(\Omega), 1 \leq t < \infty$.

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ISSN 0944-6532 / $ 2.50

Differential equations involving anisotropic operators arise in several areas of science and technology, for example in physics where such an operator is related with models that describe the dynamics of fluids with different conductivities in different directions. Another example is in biology in models related to the spread of an epidemic disease in heterogeneous environments. We also quote the applicability of anisotropic operators in the study of image noise reduction. Regarding the mentioned applications we point out [5, 6, 9, 45] and the references therein.

In the last century the operator (1) and the associated anisotropic spaces attracted the attention of several researchers, see for instance the studies considered in the references [1, 3, 9, 25, 30, 31, 39, 40, 41, 43, 46, 48, 49, 50, 51, 52, 53, 54, 56, 57, 58].

The nonlocal term $\|\cdot\|_{L^r}$ with $r = p_i = 2, i = 1, ..., N$ arises in the study of the vibrations of an elastic string. More specifically in the Carrier's equation

$$\rho u_{tt} - a(x, t, \|u\|_{L^2}^2)\Delta u = 0.$$

Regarding the Carrier's equation we quote the reference [11].

Important contributions related to problems (P_1) and (P_2) can be found in the literature, for example in the manuscript [17] where the authors considered the existence of solutions for the problem

$$\begin{cases} -a\left(\displaystyle\int_\Omega u\right)\Delta u &= f, \quad \text{in } \Omega, \\ u = 0, & \quad \text{on } \partial\Omega, \end{cases}$$

where a is a function satisfying certain properties and $f \in H^{-1}(\Omega)$.

In the reference [2] the authors proved existence results, via sub-supersolutions arguments, for the nonlocal problem

$$\begin{cases} -a\left(\displaystyle\int_\Omega |u|^q\right)\Delta u = g_1(x,u)f\left(\displaystyle\int_\Omega |u|^p\right) + g_2(x,u)l\left(\displaystyle\int_\Omega |u|^r\right), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

with q, p, r numbers satisfying some hypothesis and a, g_1, g_2, f and l are continuous functions satisfying certain conditions.

We also quote the paper [32] where the authors, in the variable exponents spaces setting, used fixed point arguments and sub-supersolutions to study the existence of solutions for a wide class of problems which includes the case

$$\begin{cases} -\mathcal{A}(x, \|u\|_{L^{r(x)}})\Delta u = \|u\|_{L^{q(x)}}^{\alpha(x)} + \|u\|_{L^{s(x)}}^{\gamma(x)}, & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{A}, r, q, s, \alpha$ and β are functions satisfying certain conditions.

In[26] related problems with the $p-$Laplacian operator $\Delta_p u := -div(|\nabla u|^{p-2}\nabla u)$ were studied.

The authors considered respectively by means of sub-supersolutions and topological arguments the existence of positive solutions for the scalar problem

$$\begin{cases} -\Delta_p u = \|u\|_{L^{t(x)}}^{\alpha(x)}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{P}$$

and its system version

$$\begin{cases} -\Delta_p u = \|v\|_{L^{t(x)}}^{\alpha(x)}, & \text{in } \Omega, \\ -\Delta_p v = \|u\|_{L^{q(x)}}^{\beta(x)}, & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases}$$

in the context of the variable spaces setting. Here $\alpha, \beta : \overline{\Omega} \to \mathbb{R}$ are functions that satisfy certain conditions. We point out that the homegeneity of the $p-$Laplacian operator and the eigenfunction associated with the first eigenvalue of the $p-$Laplacian operator played an important role in their arguments.

For other works related to (P_1) and (P_2) we mention the papers [7, 8, 10, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 27, 28, 29, 34, 35, 38, 55].

We also point out that for the study of (P_1) and (P_2) it is needed several topics of nonlinear functional analysis, mainly the theory of anisotropic spaces, see for instance [41] and its references. An additional difficulty is imposed due to the nonlocal caracter of the problems considered, that is, the terms $\|\cdot\|_{L^{r_i}}, \|\cdot\|_{L^{t_i}}$ and $\|\cdot\|_{L^{q_i}}$ implies that no longer pointwise identities are satisfied. Another point is that there are few works that consider sub-supersolution arguments for anisotropic operators, see for instance [3, 33, 36, 52].

Therefore, based on the previous comments we propose the study of (P_1) and (P_2). To the best of our knowledge the mentioned problems were not considered in the literature. In what follows we point out the contributions of this work.

- As far we know it is the first time that a sub-supersolution approach is considered for nonlocal anisotropic problems;
- A wide class of operators, which includes the Laplacian case, is considered;
- Abstract theorems involving sub-supersolutions are obtained. Moreover, the results improve partially [32, Theorem 1] and [32, Theorem 2];
- Explicit constructions are considered to obtain sub-supersolutions, see for example the functions in (35) and (55). Besides that a Fan's type Lemma (see [37, Lemma 2.1]) was needed to construct such functions, see Lemma 4.1. We quote that such constructions can be of independent interest. This type of argument is rare in the anisotropic spaces setting. Such constructions were needed to avoid of the lack of homogeneity of the operator (1), which did not occurred [2, 26, 32].

2. Presentation of the results

Some definitions are needed to present the results of this manuscritpt. Unless otherwise stated it will be considered that $2 \le p_1 \le p_2 \le \ldots \le p_N < p^\star$ and $2 \le q_1 \le q_2 \le \ldots \le q_N < q^\star$ are real numbers with $\overline{p}, \overline{q} < N$, where $\quad p^\star := \dfrac{N\overline{p}}{N-\overline{p}},\quad q^\star := \dfrac{N\overline{q}}{N-\overline{q}},\quad \overline{p} := \dfrac{N}{\sum_{i=1}^N \frac{1}{p_i}}$ and $\overline{q} := \dfrac{N}{\sum_{i=1}^N \frac{1}{q_i}}$.

A *positive solution* u of (P_1) is a function that lies in $W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega)$ with $u(x)>0$ a.e. in Ω and, for all $\phi\in W_0^{1,\overrightarrow{p}}(\Omega)$,

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega \left(\frac{\|u\|_{L^t}^\alpha}{\mathcal{H}(x,\|u\|_{L^r})}+\frac{\|u\|_{L^s}^\gamma}{\mathcal{H}(x,\|u\|_{L^r})}\right)\phi.$$

For $u,v\in\mathcal{M}(\Omega):=\{u:\Omega\to\mathbb{R}; u \text{ is measurable}\}$, we will write $u\le v$ in Ω if we have $u(x)\le v(x)$ a.e. in Ω and we will denote by $[u,v]$ the set

$$[u,v]:=\{w\in\mathcal{M}(\Omega); u(x)\le w(x)\le v(x) \text{ a.e. in } \Omega\}.$$

Now we are ready to present the definition of a sub-supersolution pair for (P_1). An element $(\underline{u},\overline{u})\in (W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega))\times(W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega))$ is a *sub-supersolution* for (P_1) if $0\le\underline{u}\le\overline{u}$ in Ω and if for all $\phi\in W_0^{1,\overrightarrow{p}}(\Omega)$ with $\phi\ge 0$ the inequalities

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i}\frac{\partial \phi}{\partial x_i} \le \int_\Omega \left(\frac{\|\underline{u}\|_{L^{t_1}}^{\alpha_1}}{\mathcal{H}(x,\|w\|_{L^{r_1}})}+\frac{\|\underline{u}\|_{L^{s_1}}^{\gamma_1}}{\mathcal{H}(x,\|w\|_{L^{r_1}})}\right)\phi \tag{2}$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \overline{u}}{\partial x_i}\frac{\partial \psi}{\partial x_i} \ge \int_\Omega \left(\frac{\|\overline{u}\|_{L^{t_2}}^{\alpha_2}}{\mathcal{H}(x,\|w\|_{L^{r_2}})}+\frac{\|\overline{u}\|_{L^{s_2}}^{\gamma_2}}{\mathcal{H}(x,\|w\|_{L^{r_2}})}\right)\psi$$

hold for all $w\in[\underline{u},\overline{u}]$.

Regarding (P_2) we assume that $\overrightarrow{p}$ and $\overrightarrow{q}=(q_1,\ldots,q_N)\in\mathbb{R}^N$ satisfy the general condition

(H_2) $2\le q_1\le q_2\le\ldots\le q_N<q^\star$, where $q^\star:=\frac{N\overline{q}}{N-\overline{q}}$ with $\overline{q}:=\frac{N}{\sum_{i=1}^N\frac{1}{q_i}}<N$ and (H_1).

We say that $(u,v)\in(W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega))\times(W_0^{1,\overrightarrow{q}}(\Omega)\cap L^\infty(\Omega))$ is a *positive solution* of (P_2) if $u(x),v(x)>0$ a.e. in Ω and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega \left(\frac{\|v\|_{L^{t_1}}^{\alpha_1}}{\mathcal{H}(x,\|v\|_{L^{r_1}})}+\frac{\|v\|_{L^{s_1}}^{\gamma_1}}{\mathcal{H}(x,\|v\|_{L^{r_1}})}\right)\phi,$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2}\frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} = \int_\Omega \left(\frac{\|u\|_{L^{t_2}}^{\alpha_2}}{\mathcal{H}(x,\|u\|_{L^{r_2}})}+\frac{\|u\|_{L^{s_2}}^{\gamma_2}}{\mathcal{H}(x,\|u\|_{L^{r_2}})}\right)\psi,$$

for all $(\phi,\psi)\in W_0^{1,\overrightarrow{p}}(\Omega)\times W_0^{1,\overrightarrow{q}}(\Omega)$.

We say that the $[(\underline{u},\overline{u}),(\underline{v},\overline{v})]$ is a *sub-supersolution pair* for (P_2) if

$$(\underline{u},\overline{u})\in(W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega))\times(W_0^{1,\overrightarrow{p}}(\Omega)\cap L^\infty(\Omega)),$$
$$(\underline{v},\overline{v})\in(W_0^{1,\overrightarrow{q}}(\Omega)\cap L^\infty(\Omega))\times(W_0^{1,\overrightarrow{q}}(\Omega)\cap L^\infty(\Omega)),$$

$0 \le \underline{u} \le \overline{u}$ and $0 \le \underline{v} \le \overline{v}$ in Ω, and if the following inequalities hold:

$$\begin{cases} \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i}\frac{\partial \phi}{\partial x_i} \le \int_\Omega \frac{\|\underline{v}\|_{L^{t_1}}^{\alpha_1} + \|\underline{v}\|_{L^{s_1}}^{\gamma_1}}{\mathcal{H}(x,\|w\|_{L^{r_1}})}\phi, \forall w \in [\underline{v},\overline{v}] \\ \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \underline{v}}{\partial x_i}\right|^{q_i-2} \frac{\partial \underline{v}}{\partial x_i}\frac{\partial \phi}{\partial x_i} \le \int_\Omega \frac{\|\underline{u}\|_{L^{t_2}}^{\alpha_2} + \|\underline{u}\|_{L^{s_2}}^{\gamma_2}}{\mathcal{H}(x,\|w\|_{L^{r_2}})}\psi, \forall w \in [\underline{u},\overline{u}] \end{cases}$$

and

$$\begin{cases} \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{u}}{\partial x_i}\right|^{p_i-2} \frac{\partial \overline{u}}{\partial x_i}\frac{\partial \phi}{\partial x_i} \ge \int_\Omega \frac{\|\overline{v}\|_{L^{t_1}}^{\alpha_1} + \|\overline{v}\|_{L^{s_1}}^{\gamma_1}}{\mathcal{H}(x,\|w\|_{L^{r_1}})}\phi, \forall w \in [\underline{v},\overline{v}] \\ \displaystyle\int_\Omega \sum_{i=1}^N \left|\frac{\partial \overline{v}}{\partial x_i}\right|^{q_i-2} \frac{\partial \overline{v}}{\partial x_i}\frac{\partial \psi}{\partial x_i} \ge \int_\Omega \frac{\|\overline{u}\|_{L^{t_2}}^{\alpha_2} + \|\overline{u}\|_{L^{s_2}}^{\gamma_2}}{\mathcal{H}(x,\|w\|_{L^{r_2}})}\psi, \forall w \in [\underline{u},\overline{u}] \end{cases}$$

for all $(\phi,\psi) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ with $\phi,\psi \ge 0$ in Ω.

Below we describe the results of this work. The first ones below are considered with the justificative of the existence of solution for (P_1) and (P_2) when one obtained sub-supersolutions for the mentioned problems.

Theorem 2.1. *Suppose that $(\underline{u},\overline{u})$ is a sub-supersolution for (P_1) with $\underline{u} > 0$ in Ω, where $\mathcal{H} : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ is a continuous function with $\mathcal{H}(x,t) > 0$ in $\overline{\Omega} \times \big[\|\underline{u}\|_{L^r},\ \|\overline{u}\|_{L^r}\big]$. Then (P) has a positive solution $u \in [\underline{u},\overline{u}]$.*

Theorem 2.2. *Suppose that $[(\underline{u},\overline{u}),(\underline{v},\overline{v})]$ is a sub-supersolution pair for (P_2) with $\underline{u},\underline{v} > 0$ in Ω, and $\mathcal{H} : \overline{\Omega} \times (0,\infty) \to \mathbb{R}$ is a continuous function with $\mathcal{H}(x,t) > 0$ in $\overline{\Omega} \times \big[\underline{\sigma},\overline{\sigma}\big]$, where $\underline{\sigma} := \min\big\{\|\underline{w}\|_{L^{r_i}}, i = 1,2\big\}$, $\overline{\sigma} := \max\big\{\|\overline{w}\|_{L^{r_i}}, i = 1,2\big\}$, $\underline{w}(x) := \min\{\underline{u}(x),\underline{v}(x)\}$ and $\overline{w}(x) := \max\{\overline{u}(x),\overline{v}(x)\}$ for $x \in \Omega$. Then (P_2) has a positive solution (u,v) with $u \in [\underline{u},\overline{u}]$ and $v \in [\underline{v},\overline{v}]$.*

Some applications of Theorem 2.1 are considered. The first result is concerned with a sublinear scalar problem of the form

$$\begin{cases} -\mathcal{H}(x,\|v\|_{L^r})\Delta_{\overrightarrow{p}} u &= \|u\|_{L^t}^{\alpha} \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{E_s}$$

The equation (E_s) under the hypothesis $p_i \equiv 2, i = 1,...,N$, was considered in the recent paper [32]. The next result generalizes partially [32, Theorem 3] in the case of constant exponents due to the general structure of (1). The following result is obtained.

