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On Rockafellar's Sum Theorem in General Banach Spaces

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Following some older ideas of ours and some more recent ones due to Yao, we give a self contained (except some well known results) and relatively short proof of the Rockafellar's sum theorem for the largest possible class of maximally monotone operators on a non reflexive Banach space.

Keywords: C_0-maximally monotone operator, convex function, convex set, Fitzpatrick function, maximally monotone operator, monotone operator, operator of type (FPV), sum theorem.

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1. Introduction

Throughout this note X will denote a non zero Banach space with topological dual X^*. For $(x, x^*) \in X \times X^*$, $\langle x, x^* \rangle$ (or $\langle x^*, x \rangle$) will denote the natural evaluation map. If $T : X \rightrightarrows X^*$ is a (multivalued) operator then $G(T) = \{(x, x^*) \in X \times X^* : x^* \in T(x)\}$ is called its *graph*, $D_T = \{x \in X : T(x) \neq \emptyset\}$ is called its *domain* and the set $R_T = \{x^* \in X^*;$ there exists $x \in X$ such that $x^* \in T(x)\}$ is called its *range*. The operator T is called *monotone* if $\langle x^* - y^*, x - y \rangle \geq 0$ for all $(x, x^*), (y, y^*) \in G(T)$; T is called *maximally monotone* if there exists no monotone operator S such that $G(S)$ strictly contains $G(T)$. A pair $(x, x^*) \in X \times X^*$ is called *monotonically related to* the monotone operator T if $\langle x^* - z^*, x - z \rangle \geq 0$ for any $(z, z^*) \in G(T)$. Clearly the (point wise) sum of two monotone operators is monotone. However the sum of two maximally monotone operators is not necessarily maximally monotone. In reflexive Banach spaces Rockafellar [7] proved that if T and S are maximally monotone operators satisfying the following constraint qualification

$$(*) \qquad D_T \cap \operatorname{int}(D_S) \neq \emptyset$$

then their sum is maximally monotone. In the context of reflexive Banach spaces, this result was generalized by weakening $(*)$ (see [8] for references). In the case of non reflexive Banach spaces the situation is more complicated and only partial results were obtained (see [2], [11], [12], [13] and for more references [8]). Before we can state our result, we need to recall some more definitions.

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Given a closed convex subset C of X we shall denote by N_C its *normal cone*, i.e. $N_C(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \le 0, y \in C\}$ if $x \in C$ and $N_C(x) = \emptyset$ if $x \notin C$. Recall that N_C is the subdifferential of I_C, the indicator function of C, i.e. $I_C(x) = 0$ if $x \in C$ and $I_C(x) = \infty$ if $x \notin C$. A maximally monotone operator T is called C_0-*maximally monotone* if $T + N_C$ is maximally monotone for any closed convex set C satisfying $D_T \cap \operatorname{int} C \neq \emptyset$. Our aim in this paper is to present a short and self contained (except some well known results) proof of the following:

Main Result 1.1. Let T and S be maximally monotone operators on X satisfying $(*)$.

(a) If $T + N_{\overline{D_S}}$ is C_0-maximally monotone then $T + S$ is maximally monotone.

(b) In particular, if T is C_0-maximally monotone then $T + S$ is maximally monotone (in fact C_0-maximally monotone) .

Remark 1.2 (1) Notice first that for (b) to be true for any S satisfying $(*)$ the operator T must be C_0-maximal (indeed, take $S = N_C$ for any $C \subseteq X$ closed convex with $D_T \cap \operatorname{int} C \neq \emptyset$).

(2) In [11] and [12] we proved the maximality of $T + S$ under more restrictive conditions, namely T must satisfy a certain regularity condition and $D_T \cap \overline{D_S} \subseteq D_S$. In the current paper we prove that C_0-maximally monotone operators satisfy that regularity condition (Proposition 3.10) and, using ideas from [14], show that the second condition is not necessary.

(3) A slightly more general result than (a) was proved in [14], Theorem 3.2. However, the proof given there is considerably longer and very computational. Assertion (b) is also mentioned in the comments at the end of that paper.

2. Notations, definitions, and known facts

Definition 2.1. (a) If T is a monotone operator on X, we shall denote by $C_0(T)$ the family of all closed convex subsets $C \subseteq X$ such that $D_T \cap \operatorname{int} C \neq \emptyset$.

