Journal of Convex Analysis
Volume 29 (2022), No. 2, 333–344

Representing Diagrams and M-Ideals in L_1-Predual Spaces

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Received: September 21, 2020
Accepted: February 6, 2021

We continue the study of diagrammatic representations of separable L_1-predual spaces and show that given an M-ideal M in a $C(K)$-space, there exists a diagram of $C(K)$ such that M is given by a directed sub-diagram. Similar results are true also for the spaces $C_0(T)$, $\mathbf{c}$ and $\mathbf{c}_0$. This allows one to find a representing matrix of an M-ideal in these spaces. We also obtain some partial results for hyperplanes that are M-ideals in separable L_1-predual spaces.

Keywords: M-ideal, L_1-predual space, $C(K)$-space, directed sub-diagram.

2010 Mathematics Subject Classification: 46B25, 46B20.

1. Introduction

Definition 1.1. A Banach space X is called an L_1*-predual space* if $X^* = L_1(\mu)$ for some measure μ.

L_1-preduals were the subject of an extensive study in the late 1960's and early 1970's. Seminal work on the structure of L_1-predual spaces was done by Lindenstrauss in [7] producing a long list of properties equivalent to a space being an L_1-predual space.

In a recent paper [1], motivated by Bratteli diagrams of Approximately Finite Dimensional C^*-algebras, the authors defined a representing diagram of a separable L_1-predual space from its representing matrix. Extending the analogy further, they showed that every directed sub-diagram represents an M-ideal in X. They also proved the converse in some special cases.

In this paper, we continue to study the converse — given any M-ideal M in a separable L_1-predual space X, do there exist a diagram $\mathcal{D}$ and a directed sub-diagram $\mathcal{S}$ such that $\mathcal{D}$ represents X and $\mathcal{S}$ represents M — and modifying a construction due to Michael and Pełczyński [9], we prove that the converse is true in any $C(K)$ space, where K is a compact metric space. As a consequence, we also establish the converse in the spaces $C_0(T)$, $\mathbf{c}$ and $\mathbf{c}_0$.

* This paper forms a part of the second author's Ph. D. Thesis.

ISSN 0944-6532 / $ 2.50

In [10, Proposition 2.4], W. Lusky observed that if $X = Y \oplus_\infty Z$, where Y and Z are separable L_1-predual spaces with representing matrices $B = (b^i_n)$ and $C = (c^i_n)$ respectively, then X has a representing matrix of the form

$$A = \begin{bmatrix} 0 & b^1_1 & 0 & b^1_2 & 0 & b^1_3 & \dots \\ & 0 & c^1_1 & 0 & c^1_2 & 0 & \dots \\ & & 0 & b^2_2 & 0 & b^2_3 & \dots \\ & & & 0 & c^2_2 & 0 & \dots \\ & & & & 0 & b^3_3 & \dots \\ & & & & & 0 & \dots \\ & & & & & & \dots \end{bmatrix}$$

and conversely, given an M-summand M of a separable L_1-predual space X, X has a representing matrix of the above form from which one can extract a representing matrix, namely, B, of M. It follows from our results that if X is one of the above spaces, given an M-ideal M in X, we can find a representing matrix of X from which we can read off a representing matrix of M.

We also obtain some partial results for hyperplanes that are M-ideals in general separable L_1-predual spaces.

Acknowledgements. Aryaman Sensarma wishes to acknowledge the support received from CSIR, India, Senior Research Fellowship (Award letter no.09/092(0872)/2013-EMR-I). We also thank the referee for some helpful comments that have considerably improved the paper's readability.

2. The problem

Our results hold for both real and complex scalars which we denote by $\mathbb{K}$. For a Banach space X, B_X and S_X will denote the closed unit ball and the unit sphere of X respectively. All subspaces we consider are norm closed and *separable*. The set of extreme points of a convex set D is denoted by $\operatorname{ext}(D)$.

For an L_1-predual space X, it is well-known [4, Proposition II.5.1] that if $x^* \in S_{X^*}$, then $\ker x^*$ is an M-ideal in X if and only if $x^* \in \operatorname{ext}(B_{X^*})$. In Subsection 3.3, we obtain sub-diagrams corresponding to a special class of extreme points of B_{X^*} introduced by Lusky [10]. In Section 5, we also obtain some partial results for the remaining class of extreme points.

Study of the structure of separable L_1-predual spaces in the 60's and 70's led to the following result :

Theorem 2.1. ([5, Theorem 1], [8, Theorem 1.1]) *Let X be a separable Banach space. Then the following statements are equivalent :*

(a) *X is an L_1-predual space.*

(b) *X has an increasing sequence of subspaces $X_1 \subseteq X_2 \subseteq \dots$, such that $X = \overline{\cup_{n=1}^\infty X_n}$ and each X_n is isometrically isomorphic to some $\ell_\infty^{m_n}$.*

(c) *X has an increasing sequence of subspaces $E_1 \subseteq E_2 \subseteq \dots$, such that $X = \overline{\cup_{n=1}^\infty E_n}$ and each E_n is isometrically isomorphic to ℓ_∞^n.*

From the embedding of $E_n \subseteq E_{n+1}$ in (c) above, Lazar and Lindenstrauss [6] introduced notion of representing matrices for separable L_1-predual spaces.