Theorem 2.3. *Consider that $0 < \alpha < p_1 - 1$ and let $h_0 > 0$ be a constant. Suppose that one of the conditions below holds.*

(i) $\mathcal{H}(x,t) \ge h_0$ *in* $\overline{\Omega} \times [0,\infty)$,

(ii) $0 < \mathcal{H}(x,t) \le h_0$ *in* $\overline{\Omega} \times (0,\infty)$ *and* $\lim_{t\to+\infty} \mathcal{H}(x,t) = h_\infty$ *uniformly in* Ω.

Then (E_s) has a positive solution.

The second application of Theorem 2.1 is related to a nonlocal concave-convex problem given by

$$\begin{cases} -\mathcal{H}(x,|v|_{L^r})\Delta_{\overrightarrow{p}}u &= \lambda\|u\|_{L^t}^{\alpha}+\theta\|u\|_{L^s}^{\gamma} \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{cases} \tag{$(E)_{\lambda,\theta}$}$$

The local version of $(E)_{\lambda,\theta}$ with $p_i = 2, i = 1, ..., N$ was studied in a classical paper by Ambrosetti-Brezis-Cerami [4] in which a sub-supersolution argument is used. Since we consider the anisotropic setting, the result below generalizes partially [32, Theorem 7] in the case of constant exponents. The result proved is the following:

Theorem 2.4. *Consider α and γ nonnegative numbers. Consider also the condition $0 < \alpha < p_1 - 1$. Let $a_0, b_0 > 0$ positive numbers. The following assertions hold:*

(i) *If $p_N - 1 < \gamma$ and $\mathcal{H}(x,t) \geq a_0$ in $\overline{\Omega}\times[0,b_0]$, then given $\theta > 0$ there exists $\lambda_0 > 0$ such that for each $\lambda \in (0,\lambda_0)$, the problem $(E)_{\lambda,\theta}$ has a positive solution $u_{\lambda,\theta}$.*

(ii) *Consider that $p_1 - 1 < \gamma$.*

Suppose that $0 < \mathcal{H}(x,t) \leq a_0$ in $\overline{\Omega}\times(0,\infty)$ and $\lim_{t\to\infty}\mathcal{H}(x,t) = b_0$ uniformly in $\overline{\Omega}$. Then given $\lambda > 0$, there exists $\theta_0 > 0$ such that for each $\theta \in (0,\theta_0)$ the problem $(E)_{\lambda,\theta}$ has a positive solution $u_{\lambda,\theta}$.

Regarding Theorem 2.2 two applications are considered. The first one consists in solving a problem with the form below:

$$\begin{cases} -\mathcal{H}(x,\|v\|_{L^{r_1}})\Delta_{\overrightarrow{p}}u = \|v\|_{L^{t_1}}^{\alpha_1}, & \text{in } \Omega, \\ -\mathcal{H}(x,\|u\|_{L^{r_2}})\Delta_{\overrightarrow{q}}v = \|u\|_{L^{t_2}}^{\alpha_2}, & \text{in } \Omega, \\ u,v = 0, & \text{on } \partial\Omega, \end{cases} \tag{(S_s)}$$

The system (S_s) with the condition $p_i \equiv 2, i = 1, ..., N$, was also studied in [32]. The next theorem generalizes partially [32, Theorem 3] in the case of constant exponents. The following result is proved.

Theorem 2.5. *Suppose that $0 < \alpha_i < \min\{p_1-1, q_1-1\}$, $i = 1, 2$. Suppose also that $\mathcal{H} : \overline{\Omega}\times[0,+\infty) \to \mathbb{R}$ satisfies one of the conditions below:*

(i) *$\mathcal{H}(x,t) \geq h_0$ in $\overline{\Omega}\times[0,+\infty)$,*

(ii) *$0 < \mathcal{H}(x,t) \leq h_0$ in $\overline{\Omega}\times(0,+\infty)$ and $\lim_{t\to+\infty}\mathcal{H}(x,t) = h_\infty$ uniformly in Ω.*

Then (S_s) has a positive solution.

The second application is concerned with a concave-convex system of the form below:

$$\begin{cases} -\mathcal{H}(x,\|v\|_{L^{r_1}})\Delta_{\overrightarrow{p}}u = \lambda\|v\|_{L^{t_1}}^{\alpha_1}+\theta\|v\|_{L^{s_1}}^{\gamma_1}, & \text{in } \Omega, \\ -\mathcal{H}(x,\|u\|_{L^{r_2}})\Delta_{\overrightarrow{q}}v = \lambda\|u\|_{L^{t_2}}^{\alpha_2}+\theta\|v\|_{L^{s_2}}^{\gamma_2}, & \text{in } \Omega, \\ u,v = 0, & \text{on } \partial\Omega. \end{cases} \tag{$(S_{\lambda,\theta})$}$$

The result obtained is the following.

Theorem 2.6. *Consider α_i, γ_i, $i = 1, 2$ nonnegative numbers and the conditions $0 < \alpha_1 < p_1 - 1$ and $0 < \alpha_2 < q_1 - 1$. Let $a_0, b_0 > 0$ be positive numbers. The following assertions hold:*

(i) *If $q_N - 1 < \gamma_1, p_N - 1 < \gamma_2$ and $\mathcal{H}(x,t) \geq a_0$ in $\overline{\Omega} \times [0, b_0]$, then given $\theta > 0$ there exists $\lambda_0 > 0$ such that for each $\lambda \in (0, \lambda_0)$, the problem $(S_{\lambda,\theta})$ has a positive solution $(u_{\lambda,\theta}, v_{\lambda,\theta})$.*

(ii) *Consider $q_N - 1 < \gamma_1$, $p_N - 1 < \gamma_2$, $0 < \alpha_1 < q_1 - 1$ and $0 < \alpha_2 < p_1 - 1$. Suppose that $0 < \mathcal{H}(x,t) \leq a_0$ in $\overline{\Omega} \times (0, \infty)$ and $\lim_{t\to\infty} \mathcal{H}(x,t) = b_0$ uniformly in $\overline{\Omega}$. Then given $\lambda > 0$, there exists $\theta_0 > 0$ such that for each $\theta \in (0, \theta_0)$ the problem $(S_{\lambda,\theta})$ has a positive solution $(u_{\lambda,\theta}, v_{\lambda,\theta})$.*

The rest of the manuscript is organized as follows: In Section 3 we recall some facts regarding the anisotropic spaces and some results related to the operator (1). In Section 4 we prove an auxiliary L^∞ regularity result which will play an important role in this work. In Section 5 the proofs of Theorem 2.1 and 2.2 are presented and applied in Section 6 to some classes of problems.

3. Preliminaries

In what follows we denote by Ω a bounded domain in $\mathbb{R}^N$ with $N \geq 2$. Let

$$1 < p_1 \leq p_2 \leq \dots \leq p_N$$

be real numbers and denote by $\overrightarrow{p}$ the vector $\overrightarrow{p} := (p_1, \dots, p_N) \in \mathbb{R}^N$. We denote by $W^{1,\overrightarrow{p}}(\Omega)$ the space defined by

$$W^{1,\overrightarrow{p}}(\Omega) := \left\{ u \in W^{1,1}(\Omega); \frac{\partial u}{\partial x_i} \in L^{p_i}(\Omega), i = 1, \dots, N \right\},$$

which is a Banach space when equipped with the norm

$$\|u\|_{1,\overrightarrow{p}} := \|u\|_{L^1} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}, \tag{3}$$

where $\|\cdot\|_{L^{p_i}}$ denotes the usual norm of $L^{p_i}(\Omega)$. The Banach space defined by the closure of $C_0^\infty(\Omega)$ in $W^{1,\overrightarrow{p}}(\Omega)$ with respect to the norm $\|\cdot\|_{1,\overrightarrow{p}}$ will be denoted by $W_0^{1,\overrightarrow{p}}(\Omega)$. Consider the *harmonic mean* $\overline{p}$ of $p_i, i = 1, \dots, N$, given by

$$\overline{p} := \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i}.$$

Suppose that $\overline{p} < N$ and define the symbol $\overline{p}^\star := \frac{N\overline{p}}{N - \overline{p}}$. If $p_N < p^\star$ then there exists an embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^q(\Omega)$ which is continuous for $q \in [1, \overline{p}^\star]$ and compact in the case $q \in [1, \overline{p}^\star)$, see [41]. Thus the norm

$$\|u\| := \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{p_i}}, \; u \in W_0^{1,\overrightarrow{p}}(\Omega)$$

is equivalent to the norm given in (3).

The next two results can be found in [33, Lemma 2.4] (or [52, Lemma 2.4]) and [33, Lemma 2.2] (or [52, Lemma 2.2]), respectively. See also [36, Lemma 2.2] and [36, Lemma 2.5].

Lemma 3.1. *Let $a \in L^\infty(\Omega)$. Then the problem*

$$\begin{cases} -\Delta_{\overrightarrow{p}} u &= a \ \ in \ \ \Omega, \\ \qquad u &= 0 \ \ on \ \ \partial\Omega, \end{cases}$$

has a unique solution in $W_0^{1,\overrightarrow{p}}(\Omega)$.

Lemma 3.2. *Let Ω be a bounded domain and consider $u, v \in W_0^{1,\overrightarrow{p}}(\Omega)$ satisfying*

$$\begin{cases} -\Delta_{\overrightarrow{p}} u &\leq -\Delta_{\overrightarrow{p}} v \ \ in \ \ \Omega, \\ \qquad u &\leq v \ on \ \ \partial\Omega, \end{cases}$$

where $u \leq v$ on $\partial\Omega$ means that $(u-v)^+ := \max\{0, u-v\} \in W_0^{1,\overrightarrow{p}}(\Omega)$. Then $u \leq v$ in Ω.

4. An auxiliary $L^\infty(\Omega)$ estimate

In what follows we present an auxiliary estimate that will play an important role in the construction of sub-supersolutions.

Let $\Omega \subset \mathbb{R}^N (N \geq 2)$ be a bounded and admissible domain, that is, there exists a continuous embedding $W_0^{1,1}(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$. The best constant of the mentioned embedding will be denoted by C_0. Then we have that

$$\|u\|_{W_0^{1,1}(\Omega)} \leq C_0 \|u\|_{L^{\frac{N}{N-1}}}(\Omega), \forall u \in W_0^{1,1}(\Omega), \tag{4}$$

where $\|u\|_{W_0^{1,1}(\Omega)} := \||\nabla u|\|_{L^1}, u \in W_0^{1,1}(\Omega)$. Adapting the ideas of [37, Lemma 2.1] to the anisotropic setting we obtain the result below which will play an important role in this work.

Lemma 4.1. *Let $\lambda > 0$ and consider $u_\lambda \in W_0^{1,\overrightarrow{p}}(\Omega)$ the unique solution of the problem*

$$\begin{cases} -\Delta_{\overrightarrow{p}} u &= \lambda \ \ in \ \ \Omega, \\ \qquad u &= 0 \ \ in \ \ \Omega \end{cases} \tag{P_λ}$$

Define $h := \frac{p_1}{2|\Omega|^{\frac{1}{N}}} C_0$. If $\lambda \geq h$ then $u \in L^\infty(\Omega)$ with $\|u\|_{L^\infty(\Omega)} \leq C^\star \lambda^{\frac{1}{p_1 - 1}}$ and $\|u\|_{L^\infty(\Omega)} \leq C_\star \lambda^{\frac{1}{p_N - 1}}$ when $\lambda < h$, where $C^\star$ and $C_\star$ are positive constants that depend only on $p_1, p_N, |\Omega|$ and C_0, with C_0 given in (4).

Proof. Let u_λ be a solution of (P_λ), then it is clear that u_λ is a nontrivial function with $u > 0$ in Ω, see for instance [36, Lemma 3.1]. For $k \geq 0$, we define the set $A_k := \{x \in \Omega; u(x) > k\}$. Let $\epsilon > 0$. In problem (P_λ) we use the test function $(u-k)^+ \in W_0^{1,\overrightarrow{p}}(\Omega)$ and obtain from (4) and Young's inequality that

$$\begin{aligned}
&\int_{A_k}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx=\lambda\int_{A_k}(u-k)dx\le\lambda|A_k|^{\frac{1}{N}}\|(u-k)^+\|_{L^{\frac{N}{N-1}}(\Omega)}\\
&\le\lambda|A_k|^{\frac{1}{N}}C_0\int_{A_k}|\nabla u|dx\le\lambda|A_k|^{\frac{1}{N}}C_0\int_{A_k}\sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|dx \qquad (5)\\
&\le\lambda|A_k|^{\frac{1}{N}}C_0\sum_{i=1}^{k}\int_{A_k}\frac{\left(\left|\epsilon\frac{\partial u}{\partial x_i}\right|\right)^{p_i}}{p_i}dx+\lambda|A_k|^{\frac{1}{N}}C_0\sum_{i=1}^{k}\int_{A_k}\frac{(\epsilon^{-1})^{(p_i)'}}{p_i{}'}dx.
\end{aligned}$$

Note that

$$\begin{aligned}
\lambda|A_k|^{\frac{1}{N}}C_0\sum_{i=1}^{k}\int_{A_k}\frac{\left(\epsilon\left|\frac{\partial u}{\partial x_i}\right|\right)^{p_i}}{p_i}&\le\frac{M|A_k|^{\frac{1}{N}}C_0}{p_1}\int_{A_k}\epsilon^{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\\
&\le\frac{M|\Omega|^{\frac{1}{N}}C_0}{p_1}\sum_{i=1}^{k}\int_{A_k}\epsilon^{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx.
\end{aligned}$$

Let $h:=\dfrac{p_1}{2|\Omega|^{\frac{1}{N}}C_0}$. Suppose that $M\ge h$. Define $\epsilon:=\left(\dfrac{p_1}{2M|\Omega|^{\frac{1}{N}}C_0}\right)^{\frac{1}{p_1}}$.