(b) A monotone operator $T : X \rightrightarrows X^*$ is called C_0-*maximal* if $T + N_C$ is a maximally monotone operator for any $C \in C_0(T)$.

(c) A monotone operator $T : X \rightrightarrows X^*$ is of *type (FPV)* if for any open convex subset $C \subseteq X$ such that $D_T \cap C \neq \emptyset$ and any $(x, x^*) \in C \times X^*$ we have

if $\langle y^* - x^*, y - x\rangle \ge 0$ for any $y \in C$ and $y^* \in T(y)$ then $x \in D_T$ and $x^* \in T(x)$.

(d) For a monotone operator $T : X \rightrightarrows X^*$, $x \in X$ and $x^* \in X^*$ let

$$L(x, x^*, T) = \max\left\{0, \sup\left\{\frac{\langle y^* - x^*, x - y\rangle}{\|x - y\|}, \, y \neq x, (y, y^*) \in G(T)\right\}\right\}.$$

It is easy to check that if $L(x, x^*, T) < \infty$ then $L(x, y^*, T) < \infty$ for any $y^* \in X^*$. A maximally monotone operator T is called *regular* if $L(x, x^*, T) < \infty$ implies that $x \in D_T$.

It is easy the check (see [12]) that the following is true:

Lemma 2.2. *If T is C_0-maximal, $C_1 \in C_0(T)$ and $C_2 \in C_0(T + N_{C_1})$ then we have $C_1 \cap C_2 \in C_0(T)$ and $T + N_{C_1}$ is C_0-maximal.*

The next result is also quite well known (see [8], Theorem 44.1).

Proposition 2.3. *A C_0-maximally monotone operator is of type FPV.*

Definition 2.4. The *Fitzpatrick function* associated with the operator $T : X \rightrightarrows X^*$ is

$$\varphi_T(x, x^*) = \sup\{\langle y^* - x^*, x - y\rangle + \langle x^*, x\rangle; (y, y^*) \in G(T)\}, (x, x^*) \in X \times X^*.$$

As an immediate consequences of the definition we have

Corollary 2.5. *Let $T : X \rightrightarrows X^*$ be monotone and $(x, x^*) \in X \times X^*$.*
(a) *The pair (x, x^*) is monotonically related to T if and only if $\varphi_T(x, x^*) \leq \langle x^*, x\rangle$.*
(b) *If $x \in D_T$ then $\varphi_T(x, x^*) \geq \langle x^*, x\rangle$.*

The following result is well known (see [8], Theorem 44.2).

Theorem 2.6. *If T is a maximally monotone operator of type FPV then*
(a) $\overline{\pi_1(\operatorname{dom}\varphi_T)} = \overline{D_T}$.
(b) $\overline{D}_T$ *is convex.*

The next result is a particular case of a more general one (see [13] Theorem 3.4 or [8] Theorem 24.1(b)).

Theorem 2.7. *Let T and S be maximally monotone operators such that we have $D_T \cap \operatorname{int}(D_S) \neq \emptyset$. If $\varphi_{T+S}(x, x^*) \geq \langle x^*, x\rangle$ for all $(x, x^*) \in X \times X^*$ then $T + S$ is maximal.*

Notation. If $x \in X$ and $D \subset X$ we shall denote by $K_{x,D}$ *the cone with vertex x and base D*, i.e. $K_{x,D} = \{z \in X : z = (1 - t)x + ty, 0 \leq t \leq 1, y \in D\}$. Let also $K^*_{x,D} = K_{x,D} \setminus \{x\}$.

3. Preliminary results

Lemma 3.1. *Assume that T and S are monotone operators with D_T and R_S both bounded and $D_T \subseteq D_S$. Then $\pi_1(\operatorname{dom}\varphi_{T+S}) = \pi_1(\operatorname{dom}\varphi_T)$.*

Proof. Choose $M \geq \operatorname{diam}(D_T)$, $\operatorname{diam}(R_S)$. Let $x \in X$, $x^* \in X^*$, $y \in D_T$, $y^* \in T(y), z^* \in S(y)$. Since

$$-M^2 \leq -\|z^*\|\|x - y\| \leq \langle z^*, x - y\rangle \leq \|z^*\|\|x - y\| \leq M^2$$

it follows that

$$\begin{aligned}\langle y^* - x^*, x - y\rangle + \langle x^*, x\rangle - M^2 &\leq \langle y^* + z^* - x^*, x - y\rangle + \langle x^*, x\rangle \\ &\leq \langle y^* - x^*, x - y\rangle + \langle x^*, x\rangle + M^2\end{aligned}$$

and therefore $\varphi_T(x, x^*) - M^2 \leq \varphi_{T+S}(x, x^*) \leq \varphi_T(x, x^*) + M^2$.