Definition 2.2. A set of elements $\{e_i\}_{i=1}^n \subseteq E$ is called an *admissible basis* of E if there is an admissible basis $\{u_i\}_{i=1}^n$ of ℓ_∞^n (see [6] for definition) and an isometric isomorphism $T : \ell_\infty^n \to E$ such that $e_i = T(u_i)$ for every $1 \leq i \leq n$.

Definition 2.3. [6] If a separable L_1-predual space has the representation as in Theorem 2.1 (c), for every $n \geq 1$, we can inductively choose an admissible basis $\{e_n^i\}_{i=1}^n$ of E_n so that

$$e_n^i = e_{n+1}^i + a_n^i e_{n+1}^{n+1}, \quad 1 \leq i \leq n, \quad n = 1, 2, 3, \ldots$$

and

$$\sum_{i=1}^n |a_n^i| \leq 1, \quad n = 1, 2, 3, \ldots .$$

The triangular matrix $A = \{a_n^i\}_{n\geq 1}^{1\leq i\leq n}$ which we associate with X in this manner is called a *representing matrix* of X.

Definition 2.4. For a separable L_1-predual space X with a representing matrix A, the authors of [1] define the corresponding *representing diagram* as follows :

A representing diagram $\mathcal{D}$ of X consists of nodes and weighted arrows.

The nodes at the n-th level of the diagram then are the admissible basis vectors $\{e_n^i : 1 \leq i \leq n\}$ of E_n. For a node e_n^i, there can be at most two arrows from e_n^i; one reaching to e_{n+1}^i with weight 1 and another to e_{n+1}^{n+1} with weight a_n^i. If $a_n^i = 0$, we do not put any arrow from e_n^i to e_{n+1}^{n+1}.

Note that every representing matrix A corresponds to a unique diagram $\mathcal{D}$ and vice-versa. For a given diagram $\mathcal{D}$, we will denote the corresponding space by $X_{\mathcal{D}}$. See [1] for some examples and further details.

Definition 2.5. [1] A sub-diagram $\mathcal{S} \subseteq \mathcal{D}$ will be called a *directed sub-diagram* if whenever $e_n^i \in \mathcal{S}$ for some $n, i \in \mathbb{N}$, $i \leq n$ then

(a) $e_{n+1}^i \in \mathcal{S}$,

(b) if $a_n^i \neq 0$, $e_{n+1}^{n+1} \in \mathcal{S}$.

In other words, a sub-diagram $\mathcal{S} \subseteq \mathcal{D}$ is directed if $e_n^i \in \mathcal{S}$ for some $n, i \in \mathbb{N}$, $i \leq n$, then the entire branch of $\mathcal{D}$ that starts with e_n^i is in $\mathcal{S}$.

Definition 2.6. A subspace M of a Banach space X is said to be an *M-ideal* in X if there is a projection P on X^* with $\ker(P) = M^\perp$ and for all $x^* \in X^*$, $\|x^*\| = \|Px^*\| + \|x^* - Px^*\|$.

One major observation of [1] is the following theorem.

Theorem 2.7. [1, Theorem 2.2] *Let X be an L_1-predual space with a diagram $\mathcal{D}$. Then for any directed sub-diagram $\mathcal{S}$ of $\mathcal{D}$, $X_{\mathcal{S}}$ is an M-ideal in $X_{\mathcal{D}}$.*

They also proved the converse of Theorem 2.7 in some special cases, see [1, Proposition 2.10, Proposition 2.11, Theorem 3.4].

In this paper, we continue that investigation. That is, we study the following

Problem 2.8. *Let X be a separable L_1-predual space and M be an M-ideal in X. Then do there exist a diagram $\mathcal{D}$ representing X and a directed sub-diagram $\mathcal{S}$ of $\mathcal{D}$ such that $M = X_{\mathcal{S}}$?*

In this paper, modifying a construction due to Michael and Pełczyński [9], we prove that the converse is true for any M-ideal in any $C(K)$ space, which implies the above results of [1]. As a consequence, we also establish the converse in the spaces $C_0(T)$, $\mathbf{c}$ and $\mathbf{c}_0$.

Going back to the representing matrices, it follows that if X is one of the above spaces, given any M-ideal M in X, it is possible to find a representing matrix of X from which one can extract a representing matrix of M.

We also obtain some partial results for hyperplanes that are M-ideals in general separable L_1-predual spaces.

3. The tools

In this section, we develop the tools needed for our main results.

3.1. Isometric copy of ℓ_∞^n

We begin with a criterion for identifying an isometric copy of ℓ_∞^n in any Banach space which is of independent interest and possibly new.

Notation. For $x \in X$, $N(x) = \{x^* \in S_{X^*} : |x^*(x)| = \|x\|\}$.
Define $sgn : \mathbb{K} \to \mathbb{T}$ by

$$sgn(z) = \begin{cases} 1 & z = 0, \\ |z|/z & z \neq 0. \end{cases}$$

That is, for any $z \in \mathbb{K}$, $|sgn(z)| = 1$ and $sgn(z) \cdot z = |z|$.

