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Fixed Point Results for p-Convex Subsets in Quasi-Normed Spaces and Applications

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We prove some fixed point results for p-convex subsets in quasi-normed spaces which are not assumed to be p-normed spaces. Then we present certain applications to game theory and a new class of integral equations of the non-continuous variable.

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1. Introduction

The class of quasi-normed spaces is an extension of the class of normed spaces by multiplying $\kappa \geq 1$ to the right side of triangle inequality. Such spaces have attracted the attention of many authors [4], [10], [11].

Definition 1.1. ([3], Definition 3) Let X be a vector space over the field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), $\kappa \geq 1$, and $\|.\| : X \to [0;\infty)$ be a function such that for all $x, y \in X$, $a \in \mathbb{K}$,

(1) $\|x\| = 0$ if and only if $x = 0$.

(2) $\|ax\| = |a|\|x\|$.

(3) $\|x + y\| \leq \kappa\big(\|x\| + \|y\|\big)$.

Then

(a) $\|.\|$ is called a *quasi-norm* on X, the smallest possible κ is called the *modulus of concavity*, and $(X, \|.\|, \kappa)$ is called a *quasi-normed space*. For a quasi-normed space $(X, \|.\|, \kappa)$, without loss of generality, we can assume κ is the modulus of concavity.

(b) $\|.\|$ is called a *p-norm* on X, and $(X, \|.\|, \kappa)$ is called a *p-normed space* if

$$\|x + y\|^p \leq \|x\|^p + \|y\|^p \tag{1}$$

for some $0 < p \leq 1$ and for all $x, y \in X$. □

Remark 1.2. (1) In [20], Xiao and Zhu used another definition of a *$\tilde{p}$-normed space* $(X, \|.\|_{\tilde{p}})$ following [13, page 160] in the sense that the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$ satisfies the following for all $x, y \in X$ and all $a \in \mathbb{K}$.

(a) $\|x\|_{\tilde{p}} = 0$ if and only if $x = 0$.

(b) $\|ax\|_{\tilde{p}} = |a|^p\|x\|_{\tilde{p}}$.

(c) $\|x + y\|_{\tilde{p}} \leq \|x\|_{\tilde{p}} + \|y\|_{\tilde{p}}$.

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For $0 < p \leq 1$, put $\|x\|_{\tilde{p}} = \|x\|^p$ for all $x \in X$. We find that

(2) (a) If $\|.\|$ is a p-norm on X in sense of Definition 1.1.(b) then $\|.\|_{\tilde{p}}$ is a $\tilde{p}$-norm on X in Xiao-Zhu's sense.

(b) If $\|.\|_{\tilde{p}}$ is a $\tilde{p}$-norm on X in Xiao-Zhu's sense then $\|.\|$ is a p-norm on X in sense of Definition 1.1.(b) with the modulus of concavity $\kappa = 2^{\frac{1}{p}-1}$.

(3) For $p = 1$, a p-normed space is a usual normed space. Moreover, a p-normed space is also a metric linear space with the translation-invariant metric defined by $d_p(x, y) = \|x-y\|^p$ for all $x, y \in X$. Regarding a p-normed space as a metric space, we have the notions of completeness, compactness, total boundedness, etc. [10, page 1102].

Definition 1.3. ([11], pages 6-7) Let $(X, \|.\|, \kappa)$ be a quasi-normed space.

(1) The sequence $\{x_n\}$ is called *convergent* to x if $\lim\limits_{n\to\infty} \|x_n - x\| = 0$, written $\lim\limits_{n\to\infty} x_n = x$.

(2) The sequence $\{x_n\}$ is called *Cauchy* if $\lim\limits_{n,m\to\infty} \|x_n - x_m\| = 0$.

(3) $(X, \|.\|, \kappa)$ is called *quasi-Banach* if each Cauchy sequence is a convergent sequence.

(4) $(X, \|.\|, \kappa)$ is called *p-Banach* if it is p-normed and quasi-Banach. □

Since the modulus of concavity $\kappa \geq 1$, there are some difficulties in studying quasi-normed spaces as follows:

(1) quasi-norms are not necessarily continuous as norms;

(2) the triangle inequality is not applied consecutively;

(3) Hahn-Banach Theorem does not hold in quasi-normed spaces [10, page 1102];

(4) there exist p-Banach spaces which are not normable.

Example 1.4. ([14], Examples 1-2) (1) The Lebesgue spaces L^p with the quasi-norms

$$\|f\|_p = \Big(\int_\Omega |f(x)|^p d\mu(x)\Big)^{\frac{1}{p}}, \qquad f \in L^p$$

are Banach spaces for $1 \leq p < \infty$. For $0 < p < 1$, they are p-Banach spaces with the modulus of concavity $\kappa = 2^{\frac{1}{p}-1}$.

(2) The Lorentz spaces $L^{p,q}$ for $0 < p, q < \infty$ and Marcinkiewicz or weak L^p-spaces $L^{p,\infty}$ for $0 < p \leq \infty$ are quasi-Banach spaces determined by the quasi-norms

$$\|f\|_{p,q} = \begin{cases} \Big(\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \Big)^{\frac{1}{q}} & \text{if } 0 < q < \infty \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if } q = \infty \end{cases}$$

where $f^*(t) = \inf\{\lambda > 0 : \mu\big(x \in \Omega : |f(x)| > \lambda\big) \leq t\}$.

Moreover, the following assertions hold.

(1) $\|.\|_{p,q}$ is a norm if and only if either $1 \leq q \leq p$ or $p = q = \infty$.

(2) The spaces $L^{p,q}$ are not normable if one of the following conditions holds.
   (a) $0 < p < \infty$ and $0 < q < 1$.
   (b) $0 < p < 1$ and $1 \leq q \leq \infty$.
   (c) $p = 1$ and $1 < q \leq \infty$. $\square$

The following result, usually called the Aoki-Rolewicz theorem, asserts that for any given quasi-norm there exists an equivalent p-norm. The theorem was published in [3] and [18]. Here we present an explicit version following [14].