Thus $\epsilon\le 1$ and

$$\begin{aligned}
\frac{\lambda|\Omega|^{\frac{1}{N}}C_0}{p_1}\sum_{i=1}^{k}\int_{A_k}\epsilon^{p_i}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx&\le\frac{\lambda|\Omega|^{\frac{1}{N}}C_0\epsilon^{p_1}}{p_1}\sum_{i=1}^{k}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\\
&=\frac{1}{2}\sum_{i=1}^{k}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx.
\end{aligned}\qquad (6)$$

From (5) and (6) we get

$$\sum_{i=1}^{k}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\le\frac{2\lambda|A_k|^{\frac{1}{N}}C_0}{(p_N)'}\sum_{i=1}^{k}\int_{A_k}\epsilon^{-(p_i)'}dx\le\gamma|A_k|^{1+\frac{1}{N}},$$

where $\gamma:=\dfrac{2\lambda N|A_k|^{\frac{1}{N}}\epsilon^{-(p_1)'}C_0}{(p_N)'}$, which implies that

$$\int_{A_k}(u-k)dx=\frac{1}{\lambda}\sum_{i=1}^{N}\int_{A_k}\left|\frac{\partial u}{\partial x_i}\right|^{p_i}dx\le\gamma|A_k|^{1+\frac{1}{N}}.$$

Therefore by the L^∞ estimates in [47, Lemma 5.1-Chapter 2] we obtain

$$\|u\|_{L^\infty(\Omega)}\le\gamma(N+1)|\Omega|^{\frac{1}{N}}=C^\star\lambda^{\frac{1}{p_1-1}},$$

where $C^\star$ is a constant that has the dependence described in the statement of the result. In the other case, that is, when $\lambda<h$ the result follows by repeating the previous arguments with

$$\epsilon:=\left(\frac{p_1}{2\lambda|\Omega|^{\frac{1}{N}}C_0}\right)^{\frac{1}{p_N}}. \qquad \square$$

5. Proof of Theorems 2.1 and 2.2

The goal of this section is to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1:

Let $T : L^1(\Omega) \to L^1(\Omega)$ be the operator given by the formula

$$(Tu)(x) = \begin{cases} \underline{u}(x), & \text{if} \quad u(x) \le \underline{u}(x), \\ u(x), & \text{if} \quad \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \overline{u}(x), & \text{if} \quad \overline{u}(x) \le u(x). \end{cases}$$

The operator T is well defined because $\underline{u}, \overline{u} \in L^\infty(\Omega)$, $Tu \in [\underline{u}, \overline{u}]$ and the measure of Ω is finite. Define the operator $H : [\underline{u}, \overline{u}] \to L^1(\Omega)$ given by

$$H(v)(x) = \frac{\|v\|_{L^t}^\alpha}{\mathcal{H}(x, \|v\|_{L^r})} + \frac{\|v\|_{L^s}^\gamma}{\mathcal{H}(x, \|v\|_{L^r})}.$$

Note that the operators H and $u \mapsto (H \circ T)(u)$ are well defined. In fact, we have that f and g are continuous functions and $\mathcal{H}(x, t) > 0$ in the compact set $\overline{\Omega} \times [\|\underline{u}\|_{L^r}, \|\overline{u}\|_{L^r}]$, then there is a constant K_0 such that

$$|H(v)(x)| \le K_0 \text{ a.e. in } \Omega, \tag{7}$$

for all $v \in [\underline{u}, \overline{u}]$. The inclusion $Tu \in [\underline{u}, \overline{u}]$ implies $|(H \circ T)(u)(x)| \le K_0$ a.e. in Ω.

Let $v \in L^1(\Omega)$ be an arbitrary function. From (7) it follows that $(HoT)(v) \in L^\infty(\Omega)$. Then we have by Lemma 3.1 that the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u &= H(Tv) \;\; \text{in} \;\; \Omega, \\ u &= 0 \qquad\;\; \text{on} \;\; \partial\Omega. \end{cases} \tag{P_L}$$

has a unique solution. Therefore we can define an operator $\widetilde{S} : L^1(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega)$, given by the equation $\widetilde{S}(v) = u$, where $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ is the unique solution of (P_L). Consider the compact embedding $i : W_0^{1,\overrightarrow{p}}(\Omega) \to L^1(\Omega)$. We claim that the operator $S := i \circ \widetilde{S}$ is compact. In order to prove this claim let (v_n) be a sequence with $v_n \to v$ in $L^1(\Omega)$ and consider $u_n := S(v_n), n \in \mathbb{N}$. From the definition of the operator S it follows that

$$\int_\Omega \sum_{i=1}^N \left| \frac{\partial u_n}{\partial x_i} \right|^{p_i - 2} \frac{\partial u_n}{\partial x_i} \frac{\partial \phi}{\partial x_i} = \int_\Omega (H \circ T)(v_n) \phi, \tag{8}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Using the test function $\phi = u_n, n \in \mathbb{N}$ in (8) and in the inequality (7) we get

$$\sum_{i=1}^N \left\| \frac{\partial u_n}{\partial x_i} \right\|_{L^{p_i}}^{p_i} \le K_0 \int_\Omega |u_n| \le K \|u_n\|, \tag{9}$$

where $K > 0$ is a constant that does not depend on $n \in \mathbb{N}$. The inequality

$$|t|^{p_1} \le 1 + |t|^{p_i}, \;\; t \in \mathbb{R}, \;\; i = 1, ..., N,$$

provides that
$$\sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}^{p_1} \le \widetilde{K}(\|u_n\|+1), \tag{10}$$

for all $n \in \mathbb{N}$, where $\widetilde{K} > 0$ is a constant that does not depend on $n \in \mathbb{N}$. Since $(a_1 + ... + a_N)^b \le C(a_1^b + ... + a_N^b)$ for $a_i \ge 0, i = 1, ..., N$ and $b \ge 1$, where $C > 0$ depends only on N and b, it follows that

$$\left(\sum_{i=1}^{N}\left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}\right)^{p_1} \le \overline{K}(\|u_n\|+1), \tag{11}$$

for all $n \in \mathbb{N}$ with $\overline{K} > 0$ being a constant that does not depend on $n \in \mathbb{N}$. From (11) it follows that the sequence u_n is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$. Since the embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^1(\Omega)$ is compact, the boundness of (u_n) in $W_0^{1,\overrightarrow{p}}(\Omega)$ implies that there exists $u \in L^1(\Omega)$ with $u_n \to u$ in $L^1(\Omega)$. Then it is verified that S is compact.

We affirm that S is continuous. To verify the mentioned continuity, let (v_n) be a sequence in $L^1(\Omega)$ such that $v_n \to v$ in $L^1(\Omega)$. Considering $u_n = S(v_n), n \in \mathbb{N}$ and $u = S(v)$, thus we obtain from the definition of S that

$$\int_\Omega \sum_{i=1}^{N}\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (H\circ T)(v_n)\phi \tag{12}$$

and
$$\int_\Omega \sum_{i=1}^{N}\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (H\circ T)(v)\phi \tag{13}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Using $\phi = u_n$ as a test function and subtracting (13) from (12) we get

$$\begin{aligned}\int_\Omega \sum_{i=1}^{N}\left[\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\right] \\ = \int_\Omega ((H\circ T)(v_n) - (H\circ T)(v))(u_n - u). \end{aligned}\tag{14}$$

Thus by using (7) and Lebesgue's Dominated Convergence Theorem it follows that the right-hand side of the equation (14) converges to zero. Since

$$\begin{aligned}\left(\left|\frac{\partial u_n(x)}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n(x)}{\partial x_i} - \left|\frac{\partial u(x)}{\partial x_i}\right|^{p_i-2}\frac{\partial u(x)}{\partial x_i}\right)\left(\frac{\partial u_n(x)}{\partial x_i} - \frac{\partial u(x)}{\partial x_i}\right) \\ \ge C_i\left|\frac{\partial u_n(x)}{\partial x_i} - \frac{\partial u(x)}{\partial x_i}\right|^{p_i} \text{ a.e. in } \Omega,\end{aligned}\tag{15}$$

where $C_i > 0$ is a constant that does not depend on $n \in \mathbb{N}$. Thus we can infer from (15) that $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$, which verifies the continuity of S.

We claim that there is $R > 0$ such that $u = \sigma S(u)$ with $\sigma \in [0,1]$ then $\|u\|_{L^1} < R$, where R is a constant that does not depend on u and σ. If $\sigma = 0$ then $u = 0$.

Suppose that $\sigma \neq 0$. Then $S(u) = \frac{u}{\sigma}$, which implies the identitiy

$$\int_\Omega \sum_{i=1}^N \left|\frac{1}{\sigma}\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{1}{\sigma}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (H\circ T)(u)\phi, \tag{16}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Using the test function $\phi = \frac{u}{\sigma}$ in (16) and the inequality (7) we obtain that

$$\frac{1}{\sigma^{p_1}}\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \le \int_\Omega \sum_{i=1}^N \frac{1}{\sigma^{p_i}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} \le \frac{K_0}{\sigma}\int_\Omega |u|,$$

which implies that
$$\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}}^{p_i} \le K_0\sigma^{p_1-1}\int_\Omega |u|. \tag{17}$$

Using again the inequalities $|t|^{p_1} \le 1+|t|^{p_i}$, $t\in\mathbb{R}$ and $(a_1+...+a_N)^b \le C(a_1^b+...+a_N^b)$ for $a_i \ge 0$, $i = 1, ..., N$ we obtain from (17) that

$$\left(\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{L^{p_i}}\right)^{p_1} \le K_1 + K_2\sigma^{p_1-1}\int_\Omega |u|, \tag{18}$$

where $K_i > 0, i = 1, 2$ are constants that does not depend on u and σ. Since $\sigma \in [0, 1]$ we obtain by applying the continuous embedding $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^1(\Omega)$ that

$$\|u\|_{L^1}^{p_1} \le K_1 + K_2\sigma^{p_1-1}\int_\Omega |u| \le K_3 + K_4\|u\|_{L^1}, \tag{19}$$

where $K_i, i = 1, ..., 4$ are constants that does not depend on u. If $\|u\|_{L^1} \le 1$ then $\|u\|_{L^1} \le (K_3 + K_4)^{\frac{1}{p_1}}$. In the case $\|u\|_{L^1} > 1$ we obtain that

$$\|u\|_{L^1}^{p_1-1} \le \frac{K_3}{\|u\|_{L^1}} + K_4 \le K_3 + K_4.$$

Therefore $\|u\|_{L^1} \le R$, where $R := \max\{(K_3 + K_4)^{\frac{1}{p_1}}, (K_3 + K_4)^{\frac{1}{p_1-1}}\}$. Thus by the Schaefer's Fixed Point Theorem there exists $u \in L^1(\Omega)$ such that $u = S(u)$. Note also that by the definition of S we obtain $u \in W_0^{1,\overrightarrow{p}}(\Omega)$ with

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (H\circ T)(u)\phi, \tag{20}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$. We claim that $u \in [\underline{u}, \overline{u}]$. In order to verify this claim consider $w = T(u)$ in (2), and subtracting from (20) we have

$$\int_\Omega \sum_{i=1}^N \left[\left(\left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial \underline{u}}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\right]$$
$$\le \int_\Omega \left(\frac{\|\underline{u}\|_{L^t}^\alpha - \|T(u)\|_{L^t}^\alpha}{\mathcal{H}(x, \|T(u)\|_{L^r})}\right)\phi + \int_\Omega \left(\frac{\|\underline{u}\|_{L^s}^\gamma - \|T(u)\|_{L^s}^\gamma}{\mathcal{H}(x, \|T(u)\|_{L^r})}\right)\phi, \tag{21}$$

for all $\phi \in W_0^{1,\overrightarrow{p}}(\Omega)$ with $\phi \ge 0$.

Consider the test function $\phi := (\underline{u} - u)_+ = \max\{\underline{u} - u, 0\} \in W_0^{1,\overrightarrow{p}}(\Omega)$. Using (21) and that $T(u) \in [\underline{u}, \overline{u}]$, we obtain that

$$\begin{aligned}
&\int_{\{\underline{u}\geq u\}} \sum_{i=1}^{N} \left[\left(\left| \frac{\partial \underline{u}}{\partial x_i} \right|^{p_i-2} \frac{\partial \underline{u}}{\partial x_i} - \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right) \left(\frac{\partial \underline{u}}{\partial x_i} - \frac{\partial u}{\partial x_i} \right) \right] \\
&\leq \int_{\{\underline{u}\geq u\}} \left(\frac{\|\underline{u}\|_{L^q}^{\alpha} - \|T(u)\|_{L^q}^{\alpha}}{\mathcal{H}(x, \|T(u)\|_{L^r})} \right) (\underline{u} - u) + \int_{\{\underline{u}\geq u\}} \left(\frac{\|\underline{u}\|_{L^s}^{\gamma} - \|T(u)\|_{L^s}^{\gamma}}{\mathcal{H}(x, \|T(u)\|_{L^r})} \right) (\underline{u} - u) \\
&\leq 0,
\end{aligned}$$

which implies that $\underline{u} \leq u$ in Ω. Now, a similar reasoning applied to the function $\phi := (u - \overline{u})^+ \in W_0^{1,\overrightarrow{p}}(\Omega)$ implies the result.