Lemma 3.1. *A Banach space X contains an isometric copy of ℓ_∞^n if and only if there exist $x_1, x_2, \ldots, x_n \in X$ such that*

(a) $\|x_i\| = 1$ *for all* $1 \leq i \leq n$, *and*

(b) $\displaystyle\sum_{i=1}^{n} |x^*(x_i)| \leq 1$ *for all* $x^* \in B_{X^*}$.

Proof. Let $\{u_n^k : 1 \leq k \leq n\}$ be the canonical basis of ℓ_∞^n.

(Only if:) Let $T : \ell_\infty^n \to X$ be a linear isometry. Consider $Tu_n^i = x_i$ for $1 \leq i \leq n$. Since T is an isometry, (a) follows.

For any $x^* \in B_{X^*}$, and $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{K}$,

$$\left|\sum_{i=1}^{n} \alpha_i x^*(x_i)\right| = \left|x^*\left(\sum_{i=1}^{n} \alpha_i x_i\right)\right| \leq \left\|\sum_{i=1}^{n} \alpha_i x_i\right\| = \left\|T\left(\sum_{i=1}^{n} \alpha_i u_n^i\right)\right\| = \max_{1 \leq i \leq n} |\alpha_i|.$$

Putting $\alpha_i = sgn(x^*(x_i))$, we get (b).

(If:) Conversely, given $x_1, x_2, \dots, x_n \in S_X$ as above, define $T : \ell_\infty^n \to X$ by

$$T\left(\sum_{i=1}^{n} \alpha_i u_n^i\right) = \sum_{i=1}^{n} \alpha_i x_i.$$

We will show that T is an isometry. Note that (b) implies

(c) *For $1 \leq i \leq n$, if $x^* \in N(x_i)$, then $x^*(x_j) = 0$ for all $j \neq i$.*

Thus, for $1 \leq i \leq n$, if we choose any $x_i^* \in N(x_i)$, then $|x_i^*(x_j)| = \delta_{ij}$, $1 \leq i, j \leq n$.

Let $u = \sum_{i=1}^{n} \alpha_i u_n^i \in \ell_\infty^n$. Then there exists $1 \leq j \leq n$ such that $|\alpha_j| = \|u\|$. By the above observation,

$$|x_j^*(Tu)| = \left|\sum_{i=1}^{n} \alpha_i x_j^*(x_i)\right| = |\alpha_j| = \|u\|.$$

Therefore, $\|Tu\| \geq \|u\|$. On the other hand, by (b), for all $x^* \in B_{X^*}$,

$$|x^*(Tu)| = \left|\sum_{i=1}^{n} \alpha_i x^*(x_i)\right| \leq \max_{1\leq i\leq n} |\alpha_i| \sum_{i=1}^{n} |x^*(x_i)| = \|u\| \sum_{i=1}^{n} |x^*(x_i)| \leq \|u\|.$$

Hence $\|Tu\| \leq \|u\|$. Thus, T is an isometry. □

Definition 3.2. Let X be a Banach space. A set $B \subseteq B_{X^*}$ is said to be a *boundary* for X if for every $x \in X$, $N(x) \cap B \neq \emptyset$.

Remark 3.3. (a) [8, Lemma 2.3] follows as a corollary.

(b) It is enough if Lemma 3.1.(b) holds for all x^* in some boundary $B \subseteq B_{X^*}$.

(c) For any $x_1^*, x_2^*, \dots, x_n^* \in S_{X^*}$ as above, it is easy to see that

$$\left\|\sum_{i=1}^{n} \alpha_i x_i^*\right\| = \sum_{i=1}^{n} |\alpha_i|.$$

That is, the span of $\{x_1^*, x_2^*, \dots, x_n^*\}$ is isometric to ℓ_1^n.

3.2. Fill in the gaps

We now obtain a representing matrix of an L_1-predual space X with a representation as in Theorem 2.1 (b). Our technique described here is much simpler than the "Fill in the Gaps" method described in [1].

Let $X = \overline{\cup_{n=1}^{\infty} X_n}$ and X_n is isometric to $\ell_\infty^{m_n}$ for an increasing sequence (m_n). Let $\{e_i : 1 \leq i \leq m_n\}$ be an admissible basis of X_n. By [8, Lemma 2.3], arguing as in [6, p. 179], there exist an admissible basis $\{v_j\}_{j=1}^{m_{n+1}}$ of X_{n+1} and scalars a_{ij} with $\sum_{i=1}^{m_n} |a_{ij}| \leq 1$ for every $m_n + 1 \leq j \leq m_{n+1}$, such that for every $1 \leq i \leq m_n$,

$$e_i = v_i + \sum_{j=m_n+1}^{m_{n+1}} a_{ij} v_j. \tag{1}$$

Define $$f_i = v_i + \sum_{j=m_n+2}^{m_{n+1}} a_{ij} v_j \quad \text{for } 1 \le i \le m_n, \quad f_{m_n+1} = v_{m_n+1} \tag{2}$$