Theorem 1.5. ([14], Theorem 1) *Let $(X, \|.\|, \kappa)$ be a quasi-normed space, assume $p = \log_{2\kappa} 2$, and*

$$|||x||| = \inf\Big\{\Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}} : x = \sum_{i=1}^{n} x_i, x_i \in X, n \geq 1\Big\} \tag{2}$$

for all $x \in X$. Then $|||.|||$ is a quasi-norm on X satisfying

$$|||x + y|||^p \leq |||x|||^p + |||y|||^p \tag{3}$$

and

$$\frac{1}{2\kappa}\|x\| \leq |||x||| \leq \|x\| \tag{4}$$

for all $x, y \in X$. In particular,

(1) *The quasi-norm $|||.|||$ is a p-norm and continuous.*

(2) *If $\|.\|$ is a norm then $p = 1$ and $|||.||| = \|.\|$.*

There have been progresses in the non-convex theory in quasi-normed spaces. Xiao and Zhu proved the types of many fixed point results for s-convex sets in $\tilde{p}$-normed spaces where $0 < s \leq p$ [20]. Alghamdi et al. proved Krasnoselskii type fixed point theorem for the non-convex domain in $\tilde{p}$-normed spaces and applied to the existence of solutions for perturbed integral equation [2]. Petre and Petrusel also extended Krasnoselskii type fixed point theorem in generalized Banach spaces and applied to the abstract system of Fredholm-Volterra differential equations and inclusions [16]. Recently, Xiao and Lu used the metric retraction and calculation of measures of noncompactness to prove some fixed point theorems for s-convex subsets are with respect to single-valued and set-valued condensing maps on $\tilde{p}$-normed spaces [19].

Recall that, for a given quasi-norm $\|.\|$ on X, the formula $d(x, y) = \|x - y\|$ for all $x, y \in X$ defines a so-called *b-metric* on X. b-metric is an interesting extension of a metric, see [1], [8], [12, Chapter 12], [17] for example. The Banach contraction principle in b-metric spaces was stated as follows.

Theorem 1.6. ([7], Theorem 2.1) *Let (X, d, κ) be a complete b-metric space and $T : X \to X$ be a map such that $d(Tx, Ty) \leq \lambda d(x, y)$ for some $\lambda \in [0, 1)$ and for all $x, y \in X$. Then T has a unique fixed point x^* and $\lim_{n\to\infty} T^n x = x^*$ for all $x \in X$.*

Motivated by fixed point results for s-convex sets in $\tilde{p}$-normed [20], by using the Aoki-Rolewicz theorem on p-normalizing a quasi-normed space, we prove some fixed point results for p-convex subsets in quasi-normed spaces which are not assumed to be p-normed spaces. Then we present certain applications to game theory and a new class of integral equations of the non-continuous variable.

2. Fixed point results in quasi-normed spaces

We introduce the following notions for vector spaces, following [20, Definition 1.3].

Definition 2.1. Let $(X, \|.\|, \kappa)$ be a vector space, $p = \log_{2\kappa} 2$, and C a subset of X.

(1) C is called *p-convex* if for all $x, y \in C$ and all $t \in [0; 1]$,

$$(1-t)^{\frac{1}{p}} x + t^{\frac{1}{p}} y \in C.$$

(2) The smallest p-convex subset of X containing A is called the *p-convex hull* of A and denoted by $\mathrm{co}_p A$.

(3) The smallest closed p-convex subset of X containing A is called the *closed p-convex hull* of A and denoted by $\overline{\mathrm{co}_p} A$.

(4) The functional $f : C \to \mathbb{R}$ is called *p-convex* if C is p-convex and for all $x, y \in C$, all $t \in [0; 1]$,

$$f\big((1-t)^{\frac{1}{p}} x + t^{\frac{1}{p}} y\big) \le (1-t) f(x) + t f(y).$$

(5) The functional $f : C \to \mathbb{R}$ is called *p-concave* if C is p-convex and for all $x, y \in C$, all $t \in [0; 1]$,

$$f\big((1-t)^{\frac{1}{p}} x + t^{\frac{1}{p}} y\big) \ge (1-t) f(x) + t f(y).$$

(6) For the p-convex set C with $0 \in C$, the functional $q_C : X \to [0, \infty]$ defined by

$$q_C(x) = \inf\{t > 0 : x \in t^{\frac{1}{p}} C\}, x \in X$$

is called the *Minkowski p-functional* of C.

We mention the following remarks concerning these notions, for instance, see [20, page 1740]. Note that the formula for $\mathrm{co}_p\{x\}$ was proved in [5].

Remark 2.2. Let $(X, \|.\|, \kappa)$ be a quasi-normed space, $p = \log_{2\kappa} 2$, and C be a subset of X.

(1) C is p-convex if and only if for all $t_1, t_2 \ge 0$, $t_1^p + t_2^p = 1$: $t_1 x + t_2 y \in C$.

(2) If C is p-convex and $x_0 \neq 0$, then $C - \{x_0\}$ is not p-convex in general.

(3) The p-convex hull of C is given by

$$\mathrm{co}_p C = \Big\{ \sum_{i=1}^{n} t_i x_i : t_i \ge 0, \sum_{i=1}^{n} t_i^p = 1, x_i \in C, n \ge 2 \Big\}.$$

(4) For $x \in X$, we have

$$\mathrm{co}_p\{x\} = \begin{cases} \{tx : 0 < t \le 1\} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and

$$\overline{\mathrm{co}_p}\{x\} = \begin{cases} \{tx : 0 \le t \le 1\} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

The completely continuous map in the next definition is stated in the manner of [20], and it is exactly the compact map in [15] and many others.

Definition 2.3. ([20], Definition 1.1) Let X and Y be two metric spaces and $T : X \to Y$.

(1) T is called *completely continuous* if T is continuous and $\overline{T(A)}$ is compact for each bounded subset A of X.

(2) T is called *k-Lipschitz* if for some $k \geq 0$ and for all $x, y \in X$, we have $d(Tx, Ty) \leq kd(x, y)$. T is called *nonexpansive* if $k = 1$, and T is called *contractive* if $k < 1$. □

The following remark is useful in the next.