Proof of Theorem 2.2:

Consider $T, S : L^1(\Omega) \to L^1(\Omega)$ the operators provided by the formulas

$$(Tz)(x) = \begin{cases} \underline{u}(x), & \text{if } \ z(x) \leq \underline{u}(x), \\ u(x), & \text{if } \ \underline{u}(x) \leq z(x) \leq \overline{u}(x), \\ \overline{u}(x), & \text{if } \ \overline{u}(x) \leq z(x). \end{cases}$$

$$(Sw)(x) = \begin{cases} \underline{v}(x), & \text{if } \ w(x) \leq \underline{v}(x), \\ u(x), & \text{if } \ \underline{v}(x) \leq w(x) \leq \overline{v}(x), \\ \overline{v}(x), & \text{if } \ \overline{v}(x) \leq w(x). \end{cases}$$

The boundedness of $\underline{u}, \overline{u}, \underline{v}$ and $\overline{v}$ and the finite measure of Ω imply that the operators T and S are well defined. Define the operators

$$H : [\underline{v}, \overline{v}] \to L^1(\Omega) \text{ and } \widetilde{H} : [\underline{u}, \overline{u}] \to L^1(\Omega)$$

given respectively by

$$H(v)(x) = \frac{\|v\|_{L^{t_1}}^{\alpha_1}}{\mathcal{H}(x, \|v\|_{L^{r_1}})} + \frac{\|v\|_{L^{s_1}}^{\gamma_1}}{\mathcal{H}(x, \|v\|_{L^{r_1}})}$$

and

$$\widetilde{H}(u)(x) = \frac{\|u\|_{L^{t_2}}^{\alpha_2}}{\mathcal{A}(x, \|u\|_{L^{r_2}})} + \frac{\|u\|_{L^{s_2}}^{\gamma_2}}{\mathcal{A}(x, \|u\|_{L^{r_2}})}$$

Note that the operators $H, \widetilde{H}$ and $w \mapsto (H \circ S)(w), z \mapsto (\widetilde{H} \circ T)(z)$ are well defined. In fact, since $\mathcal{H}$ is continuous and $\mathcal{H}(x,t) > 0$ in the compact set $\overline{\Omega} \times [\underline{\sigma}, \overline{\sigma}]$, then there is a constant K_0 such that

$$|H(v)(x)|, |\widetilde{H}(u)(x)| \leq K_0 \text{ a.e. in } \Omega, \tag{22}$$

for all $v \in [\underline{v}, \overline{v}]$ and $u \in [\underline{u}, \overline{u}]$. The inclusions $Tu \in [\underline{u}, \overline{u}]$, $u \in L^1(\Omega)$ and $Sv \in [\underline{v}, \overline{v}]$, $v \in L^1(\Omega)$ imply that $|(H \circ S)(w)(x)|$, $|(\widetilde{H} \circ T)(z)(x)| \leq K_0$ a.e. in Ω for all $w, z \in L^1(\Omega)$. Since the measure of Ω is finite it follows that the operators are well defined.

Let $(z, w) \in L^1(\Omega) \times L^1(\Omega)$ be arbitrary functions. From (22) it follows that

$$(H \circ S)(w), \ (\widetilde{H} \circ T)(z) \in L^\infty(\Omega).$$

Thus by Lemma 3.1 the problem

$$\begin{cases} -\Delta_{\overrightarrow{p}} u &= (H \circ S)(w) \text{ in } \Omega, \\ -\Delta_{\overrightarrow{q}} v &= (\widetilde{H} \circ T)(z) \text{ in } \Omega, \\ u = v &= 0 \text{ on } \partial\Omega. \end{cases} \qquad (\widetilde{P_L})$$

has a unique solution in $W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$. Therefore we can define the operator $\overline{S} : L^1(\Omega) \times L^1(\Omega) \to W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$, given by the equation $\overline{S}(z,w) = (u,v)$, where $(u,v) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$ is the unique solution of $(\widetilde{P_L})$. Consider in $L^1(\Omega) \times L^1(\Omega)$ the norm $|(z,w)| := \|z\|_{L^1(\Omega)} + \|w\|_{L^1(\Omega)}$, $(z,w) \in L^1(\Omega) \times L^1(\Omega)$. Then consider the compact embeddings

$$i_{\overrightarrow{p}} : W_0^{1,\overrightarrow{p}}(\Omega) \to L^1(\Omega) \text{ and } i_{\overrightarrow{q}} : W_0^{1,\overrightarrow{q}}(\Omega) \to L^1(\Omega)$$

given in [41]. We affirm that the operator

$$\widetilde{S}(z,w) = (i_{\overrightarrow{p}}(u), i_{\overrightarrow{q}}(v)) = (u,v), (z,w) \in L^1(\Omega) \times L^1(\Omega),$$

where $\overline{S}(z,w) = (u,v)$ is compact. In fact let (z_n, w_n) be a sequence with $z_n \to z$ and $v_n \to v$ in $L^1(\Omega)$. Define $(u_n, v_n) := \widetilde{S}(w_n, z_n), n \in \mathbb{N}$. From the definition of the operator $\widetilde{S}$ it follows that

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (H \circ S)(w_n)\phi, \tag{23}$$

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial v_n}{\partial x_i}\right|^{q_i-2} \frac{\partial v_n}{\partial x_i}\frac{\partial \psi}{\partial x_i} = \int_\Omega (\widetilde{H} \circ T)(z_n)\psi, \tag{24}$$

for all $(\phi,\psi) \in W_0^{1,\overrightarrow{p}}(\Omega) \times W_0^{1,\overrightarrow{q}}(\Omega)$. Using the inequality (22) and the test functions $\phi = u_n$ and $\psi = v_n, n \in \mathbb{N}$ in (23) and (24) and arguing as in (9) and (10) we obtain

$$\begin{aligned} \left(\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{L^{p_i}}\right)^{p_1} &\le K(\|u_n\|_{1,\overrightarrow{q}} + 1), \\ \left(\sum_{i=1}^N \left\|\frac{\partial v_n}{\partial x_i}\right\|_{L^{q_i}}\right)^{q_1} &\le K(\|v_n\|_{1,\overrightarrow{q}} + 1), \end{aligned} \tag{25}$$

for all $n \in \mathbb{N}$ where $K > 0$ is a constant that does not depend on n. Thus by (25) it follows that u_n and v_n are bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$ and $W_0^{1,\overrightarrow{q}}(\Omega)$ respectively. Using the compactness of the embeddings $W_0^{1,\overrightarrow{p}}(\Omega) \hookrightarrow L^1(\Omega)$, $W_0^{1,\overrightarrow{q}}(\Omega) \hookrightarrow L^1(\Omega)$ we obtain that the operator $\widetilde{S}$ is compact.

We claim that $\widetilde{S}$ is continuous. To verify the mentioned continuity let (w_n, z_n) be a sequence in $L^1(\Omega) \times L^1(\Omega)$ such that $w_n \to w$ and $z_n \to z$ in $L^1(\Omega)$ with $w, z \in L^1(\Omega)$. Considering $(u_n, v_n) = \widetilde{S}(w_n, z_n), n \in \mathbb{N}$ and $(u,v) = \widetilde{S}(w,z)$, we obtain from the definition of $\widetilde{S}$ that

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2} \frac{\partial u_n}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (H\circ S)(w_n)\phi,$$

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial v_n}{\partial x_i}\right|^{q_i-2} \frac{\partial v_n}{\partial x_i}\frac{\partial \psi}{\partial x_i} = \int_\Omega (\widetilde{H}\circ T)(z_n)\psi \tag{26}$$

and

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} = \int_\Omega (H\circ S)(w)\phi,$$

$$\int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} = \int_\Omega (\widetilde{H}\circ T)(z)\psi \tag{27}$$

for all $(\phi,\psi)\in W_0^{1,\overrightarrow{p}}(\Omega)\times W_0^{1,\overrightarrow{q}}(\Omega)$. Using (26), (27) and $\phi = u_n$ as a test function we obtain that

$$\int_\Omega \sum_{i=1}^N \left[\left(\left|\frac{\partial u_n}{\partial x_i}\right|^{p_i-2}\frac{\partial u_n}{\partial x_i} - \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u_n}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\right]$$
$$= \int_\Omega ((H\circ T)(v_n) - (H\circ T)(v))(u_n - u). \tag{28}$$

Thus by using (22) and the Lebesgue's Dominated Convergence Theorem it follows that the right-hand side of the equation (28) converges to zero. Using (15) it follows that $u_n \to u$ in $W_0^{1,\overrightarrow{p}}(\Omega)$. A similar reasoning implies that $v_n \to v$ in $W_0^{1,\overrightarrow{q}}(\Omega)$. Thus it follows that S is continuous.

We claim that there is $R > 0$ such that if $(u,v) = \sigma S(u,v)$ with $\sigma \in [0,1]$ then $|(u,v)| \le R$, where R is a constant that does not depend on $u, v \in L^1(\Omega)$ and σ. If $\sigma = 0$ then $u = v = 0$. Suppose that $\sigma \neq 0$. Then $S(u,v) = \frac{(u,v)}{\sigma}$ and it follows that

$$\begin{aligned} \int_\Omega \sum_{i=1}^N \left|\frac{1}{\sigma}\frac{\partial u}{\partial x_i}\right|^{p_i-2} \frac{1}{\sigma}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i} &= \int_\Omega (H\circ S)(v)\phi,\\ \int_\Omega \sum_{i=1}^N \left|\frac{1}{\sigma}\frac{\partial v}{\partial x_i}\right|^{q_i-2} \frac{1}{\sigma}\frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i} &= \int_\Omega (\widetilde{H}\circ T)(u)\psi, \end{aligned} \tag{29}$$

for all $(\phi,\psi)\in W_0^{1,\overrightarrow{p}}(\Omega)\times W_0^{1,\overrightarrow{q}}(\Omega)$. Using the test functions $\phi = \frac{u}{\sigma}$ and $\psi = \frac{v}{\sigma}$ in (29) and the inequality (22) we obtain that

$$\frac{1}{\sigma^{p_1}}\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i} \le \int_\Omega \sum_{i=1}^N \frac{1}{\sigma^{p_i}}\left|\frac{\partial u}{\partial x_i}\right|^{p_i} \le \frac{K_0}{\sigma}\int_\Omega |u|$$

and

$$\frac{1}{\sigma^{q_1}}\int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^{q_i} \le \int_\Omega \sum_{i=1}^N \frac{1}{\sigma^{q_i}}\left|\frac{\partial v}{\partial x_i}\right|^{q_i} \le \frac{K_0}{\sigma}\int_\Omega |v|.$$

Arguing as in (17), (18) and (19) we obtain that

$$\|u\|_{L^1}^{p_1} \le A_1 + A_2\|u\|_{L^1} \quad \text{and} \quad \|v\|_{L^1}^{q_1} \le A_1 + A_2\|v\|_{L^1},$$

where A_1 and A_2 are constants that does not depend on $n \in \mathbb{N}$.

If $\|u\|_{L^1}\le 1$ then $\|u\|_{L^1}\le (A_1+A_2)^{\frac{1}{p_1}}$. In the case $\|u\|_{L^1}>1$ we obtain that

$$\|u\|_{L^1}^{p_1-1}\le \frac{A_2}{\|u\|_{L^1}}+A_2\le A_1+A_2.$$

Therefore $\|u\|_{L^1}\le R_1$, where $R_1:=\max\{(A_1+A_2)^{\frac{1}{p_1}},(A_1+A_2)^{\frac{1}{p_1-1}}\}$. A similar reasoning implies that $\|v\|_{L^1}\le R_2$, where $R_2:=\max\{(A_1+A_2)^{\frac{1}{q_1}},(A_1+A_2)^{\frac{1}{q_1-1}}\}$. Thus by the Schaefer's Fixed Point Theorem there exists $(u,v)\in L^1(\Omega)\times L^1(\Omega)$ such that $(u,v)=S(u,v)$. Note that in consequence of the definition of S we have $(u,v)\in W_0^{1,\overrightarrow{p}}(\Omega)\times W_0^{1,\overrightarrow{q}}(\Omega)$ with

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^N \left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\frac{\partial \phi}{\partial x_i}=\int_\Omega (H\circ S)(v)\phi,\\
&\int_\Omega \sum_{i=1}^N \left|\frac{\partial v}{\partial x_i}\right|^{q_i-2}\frac{\partial v}{\partial x_i}\frac{\partial \psi}{\partial x_i}=\int_\Omega (\widetilde{H}\circ T)(u)\psi,
\end{aligned}\tag{30}$$

for all $(\phi,\psi)\in W_0^{1,\overrightarrow{p}}(\Omega)\times W_0^{1,\overrightarrow{q}}(\Omega)$. We claim that $u\in[\underline{u},\overline{u}]$. In order to verify such claim consider $w=S(v)$ in (2) and subtracting from (30) we have

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^N \left[\left(\left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\frac{\partial \phi}{\partial x_i}\right]\\
&\le \int_\Omega\left(\frac{\|\underline{v}\|_{L^{t_1}}^{\alpha_1}-\|S(v)\|_{L^{t_1}}^{\alpha_1}}{\mathcal{H}(x,\|S(v)\|_{L^{r_1}})}\right)\phi+\int_\Omega\left(\frac{\|\underline{u}\|_{L^{s_1}}^{\gamma_1}-\|S(v)\|_{L^{s_1}}^{\gamma}}{\mathcal{H}(x,\|S(v)\|_{L^{r_1}})}\right)\phi,
\end{aligned}\tag{31}$$

for all $\phi\in W_0^{1,\overrightarrow{p}}(\Omega)$ with $\phi\ge 0$. Consider the test function

$$\phi:=(\underline{u}-u)_+=\max\{\underline{u}-u,0\}\in W_0^{1,\overrightarrow{p}}(\Omega).$$

Using the inequality (31) and the fact that $S(v)\in[\underline{v},\overline{v}]$ we obtain that

$$\begin{aligned}
&\int_{\{\underline{u}\ge u\}} \sum_{i=1}^N \left[\left(\left|\frac{\partial \underline{u}}{\partial x_i}\right|^{p_i-2}\frac{\partial \underline{u}}{\partial x_i}-\left|\frac{\partial u}{\partial x_i}\right|^{p_i-2}\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial \underline{u}}{\partial x_i}-\frac{\partial u}{\partial x_i}\right)\right]\\
&\le \int_{\{\underline{u}\ge u\}}\left(\frac{\|\underline{v}\|_{L^{t_1}}^{\alpha_1}-\|S(v)\|_{L^{t_1}}^{\alpha_1}}{\mathcal{H}(x,\|S(v)\|_{L^{r_1}})}\right)(\underline{u}-u)+\int_{\{\underline{u}\ge u\}}\left(\frac{\|\underline{v}\|_{L^{s_1}}^{\gamma_1}-\|S(v)\|_{L^{s_1}}^{\gamma_1}}{\mathcal{H}(x,\|S(v)\|_{L^{r_1}})}\right)(\underline{u}-u)\le 0,
\end{aligned}$$

which implies that $\underline{u}\le u$ in Ω. A similar reasoning applied to the function ψ defined by $\psi:=(u-\overline{u})^+\in W_0^{1,\overrightarrow{p}}(\Omega)$ implies that $u\le\overline{u}$ in Ω. The previous arguments also imply that $\underline{v}\le v\le\overline{v}$ in Ω.