Thus $\varphi_{T+S}(x, x^*)$ is finite if and only if $\varphi_T(x, x^*)$ is finite, proving the lemma. □

Lemma 3.2. *Assume that T and S are monotone operators such that $D_T \cap \operatorname{int}(D_S) \neq \emptyset$, $T + N_{\overline{D_S}}$ is of type FPV, and S is maximally monotone. Then $\varphi_{T+S}(x, x^*) \geq \langle x, x^*\rangle$ for any $(x, x^*) \in \overline{D}_T \times X^*$.*

Proof. Since we have $T + S = T + S + N_{\overline{D_S}} = T + N_{\overline{D_S}} + S$ we can replace T with $T + N_{\overline{D_S}}$ and assume that T is of type FPV and $D_T \subseteq \overline{D}_S$. First we shall prove that $\varphi_{T+S}(tx, tx^*) \geq \langle tx, tx^* \rangle$ for any $t \in (0, 1)$. We can assume that $0 \in D_T \cap D_S$. Since $\overline{D}_T$ and $\overline{D}_S$ are convex, $tx \in \overline{D}_T \cap \operatorname{int}(D_S)$ for any $t \in (0, 1)$. If $tx \in D_T$, then $\varphi_{T+S}(tx, tx^*) \geq \langle tx, tx^* \rangle$ from the definition of φ_{T+S}. Assume now that $tx \notin D_T$. Since $tx \in \operatorname{int}(D_S)$, there is an open ball $B = B(tx, r) \subset D_S$ and a positive number M such that $\|y^*\| < M$ for any $y \in D_S \cap B$ and $y^* \in S(y)$. Let n be a positive integer larger than r. Since $tx \notin D_T$, $B(tx, r/n) \cap D_T \neq \emptyset$ and T is of type FPV there exists $y_n \in D_T \cap B(tx, r/n)$ and $y_n^* \in T(y_n)$ such that $\langle y_n^* - tx^*, tx - y_n \rangle > 0$. Using this, for any $z^* \in S(y_n)$ and the fact tat $\|y^*\| \leq M$, we have

$$\begin{aligned}\varphi_{T+S}(tx, tx^*) &\geq \langle y_n^* + z^* - tx^*, tx - y_n \rangle + \langle tx^*, tx \rangle \\ &= \langle y_n^* - tx^*, tx - y_n \rangle + \langle z^*, tx - y_n \rangle + \langle tx^*, tx \rangle \\ &\geq 0 - M\|tx - y_n\| + \langle tx^*, tx \rangle.\end{aligned}$$

Since y_n converges to tx, this proves that $\varphi_{T+S}(tx, tx^*) \geq \langle tx, tx^* \rangle$ for any element $t \in (0, 1)$. From the convexity of φ_{T+S} it follows that

$$t\varphi_{T+S}(x, x^*) \geq \varphi_{T+S}(tx, tx^*) \geq \langle tx, tx^* \rangle = t^2 \langle x^*, x \rangle \quad \text{for any} \quad t \in (0, 1).$$

Letting $t \to 1$ we obtain that $\varphi_{T+S}(x, x^*) \geq \langle x, x^* \rangle$. □

Remark 3.3. The above proof is the second part of the proof of a more general result ([see [1] Proposition 3.1; see also our Proposition 3.12 below).

Lemma 3.4. *Assume that T and S are maximally monotone operators such that D_T and R_S are bounded, $D_T \cap \operatorname{int}(D_S) \neq \emptyset$, $D_T \subseteq D_S$ and T is of type FPV. Then $\varphi_{T+S}(x, x^*) \geq \langle x^*, x \rangle$ for all $x \in X, x^* \in X^*$.*

Proof. Clearly $D_T = D_{T+S}$ and $\overline{D_T} = \overline{D_{T+S}}$. We shall consider two cases.