Remark 2.4. By Theorem 1.5, $|||.|||$ is a p-norm with $p = \log_{2\kappa} 2$. It follows from Remark 1.2.(2) that we have a $\tilde{p}$-norm $\|.\|_{\tilde{p}}$ defined for all $x \in X$ by

$$\|x\|_{\tilde{p}} = |||x|||^p. \tag{5}$$

By the inequality (4) and the equality (5), a quasi-normed space $(X, \|.\|, \kappa)$ and the corresponding p-normed space $(X, |||.|||, \kappa)$ and the $\tilde{p}$-normed space $(X, \|.\|_{\tilde{p}})$ are topologically and vector isomorphic. Moreover, many notions and properties between these spaces are equivalent such as convergence, Cauchy sequence, completeness, compactness, boundedness, closedness, semi-continuity, complete continuity, closed graph property. As consequences, for a subset D of X, the closure $\overline{D}$, the boundary ∂D and the interior D^0 of D in $(X, \|.\|, \kappa)$, $(X, |||.|||, \kappa)$ and $(X, \|.\|_{\tilde{p}})$ are equal. □

The two following lemmas are well-known, see [11] for instance. They can be also deduced directly from [20, Lemma 1.2] and [20, Lemma 2.1] by using Remark 2.4 and [20, Lemma 2.1].

Lemma 2.5. *Assume that $(X, \|.\|_X, \kappa_X)$ and $(Y, \|.\|_X, \kappa_Y)$ are quasi-normed spaces. Then*

(1) *A linear map $T : X \to Y$ is continuous if and only if T is bounded.*

(2) *If X is a finite-dimensional space then every linear map $T : X \to Y$ is continuous.*

Lemma 2.6. *Let $(X, \|.\|, \kappa_X)$ be a quasi-Banach space and A be a totally bounded subset of X. Then $\overline{\mathrm{co}_p}A$ is compact.*

The following result is an extension of [20, Lemma 1.4] in $\tilde{p}$-normed spaces to quasi-normed spaces.

Lemma 2.7. *Assume that $(X, \|.\|, \kappa)$ is a quasi-normed space and $p = \log_{2\kappa} 2$. Then*

(1) *The ball $B(0, r)$ is p-convex for all $r > 0$.*

(2) *If $C \subset X$ is p-convex and $\alpha \in \mathbb{K}$, then αC is p-convex.*

(3) *If $C_1, C_2 \subset X$ are p-convex, then $C_1 + C_2$ is p-convex.*

(4) *If $C_i \subset X$, $i = 1, 2, \ldots$ are p-convex then $\bigcap_{i=1}^{\infty} C_i$ is p-convex.*

(5) *If $A \subset X$ and $0 \in A$, then $\mathrm{co}_p A \subset \mathrm{co}A$, where $\mathrm{co}A$ is the convex hull of A.*

(6) *If C is a closed p-convex set and $0 < k < p$, then C is a closed k-convex set.*

Proof. By using Remark 2.4 and [20, Lemma 1.4], the conclusions are proved with a similar argument as in the proof of Lemma 2.5. Note that for the cases (2) and (3) the scalar multiplication and the addition are also continuous since every quasi-normed space is a topological vector space. □

All following properties of p-convex sets and the Minkowski gauge function are treated in detail in [9, Chapter 6]. They are also extensions of that in [20, Lemma 1.5].

Lemma 2.8. *Assume that $(X, \|.\|, \kappa)$ is a quasi-normed space, $p = \log_{2\kappa} 2$, C is an p-convex subset of X with $0 \in C$, and q_C is the Minkowski p-functional of C. Then*

(1) $q_C(0) = 0$.

(2) *q_C is positively p-homogeneous, that is, $q_C(tx) = t^p q_C(x)$ for $x \in X$ and $t > 0$.*

(3) *q_C is sub-additive, that is, $q_C(c + y) \leq q_C(x) + q_C(y)$ for all all $x, y \in X$.*

(4) *If C is bounded, then $q_C(x) > 0$ for all $x \neq 0$.*

(5) *If C is closed, then $q_C(x)$ is lower semi-continuous and $\overline{C} = \{x \in X : q_C(x) \leq 1\}$.*

(6) *If C is absorbing, then $q_C(x) < \infty$ for all $x \in X$.*

(7) *If $0 \in C^0$, then q_C is continuous and*

$$C^0 = \{x \in X : q_C(x) < 1\}, \quad \overline{C} = \{x \in X : q_C(x) \leq 1\}.$$

Next, we prove some fixed point results in quasi-Banach spaces. Note that the completely continuous map in the next is used in sense of Definition 2.3, and it is the same as the compact map in [15] and many others.

Proposition 2.9. *Let $(X, \|.\|, \kappa)$ be a quasi-Banach space, and $p = \log_{2\kappa} 2$.*

(1) *If X is finite-dimensional, C is a bounded closed p-convex of X and $T : C \to C$ is continuous, then T has a fixed point.*

(2) *If C is a compact p-convex subset of X and $T : C \to C$ is continuous, then T has a fixed point.*

(3) *If C is a bounded closed p-convex subset of X and $T : C \to C$ is completely continuous, then T has a fixed point.*

(4) *If C is a compact p-convex subset of X, $T : C \rightrightarrows C$ is upper semi-continuous and Tx is closed p-convex for all $x \in C$, then T has a fixed point.*

(5) *If C is a compact p-convex subset of X, $T : C \rightrightarrows C$ has the closed graph and Tx is closed p-convex for all $x \in C$, then T has a fixed point.*

Proof. We prove the statement (1), the others are proved similarly. By using Remark 2.4, $(X, \|.\|_{\tilde{p}})$ is a $\tilde{p}$-normed space. Moreover, X is complete, C is bounded, closed, p-convex and T is continuous with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$. So all assumptions of [20, Theorem 2.13] hold. This implies that T has a fixed point. □

In the following result, the function α is continuous in $\tilde{p}$-Banach spaces but we do not know whether α is continuous or not in quasi-Banach spaces. Note that, following the proof of [20, Lemma 2.3], we see that the function $f_y(x) = \frac{\|x-y\|}{\|x\|_0}$ is continuous in p-Banach spaces since the p-norm $\|.\|$ is continuous.