6. Applications

The goal of this section is the application of Theorems 2.1 and 2.2 to the nonlocal problems $(E_s),(E_{\lambda,\theta}),(S_s)$ and $(S_{\lambda,\theta})$.

Proof of Theorem 2.3:

We will begin the proof with the condition (i), that is, $\mathcal{H}(x,t) \geq h_0$ in $\overline{\Omega} \times [0,+\infty)$. First it will be considered the construction of the function $\overline{u}$. Consider $\lambda > 0$ and let $z_\lambda \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ be the unique solutions of the problem below

$$\begin{cases} -\Delta_{\overrightarrow{p}} u &= \lambda \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ will be chosen later. For $\lambda \geq 1$ large enough we have by Lemma 4.1 that there is a constant $K > 0$, that does not depend on λ, such that

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{p_1-1}} \text{ in } \Omega. \tag{32}$$

Since $0 < \alpha < p_1 - 1$, there exists $\lambda \geq 1$ large enough such that (32) occurs and

$$\frac{1}{h_0}\lambda^{\frac{\alpha}{p_1-1}}\|K\|^\alpha_{L^t} \leq \lambda. \tag{33}$$

The inequalities (32) and (33) imply that $\frac{1}{h_0}\ \|z_\lambda\|^\alpha_{L^t} \leq \lambda$. Thus for an arbitrary function $w \in L^\infty(\Omega)$ we get

$$\begin{cases} -\Delta_{\overrightarrow{p}} z_\lambda &\geq \dfrac{1}{\mathcal{H}(x,\|w\|_{L^r})}\|z_\lambda\|^\alpha_{L^t} \text{ in } \Omega, \\ z_\lambda &= 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega. \end{cases}$$

Let $\mathcal{H}_\lambda := \max\{\mathcal{H}(x,t); (x,t) \in \overline{\Omega} \times [0,\|z_\lambda\|_{L^r}]\}$. Therefore

$$h_0 \leq \mathcal{H}(x,t) \leq \mathcal{H}_\lambda \text{ in } \Omega, \forall w \in [0,z_\lambda]. \tag{34}$$

Now it will be considered the function $\underline{u}$. Since $\partial\Omega$ is C^2, there exist a constant $\delta > 0$ such that $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d| \equiv 1$ in $\overline{\Omega}_{3\delta}$, where $d(x) := dist(x,\partial\Omega)$ and $\overline{\Omega_{3\delta}} := \{x \in \overline{\Omega}; d(x) \leq 3\delta\}$, see [44, Lemma 14.16] and its proof.

Let $k > 0$ and $\sigma \in (0,\delta)$. Motivated by [36, 59] we introduce the function

$$\phi(x) = \begin{cases} e^{kd(x)} - 1, & \text{if } \ d(x) < \sigma, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^\theta dt, & \text{if } \ \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^\theta dt, & \text{if } \ 2\delta \leq d(x), \end{cases} \tag{35}$$

where $\theta > \frac{1}{p_1-1}$. Note that $\phi \in C^1(\overline{\Omega})$ and $\phi = 0$ on $\partial\Omega$. Consider $\mu > 0$. Direct computations imply that if $x \in \Omega$ satisfy $d(x) < \sigma$ with $\frac{\partial d(x)}{\partial x_i} \neq 0$ then

$$\begin{aligned} &-\sum_{i=1}^N \frac{\partial}{\partial x_i}\left(\left|\frac{\partial(\mu\phi)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\mu\phi)}{\partial x_i}\right) \\ &= -\sum_{i=1}^N (k\mu)^{p_i-1}\left(e^{kd(x)(p_i-1)}k(p_i-1)\left|\frac{\partial d}{\partial x_i}\right|^{p_i} + e^{kd(x)(p_i-1)}\left|\frac{\partial d}{\partial x_i}\right|^{p_i-2}\frac{\partial^2 d}{\partial x_i^2}\right) \\ &:= B(x). \end{aligned} \tag{36}$$

In the case $\sigma < d(x) < \delta$ we have

$$\begin{aligned}
&-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial(\mu\phi)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\mu\phi)}{\partial x_i}\right)\\
&=\sum_{i=1}^{n}(\mu k e^{k\sigma})^{p_i-1}\theta(p_i-1)\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\theta(p_i-1)-1}\frac{1}{2\delta-\sigma}\left|\frac{\partial d}{\partial x_i}\right|^{p_i-1}+\\
&\qquad+\sum_{i=1}^{n}(\mu k e^{k\sigma})^{p_i-1}\left(\frac{2\delta-d(x)}{2\delta-\sigma}\right)^{\theta(p_i-1)}\left|\frac{\partial d}{\partial x_i}\right|^{p_i-2}\left(-\frac{\partial^2 d}{\partial x_i^2}\right)\\
&:= C(x). \qquad (37)
\end{aligned}$$

From (36) and (37) we get

$$\int_\Omega \sum_{i=1}^{N}\left|\frac{\partial(\mu\phi)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\mu\phi)}{\partial x_i}\frac{\partial\varphi}{x_i}=\int_\Omega A\varphi,\ \forall\varphi\in W_0^{1,\overrightarrow{p}}(\Omega),$$

where $A(x)$ is equal to $B(x)$ when $d(x) < \sigma$ with $\frac{\partial d(x)}{\partial x_i} \neq 0$, and equal to $C(x)$ when $\sigma < d(x) < 2\delta$ with $\frac{\partial d(x)}{\partial x_i} \neq 0$, and equal to zero when one of the following conditions is satisfied: $\frac{\partial d(x)}{\partial x_i} = 0$, $d(x) \geq 2\delta$ or $d(x) = \sigma$.

Consider that $\mu = e^{-k}$. Since $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d(x)| \equiv 1$ in $\overline{\Omega_{2\delta}}$ we have from (36) and (37) that

$$\begin{aligned}
|A(x)| &\leq \max_{i=1,\dots,n}\{(\mu k e^{k\sigma})^{p_i-1}\}\frac{C_1}{2\delta-\sigma} \quad \text{a.e. in } \ \Omega_{2\delta},\\
&\leq \frac{C_1}{2\delta-\sigma}(\mu k)^{p_1-1}, \qquad (38)
\end{aligned}$$

for $k > 0$ large enough and for all $x \in \Omega$, where $C_1 > 0$ is a constant that does not depend on k and σ.

Consider $\sigma := (\ln 2)/k$. Note that $\phi(x) \geq e^{k\sigma} - 1 \geq 2 - 1 \geq 1$ for all $k > 0$ and for $x \in \Omega$ with $\sigma \leq d(x) < 2\delta$. Then there exists a constant $C_2 > 0$, which does not depend on $k > 0$, such that

$$\|\mu\phi\|_{L^t}^{\alpha} \geq \mu^\alpha C_2. \qquad (39)$$

Using the fact $0 < \alpha < p_1 - 1$, and the L'Hospital's rule we obtain that

$$\lim_{k\to+\infty}\frac{k^{p_1-1}}{e^{k(p_1-1-\alpha)}} = 0,$$

which implies that there exists k_0 large enough such that

$$\frac{C_2}{\mathcal{H}_\lambda C_1} \geq \frac{k^{p_1-1}}{e^{k(p_1-1-\alpha)}}\frac{1}{2\delta-\frac{ln2}{\sigma}}, \qquad (40)$$

for all $k \geq k_0$, where $\mathcal{H}_\lambda$ is given in (34).

Then it is possible to conclude by (38), (39) and (40) that

$$-\sum_{i=1}^{N}\left|\frac{\partial(\mu\phi)}{\partial x_i}\right|^{p_i-2}\frac{\partial(\mu\phi)}{\partial x_i}\leq\frac{1}{\mathcal{H}_\lambda}\|\mu\phi\|_{L^t}^{\alpha}\leq\frac{1}{\mathcal{H}(x,\|w\|_{L^r})}\|\mu\phi\|_{L^t}^{\alpha}$$

in Ω for all $w\in[0,z_\lambda]$. Note that from (38) if k is large enough we also have

$$-\Delta_{\overrightarrow{p}}(\mu\phi)\leq 1\leq\lambda=-\Delta_{\overrightarrow{p}}z_\lambda \text{ in } \Omega.$$

Thus by Lemma 3.2 we have $\mu\phi\leq z_\lambda$ in Ω. Then by Theorem 2.1 we have the result.

In order to prove the second part of the result suppose that $0<\mathcal{H}(x,t)\leq h_0$ in $\overline{\Omega}\times(0,\infty)$ and that $\lim_{t\to+\infty}\mathcal{H}(x,t)=h_\infty$ uniformly in Ω. Let $\delta,\sigma,\mu,M,z_\lambda$ and ϕ as before. Repeating the previous arguments it follows that there exist $k>0$ large and $\mu>0$ small enough such that

$$-\Delta_{\overrightarrow{p}}(\mu\phi)\leq 1 \text{ in } \Omega \quad\text{ and }\quad -\Delta_{\overrightarrow{p}}(\mu\phi)\leq\frac{1}{h_0}\|\mu\phi\|_{L^t}^{\alpha} \text{ in } \Omega.$$

Thus for $w\in L^\infty(\Omega)$ we obtain that

$$-\Delta_{\overrightarrow{p}}(\mu\phi)\leq\frac{1}{\mathcal{H}(x,\|w\|_{L^r})}\|\mu\phi\|_{L^t}^{\alpha} \text{ in } \Omega. \tag{41}$$

Using the fact that $\lim_{t\to+\infty}\mathcal{H}(x,t)=h_\infty$ we can infer that there exists a constant $h_1>0$ such that $H(x,t)\geq\frac{h_\infty}{2}$ in $\overline{\Omega}\times(h_1,\infty)$. Let

$$m_k=\min\{\mathcal{H}(x,t),\overline{\Omega}\times[\|\mu\phi\|_{L^r},h_1]\} \text{ and } \mathcal{H}_k:=\min\{m_k,\tfrac{h_\infty}{2}\}.$$

Then $\mathcal{H}(x,t)\geq\mathcal{H}_k$ in $\overline{\Omega}\times[\|\mu\phi\|_{L^r},\infty]$. Fix $k>0$ such that (41) occurs. Suppose also that $\lambda>0$ is large enough such that (32) is satisfied and

$$\frac{1}{\mathcal{H}_k}\lambda^{\frac{\alpha}{p_1-1}}\|K\|_{L^t}^{\alpha}\leq\lambda,$$

where $K>0$ is a constant which does not depend on k and λ (see Lemma 4.1). Then we can infer that for all $w\in L^\infty(\Omega)$ with $\mu\phi\leq w$ in Ω we have

$$-\Delta_{\overrightarrow{p}}z_\lambda\geq\frac{1}{\mathcal{H}(x,\|w\|_{L^r})}\|z_\lambda\|_{L^t}^{\alpha} \text{ in } \Omega.$$

By using Lemma 3.2 we obtain that $\mu\phi\leq z_M$ in Ω. Then we have that $(\mu\phi,z_\lambda)$ is a sub-supersolution for (E_s).

Proof Theorem 2.4:

Suppose initially that (i) occurs. Consider $z_\lambda\in W_0^{1,\overrightarrow{p}}(\Omega)\bigcap L^\infty(\Omega)$ to be the unique solution of

$$\begin{cases}-\Delta_{\overrightarrow{p}}u &= \lambda \text{ in } \Omega,\\ u &= 0 \text{ in } \partial\Omega\end{cases}$$

where $\lambda\in(0,1)$ will be chosen later.

Lemma 4.1 implies that for $\lambda > 0$ small there is a constant $K > 1$ which does not depend on λ that satisfies

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{p_N-1}} \text{ in } \Omega. \tag{42}$$

We will construct the function $\overline{u}$. Consider $\overline{K} := \max\{\|K\|^\alpha_{L^q}, \|K\|^\gamma_{L^s}|\}$. Let $\theta > 0$ be an arbitrary number. Then it is possible choose $\lambda_0 > 0$, which depends on θ, such that

$$\frac{1}{a_0}\lambda^{\frac{\alpha+p_N-1}{p_N-1}}\overline{K} + \theta\lambda^{\frac{\gamma}{p_N-1}}\overline{K} \leq \lambda, \text{ for all } \lambda \in (0, \lambda_0), \tag{43}$$

where a_0 is given in (i). Thus by using (42) and (43) we obtain the inequality

$$\frac{1}{a_0}(\lambda\|z_\lambda\|^\alpha_{L^t} + \theta\|z_\lambda\|^\gamma_{L^s}) \leq \lambda. \tag{44}$$

If necessary, consider a smaller $\lambda_0 > 0$ satisfying $\|z_\lambda\|_{L^r} \leq \|K\|_{L^r}\lambda^{\frac{1}{p_N-1}} \leq b_0$ for all $\lambda \in (0, \lambda_0)$. Then we obtain that $\mathcal{H}(x, |w|_{L^r}) \geq a_0, w \in [0, z_\lambda]$. Thus from (44) we can infer that

$$-\Delta_{\overrightarrow{p}} z_\lambda \geq \frac{1}{\mathcal{H}(x, |w|_{L^r})}\left(\lambda\|z_\lambda\|^\alpha_{L^t} + \theta\|z_\lambda\|^\gamma_{L^s}\right), \tag{45}$$

for all $w \in [0, z_\lambda]$.