Case 1: $x \notin \overline{D_T} = \overline{D_{T+S}}$. Then, following Theorem 2.6 and Lemma 3.1, we have $x \notin \pi_1(\operatorname{dom} \varphi_T) = \pi_1(\operatorname{dom} \varphi_{T+S})$ and the assertion follows.

Case 2: $x \in \overline{D_T}$. Since $T = T + N_{\overline{D_S}}$, the assertion follows from Lemma 3.2. □

Corollary 3.5. *Assume that T is a maximally monotone operator of type FPV with D_T bounded and S is a maximally monotone operator with R_S bounded and $D_S = X$. Then $T + S$ is maximally monotone.*

Proof. This is a straightforward consequence of Lemma 3.4 and Theorem 2.7. □

Remark 3.6. As a matter of fact the assumption that $D_S = X$ is not necessary; it follows from [3].

Lemma 3.7. *For any $r > 0$ let B_r be the closed ball in X centered at 0 and with radius r. Let T be a monotone operator and assume that $T + N_{B_r}$ is maximally monotone for any r sufficiently large. Then T is maximally monotone.*

Proof. Assume that (x, x^*) is monotonically related to T and choose $r > \|x\|$ large enough such that $T + N_{B_r}$ is maximally monotone. It is straightforward to check that (x, x^*) is monotonically related to $T + N_{B_r}$ and thus $x^* \in (T + N_{B_r})(x)$. Since $N_{B_r}(x) = \{0\}$, the lemma is proved. $\square$

Proposition 3.8. *Assume that T is a C_0-maximally monotone operator and S is a maximally monotone operator with R_S bounded and $D_S = X$. Then $T + S$ is maximally monotone.*

Proof. In view of Lemma 3.7 it is enough to prove that $T + S + N_{B_r}$ is maximally monotone for any r sufficiently large. Let $r > 0$ be such that D_T intersects the interior of B_r. Then $T + S + N_{B_r} = (T + N_{B_r}) + S$. Since $T + N_{B_r}$ is C_0-maximally monotone (thus of type FPV) and has bounded domain it follows from Corollary 3.5 that $T + S + N_{B_r} = T + N_{B_r} + S$ is maximally monotone. Lemma 3.7 implies now that $T + S$ is maximally monotone. $\square$

Lemma 3.9. *Assume that $S : X \rightrightarrows X^*$ is maximally monotone and* $\operatorname{int}(D_S) \neq \emptyset$. *Then for any $s_0 \in \operatorname{int}(D_S)$ there exist $\rho, M > 0$ such that for any $x \in X$ we have*

$$\langle s^*, s - x\rangle \leq M\|s - x\|, \text{ for any } s \in D_S \cap K_{x,B(s_0,\rho)},\ s^* \in S(s).$$

Proof. Let $s_0 \in \operatorname{int}(D_T)$. It is well known (see for example [6] or [8, Theorem 26.1] and the references therein) that T is locally bounded at such a point and thus there exists $\rho > 0$ and $M > 0$ such that $B[s_0, \rho] \subset D_S$ and for any $u \in B(s_0, \rho)$ and $u^* \in S(u)$ we have $\|u^*\| \leq M$.

Let $x \in X$, $s \in K_{x,B(s_0,\rho)} \cap D_S$, $s \neq x$ and $s^* \in S(s)$.

Then there exists $u \in B(s_0, \rho)$ and $\varepsilon \in (0, 1]$ such that $s = (1 - \varepsilon)x + \varepsilon u$. If $s = u$ then $\|s^*\| \leq M$ and $\langle s^*, s - x\rangle \leq M\|s - x\|$. If $s \neq u$ then $\varepsilon \neq 1$. Choose $u^* \in S(u)$. An easy computation shows that $s - x = \dfrac{\varepsilon}{1-\varepsilon}(u - s)$ and therefore

$$\frac{\langle s^*, s - x\rangle}{\|s - x\|} = \frac{\langle s^*, u - s\rangle}{\|u - s\|} = \frac{\langle s^* - u^*, u - s\rangle}{\|u - s\|} + \frac{\langle u^*, u - s\rangle}{\|u - s\|} \leq 0 + M$$

proving the lemma. $\square$

Proposition 3.10. *Any C_0-maximally monotone operator is regular.*

Proof. Let T be a C_0-maximally monotone operator, $x \in X, x^* \in X^*$ and assume that $\lambda = L(x, x^*, T)$ is finite. We have to show that $x \in D_T$.