and let $F \subseteq X_{n+1}$ be the subspace spanned by $\{f_i : 1 \le i \le m_n + 1\}$. As in the proof of [8, Lemma 3.2], $X_n \subseteq F$, F is isometric to $\ell_\infty^{m_n+1}$ with $\{f_i : 1 \le i \le m_n+1\}$ an admissible basis and for $1 \le i \le m_n$,

$$e_i = f_i + a_{i(m_n+1)} f_{m_n+1}.$$

It follows that the m_n-th column of the representing matrix of X with respect to these admissible bases is given by $a_{i(m_n+1)}$ for $1 \le i \le m_n$. Comparing the expression of e_i and f_i from equations (1) and (2), it follows that we can define the admissible basis $\{g_i : 1 \le i \le m_n + 2\}$ at the next level similarly to get that the $(m_n + 1)$-st column of the representing matrix is given by $a_{i(m_n+2)}$ for $1 \le i \le m_n$, with 0 as the $(m_n + 1)$-st entry and so on.

In other words, for $m_n \le j \le m_{n+1} - 1$, the j-th column of the representing matrix of X is given precisely by the coefficients a_{ij} appearing in (1) followed by number of 0's.

Remark 3.4. It is worth noting that using this "Fill in the Gaps" technique, we can get a simpler proof of [10, Proposition 2.4] mentioned in the Introduction.

3.3. Diagram of some hyperplanes

We need to recall some notation and a result from [10].

Let X be an L_1-predual space with the representation $X = \overline{\cup_{n=1}^\infty E_n}$, where $E_n \subseteq E_{n+1}$ and E_n isometrically isomorphic to ℓ_∞^n for every n. Let $\{e_n^i : 1 \le i \le n\}$ be an admissible basis of E_n, $n \in \mathbb{N}$. Define $e_j^* \in X^*$, $j \in \mathbb{N}$, by

$$e_j^*(e_n^i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j; \end{cases} \quad 1 \le i \le n;\ j \le n;\ n \in \mathbb{N}. \tag{3}$$

Lemma 3.5. [10, Lemma 1.2] *Let X and $\{e_j^*\}$ be as above. Then*

(a) $e_j^* \in \operatorname{ext}(B_{X^*})$ *for all* $j \in \mathbb{N}$ *and*

(b) $\overline{\{\alpha e_i^* : i \in \mathbb{N}, \alpha \in \mathbb{T}\}}^{w^*} = \overline{\operatorname{ext}(B_{X^*})}^{w^*}$.

We now show that in any L_1-predual space, the M-ideal $\ker e_i^*$ corresponds to a directed sub-diagram.

Definition 3.6. We say that a representing matrix A is *nonnegative* if $a_n^i \ge 0$ for all $n \ge 1$, $1 \le i \le n$.

Lemma 3.7. *Let X be a separable L_1-predual with a nonnegative representing matrix. Let $\mathcal{D}$ be the corresponding representing diagram of X. Then for each j,* $\ker e_j^*$ *represents the space $X_{\mathcal{S}_j}$, where $\mathcal{S}_j$ is precisely the directed sub-diagram of $\mathcal{D}$ that avoids the entire line $e_j^j \to e_{j+1}^j \to e_{j+2}^j \to \cdots$.*

Proof. From the definition of e_j^*, we get $e_j^*(e_n^j) = 1$ for all $n \geq j$. Hence the line $e_j^j \to e_{j+1}^j \to e_{j+2}^j \to \cdots$ is not in $\ker e_j^*$.

Case I: $n \geq j$. Then for all $i \neq j$, $e_j^*(e_n^i) = 0$. That is, $e_n^i \in \ker e_j^*$ for all $i \neq j$, $n \geq j$. Clearly, there can be no path from any such e_n^i to any e_m^j for $m \geq j$.

Case II: Suppose $n < j$. If $e_n^i \in \ker e_j^*$, then $i \leq n < j$. By the definition of representing matrix, we can write

$$\begin{aligned} e_n^i &= e_{n+1}^i + a_n^i e_{n+1}^{n+1} = e_{n+2}^i + a_n^i e_{n+2}^{n+1} + (a_{n+1}^i + a_n^i a_{n+1}^{n+1}) e_{n+2}^{n+2} \\ &= e_{n+3}^i + a_n^i e_{n+3}^{n+1} + (a_{n+1}^i + a_n^i a_{n+1}^{n+1}) e_{n+3}^{n+2} \\ &\quad + (a_{n+2}^i + a_{n+1}^i a_{n+2}^{n+2} + a_n^i a_{n+2}^{n+1} + a_{n+1}^{n+1} a_n^i a_{n+2}^{n+2}) e_{n+3}^{n+3} \quad \text{etc., etc.} \end{aligned}$$

It follows that if we write e_n^i as linear combinations of e_j^k's, then the coefficient of e_j^k is the sum over all paths connecting e_n^i to e_j^k of the products of the weights of each arrow along the path.