Regarding the construction of the function $\underline{u}$ let $\phi, \delta, \sigma, \mu$ as in the proof of Theorem 2.3. Let $\lambda > 0$ such that (45) occurs. Define the quantity

$$\mathcal{H}_0 := \max\{\mathcal{H}(x, t) : (x, t) \in \overline{\Omega} \times [0, b_0]\}.$$

Using the inequality $0 < \alpha < p_1 - 1$ and the arguments of Theorem 2.3 is is possible to infer that there exists $\mu = \mu(\lambda) > 0$ such that $-\Delta_{\overrightarrow{p}}(\mu\phi) \leq \lambda$ in Ω and

$$-\Delta_{\overrightarrow{p}}(\mu\phi) \leq \frac{\lambda}{\mathcal{H}_0}\|\mu\phi\|^\alpha_{L^t} \leq \frac{\lambda}{\mathcal{H}(x, \|w\|_{L^r})}\|\mu\phi\|^\alpha_{L^t}$$

for all $w \in [0, z_\lambda]$. From Lemma 3.2 we have $\mu\phi \leq z_\lambda$ in Ω. The first part of the result is proved.

Now we will prove the second part of the result. Consider $\phi, \delta, \sigma, \mu$ as in the proof of Theorem 2.3 and let $\lambda \in (0, +\infty)$. Since $0 < \alpha < p_1 - 1$, it is possible to repeat the arguments of Theorem 2.3 to obtain $\mu > 0$, depending on λ, such that

$$-\Delta_{\overrightarrow{p}}(\mu\phi) \leq 1 \quad \text{and} \quad -\Delta_{\overrightarrow{p}}(\mu\phi) \leq \frac{\lambda}{\mathcal{H}_0}\|\mu\phi\|^\alpha_{L^t} \text{ in } \Omega.$$

Denote by $z_M \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ the unique solution of (4.1), where M will be chosen later. By Lemma 4.1 for $M \geq 1$ large enough there is a constant $K > 1$, which does not depend on M, such that

$$0 < z_M(x) \leq KM^{\frac{1}{p_1-1}} \text{ in } \Omega. \tag{46}$$

We must find a number $M \geq 1$ such that for all $w \in L^\infty(\Omega)$ with $\mu\phi \leq w$ the inequality

$$M \geq \frac{1}{\mathcal{H}(x, \|w\|_{L^r})}(\lambda \|z_M\|^\alpha_{L^t} + \theta \|z_M\|^\gamma_{L^s}) \text{ in } \Omega \tag{47}$$

holds. The continuity of $\mathcal{H}$ and the fact that $\lim_{t\to\infty} \mathcal{H}(x,t) = b_0 > 0$ uniformly in Ω implies that there is a number $a_1 > 0$ such that $\mathcal{H}(x,t) \geq \frac{b_0}{2}$ in $\overline{\Omega} \times (a_1, +\infty)$.

Define $$m_\lambda := \min\{\mathcal{H}(x,t); (x,t) \in \overline{\Omega} \times [\|\mu\phi\|_{L^r}, a_1]\} > 0$$

and $\mathcal{H}_\lambda := \min\{m_\lambda, \frac{b_0}{2}\}$. Then $\mathcal{H}(x,t) \geq \mathcal{H}_\lambda$ in the set $\overline{\Omega} \times [\|\mu\phi\|_{L^r}, +\infty)$. Therefore $0 < \mathcal{H}_\lambda \leq \mathcal{H}(x, \|w\|_{L^r}) \leq a_0$ for all $w \in L^\infty(\Omega)$ with $\mu\phi \leq w$. Using the inequality (46) we obtain that

$$\frac{(\lambda \|z_M\|^\alpha_{L^t} + \theta \|z_M\|^\gamma_{L^s})}{\mathcal{H}(x, \|w\|_{L^r})} \leq \frac{\lambda M^{\frac{\alpha}{p_1-1}} \overline{K} + \theta M^{\frac{\gamma}{p_1-1}} \overline{K}}{\mathcal{H}_\lambda}, \tag{48}$$

where $\overline{K} := \max\{\|K\|^\alpha_{L^q}, \|K\|^\gamma_{L^s}|\}$. Denote by J the right-hand side of (48). Note that we have $J \leq M$ if and only if

$$1 \geq \frac{1}{\mathcal{H}_\lambda}\left(\lambda \overline{K} M^{\frac{\alpha}{p_1-1}-1} + \theta \overline{K} M^{\frac{\gamma}{p_1-1}-1}\right). \tag{49}$$

The conditions $0 < \alpha < p_1 - 1 < \gamma$ imply that the function

$$\Phi(t) := \frac{\lambda \overline{K} t^{\frac{\alpha}{p_1-1}-1} + \theta \overline{K} t^{\frac{\gamma}{p_1-1}-1}}{\mathcal{H}_\lambda}, t > 0$$

which belongs to $C^1((0,\infty), \mathbb{R})$ attains a global minimum at

$$M_{\lambda,\theta} := M(\lambda, \theta) = W \left(\frac{\lambda}{\theta}\right)^{\frac{p_1-1}{\gamma-\alpha}} \tag{50}$$

with $W := \left(\frac{(p_1-1)-\alpha)}{\gamma-(p_1-1)}\right)^{\frac{p_1-1}{\gamma-\alpha}}$. The inequality (49) is equivalent to find a number $M_{\lambda,\theta} > 0$ such that $\Phi(M_{\lambda,\theta}) \leq 1$. Using (50) we get that $\Phi(M_{\lambda,\theta}) \leq 1$ if and only if

$$\frac{\lambda \overline{K} R}{\mathcal{H}_\lambda}\left(\frac{\lambda}{\theta}\right)^{\frac{\gamma-(p_1-1)}{\gamma-\alpha}} + \frac{\theta \overline{K} S}{\mathcal{H}_\lambda}\left(\frac{\lambda}{\theta}\right)^{\frac{\gamma-(p_1-1)}{\gamma-\alpha}} \leq 1,$$

with $R := W^{\frac{\alpha-(p_1-1)}{p_1-1}}$ and $S := W^{\frac{\gamma-(p_1-1)}{p_1-1}}$. Note that the previous inequality holds for a fixed $\lambda > 0$ for $\theta > 0$ small, depending on λ, because of the inequalities $0 < \alpha < p_1 - 1 < \gamma$. Then for a fixed $\lambda > 0$ there exists a $\theta_0 = \theta_0(\lambda)$ such that for each $\theta \in (0, \theta_0)$ there is a number $M_{\lambda,\theta} \geq 0$ such that (49) holds. Thus we have that (47) also holds. Then we can infer that

$$-\Delta_{\overrightarrow{p}} z_M \geq \frac{1}{\mathcal{H}(x, \|w\|_{L^r})}(\lambda \|z_M\|^\alpha_{L^t} + \theta \|z_M\|^\gamma_{L^s}) \text{ in } \Omega.$$

Note also that we can consider $M_{\lambda,\theta} \geq 1$ for all $\theta \in (0, \theta_0)$ because of (50). Thus we obtain that $-\Delta_{\overrightarrow{p}}(\mu\phi) \leq -\Delta_{\overrightarrow{p}} z_M$ in Ω. From Lemma 3.2 we get $\mu\phi \leq z_M$ in Ω, which finishes the proof.

Proof of Theorem 2.5:

Consider the condition (i). We will begin the proof by considering the functions $\overline{u}$ and $\overline{v}$. Let $\lambda > 0$, which will be suitably chosen later and denote by

$$z_\lambda \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega) \quad \text{and} \quad y_\lambda \in W_0^{1,\overrightarrow{q}}(\Omega) \cap L^\infty(\Omega)$$

the unique solutions of the problems

$$\left\{ \begin{array}{rcl} -\Delta_{\overrightarrow{p}} z_\lambda & = & \lambda \ \text{ in } \ \Omega \\ z_\lambda & = & 0 \ \text{ on } \ \partial\Omega \end{array} \right. \qquad \left\{ \begin{array}{rcl} -\Delta_{\overrightarrow{q}} y_\lambda & = & \lambda \ \text{ in } \ \Omega \\ y_\lambda & = & 0 \ \text{ on } \ \partial\Omega \end{array} \right. \tag{51}$$

By Lemma 4.1 we can infer that for $\lambda > 0$ large enough we have

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{p_1-1}} \text{ in } \Omega \tag{52}$$

and

$$0 < y_\lambda(x) \leq K\lambda^{\frac{1}{q_1-1}} \text{ in } \Omega, \tag{53}$$

where $K > 0$ is a constant that does not depend on $\lambda > 0$. Since $0 < \alpha_1 < q_1 - 1$ it is possible to choose $\lambda > 0$ large enough such that

$$\frac{1}{h_0}\|K\|_{L^{q_1}}^{\alpha_1}\lambda^{\frac{\alpha_1}{q_1-1}} \leq \lambda. \tag{54}$$

From (52) and (54) we have that

$$\left\{ \begin{array}{rcll} -\Delta_{\overrightarrow{p}} z_\lambda & \geq & \dfrac{1}{\mathcal{H}(x, \|w\|_{L^{r_1}})}\|y_\lambda\|_{L^{q_1}}^{\alpha_1} & \text{in } \Omega, \\ z_\lambda & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

for all $w \in L^\infty(\Omega)$. If necessary consider a larger $\lambda > 0$, which is possible because $0 < \alpha_2 < p_1 - 1$, that satisfies

$$\left\{ \begin{array}{rcll} -\Delta_{\overrightarrow{q}} y_\lambda & \geq & \dfrac{1}{\mathcal{H}(x, \|w\|_{L^{r_2}})}\|z_\lambda\|_{L^{q_2}}^{\alpha_2} & \text{in } \Omega, \\ y_\lambda & = & 0 & \text{on } \partial\Omega, \end{array} \right.$$

for all $w \in L^\infty(\Omega)$. We consider the functions $\underline{u}$ and $\underline{v}$. Let $\delta > 0$ such that $d \in C^2(\overline{\Omega_{3\delta}})$ and $|\nabla d| \equiv 1$ in $\overline{\Omega_{3\delta}}$, with $d(x) := dist(x, \partial\Omega)$ and $\overline{\Omega_{3\delta}} := \{x \in \overline{\Omega};\ d(x) \leq 3\delta\}$. Let $k > 0$ and $\sigma \in (0, \delta)$. Consider the function

$$\phi(x) = \begin{cases} e^{kd(x)} - 1, & \text{if } \ d(x) < \sigma, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{d(x)} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^\xi dt, & \text{if } \ \sigma \leq d(x) < 2\delta, \\ e^{k\sigma} - 1 + \displaystyle\int_\sigma^{2\delta} ke^{k\sigma}\left(\frac{2\delta - t}{2\delta - \sigma}\right)^\xi dt, & \text{if } \ 2\delta \leq d(x), \end{cases} \tag{55}$$

where
$$\xi > \max\left\{\frac{1}{p_1-1}, \frac{1}{q_1-1}\right\}.$$

Note also that $\phi \in C^1(\overline{\Omega})$ and $\phi = 0$ on $\partial\Omega$. Let $\mu = e^{-k}$ with $k > 0$. Arguing as in (38) we obtain that
$$-\Delta_{\overrightarrow{p}}(\mu\phi) \le \frac{C_1}{2\delta - \sigma}(\mu k)^{p_1-1} \tag{56}$$
and
$$-\Delta_{\overrightarrow{q}}(\mu\phi) \le \frac{C_1}{2\delta - \sigma}(\mu k)^{q_1-1}, \tag{57}$$

for $k > 0$ large enough where $C_1 > 0$ is a constant which does not depend on k. Consider $\sigma := \frac{\ln 2}{k}$. Thus $\phi(x) \ge 1$ for $x \in \Omega$ with $\sigma \le d(x) < 2\delta$. Thus there is a constant $C_2 > 0$, which does not depend on k, such that
$$\|\mu\phi\|_{L^{q_1}}^{\alpha_1} \ge \mu^{\alpha_1} C_2. \tag{58}$$

Since $0 < \alpha_1 < p_1 - 1$ we have
$$\lim_{k\to+\infty} \frac{k^{p_1-1}}{e^{k(p_1-1-\alpha_1)}} = 0. \tag{59}$$

Define $\mathcal{H}_\lambda := \max\{\mathcal{H}(x,t); (x,t) \in \overline{\Omega} \times [0, \max\{\|y_\lambda\|_{L^{r_1}}, \|y_\lambda\|_{L^{r_2}}\}]\}$. From (59) it is possible to choose $k > 0$ large enough such that
$$\frac{C_2}{\mathcal{H}_\lambda C_1} \ge \frac{k^{p_1-1}}{e^{k(p_1-1-\alpha)}} \frac{1}{2\delta - \frac{\ln 2}{k}}. \tag{60}$$

Thus we can infer from (56),(58) and (60) that
$$-\Delta_{\overrightarrow{p}}(\mu\phi) \le \frac{1}{\mathcal{H}_\lambda}\|\mu\phi\|_{L^{q_1}}^{\alpha_1} \le \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_1}})}\|\mu\phi\|_{L^{q_1}}^{\alpha_1},$$

for all $w \in L^\infty(\Omega)$. If necessary consider a larger $k > 0$ such that $-\Delta_{\overrightarrow{p}}(\mu\phi) \le 1$. Thus $-\Delta_{\overrightarrow{p}}(\mu\phi) \le -\Delta_{\overrightarrow{p}} z_\lambda$, which implies that $\mu\phi \le z_\lambda$ in Ω. Since $0 < \alpha_2 < q_1 - 1$, by using (57) and the previous reasonings we obtain that there is $\mu > 0$ small enough such that
$$-\Delta_{\overrightarrow{q}}(\mu\phi) \le \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_2}})}\|\mu\phi\|_{L^{q_2}}^{\alpha_2},$$

for all $w \in L^\infty(\Omega)$ and $\mu\phi \le y_\lambda$ in Ω. Thus the first part of the result is proved.