First, consider the function $g_{\lambda,x}(y) = \lambda\|x - y\|, y \in X$. It is known (and easy to check) that

$$\partial g_{\lambda,x}(y) = \{z^* \in X^*; \|z^*\| \leq \lambda \text{ and } \langle z^*, y - x\rangle = \lambda\|y - x\|\}.$$

Notice that

$$\begin{aligned}
&L(x, x^*, T + \partial g_{\lambda,x})\\
&\quad = \max\left\{0, \sup\left\{\frac{\langle y^* + z^* - x^*, x - y\rangle}{\|x - y\|}, y^* \in T(y), z^* \in \partial g_{\lambda,x}(y), y \neq x\right\}\right\}\\
&\quad = \max\left\{0, \sup\left\{\frac{\langle y^* - x^*, x - y\rangle}{\|x - y\|} - \lambda, y^* \in T(y), y \neq x\right\}\right\}\\
&\quad = L(x, x^*, T) - \lambda = 0
\end{aligned}$$

This means that (x, x^*) is monotonically related to $T + \partial g_{\lambda,x}$ which is maximal by Proposition 3.8. Thus $x \in D_T$.

Assume now that $S : X \rightrightarrows X^*$ is a maximally monotone operator such that we have $0 \in \operatorname{int}(D_S)$. Then S is locally bounded at 0 and therefore there exists positive numbers r and M such that $B = B[0, r] \subseteq \operatorname{int}(D_S)$ and $\|s^*\| \leq M$ for any $s^* \in S(s)$ and $s \in B$. For $x \in X$ let $K = K_{x,B}$.

Lemma 3.11. *Let S be as above and let $T : X \rightrightarrows X^*$ be a maximally monotone operators such that $0 \in D_T \cap \operatorname{int}(D_S)$ and $T + N_{\overline{D_S}}$ is C_0-maximal. Assume that for some $(x, x^*) \in X \times X^*$ we have $\varphi_{T+S}(x, x^*) \leq \langle x, x^* \rangle$ (i.e. (x, x^*) is monotonically related to $T + S$) and $D_T \cap K^*_{x,B} \subseteq D_S$. Then $x \in D_T$ and $\varphi_{T+S}(x, x^*) = \langle x, x^* \rangle$.*

Proof. Let $K^* = K \setminus \{x\}$. Let $y \in D_T \cap K^* \subseteq D_S$, $y^* \in T(y)$, $i^* \in N_{\overline{D_S}}(y)$, $j^* \in N_K(y)$ and $s^* \in S(y)$. Then $u^* = s^* + i^* \in S(y)$, $\langle j^*, x - y \rangle \leq 0$ and, from Lemma 3.9, $\langle u^*, y - x \rangle \leq M\|x - y\|$. It follows that

$$\begin{aligned}\langle y^* + i^* + j^* - x^*, x - y \rangle &= \langle y^* + u^* - x^*, x - y \rangle + \langle j^*, x - y \rangle + \langle s^*, y - x \rangle \\ &\leq M\|x - y\|\end{aligned}$$

and therefore $L(x, x^*, T + N_{\overline{D_S}} + N_K) \leq M$. Since $T + N_{\overline{D_S}} + N_K$ is C_0-maximal (Lemma 2.2) it is regular (Proposition 3.10) and this shows that $x \in D_T \cap \overline{D_S} \cap K$ and thus $x \in D_T$. From Lemma 3.2 we obtain that $\varphi_{T+S}(x, x^*) = \langle x, x^* \rangle$.

Proposition 3.12. *Assume that T and S are maximally monotone operators such that $D_T \cap \operatorname{int}(D_S) \neq \emptyset$ and $T + N_{\overline{D_S}}$ is C_0-maximally monotone.*
Then $\varphi_{T+S}(x, x^*) \geq \langle x, x^* \rangle$ for any $(x, x^*) \in \overline{D}_S \times X^*$.