Proposition 2.10. *Assume that $(X, \|.\|, \kappa)$ is a quasi-normed space, C is a compact p-convex subset of X with $p = \log_{2\kappa} 2$, $\alpha(x) = \min_{y \in C_x} \frac{\|x-y\|}{\|x_0\|}$ for all $x \in D$ where $0 \neq x_0 \in C$, $D = C - \overline{\text{co}_p}\{x_0\}$, $C_x = \{y \in C : x = y - tx_0$ for some $t \in [0; 1]\}$. Then*

(1) *D is compact p-convex set, $0 \in D$ and $C \subset D$.*

(2) *$\alpha(x) \in [0;1]$, and $x \in D$ has the decomposition $x = y(x) - \alpha(x)x_0$ for some $y(x) \in C$.*

(3) *If $T : C \to C$ and $T_0 : D \to D$ is defined by*

$$T_0 x = T[x + \alpha(x)x_0] - \alpha(x)x_0 \tag{6}$$

for all $x \in D$, then T_0 has a fixed point if and only if T has a fixed point.

Proof. (1) We find that C and $\overline{\mathrm{co}_p}\{x_0\}$ are all compact and p-convex. So D is compact and p-convex by Lemma 2.8. Note that $0 = x_0 - x_0 \in D$. Since $0 \in \overline{\mathrm{co}_p}\{x_0\}$, we get $C = C - 0 \subset C - \overline{\mathrm{co}_p}\{x_0\} = D$.

(2) Since C is compact, C_x is a compact subset of C. So C_x is also closed. For $y = x + tx_0$, we find that $t = \frac{\|x-y\|}{\|x_0\|}$. Since C_x is compact, the value $\alpha(x)$ is well defined and $\alpha(x) \in [0;1]$. We find that $\alpha(x) = \lim_{n\to\infty} t_n$ where $\{y_n\} \subset C_x$, $y_n = x - t_n x_0$ and $t_n = \frac{\|x-y_n\|}{\|x_0\|} \in [0,1]$. So there exists

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y(x - t_n x_0) = x - \alpha(x)x_0.$$

Let $y(x) = x + \alpha(x)x_0$. We set $x = y(x) - \alpha(x)x_0$. Note that $y(x) \in C_x$ since C_x is closed and $\{y_n\} \subset C_x$.

(3) Let T have a fixed point $z \in C$. Then $z = Tz = T(z+0) - 0 = T_0 z$. Then z is also a fixed point of T_0.

Let T_0 have a fixed point $z \in D$. Then $z = T_0 z = T[z + \alpha(z)x_0] - \alpha(z)x_0$. This implies that $z + \alpha(z)x_0 = T[z + \alpha(z)x_0]$. This proves that $z + \alpha(z)x_0$ is a fixed point of T. □

Lemma 2.11. *Assume that $(X, \|.\|, \kappa)$ is a quasi-normed space; C is a closed bounded p-convex subset of X with $0 \in C^0$ and $p = \log_{2\kappa} 2$; and for each $x \in X$, $r(x) = \frac{x}{\max\{1, (q_C(x))^{1/p}\}}$ where q_C is the Minkowski p-functional of C. Then*

(1) *C is a retract of X in the sense that $r : X \to C$ is a continuous map, and if $x \in C$ then $r(x) = x$.*

(2) *If $x \notin C$ then $r(x) \in \partial C$.*

Proof. (1) By Lemma 2.8, since C is closed and p-convex, and $0 \in C^0$, we find that the Minkowski p-functional q_C is a positively p-homogeneous, sub-additive and continuous functional with $C = \{x \in X : q_C(x) \leq 1\}$ and $C^0 = \{x \in X : q_C(x) < 1\}$. Note that r is continuous. If $x \in C$, then $q_C(x) \leq 1$. So $r(x) = x \in C$. If $x \notin C$, then $q_C(x) > 1$. Note that C is p-convex and $0, x \in C$. So

$$r(x) = \big(\frac{1}{q_C(x)}\big)^{1/p} x + \big(1 - \frac{1}{q_C(x)}\big)^{1/p} .0 \in C.$$

Then $r : X \to C$ is a continuous map and $r(x) = x$ for all $x \in X$. This proves that C is a retract of X.

(2) If $x \notin C$, then by $q_C(x) > 1$ and Lemma 2.8.(2) we have

$$q_C(r(x)) = q_C\big(\frac{x}{(q_C(x))^{1/p}}\big) = \frac{q_C(x)}{q_C(x)} = 1.$$

By Lemma 2.8.(7) we get $r(x) \in \{x \in X : q_C(x) = 1\} = \overline{C} \setminus C^0 = \partial C$. □

Now we prove Krasnoselskii type in quasi-normed spaces. Note that we must suppose the map $S : X \to X$ replacing for $S : C \to X$ as in [20, Theorem 2.17].

Theorem 2.12. *Assume that $(X, \|.\|, \kappa)$ is a quasi-Banach space; C is a bounded closed p-convex subset of X with $p = \log_{2\kappa} 2$; and $S : X \to X$ is a contraction map, $T : C \to X$ is completely continuous, and $Sx + Ty \in C$ for all $x, y \in C$. Then $T + S$ has a fixed point.*

Proof. By using Remark 2.4 and a similar argument as in the proof of Lemma 2.5, we find that

(1) $(X, \|.\|_{\tilde{p}})$ is a $\tilde{p}$-Banach space.

(2) C is a bounded closed p-convex subset of $(X, \|.\|_{\tilde{p}})$.

(3) T is completely continuous with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$.