Clearly, $e_n^i \in \ker e_j^*$ if and only if the coefficient of e_j^j in the expansion of e_n^i is 0. Then, since all the terms of the sum are nonnegative, each of them is 0. It follows that at least one weight along each path is zero. Thus, no path from e_n^i reaches e_j^j. That is, $e_n^i \in \ker e_j^*$ if and only if there is no path from e_n^i to e_j^j. This completes the proof. $\square$

Remark 3.8. (a) Our assumption that A is nonnegative is needed only for the case $n \leq j - 2$. Therefore, what we get from j-th stage onwards is always a sub-diagram representing $\ker e_j^*$, even when A is not nonnegative.

(b) It is proved in [6, Theorem 5.3] that a separable infinite-dimensional L_1-predual space X admits a nonnegative representing matrix if and only if X is a simplex space [2].

It follows from [6, p. 181] that $C(K)$, $C_0(T)$ are simplex spaces.

We now give an example to show that without some additional assumptions, this result in general is not true.

Example 3.9. Let us take $a_1^1 = 1$, $a_2^1 = \frac{1}{2}$, $a_2^2 = -\frac{1}{2}$ and $a_3^i = \frac{1}{3}$, $1 \leq i \leq 3$. Then

$$e_2^1 = e_3^1 + \frac{1}{2} e_3^3, \qquad e_2^2 = e_3^2 - \frac{1}{2} e_3^3 \quad \text{and}$$
$$e_1^1 = e_2^1 + e_2^2 = e_3^1 + e_3^2 + (\frac{1}{2} - \frac{1}{2}) e_3^3 = e_3^1 + e_3^2.$$

Clearly $e_1^1 \in \ker e_3^*$ but $\{e_2^1, e_2^2\} \notin \ker e_3^*$.

4. M-ideals in $C(K)$

Let K be a compact metric space and let $C(K)$ be the Banach space of scalar-valued continuous functions on K. It is well known that any M-ideal in $C(K)$ is of the form $M = J_D = \{f \in C(K) : f(t) = 0 \text{ for all } t \in D\}$, where D is a closed subset of K [4, Example 1.4 (a)]. Any unexplained terminology in this section can be found in [9].

Definition 4.1. [9] A family $\Phi = \{\phi_1, \dots, \phi_n\} \subseteq C(K)$ is called a *peaked partition of unity* on K if

(a) $\phi_i \geq 0$ for all $i = 1, \dots, n$,

(b) $\sum_{i=1}^{n} \phi_i(x) = 1$ for all $x \in K$,

(c) $\|\phi_i\| = 1$ for all $i = 1, \dots, n$.

Definition 4.2. For a peaked partition of unity $\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ on K, let us denote by $[\Phi]$ the linear subspace of $C(K)$ spanned by Φ and by $P(\Phi)$ the set of points $x_1, x_2, \dots, x_n \in K$ such that $\phi_i(x_i) = 1$, $i = 1, 2, \dots, n$.

We will need the following strengthening of [9, Lemma 2.2].

Lemma 4.3. *Let $\Phi = \{\phi_1, \dots, \phi_n\}$ be a peaked partition of unity on K, $d \in K$, $\mathcal{U}$ an open covering of K and $\varepsilon > 0$, then there exists a peaked partition of unity $\Psi = \{\psi_1, \dots, \psi_m\}$ on K which is ε-subordinated to $\mathcal{U}$, $[\Phi] \subseteq [\Psi]$, and $d \in P(\Psi)$.*

Proof. Let $\Phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ be a peaked partition of unity on K, let $\mathcal{U}$ be an open covering of K, and let $0 < \varepsilon < 1$.

Choose $x_1, x_2, \dots, x_n \in K$ such that $\phi_i(x_i) = 1$, $i = 1, 2, \dots, n$.

For each $z \in K$, pick $U_z \in \mathcal{U}$ such that $z \in U_z$. Define

$$U_{z,i} = \begin{cases} U_z, & \text{if } \ \phi_i(z) = 0, \\ U_z \cap \{x \in K : \phi_i(x) > (1-\varepsilon)\phi_i(z)\}, & \text{if } \ \phi_i(z) > 0. \end{cases} \tag{4}$$

Define $U(z) = \bigcap_{i=1}^{n} U_{z,i}$. Clearly $U(z)$ is an open set containing z. Since K is compact, we can find $z_1, z_2, \dots, z_m \in K$ such that $\{U(z_j) : j = 1, 2, \dots, m\}$ covers K. Clearly, we can throw in additional points to ensure that $m > n$, $z_j = x_j$ for $1 \leq j \leq n$ and $z_{n+1} = d$.