Now suppose that (ii) holds. Consider $\delta, \sigma, \mu, \lambda, z_\lambda, y_\lambda$ and ϕ as in the first part of the result. Repeating the reasonings of the first part of the Theorem it is possible to find $\mu > 0$ small enough such that $-\Delta_p(\mu\phi) \le 1$ in Ω and
$$-\Delta_{\overrightarrow{p}}(\mu\phi) \le \frac{1}{h_0}\|\mu\phi\|_{L^{q_1}}^{\alpha_1} \le \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_1}})}\|\mu\phi\|_{L^{q_1}}^{\alpha_1} \text{ in } \Omega,$$

for all $w \in L^\infty(\Omega)$ with $-\Delta_{\overrightarrow{q}}(\mu\phi) \le 1$ in Ω and
$$-\Delta_{\overrightarrow{q}}(\mu\phi) \le \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_2}})}\|\mu\phi\|_{L^{q_2}}^{\alpha_2} \text{ in } \Omega,$$

for all $w \in L^\infty(\Omega)$.

Since $\lim_{t\to+\infty} \mathcal{H}(x,t) = h_\infty$ uniformly in Ω there exists $h_1 > 0$ large enough such that $\mathcal{H}(x,t) \geq \frac{h_\infty}{2}$ in $\overline{\Omega} \times (h_1, +\infty)$. Define

$$m_k := \min\{\mathcal{H}(x,t);\ (x,t) \in \overline{\Omega} \times [\min\{\|\mu\phi\|_{L^{r_1}}, \|\mu\phi\|_{L^{r_2}}\}, h_1]\}.$$

Therefore $\mathcal{H}(x,t) \geq \mathcal{H}_k$ in $\overline{\Omega} \times [\min\{\|\mu\phi\|_{L^{r_1}}, \|\mu\phi\|_{L^{r_2}}\}, +\infty)$. Let $\lambda > 1$ such that (52) and (53) occurs and

$$\frac{1}{\mathcal{H}_k}(\lambda^{\frac{\alpha_1}{q_1-1}}\|K\|_{L^{q_1}}^{\alpha_1}) \leq \lambda \quad \text{and} \quad \frac{1}{\mathcal{H}_k}(\lambda^{\frac{\alpha_2}{p_1-1}}\|K\|_{L^{q_2}}^{\alpha_2}) \leq \lambda.$$

Therefore we have

$$\frac{1}{\mathcal{H}(x, \|w\|_{L^{r_1}})}\|y_\lambda\|_{L^{t_1}}^{\alpha_1} \leq -\Delta_{\overrightarrow{p}} z_\lambda \quad \text{and} \quad \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_2}})}\|z_\lambda\|_{L^{t_2}}^{\alpha_2} \leq -\Delta_{\overrightarrow{q}} y_\lambda \text{ in } \Omega,$$

for all $w \in L^\infty(\Omega)$ with $\mu\phi \leq w$ in Ω. By Lemma 3.2 we have $\mu\phi \leq z_\lambda$ and $\mu\phi \leq y_\lambda$ in Ω. Thus the result is verified.

Proof of Theorem 2.6:

Consider the condition (i). Let $\lambda \in (0,1)$, which will be suitable chosen and consider z_λ and y_λ the functions given in (51). If $\lambda > 0$ is small enough we have by Lemma 4.1

$$0 < z_\lambda(x) \leq K\lambda^{\frac{1}{p_1-1}} \text{ in } \Omega \tag{61}$$

and

$$0 < y_\lambda(x) \leq K\lambda^{\frac{1}{q_1-1}} \text{ in } \Omega. \tag{62}$$

First it will be considered the functions $\overline{u}$ and $\overline{v}$. It will be proved that for each $\theta > 0$ there exists $\lambda_0 > 0$, which depends on θ, such that

$$\frac{1}{a_0}(\lambda\|y_\lambda\|_{L^{t_1}}^{\alpha_1} + \theta\|y_\lambda\|_{L^{s_1}}^{\gamma_1}) \leq \lambda \text{ and } \frac{1}{a_0}(\lambda\|y_\lambda\|_{L^{t_2}}^{\alpha_2} + \theta\|y_\lambda\|_{L^{s_2}}^{\gamma_2}) \leq \lambda,$$

for all $\lambda \in (0, \lambda_0)$. Define

$$\overline{K} := \max\{\|K\|_{L^{t_1}}^{\alpha_1}, \|K\|_{L^{s_1}}^{\gamma_1}, \|K\|_{L^{t_2}}^{\alpha_2}, \|K\|_{L^{s_2}}^{\gamma_2}\}. \tag{63}$$

Since $0 < \alpha_1$ and $q_N - 1 < \gamma_1$, given $\theta > 0$ there exists $\lambda_0 > 0$, which depends on θ, such that

$$\frac{1}{a_0}(\lambda^{\frac{\alpha_1+q_N-1}{q_N-1}}\overline{K} + \theta\lambda^{\frac{\gamma_1}{q_N-1}}\overline{K}) \leq \lambda, \tag{64}$$

for all $\lambda \in (0, \lambda_0)$. Consider a smaller $\lambda_0 > 0$ that satisfies

$$\|y_\lambda\|_{L^{r_1}} \leq \|K\|_{L^{r_1}}\lambda^{\frac{1}{q_N-1}} \leq b_0$$

for all $\lambda \in (0, \lambda_0)$. Under the previous condition we have $\mathcal{H}(x, \|w\|_{L^{r_1}}) \geq a_0$ for $w \in [0, y_\lambda]$. Thus by (62) and (64) we have that

$$-\Delta_{\overrightarrow{p}} z_\lambda \geq \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_1}})}(\lambda\|y_\lambda\|_{L^{t_1}}^{\alpha_1} + \theta\|y_\lambda\|_{L^{s_1}}^{\gamma_1}),$$

for all $w \in [0, y_\lambda]$.

Consider also that $\lambda_0 > 0$ satisfies

$$\frac{1}{a_0}(\lambda^{\frac{\alpha_2+p_N-1}{p_N-1}}\overline{K} + \theta\lambda^{\frac{\gamma_2}{p_N-1}}\overline{K}) \leq \lambda, \tag{65}$$

and $\|z_\lambda\|_{L^{r_2}} \leq \|K\|_{L^{r_2}}\lambda^{\frac{1}{p_N-1}} \leq b_0$ for all $\lambda \in (0, \lambda_0)$. Thus by (61) and (65) we have

$$-\Delta_{\overrightarrow{q}} y_\lambda \geq \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_2}})}(\lambda\|z_\lambda\|^{\alpha_2}_{L^{t_2}} + \theta\|z_\lambda\|^{\gamma_2}_{L^{s_2}})$$

for all $w \in [0, z_\lambda]$.

Regarding the functions $\underline{u}$ and $\underline{v}$ consider $\phi, \delta, \sigma, \mu$ as in the proof of Theorem 2.5. Since $0 < \alpha_1 < p_1 - 1$ and $0 < \alpha_2 < q_1 - 1$ we can repeat the arguments of Theorem 2.5 to obtain $\mu > 0$ small enough such that $\mu\phi \leq z_\lambda$ and $\mu\phi \leq y_\lambda$ in Ω and

$$-\Delta_{\overrightarrow{p}}(\mu\phi) \leq \lambda \quad \text{and} \quad -\Delta_{\overrightarrow{p}}(\mu\phi) \leq \frac{\lambda}{\mathcal{H}(x, \|w\|_{L^{r_1}})}\|\mu\phi\|^{\alpha_1}_{L^{t_1}},$$

for all $w \in [0, y_\lambda]$ and

$$-\Delta_{\overrightarrow{q}}(\mu\phi) \leq \lambda \quad \text{and} \quad -\Delta_{\overrightarrow{q}}(\mu\phi) \leq \frac{\lambda}{\mathcal{H}(x, \|w\|_{L^{r_1}})}\|\mu\phi\|^{\alpha_1}_{L^{t_1}},$$

for all $w \in [0, z_\lambda]$. Then by Theorem 2.2 we have the result.

Consider now the condition (ii). Let ϕ, δ and σ as in the first part of the result and let $\lambda > 0$ fixed. Since $0 < \alpha_1 < p_1 - 1$ and $0 < \alpha_2 < q_1 - 1$ there exists $\mu > 0$, which depends only on λ, that satisfies

$$-\Delta_{\overrightarrow{p}}(\mu\phi) \leq 1 \ \text{ and } \ -\Delta_{\overrightarrow{p}}(\mu\phi) \leq \frac{\lambda}{a_0}\|\mu\phi\|^{\alpha_1}_{L^{t_1}}$$
$$-\Delta_{\overrightarrow{q}}(\mu\phi) \leq 1 \ \text{ and } \ -\Delta_{\overrightarrow{q}}(\mu\phi) \leq \frac{\lambda}{a_0}\|\mu\phi\|^{\alpha_2}_{L^{t_2}}$$

in Ω. Consider $M > 0$ and let $z_M \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ and $y_M \in W_0^{1,\overrightarrow{q}}(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\begin{cases} -\Delta_{\overrightarrow{p}} z_M &= M \text{ in } \Omega \\ z_M &= 0 \text{ on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta_{\overrightarrow{q}} y_M &= M \text{ in } \Omega \\ y_M &= 0 \text{ on } \partial\Omega \end{cases}$$

By Lemma 4.1 we have

$$0 < z_M(x) \leq KM^{\frac{1}{p_1-1}} \text{ in } \Omega, \tag{66}$$

$$0 < y_M(x) \leq KM^{\frac{1}{q_1-1}} \text{ in } \Omega, \tag{67}$$

for $M > 0$ large enough where $K > 0$ is a constant that does not depend on M.

Regarding the construction of the function $\overline{u}$ and $\overline{v}$ it will be proved that there exists $\theta_0 > 0$, which depends only on λ, that satisfies the following property: given $\theta \in (0, \theta_0)$ then there is a constant M, depending only on λ and θ, that satisfies

$$M \geq \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_1}})}(\lambda\|y_M\|^{\alpha_1}_{L^{t_1}} + \theta\|y_M\|^{\gamma_1}_{L^{s_1}}) \tag{68}$$

and $$M \geq \frac{1}{\mathcal{H}(x, \|w\|_{L^{r_2}})}(\lambda\|z_M\|^{\alpha_2}_{L^{t_2}} + \theta\|z_M\|^{\gamma_2}_{L^{s_2}}), \tag{69}$$

for $w \in L^\infty(\Omega)$ with $\mu\phi \leq w$ in Ω. Combining the continuity of $\mathcal{H}$ with the fact that $\lim_{t\to+\infty} \mathcal{H}(x,t) = b_0$ uniformly in Ω we obtain that there exists $a_1 > 0$ such that $\mathcal{H}(x,t) \geq \frac{b_0}{2}$ in $\overline{\Omega} \times (a_1, +\infty)$. Define

$$m_\lambda := \min\{\mathcal{H}(x,t); (x,t) \in \overline{\Omega} \times [\min\{|\mu\phi|_{L^{r_1}}, |\mu\phi|_{L^{r_2}}\}, a_1]\} > 0$$

and $\mathcal{H}_\lambda := \min\{m_\lambda, \frac{b_0}{2}\}$. Then $\mathcal{H}(x,t) \geq \mathcal{H}_\lambda$ in $\overline{\Omega} \times [\min\{\|\mu\phi\|_{L^{r_1}}, \|\mu\phi\|_{L^{r_2}}\ , +\infty)$. Therefore $\mathcal{H}_\lambda \leq \mathcal{H}(x, \|w\|_{L^{r_1}}) \leq a_0$ for all $w \in L^\infty(\Omega)$ with $\mu\phi \leq w$ in Ω. Note that from (66) and (67) we have that the inequalities (68) and (69) hold if the inequalities

$$\frac{1}{\mathcal{H}_\lambda}(\lambda\overline{K}M^{\frac{\alpha_1}{q_1-1}} + \theta\overline{K}M^{\frac{\gamma_1}{q_1-1}}) \leq M$$

and $$\frac{1}{\mathcal{H}_\lambda}(\lambda\overline{K}M^{\frac{\alpha_2}{p_1-1}} + \theta\overline{K}M^{\frac{\gamma_2}{p_1-1}}) \leq M,$$

hold where $\overline{K}$ is given in (63). Consider the inequality

$$\frac{1}{\mathcal{H}_\lambda}(\lambda\overline{K}M^{\rho-1} + \theta\overline{K}M^{\tau-1}) \leq 1, \tag{70}$$

where $\rho := \max\left\{\frac{\alpha_1}{q_1-1}, \frac{\alpha_2}{p_1-1}\right\}$ and $\tau := \max\left\{\frac{\gamma_1}{q_1-1}, \frac{\gamma_2}{p_1-1}\right\}$. Define

$$\Phi_{\lambda,\theta}(M) := \frac{\lambda\overline{K}}{\mathcal{H}_\lambda}M^{\rho-1} + \frac{\theta\overline{K}}{\mathcal{H}_\lambda}M^{\tau-1}, M > 0.$$

Since $0 < \rho < 1$ and $\tau > 1$ we obtain that

$$\lim_{M\to0^+} \Phi_{\lambda,\theta}(M) = \lim_{M\to+\infty} \Phi_{\lambda,\theta}(M) = +\infty.$$

We have $\Phi'_{\lambda,\theta}(M) = 0$ if and only if

$$M = M_{\lambda,\theta} := \left(\frac{\lambda}{\theta}\right)^{\frac{1}{\tau-\rho}} c, \tag{71}$$

where $c := \left(\frac{1-\rho}{\tau-1}\right)^{\frac{1}{\tau-\rho}}$. From the mentioned properties of $\Phi_{\lambda,\theta}$ we have that the global minimum of $\Phi_{\lambda,\theta}$ is attained at $M_{\lambda,\theta}$. We have that (70) is equivalent to find $M_{\lambda,\theta} > 0$ that satisfies $\Phi_{M_{\lambda,\theta}} \leq 1$. By using (71) we obtain that $\Phi_{M_{\lambda,\theta}} \leq 1$, if and only if

$$\frac{\lambda\overline{K}}{\mathcal{H}_\lambda}\left(\frac{\lambda}{\theta}\right)^{\frac{\rho-1}{\tau-\rho}} c^{\rho-1} + \theta^{1-\left(\frac{\tau-1}{\tau-\rho}\right)}\frac{\overline{K}}{\mathcal{H}_\lambda}\lambda^{\frac{\tau-1}{\tau-\rho}}c^{\tau-1} \leq 1. \tag{72}$$