Since $S : X \to X$ is a contraction map, we have $\|Sx - Sy\| \le \alpha\|x - y\|$ for some $0 \le \alpha < 1$ and for all $x, y \in X$. We will show that $S : X \to X$ is a contraction with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$. Let $x, y \in X$. For $x_i \in X$, $i = 1, \dots, n$ and $x - y = \sum_{i=1}^n x_i$, we find that

$$y + \sum_{j=1}^n x_j, y + \sum_{j=2}^n x_j, \dots, y + x_n \in X.$$

So we can put

$$y_1 = S\big(y + \sum_{j=1}^n x_j\big) - S\big(y + \sum_{j=2}^n x_j\big),$$

and $$y_2 = S\big(y + \sum_{j=2}^n x_j\big) - S\big(y + \sum_{j=3}^n x_j\big), \dots, y_n = S(y + x_n) - Sy.$$

Then we have $Sx - Sy = S\big(y + \sum\limits_{j=1}^n x_j\big) - Sy = \sum\limits_{ji=1}^n y_j$. Therefore,

$$\begin{aligned}
|||Sx - Sy||| &= \inf\Big\{\Big(\sum_{i=1}^n \|z_i\|^p\Big)^{\frac{1}{p}} : Sx - Sy = \sum_{i=1}^n z_i, z_i \in X, n \ge 1\Big\} \\
&\le \Big(\sum_{i=1}^n \|y_i\|^p\Big)^{\frac{1}{p}} = \Big(\sum_{i=1}^n \|S\big(y + \sum_{j=i}^n x_j\big) - S\big(y + \sum_{j=i+1}^n x_j\big)\|^p\Big)^{\frac{1}{p}} \\
&\le \Big(\sum_{i=1}^n \big(\alpha\|\big(y + \sum_{j=i}^n x_j\big) - \big(y + \sum_{j=i+1}^n x_j\big)\|\big)^p\Big)^{\frac{1}{p}} = \alpha\Big(\sum_{i=1}^n \|x_i\|^p\Big)^{\frac{1}{p}}.
\end{aligned}$$

This implies that $|||Sx - Sy||| \le \alpha|||x - y|||$. By (5) we have $\|Sx - Sy\|_{\tilde{p}} \le \alpha^p\|x - y\|_{\tilde{p}}$. This implies that $S : C \to X$ is a contraction with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$.

The above arguments show that all assumptions of [20, Theorem 2.17] are satisfied for the $\tilde{p}$-Banach space $(X, \|.\|_{\tilde{p}})$ and the maps $S, T : C \to X$. Then the map $S+T$ has a fixed point. $\square$

In the following result we suppose that the map $T : X \to C$ replaces the map $T : C \to C$ as in [20, Theorem 2.18].

Theorem 2.13. *Assume that $(X, \|.\|, \kappa)$ is a quasi-Banach space, C is a bounded closed p-convex subset of X with $p = \log_{2\kappa} 2$, and $T : X \to C$ is a nonexpansive map where I is an identity map and $(I-T)(C)$ is closed. Then there exists $z \in C$ such that $Tz = z$.*

Proof. By using Remark 2.4 and a similar argument as in the proof of Lemma 2.5, we find that

(1) $(X, \|.\|_{\tilde{p}})$ is a $\tilde{p}$-Banach space.

(2) C is a bounded closed p-convex subset of X and $(I-T)(C)$ is a closed subset of X.

We will show that $T : X \to C$ is a nonexpansive map with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$. Let $x, y \in X$. For $x_i \in X$, $i = 1, \ldots, n$ and $x - y = \sum\limits_{i=1}^{n} x_i$, put

$$\begin{aligned} y_1 &= T\big(y + \sum_{j=1}^{n} x_j\big) - T\big(y + \sum_{j=2}^{n} x_j\big) \\ y_2 &= T\big(y + \sum_{j=2}^{n} x_j\big) - T\big(y + \sum_{j=3}^{n} x_j\big) \\ &\vdots \\ y_n &= T(y + x_n) - Ty. \end{aligned}$$

Then we have $Tx - Ty = \sum\limits_{ji=1}^{n} y_j$. Therefore,

$$\begin{aligned} |||Tx - Ty||| &= \inf\Big\{\Big(\sum_{i=1}^{n} \|z_i\|^p\Big)^{\frac{1}{p}} : Tx - Ty = \sum_{i=1}^{n} z_i, z_i \in X, n \geq 1\Big\} \\ &\leq \Big(\sum_{i=1}^{n} \|y_i\|^p\Big)^{\frac{1}{p}} = \Big(\sum_{i=1}^{n} \|T\big(y + \sum_{j=i}^{n} x_j\big) - T\big(y + \sum_{j=i+1}^{n} x_j\big)\|^p\Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{i=1}^{n} \|\big(y + \sum_{j=i}^{n} x_j\big) - \big(y + \sum_{j=i+1}^{n} x_j\big)\|^p\Big)^{\frac{1}{p}} = \Big(\sum_{i=1}^{n} \|x_i\|^p\Big)^{\frac{1}{p}}. \end{aligned}$$

This implies that $|||Tx - Ty||| \leq |||x - y|||$. By (5) we have $\|Tx - Ty\|_{\tilde{p}} \leq \|x - y\|_{\tilde{p}}$. This implies that $T : X \to C$ is a nonexpansive map with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$.

Since $T : X \to C$ is a nonexpansive map with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$, we deduce that $T : C \to C$ is a nonexpansive map with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$.

The above arguments show that all assumptions of [20, Theorem 2.18] are satisfied for the $\tilde{p}$-Banach space $(X, \|.\|_{\tilde{p}})$ and the map $T : C \to C$. Then the map T has a fixed point in C. $\square$

Now we prove Rothe-Potter type theorem and some others in quasi-normed spaces.

Proposition 2.14. *Assume that $(X, \|.\|, \kappa)$ is a quasi-Banach space, D is a bounded open p-convex subset of X with $0 \in D$ and $p = \log_{2\kappa} 2$, and $T : \overline{D} \to X$ is completely continuous and $T(\partial D) \subset \overline{D}$. Then T has a fixed point $z \in \overline{D}$.*

Proof. By using Remark 2.4 and a similar argument as in the proof of Lemma 2.5, we find that

(1) $(X, \|.\|_{\tilde{p}})$ is a $\tilde{p}$-Banach space.

(2) D is a bounded open p-convex subset of X with $0 \in D$.

(3) The closure of D and the boundary of D with respect to the quasi-norm $\|.\|$ and the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$ are equal. So $T : \overline{D} \to X$ is completely continuous and $T(\partial D) \subset \overline{D}$ with respect to the $\tilde{p}$-norm $\|.\|_{\tilde{p}}$.