Using [9, Lemma 2.1] we get a peaked partition of unity $\Psi^* = \{\psi_1^*, \psi_2^*, \dots, \psi_m^*\}$ on K such that ψ_j^* vanishes outside $U(z_j)$ and $\psi_j^*(z_j) = 1$ for $j = 1, 2, \dots, m$. Define

$$\phi_i^*(x) = \sum_{j=1}^{m} \phi_i(z_j)\psi_j^*(x), \quad i = 1, 2, \dots, n. \tag{5}$$

For all $x \in K$ and $i = 1, 2, \dots, n$, let

$$\begin{aligned} \gamma_i(x) &= \sup\{\gamma \leq 1 : \phi_i(x) \geq \gamma\phi_i^*(x)\}, \\ \gamma(x) &= \min\{\gamma_i(x) : \ i = 1, 2, \dots, n\}. \end{aligned}$$

Define $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$ as follows :

$$\begin{aligned} \psi_j(x) &= \gamma(x)\psi_j^*(x) + (\phi_j(x) - \gamma(x)\phi_j^*(x)), \quad 1 \leq j \leq n, \\ \psi_j(x) &= \gamma(x)\psi_j^*(x), \quad n < j \leq m. \end{aligned}$$

As in [9, Lemma 2.2], it follows that $\Psi = \{\psi_1, \psi_2, \dots, \psi_m\}$ is a peaked partition of unity with $\psi_j(z_j) = 1$ for $j = 1, 2, \dots, m$. Arguing as in [9, Lemma 2.2] again, we complete the proof. □

Now we are ready to prove our main result.

Theorem 4.4. *Let K be a compact metric space and $D \subseteq K$ be a closed subset. Then there exists a diagram $\mathcal{D}$ representing $C(K)$ and a directed sub-diagram $\mathcal{S}$ of $\mathcal{D}$ such that the M-ideal $J_D = X_{\mathcal{S}}$.*

Proof. Let $D_0 = \{d_n : n \in \mathbb{N}\}$ be a countable dense set in D.

For $n \geq 1$, let $\mathcal{U}_n = \{B(x, 1/n) : x \in K\}$. We inductively define a sequence $\{\Phi_n\}$ of peaked partitions of unity on K as follows:

Let $\Phi_1 = \{1\}$. If Φ_n has been defined by Lemma 4.3, then there exists a peaked partition of unity Φ_{n+1} on K which is $1/n$-subordinated to $\mathcal{U}_n$ with $[\Phi_n] \subseteq [\Phi_{n+1}]$, $P(\Phi_n) \subseteq P(\Phi_{n+1})$ and $\{d_1, d_2, \dots, d_n\} \subseteq P(\Phi_{n+1})$.

To show $C(K) = \overline{\cup[\Phi_n]}$, let $f \in C(K)$ and $\varepsilon > 0$. By [9, Lemma 2.3], there exist $\alpha > 0$ and an open covering $\mathcal{U}$ of K such that, if Φ is a peaked partition of unity on K which is α-subordinated to $\mathcal{U}$, then $\|f - f'\| < \varepsilon$ for some $f' \in [\Phi]$.

By Lebesgue Covering Lemma, there exists $n \geq 1$ such that $1/n < \alpha$ and $\mathcal{U}_n$ is a refinement of $\mathcal{U}$. By construction, Φ_{n+1} is $1/n$-subordinated to $\mathcal{U}_n$. It follows that Φ_{n+1} is α-subordinated to $\mathcal{U}$. And hence, $\|f - f'\| < \varepsilon$ for some $f' \in [\Phi_{n+1}]$.

Let $\dim([\Phi_n]) = m_n$, $\Phi_n = \{\phi_n^i : 1 \leq i \leq m_n\}$ and $P(\Phi_n) = \{z_n^i : 1 \leq i \leq m_n\}$. Since $[\Phi_n] \subseteq [\Phi_{n+1}]$, by [9, Lemma 4.1], for every $1 \leq i \leq m_n$,

$$\phi_n^i(x) = \phi_{n+1}^i(x) + \sum_{j=m_n+1}^{m_{n+1}} \phi_n^i(z_{n+1}^j)\phi_{n+1}^j(x), \tag{6}$$

which is of the form (1) with coefficients $\phi_n^i(z_{n+1}^j) \geq 0$.

Now, if we inductively define the intermediate admissible bases between $[\Phi_n]$ and $[\Phi_{n+1}]$ using the "Fill in the gaps" technique (Subsection 3.2) as in (2), then by [8, Proposition 5.1], they actually form peaked partitions of unity. Thus, we obtain a sequence $\{\Psi_n\}$ of peaked partitions of unity such that $\dim([\Psi_n]) = n$, $[\Psi_n] \subseteq [\Psi_{n+1}]$, $C(K) = \overline{\cup[\Psi_n]}$, and $\Psi_{m_n} = \Phi_n$. It also follows that the representing matrix of $C(K)$ thus obtained is nonnegative.

Claim: $P(\Psi_n) \subseteq P(\Psi_{n+1})$.

Let $\Psi_n = \{g_n^i : i = 1, 2, \dots, n\}$ and $P(\Psi_{n+1}) = \{t_i : i = 1, 2, \dots, n+1\}$. Since $g_{n+1}^i(t_j) = \delta_{ij}$ for $i, j = 1, 2, \dots, n+1$, and

$$g_n^i = g_{n+1}^i + a_n^i g_{n+1}^{n+1}, \quad 1 \leq i \leq n,$$

it follows that

$$g_n^i(t_i) = g_{n+1}^i(t_i) + a_n^i g_{n+1}^{n+1}(t_i) = 1 \text{ for } 1 \leq i \leq n.$$

Thus, $P(\Psi_n) = \{t_i : i = 1, 2, \dots, n\}$.