Thus by (71) and (72), we can infer that given $\lambda > 0$ there exists $\theta_0 > 0$ such that for each $\theta \in (0, \theta_0)$ there exists $M_{\lambda,\theta}$ that satisfies

$$M_{\lambda,\theta} \geq 1 \text{ and } \frac{1}{\mathcal{H}_\lambda}(\lambda\overline{K}M^{\rho-1}_{\lambda,\theta} + \theta\overline{K}M^{\tau-1}_{\lambda,\theta}) \leq 1.$$

Then we have that

$$-\Delta_{\overrightarrow{p}} z_M \geq \frac{1}{\mathcal{H}_\lambda}(\lambda \|y_M\|_{L^{t_1}}^{\alpha_1} + \theta \|y_M\|_{L^{s_1}}^{\gamma_1})$$

and

$$-\Delta_{\overrightarrow{q}} y_M \geq \frac{1}{\mathcal{H}_\lambda}(\lambda \|z_M\|_{L^{t_2}}^{\alpha_2} + \theta \|z_M\|_{L^{s_2}}^{\gamma_2}),$$

for all $w \in L^\infty(\Omega)$ with $\mu\phi \leq w$ in Ω. Note that for $\theta_0 > 0$ small enough we have for $\theta \in (0, \theta_0)$ that

$$-\Delta_{\overrightarrow{p}}(\mu\phi) \leq 1 \leq M_{\lambda,\theta_0} \leq M_{\lambda,\theta},$$

because $M_{\lambda,\theta} \to +\infty$ as $\theta \to 0^+$ and $\theta \longmapsto M_{\lambda,\theta}$ is decreasing. A similar argument implies that $-\Delta_{\overrightarrow{q}}(\mu\phi) \leq M_{\lambda,\theta_0} \leq M_{\lambda,\theta}$ for all $\theta \in (0, \theta_0)$ for θ_0 small enough. From Lemma 3.2 we obtain that $\mu\phi \leq z_M$ and $\mu\phi \leq y_M$ in Ω. The proof of the result is finished.

Acknowledgements. The first author thanks for the support by Coordenação de Aperfeiçoamento de Nível Superior (CAPES), Conselho Nacional de Desenvolvimento Científico (CNPq) and Fundação de Apoio a Pesquisa do Distrito Federal (FAPDF). The authors would like to thank the referees for their valuable comments which helped to improve the manuscript.

References

[1] E. Acerbi, N. Fusco: *Partial regularity under anisotropic (p, q) growth conditions*, J. Diff. Equations 107/1 (1994) 46–67.

[2] C. O. Alves, D. P. Covei: *Existence of solutions for a class of nonlocal elliptic problem via sub-supersolution method*, Nonlinear Analysis Real World Appl. 23 (2015) 1–8.

[3] C. O. Alves, A. El Hamidi: *Existence of solution for a anisotropic equation with critical exponent*, Diff. Integral Equations 21/1-2 (2008) 25–40.

[4] A. Ambrosetti, H. Brezis, G. Cerami: *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Analysis 122/2 (1994) 519–543.

[5] S. N. Antontsev, J. I. Diaz, S. Shmarev: *Energy Methods for Free Boundary Problems*, Progress in Nonlinear Differential Equations and their Applications 48, Birkhäuser, Boston (2002).

[6] J. Bear: *Dynamics of Fluids in Porous Media*, American Elsevier, New York (1972)

[7] J. W. Bebernes, A. Bressan: *Thermal behaviour for a confined reactive gas*, J. Diff. Equations 44 (1982) 118–133.

[8] J. W. Bebernes, P. Talaga: *Nonlocal problems modelling shear banding*, Comm. Appl. Nonlinear Analysis 3/2 (1996) 79–103.

[9] M. Bendahmane, K. H. Karlsen: *Renormalized solutions of an anisotropic reaction-diffusion-advection system with L1 data*, Commun. Pure Appl. Analysis 5/4 (2006) 733–762.

[10] A. Cabada, F. J. S. A. Corrêa: *Existence of solutions of a nonlocal elliptic system via Galerkin method*, Abstr. Appl. Analysis (2012), art. no. 137379, 16 pp.

[11] G. F. Carrier: *On the non-linear vibration problem of the elastic string*, Quart. Appl. Math. 3 (1945) 157–165.

[12] N. H. Chang, M. Chipot: *Nonlinear nonlocal evolution problems*, Real Acad. Ciencias Exactas Fis. Nat. Ser. A Mat. 97/3 (2003) 423–445.

[13] N. H. Chang, M. Chipot: *On some model diffusion problems with a nonlocal lower order term*, Chinese Ann. Math. Ser. B 24/2 (2003) 147–166.

[14] N. H. Chang, M. Chipot: *On some mixed boundary value problems with nonlocal diffusion*, Adv. Math. Sci. Appl. 14/1 (2004) 1–24.

[15] Y. Chen, H. Gao: *Existence of positive solutions for nonlocal and nonvariational elliptic system*, Bull. Australian Math. Soc. 72 (2005) 271–281.

[16] M. Chipot, W. Ganbo, B. Kawohl: *On some nonlocal variational problems*, Analysis Appl. (Singapore) 4/4 (2006) 345–356.

[17] M. Chipot, B. Lovat: *Some remarks on non local elliptic and parabolic problems*, Nonlinear Analysis 30/7 (1997) 4619–4627.

[18] M. Chipot, B. Lovat: *On the asymptotic behaviour of some nonlocal problems*, Positivity 3/1 (1997) 65–81.

[19] M. Chipot, J. F. Rodrigues: *On a class of nonlinear elliptic problems*, RAIRO Modél. Math. Anal. Numérique 26/3 (1992) 447–467.

[20] M. Chipot, T. Savitska: *Nonlocal p-Laplace equations depending on the L^p norm of the gradient*, Adv. Diff. Equations 19/11-12 (2014) 997–1020.

[21] M. Chipot, T. Savitska: *Asymptotic behaviour of the solutions of nonlocal p-Laplace equations depending on the L^p norm of the gradient*, J. Elliptic Parabolic Equations 1 (2015) 63–74.

[22] M. Chipot, M. Siegwart: *On the asymptotic behaviour of some nonlocal mixed boundary value problems*, in: *Nonlinear Analysis and Applications, Vol. 1 & 2*, Kluwer, Dordrecht (2003) 431–449.

[23] M. Chipot, V. Valente, G. V. Caffarelli: *Remarks on a nonlocal problem involving the Dirichlet energy*, Rend. Sem. Mat. Univ. Padova 110 (2003) 199–220.

[24] M. Chipot, S. Zheng: *Asymptotic behavior of solutions to nonlinear parabolic equations with nonlocal terms*, Asymp. Analysis 45/3-4 (2005) 301–312.

[25] A. Cianchi: *Local boundedness of minimizers of anisotropic functionals*, Ann. Inst. H. Poincaré Anal. Non Linéaire 17/2 (2000) 147–168.

[26] F. J. S. A. Corrêa, G. M. Figueiredo, F. P. M. Lopes: *On the existence of positive solutions for a nonlocal elliptic problem involving the p-Laplacian and the generalized Lebesgue space $L^{p(x)}(\Omega)$*, Diff. Integral Equations 21/3-4 (2008) 305–324.

[27] F. J. S. A. Corrêa, F. P. M. Lopes: *Positive solutions for a class of nonlocal elliptic systems*, Comm. Appl. Nonlinear Analysis 14/2 (2007) 67–77.

[28] W. Deng, Y. Lie, C. Xie: *Existence and nonexistence of global solutions of some nonlocal degenerate parabolic equations*, Appl. Math. Letters 16/5 (2003) 803–808.

[29] W. Deng, Y. Lie, C. Xie: *Blow-up and global existence for a nonlocal degenerate parabolic system*, J. Math. Analysis Appl. 277/1 (2003) 199–217.

[30] A. Di Castro: *Elliptic Problems for some Anisotropic Operators*, Ph. D. Thesis, University of Rome “La Sapienza”, Rome (2009).

[31] A. Di Castro: *Existence and regularity results for anisotropic elliptic problems*, Adv. Nonlinear Studies 9 (2009) 367–393.

[32] G. C. G. dos Santos, G. M. Figueiredo: *Positive solutions for a class of nonlocal problems involving Lebesgue generalized spaces: scalar and system cases*, J. Elliptic Parabolic Equations 2/1-2 (2016) 235–266.

[33] G. C. G. dos Santos, G. M. Figueiredo, J. R. S. Silva: *Multiplicity of positive solutions for an anisotropic problem via Sub-supersolution method and Mountain Pass Theorem*, J. Convex Analysis 27/4 (2020) 1363–1374.

[34] G. C. dos Santos, G. M. Figueiredo, L. S. Tavares: *A sub-supersolution method for a class of nonlocal system involving the $p(x)$-Laplacian operator and applications*, Electronic J. Diff. Equations 25 (2020) 1–19.

[35] G. C. dos Santos, G. M. Figueiredo, L. S. Tavares: *A sub-supersolution method for a class of nonlocal problems involving the $p(x)$-Laplacian operator and applications*, Acta Appl. Math. 153 (2018) 171–187.

[36] G. C. dos Santos, G. M. Figueiredo, L. S. Tavares: *Existence results for some anisotropic singular problems via sub-supersolutions*, Milan J. Math. 87 (2019) 249–272.

[37] X. Fan: *On the sub-supersolution method for $p(x)$-Laplacian equations*, J. Math. Analysis Appl. 330/1 (2007) 665–682.

[38] G. M. Figueiredo, A. Moussaoui, G. C. G. dos Santos, L. S. Tavares: *A sub-supersolution approach for some classes of nonlocal problems involving Orlicz spaces*, J. Diff. Equations 267/7 (2019) 4148–4169.

[39] G. M. Figueiredo, J. R. Santos Júnior, A. Suárez: *Multiplicity results for an anisotropic equation with subcritical or critical growth*, Adv. Nonlinear Studies 15/2 (2015) 377–394.

[40] G. M. Figueiredo, J. R. S. Silva: *A critical anisotropic problem with discontinuous nonlinearities*, Nonlinear Analysis Real World Appl. 47 (2019) 364–372.

[41] I. Fragalà, F. Gazzola, B. Kawohl: *Existence and nonexistence results for anisotropic quasilinear elliptic equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire 21/5 (2004) 715–734.

[42] J. Furter, M. Grinfeld: *Local vs. nonlocal interactions in population dynamics*, J. Math. Biol. 27/1 (1989) 65–80.

[43] M. Giaquinta: *Growth conditions and regularity, a counterexample*, Manuscripta Math. 59/2 (1987) 245–248.

[44] D. Gilbarg, N. S. Trudinger: *Elliptic Partial Differential Equations of Second Order*, Springer, Berlin (2001).

[45] M. R. Hajiaboli: *An anisotropic fourth-order diffusion filter for image noise removal*, Int. J. Comput. Vis. 92/2 (2011) 177–191.

[46] S. N. Kruzhkov, I. M. Kolodii: *On the theory of embedding of anisotropic Sobolev spaces*, Russ. Math. Surveys 38 (1983) 188–189.

[47] O. A. Ladyzenskaya, N. N. Uraltseva: *Linear and Quasilinear Elliptic Equations*, Academic Press, New York (1968).

[48] M. Mihăilescu, G. Moroşanu, V. Rădulescu: *Eigenvalue problems in anisotropic Orlicz-Sobolev spaces*, C. R. Math. Acad. Sci. Paris 347/9-10 (2009) 521–526.

[49] M. Mihăilescu, G. Moroşanu, V. Rădulescu: *Eigenvalue problems for anisotropic elliptic equations: an Orlicz-Sobolev space setting*, Nonlinear Analysis 73/10 (2010) 3239–3253.

[50] M. Mihăilescu, P. Pucci, V. Rădulescu: *Nonhomogeneous boundary value problems in anisotropic Sobolev spaces*, C. R. Acad. Sci. Paris 345/10 (2007) 561–566.

[51] M. Mihăilescu, P. Pucci, V. Rădulescu: *Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent*, J. Math. Analysis Appl. 340/1 (2008) 687–698.

[52] J. R. S. Silva: *Existence and Multiplicity of Solutions for Problems of Anisotropic Type (Portuguese)*, PhD Thesis, Federal University of Pará, Belém (2018).

[53] B. Song: *Anisotropic diffusions with singular advections and absorptions. Part 1: Existence*, Appl. Math. Letters 14/7 (2001) 811–816.

[54] B. Song: *Anisotropic diffusions with singular advections and absorptions. Part 2: Uniqueness*, Appl. Math. Letters 14/7 (2001) 817–823.

[55] P. Souplet: *Uniform blow-up profiles and boundary behavior for diffusion equations with nonlocal nonlinear source*, J. Diff. Equations 153/2 (1999) 374–406.

[56] J. Vétois: *The blow-up of critical anisotropic equations with critical directions*, Nonlinear Diff. Equations Appl. 18/2 (2011) 173–197.

[57] J. Vétois: *Strong maximum principles for anisotropic elliptic and parabolic equations*, Adv. Nonlinear Studies 12/1 (2012) 101–114.

[58] M. Yu, X. Lian: *Boundedness of solutions of parabolic equations with anisotropic growth conditions*, Canad. J. Math. 49/4 (1997) 798–809.

[59] Q. Zhang, Z. Qiu, R. Dong: *Existence and asymptotic behavior of positive solutions for a variable exponent elliptic system without variational structure*, Nonlinear Analysis 72/1 (2010) 354–363.