The above shows that all assumptions of [20, Theorem 2.20] are satisfied for the $\tilde{p}$-Banach space $(X, \|.\|_{\tilde{p}})$ and the map T. Then T has a fixed point $z \in \overline{D}$. $\square$

Theorem 2.15. *Assume that $(X, \|.\|)$ is a quasi-Banach space, D is a bounded open p-convex subset of X with $0 \in D$ and $p = \log_{2\kappa} 2$, and $T : \overline{D} \to X$ is a completely continuous map satisfying one of the following conditions*

(1) $\|Tx\| \leq \|x\|$ *for all* $x \in \partial D$.

(2) $\|x - Tx\| \geq \|Tx\|$ *for all* $x \in \partial D$.

(3) $\|x - Tx\|^{\frac{2}{p}} \geq \|Tx\|^{\frac{2}{p}} - \|x\|^{\frac{2}{p}}$ *for all* $x \in \partial D$.

Then there exists $z \in \overline{D}$ such that $Tz = z$.

Proof. Define a retraction $r : X \to \overline{D}$ by

$$r(x) = \frac{x}{\max\{1, q_D(x)\}}$$

for all $x \in X$. Since D is an open p-convex subset with $0 \in D$, r is continuous by Lemma 2.8.(7). Since $T : X \to \overline{D}$ is completely continuous, we see that $rT : \overline{D} \to \overline{D}$ is completely continuous. By Proposition 2.9.(3), there exists $z \in \overline{D}$ such that $r(Tz) = z$. We consider two following cases.

Case 1. $z \in \partial D$. We will prove that

$$q_D(Tz) \leq 1. \tag{7}$$

Let condition (1) be satisfied. Then

$$\|z\| = \|r(Tz)\| = \frac{\|Tz\|}{\max\{1, q_D(Tz)\}} \leq \frac{\|Tz\|}{q_D(Tz)}.$$

Note that $z \neq 0$. So (7) holds.

Let condition (2) be satisfied. Suppose to the contrary that (7) does not hold.

Then $q_D(Tz) > 1$. It follows from

$$z = r(Tz) = \frac{Tz}{\max\{1, q_D(Tz)\}} = \frac{Tz}{q_D(Tz)}$$

that $Tz = q_D(Tz)z$. Therefore

$$q_D(Tz)\|z\| = \|Tz\| \leq \|z - Tz\| = \|z - q_D(Tz)z\| = (q_D(Tz) - 1)\|z\|.$$

Note that $z \neq 0$. So we get $q_D(Tz) \leq q_D(Tz) - 1$. This is a contradiction. So (7) holds.

Let condition (3) be satisfied. Suppose to the contrary that (7) does not hold. Then $q_D(Tz) > 1$. It follows from

$$z = r(Tz) = \frac{Tz}{\max\{1, q_D(Tz)\}} = \frac{Tz}{q_D(Tz)}$$

that $Tz = q_D(Tz)z$. We find that

$$\begin{aligned}[q_D^2(Tz) - 1]\|z\|^2 &= q_D^2(Tz)\|z\|^2 - \|z\|^2 = \|Tz\|^2 - \|z\|^2 \leq \|z - Tz\|^2 \\ &= \|z - q_D(Tz)z\|^2 = [q_D(Tz) - 1]^2\|z\|^2.\end{aligned}$$

Note that $z \neq 0$. So we get $q_D^2(Tz) - 1 \leq [q_D(Tz) - 1]^2$. This implies that $q_D(Tz) \leq 1$ – a contradiction. So (7) holds.

By (7), we find that $z = r(Tz) = \dfrac{Tz}{\max\{1, q_D(Tz)\}} = Tz.$

Case 2. $z \notin \partial D$. By Lemma 2.11.(2) we have $Tz \in \partial D$, and by Lemma 2.11.(1) we also have $z = r(Tz) = Tz$. □

Remark 2.16. It remains open if we can extend fully [20, Theorem 2.17] and [20, Theorem 2.18] to quasi-normed spaces or not. □

Now, we present some applications of Proposition 2.9.(5) and Theorem 2.12. First we recall the following notions [6, page 9].

(1) A *game* is a triple (A, B, F), where A, B are two non-empty sets, whose elements are *strategies*, and $F : A \times B \to \mathbb{R}$ is the *gain function.*

(2) The quantity $F(x, y)$ represents the gain of the player P when he chooses the strategy $x \in A$ and the player Q chooses $y \in B$, and $-F(x, y)$ represents the gain of the player Q in the same situation.

(3) The target of the player P is to maximize his gain when Q chooses the worst strategy, that is, to choose $x_0 \in A$ such that $\inf\limits_{y \in B} F(x_0, y) = \sup\limits_{x \in A} \inf\limits_{y \in B} F(x, y)$.

(4) The target of the player Q is to maximize his gain when P chooses the worst strategy, that is, to choose $y_0 \in B$ such that $\sup\limits_{x \in A} F(x, y_0) = \inf\limits_{y \in B} \sup\limits_{x \in A} F(x, y)$.

(5) We always have

$$\sup_{x \in A} \inf_{y \in B} F(x, y) = \inf_{y \in B} F(x_0, y) \leq F(x_0, y_0) \leq \sup_{x \in A} F(x, y_0) = \inf_{y \in B} \sup_{x \in A} F(x, y). \quad (8)$$

If the equality holds in (8), then $(x_0, y_0) \in A \times B$ is called a *solution* of the game (A, B, F).

The following result is an extension of [20, Theorem 3.2] in $\tilde{p}$-Banach spaces to quasi-Banach spaces. We recall here a technical lemma to ensure the continuity of certain functions.

Lemma 2.17. ([6], Lemma 3.3) *If A, B are compact Hausdorff topological spaces and $F : A \times B \to \mathbb{R}$ is continuous, then the functions $\varphi : A \to \mathbb{R}$ and $\psi : B \to \mathbb{R}$ are continuous, where φ and ψ are defined by*

$$\varphi(x) = \inf_{y\in B} F(x, y) \ \ \textit{for all} \ \ x \in A, \psi(y) = \sup_{x\in A} F(x, y) \ \ \textit{for all} \ \ y \in B. \tag{9}$$

Theorem 2.18. *Assume that $(X, \|.\|_X, \kappa_X)$ and $(Y, \|.\|_Y, \kappa_Y)$ are two quasi-Banach spaces, $A \subset X$, $B \subset Y$ are non-empty compact p-convex sets with $p = \log_{2\kappa_X} 2$ and $p = \log_{2\kappa_Y} 2$ respectively, $F : A \times B \to \mathbb{R}$ is continuous; and for every $x \in A$, $F(x, .)$ is p-convex and for every $y \in B$, $F(., y)$ is p-concave. Then*

$$\inf_{y\in B} \sup_{x\in A} F(x, y) = \sup_{x\in A} \inf_{y\in B} F(x, y)$$

and the game (A, B, F) has a solution.