Now, for $d_k \in D_0$, let $r_k \in \mathbb{N}$ be the smallest $n \in \mathbb{N}$ such that $d_k \in P(\Psi_n)$. Then $d_k \in P(\Psi_n)$ for all $n \geq r_k$ and from the above argument $g_n^{r_k}(d_k) = 1$ for all $n \geq r_k$. It follows that $\ker \delta_{d_k} = \ker e^*_{r_k}$, where e^*_n's are as in (3).

Let $\mathcal{D}$ be the diagram representing $C(K)$ given by Ψ_n. Since D_0 is dense in D, we have

$$J_D = \cap_{d \in D_0} \ker \delta_d.$$

It is easy to see that, for each $k \in \mathbb{N}$, $\ker e^*_{r_k}$ represents the space X_{S_k} for some directed sub-diagram S_k of $\mathcal{D}$. Also intersection of directed sub-diagrams is a directed sub-diagram. Let $S = \cap_{k \in \mathbb{N}} S_k$. Thus $J_D = X_{\mathcal{S}}$.

This completes the proof. □

Theorem 4.5. *Let K be a compact metric space, $d_0 \in K$ and*

$$X = J_{d_0} = \{f \in C(K) : f(d_0) = 0\}.$$

Let M be an M-ideal in X. Then there exist a diagram $\mathcal{D}$ representing X and a directed sub-diagram $\mathcal{S}$ of $\mathcal{D}$ such that $M = X_{\mathcal{S}}$.

Proof. From [4, Proposition 1.18] we get $M = J_D \cap X$, where D is a closed subset of K. As before, let $D_0 = \{d_n : n \geq 1\}$ be a countable dense set in D and let $D_1 = \{d_n : n \geq 0\}$.

Now, by Theorem 4.4, there exists a diagram $\mathcal{D}'$ representing $C(K)$ such that the M-ideal J_{D_1} corresponds to the space $X_{\mathcal{S}'}$ for some directed sub-diagram $\mathcal{S}'$ of $\mathcal{D}'$.

As in Theorem 4.4, let $r_0 \in \mathbb{N}$ be the smallest $n \in \mathbb{N}$ such that $d_0 \in P(\Psi_n)$. Then $d_0 \in P(\Psi_n)$ for all $n \geq r_0$ and as before, $X = \ker \delta_{d_0} = \ker e^*_{r_0}$.

If $\Psi_n = \{g_n^i : i = 1, 2, \ldots, n\}$ for $n \geq r_0$, then as before, $g_n^{r_0}(d_0) = 1$ for all $n \geq r_0$. It follows that $g_n^i(d_0) = 0$ for $i \neq r_0$, that is, $g_n^i \in X$ for $i \neq r_0$. Thus, $\{g_n^i : i \neq r_0\}$ is an admissible basis of $X_n = [g_n^i : i \neq r_0]$. Moreover, [9, Lemma 2.3] shows that if $f \in X$ and $\varepsilon > 0$ then we can choose the approximating function $f' \in [\Phi_n]$ with $\|f - f'\| < \varepsilon$ such that $f' \in X_n$, $n \geq r_0$.

By the above discussion, $\mathcal{S}'$ is a directed sub-diagram of the corresponding sub-diagram $\mathcal{D}$ of $X = \ker \delta_{d_0}$. This completes the proof. □

Corollary 4.6. *Let T be a locally compact Hausdorff space. Problem* 2.8 *has an affirmative answer for any M-ideal in the space $C_0(T)$.*

Proof. $J \subseteq C_0(T)$ is an M-ideal if and only if there is a closed subset $D \subseteq T$ such that $J = \{f \in C_0(T) : f(t) = 0 \text{ for all } t \in D\}$ [4, Example 1.4 (a)].

Let $T^* = T \cup \{\infty\}$ be the one point compactification of T. Consider $C(T^*)$ and $D^* = D \cup \{\infty\}$, which is a closed subset of T^*. Clearly $\ker \delta_\infty = C_0(T)$. Now the conclusion is direct from Theorem 4.5. □

As above, let $\mathbb{N}^*$ be the one point compactification of $\mathbb{N}$. Then $C(\mathbb{N}^*) = \mathbf{c}$ and $C_0(\mathbb{N}) = \mathbf{c}_0$. Thus,

Corollary 4.7. *Problem* 2.8 *has an affirmative answer for any M-ideal in the space $\boldsymbol{c}$ or $\boldsymbol{c}_0$.*

5. Hyperplanes revisited

Let X be an L_1-predual space. Let $x^* \in \mathrm{ext}(B_{X^*})$. By Lemma 3.5, any element $x^* \in \mathrm{ext}(B_{X^*})$ is either of the form αe_j^* for some $\alpha \in \mathbb{T}$, $j \in \mathbb{N}$ or is a w*-limit of such elements. The latter is the case in the following example.