Proof. Let φ and ψ be the functions defined by (9) and let

$$B_x = \{y \in B : F(x, y) = \varphi(x)\}, \quad A_y = \{x \in A : F(x, y) = \psi(y)\}.$$

We find that X, Y are Hausdorff spaces, A and B are compact, and the functions F, φ, ψ are continuous. Then A_y and B_x are non-empty and closed for each $(x, y) \in A \times B$. Since the p-convexity does not depend on either quasi-norm, p-norm or $\tilde{p}$-norm, A_y and B_x are p-convex as in the proof of [20, Theorem 3.2].

Let $C = A \times B$. Then C is also p-convex. Define the multi-valued map $T : C \rightrightarrows C$ by $T(x, y) = A_y \times B_x$ for all $(x, y) \in C$. As then the proof of [20, Theorem 3.2], the graph of T is closed with respect to the p-norms induced by the given quasi-norms on X and Y. By using Remark 2.4 we find that the graph of T is closed with respect to the given quasi-norms. By Proposition 2.9.(5) there exists $(x_0, y_0) \in C$ such that $(x_0, y_0) \in T(x_0, y_0)$.

The remaining argumentation is very similar to that in the proof of [20, Theorem 3.2] and we get the desired conclusion. □

Finally, we present an application of Krasnoselskii type theorem in quasi-normed spaces to show the existence of the solution of an integral equation. Recall that Krasnoselskii theorem in normed spaces is one of the most important technique to show the existence of the solution of integral equations. So Krasnoselskii type theorem in quasi-normed spaces may play a similar role but in the wider class of integral equations. Note that the following result is on $L^{\frac{1}{2}}[0; 1]$, a quasi-normed space which is not normable, see [10, page 1102] or [11, page viii] for example, and thus, Krasnoselskii theorem in normed spaces is not applicable. As to our best knowledge, the example is new by the non-continuity of the variable function x and the application of Theorem 1.6 in the argument.

Recall that $L^{\frac{1}{2}}[0;1]$ is a quasi-Banach space with the quasi-norm

$$\|x\| = \Big(\int_0^1 |x(t)|^{\frac{1}{2}}dt\Big)^2, x \in L^{\frac{1}{2}}[0;1]$$

and the modulus of concavity $\kappa = 2$.

It is also a $\tilde{p}$-Banach space with the $\tilde{p}$-norm

$$\|x\|_{\tilde{p}} = \int_0^1 |x(t)|^{\frac{1}{2}}dt, x \in L^{\frac{1}{2}}[0;1].$$

Theorem 2.19. *Assume that the following conditions hold.*

(1) $\lambda, \mu \in \mathbb{R}$, *and* $k \in L^{\frac{1}{2}}([0;1] \times \mathbb{R})$, $l \in C([0;1] \times \mathbb{R})$ *with*

$$M = \sup_{t \in [0;1], x \in L^{\frac{1}{2}}[0;1]} \int_0^1 |l(t, x(s))|ds < \infty.$$

(2) *There exist* $a, b > 0$ *such that* $a|\lambda| < 1$ *and for all* $(t, u), (t, v) \in [0;1] \times \mathbb{R}$,

$$|k(t, u) - k(t, v)| \leq a|u - v|, |l(t, u) - l(t, v)| \leq b|u - v|.$$

Then there exists $x_0 \in L^{\frac{1}{2}}[0;1]$ *such that the equation*

$$x(t) = \lambda \int_0^t k(t, x(s))ds + \mu \int_0^1 l(t, x(s))ds, \quad t \in [0;1] \tag{10}$$

has at least on solution $x^* \in B[x_0, r] \subset L^{\frac{1}{2}}[0;1]$, *where* $r = \left(\frac{\sqrt{b|\mu|M}}{1-\sqrt{a|\lambda|}}\right)^2$ *and*

$$B[x_0, r] = \big\{x \in L^{\frac{1}{2}}[0;1] : \Big(\int_0^1 |x(t) - x_0(t)|^{\frac{1}{2}}dt\Big)^2 \leq r\big\}.$$

Proof. For all $t \in [0;1]$ and $x \in L^{\frac{1}{2}}[0;1]$, let

$$f(x)(t) = \lambda \int_0^t k(t, x(s))ds, \qquad g(x)(t) = \mu \int_0^1 l(t, x(s))ds.$$

Since $k \in L^{\frac{1}{2}}([0;1] \times \mathbb{R})$ and $l \in C([0;1] \times \mathbb{R})$, we have $f, g : L^{\frac{1}{2}}[0;1] \to L^{\frac{1}{2}}[0;1]$ are well-defined. We find that the equation (10) becomes

$$x = f(x) + g(x). \tag{11}$$

Moreover, x^* is a solution of (10) if and only if x^* is a fixed point of $f + g$.

We show that all assumptions of Theorem 2.12 are satisfied. For all $t \in \mathbb{R}$, we have

$$\begin{aligned} |f(x)(t) - f(y)(t)| &= |\lambda \int_0^t k(t, x(s))ds - \lambda \int_0^t k(t, y(s))ds| \\ &\leq |\lambda| \int_0^t |k(t, x(s)) - k(t, y(s))|ds \leq |\lambda| \int_0^t a|x(s) - y(s)|ds \\ &\leq a|\lambda| \int_0^1 |x(s) - y(s)|ds. \end{aligned} \tag{12}$$

Then we have

$$\begin{aligned}\|f(x)-f(y)\| &= \Big(\int_0^1 |f(x)(t)-f(y)(t)|^{\frac{1}{2}}dt\Big)^2\\ &\le \Big(\int_0^1 \big(a|\lambda|\big)^{\frac{1}{2}}\Big(\int_0^1 |x(s)-y(s)|ds\Big)^{\frac{1}{2}}dt\Big)^2\\ &\le \Big(\int_0^1 \big(a|\lambda|\big)^{\frac{1}{2}}\int_0^1 |x(s)-y(s)|^{\frac{1}{2}}dsdt\Big)^2\\ &= \Big(\int_0^1 \big(a|\lambda|\big)^{\frac{1}{2}}\|x-y\|^{\frac{1}{2}}dt\Big)^2 = a|\lambda|\|x-y\|. \end{aligned} \tag{13}$$

Since $0 \le a|\lambda| < 1$, we get $f : L^{\frac{1}{2}}[0;1] \to L^{\frac{1}{2}}[0;1]$ is a contraction by (13).