Example 5.1. Consider the matrix A such that $a_n^n = 1$ for all n and $a_n^i = 0$ for all $i \neq n$ as in [6, Example 5.1], then A represents $\mathbf{c}$. Here

$$\begin{aligned} &e_1^1 = (1,1,1,1,\cdots), \quad && e_2^1 = (1,0,0,0,\cdots), \quad && e_2^2 = (0,1,1,1,\cdots), \\ &e_3^1 = (1,0,0,0,\cdots), && e_3^2 = (0,1,0,0,\cdots), && e_3^3 = (0,0,1,1,\cdots), \cdots . \end{aligned}$$

In this case, $e_k^*((x_n)) = x_k$, for all $k \geq 1$. Hence the M-ideal $\mathbf{c}_0$ is no longer given by any $\ker e_i^*$ or their intersections. Nonetheless, consider the sub-diagram that is the union over $n \geq 1$ of all the lines $e_{n+1}^n \to e_{n+2}^n \to e_{n+3}^n \to \cdots$. It is easy to see that this sub-diagram represents $\mathbf{c}_0$. Thus, Problem 2.8 still has an affirmative answer though our earlier approach no longer works.

The above example suggests the following sufficient condition.

Theorem 5.2. *Let X be an L_1-predual space with the representation $X = \overline{\cup_{n=1}^\infty X_n}$, where $X_n \subseteq X_{n+1}$ and X_n isometrically isomorphic to ℓ_∞^n for every n. Let $\mathcal{D}$ be the corresponding representing diagram.*

Let $\phi \in ext(B_{X^*})$ be such that $\phi|_{X_n} \in ext(B_{X_n^*})$ for every $n \in \mathbb{N}$. Let $M = \ker\phi$. Then there exist a directed sub-diagram $\mathcal{S}$ of $\mathcal{D}$ such that $M = X_{\mathcal{S}}$.

Proof. Fix $n \in \mathbb{N}$. Let $\{x_1, x_2, \dots, x_n\}$ be an admissible basis of X_n. Hence, by Lemma 3.1, there exists $x_i^* \in S_{X^*}$ such that $x_i^*(x_j) = \delta_{ij}$. And from the Remark 3.3(c), $\mathrm{ext}(B_{X_n^*}) = \{\alpha x_i^*|_{X_n} : \alpha \in \mathbb{T}, 1 \leq i \leq n\}$. Thus, by our assumption on ϕ, $\phi|_{X_n} = \alpha x_k^*|_{X_n}$ for some $\alpha \in \mathbb{T}$, $1 \leq k \leq n$. The choice of k is clearly unique. In particular, $\{x_i : i \neq k, 1 \leq i \leq n\} \in \ker\phi$.

Define $M_{n-1} = [x_i : i \neq k, 1 \leq i \leq n]$. Then $M_n \subseteq M$, M_n is isometrically isomorphic to ℓ_∞^n for every n. Since $M_{n-1} = X_n \cap \ker\phi$, $M_n \subseteq M_{n+1}$.

Claim : $M = \overline{\cup_{n=1}^\infty M_n}$.

Let $\varepsilon > 0$ and $m \in M$. Since $X = \overline{\cup_{n=1}^\infty X_n}$, there exist $n \in \mathbb{N}$ and $x \in X_n$ such that $\|m - x\| < \varepsilon/2$. Let $1 \leq k \leq n$ be as above. Then $x = \sum_{i=1}^n x_i^*(x) x_i$ and $|x_k^*(x)| = |\phi(x)| = |\phi(m - x)| < \varepsilon/2$. Therefore,

$$\|m - \sum_{i \neq k} x_i^*(x) x_i\| = \|m - x + x_k^*(x) x_k\| \leq \|m - x\| + |x_k^*(x)| < \varepsilon.$$

Now, to obtain the diagram of M as a directed sub-diagram of that of X, let $\{y_i : 1 \leq i \leq n+1\}$ be an admissible basis of X_{n+1} such that $x_i = y_i + a_i y_{n+1}$ for $1 \leq i \leq n$.

Thus, $\{x_1, x_2, \dots, x_n\}$ are the nodes in the diagram of X at the n-th level and $\{y_1, y_2, \dots, y_{n+1}\}$ are the nodes in the diagram of X at the $(n+1)$-th level. There is an arrow $x_i \to y_i$ of weight 1 and an arrow $x_i \to y_{n+1}$ if $a_i \neq 0$.

It follows that $\{x_i : i \neq k, 1 \leq i \leq n\}$ are the nodes in the diagram of M at the $(n-1)$-th level and there exists $1 \leq m \leq n+1$ such that $\{y_i : i \neq m, 1 \leq i \leq n+1\}$

are the nodes in the diagram of M at the n-th level. Since $x_k = y_k + a_k y_{n+1} \notin M$, m must be either k or $n+1$.

If $m = k$, then $y_{n+1} \in M$, and for $i \neq k$, all the nodes in $x_i = y_i + a_i y_{n+1}$ are in M and arrows in the diagram of X remain in that of M.

If $m = n+1$, then $y_{n+1} \notin M$. Then for $i \neq k$, the equation $x_i = y_i + a_i y_{n+1}$ implies $a_i = 0$ and there are no arrows $x_i \to y_{n+1}$ in X.

Hence the diagram of M is indeed a directed sub-diagram of the diagram of X. This completes the proof. □

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