Put $d(x,y) = \|x-y\|$ for all $x, y \in L^{\frac{1}{2}}[0;1]$. Then d defines a b-metric, and $L^{\frac{1}{2}}[0;1]$ becomes a complete b-metric space. So, following Theorem 1.6, f has a fixed point $x_0 \in L^{\frac{1}{2}}[0;1]$.

By Lemma 2.7, we find that $B[x_0,r]$ is a bounded closed p-convex subset of $L^{\frac{1}{2}}[0;1]$ with $p = \log_4 2 = \frac{1}{2}$.

We prove that the map $g : B[x_0,r] \to L^{\frac{1}{2}}[0;1]$ is completely continuous by showing that $g(B[x_0,r])$ is relatively compact in $L^{\frac{1}{2}}[0;1]$. For each $x \in B[x_0,r]$, there exists $R > 0$ such that $x(s) \in [-R;R]$ for all $s \in [0;1]$. Note that l is continuous on $[0;1]\times[-R;R]$. Then l is uniformly continuous on $[0;1]\times[-R;R]$. Then for each $\varepsilon > 0$, there exits $\delta_\varepsilon > 0$ such that for all $(t_1,s_1),(t_2,s_2) \in [0;1]\times[-R;R]$ with $\sqrt{|t_1-t_2|^2+|s_1-s_2|^2} < \delta_\varepsilon$,

$$|l(t_1,s_1) - l(t_2,s_2)| \le \varepsilon. \tag{14}$$

So for $x \in B[x_0,r]$ we have

$$\|g(x)\| = \Big(\int_0^1 |g(x)(t)|^{\frac{1}{2}}dt\Big)^2 \le \Big(\int_0^1 |\mu\int_0^1 l(t,x(s))ds|^{\frac{1}{2}}dt\Big)^2 \le |\mu|\alpha. \tag{15}$$

So g is bounded on $B[x_0,r]$. Now, for $|t_1-t_2| \le \delta_\varepsilon$, we find that

$$\begin{aligned}|g(x)(t_1)-g(x)(t_2)| &= \big|\mu\int_0^1 l(t_1,x(s))ds - \mu\int_0^1 l(t_2,x(s))ds\big|\\ &\le |\mu|\int_0^1 |l(t_1,x(s)) - l(t_2,x(s))|ds\\ &\le |\mu|\int_0^1 \varepsilon ds = |\mu|\varepsilon.\end{aligned} \tag{16}$$

This proves that $g(B[x_0,r])$ is equicontinuous. So, by Arzela-Ascoli's theorem, $g(B[x_0,r])$ is relatively compact.

Next, for all $x, y \in B[x_0, r]$ we will show that $f(x) + g(y) \in B[x_0, r]$. We find that

$$\begin{aligned}
\|f(x) - x_0\| &= \Big(\int_0^1 |f(x)(t) - x_0(t)|^{\frac{1}{2}} dt\Big)^2 \\
&= \Big(\int_0^1 |f(x)(t) - f(x_0)(t)|^{\frac{1}{2}} dt\Big)^2 \\
&\leq \Big(\int_0^1 \Big|\int_0^t k(t, x(s))ds - \int_0^t k(t, x_0(s))ds\Big|^{\frac{1}{2}} dt\Big)^2 \\
&= \Big(\int_0^1 \Big|\int_0^t \big(k(t, x(s)) - k(t, x_0(s))\big)ds\Big|^{\frac{1}{2}} dt\Big)^2
\end{aligned}$$

hence

$$\begin{aligned}
\|f(x) - x_0\| &\leq \Big(\int_0^1 \Big(\int_0^t \big|k(t, x(s)) - k(t, x_0(s))\big|ds\Big)^{\frac{1}{2}} dt\Big)^2 \\
&\leq \Big(\int_0^1 \Big(a|\lambda| \int_0^1 |x(s) - x_0(s)|ds\Big)^{\frac{1}{2}} dt\Big)^2 \\
&\leq \Big(\int_0^1 \big(a|\lambda|\big)^{\frac{1}{2}} \int_0^1 |x(s) - x_0(s)|^{\frac{1}{2}} ds dt\Big)^2 \\
&= a|\lambda| \|x - x_0\| \leq a|\lambda| r. \qquad (17)
\end{aligned}$$

We also have

$$\begin{aligned}
\|g(y)\| = \Big(\int_0^1 |g(y)(t)|^{\frac{1}{2}} dt\Big)^2 &= \Big(\int_0^1 \Big|\int_0^1 l(t, y(s))ds\Big|^{\frac{1}{2}} dt\Big)^2 \\
\leq b|\mu| \int_0^1 |l(t, y(s))|ds &\leq b|\mu| M. \qquad (18)
\end{aligned}$$

From (17) and (18) we conclude that

$$\|f(x) + g(y) - x_0\|^{\frac{1}{2}} \leq \|f(x) - x_0\|^{\frac{1}{2}} + \|g(y)\|^{\frac{1}{2}} \leq \sqrt{a|\lambda|r} + \sqrt{b|\mu|M} = \sqrt{r}.$$

This proves that $f(x) + g(y) \in B[x_0, r]$.

By the above calculations, all assumptions of Theorem 2.12 are satisfied for the maps $f : L^{\frac{1}{2}}[0; 1] \to L^{\frac{1}{2}}[0; 1]$ and $g : B[x_0, r] \to L^{\frac{1}{2}}[0; 1]$. Then $f + g$ has a fixed point $x^* \in B[x_0, r]$. It implies that x^* is a solution of (10). □

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