Proof. By contradiction, assume that we have $\varphi_{T+S}(x, x^*) < \langle x, x^* \rangle$ for some $(x, x^*) \in \overline{D}_S \times X^*$. Since $x \in \overline{D_S}$, with the notation in Lemma 3.11, it follows that $D_T \cap K^*_{x,B} \subseteq D_S$ and Lemma 3.11 shows that $\varphi_{T+S}(x, x^*) = \langle x, x^* \rangle$. This contradiction proves the Proposition. □

Lemma 3.13. (see [1, Fact 4.1]) *Assume that $S : X \rightrightarrows X^*$ is monotone, $(s, s^*) \in X \times X^*$ is monotonically related to S and $0 \in \operatorname{int}(D_S)$. Then there exist $\rho > 0$ and $M > 0$ such that*

$$\langle s^*, s \rangle \geq \rho\|s^*\| - M(\|s\| + \rho).$$

Proof. Since $0 \in \operatorname{int}(D_S)$, it is well known that there exit $\rho > 0$ and $M > 0$ such that $B[0, \rho] \subset D_S$ and $\|y^*\| \leq M$ for any $y \in B[0, \rho]$ and $y^* \in S(y)$. Let (y, y^*) be as before. We have:

$$\begin{aligned}\langle s^*, s \rangle &= \langle s^* - y^*, s - y \rangle + \langle s^*, y \rangle + \langle y^*, s - y \rangle \geq 0 + \langle s^*, y \rangle + \langle y^*, s - y \rangle \\ &\geq \langle s^*, y \rangle - \|y^*\|\,\|s - y\| \geq \langle s^*, y \rangle - \|y^*\|(\|s\| + \|y\|) \\ &\geq \langle s^*, y \rangle - M(\|s\| + \rho)\end{aligned}$$

Since this is true for any $y \in B[0, \rho]$ the lemma is proved.

4. Main result

Theorem 4.1. *Let T and S be a maximally monotone operators such that we have $D_T \cap \operatorname{int} D_S \neq \emptyset$ and $(T+N_{\overline{D_S}})$ is C_0-maximal. Then $T+S$ is a maximally monotone operator. In particular, if T is C_0-maximally monotone then $T+S$ is maximally monotone, in fact C_0-maximally monotone.*

Proof. We can and will assume that $0 \in D_T \cap \operatorname{int}(D_S)$ and $0 \in T(0)$. Thus

$$\langle t^*, t\rangle \geq 0 \quad \text{for any} \quad t^* \in T(t), \ t \in D_T. \tag{1}$$

Since $T+S = T+(S+N_{\overline{D_S}}) = (T+N_{\overline{D_S}})+S$ we can replace T with $T+N_{\overline{D_S}}$ and assume that $D_T \subseteq \overline{D_S}$ and T is C_0-maximally monotone. Assume that $T+S$ is not maximally monotone. This means that there exists $(x,x^*) \in X \times X^*$ such that $\varphi_{T+S}(x,x^*) < \langle x, x^*\rangle$. From Proposition 3.12 it follows that $x \notin \overline{D}_S$.

Let $\rho > 0$ and $M > 0$ be such that $B_1 = B[0,\rho] \subseteq D_S$ and $\|s^*\| \leq M$ for all $s^* \in S(s)$ and $s \in B_1$. Set $B_n = B[0,\rho/n]$ and let $K_n = K_{x,B_n}$ be the cone with vertex x and base B_n. Since $\varphi_{T+S}(x,x^*) < \langle x,x^*\rangle$, from Lemma 3.11

$$\text{there exists} \ \ z_n \in (D_T \cap K_n) \setminus D_S \subseteq (D_T \cap K_n) \setminus \operatorname{int}(D_S) \tag{2}$$

with $z_n = (1-\gamma_n)b_n + \gamma_n x$, $0 \leq \gamma_n \leq 1$. By replacing $\{\gamma_n\}$ with a subsequence we may assume that it converges to $\gamma \in [0,1]$ and thus z_n converges to a point

$$x_0 = \gamma x \in [0,x] \cap (\overline{D_T} \setminus \operatorname{int}(D_S)).$$

Clearly $x_0 \notin \operatorname{int}(D_S)$ and Proposition 3.12 implies that $x \notin \overline{D_S}$. Thus $0 < \gamma < 1$. Choose n such that $\|\gamma x - z_n\| < \rho(1-\gamma)/8 = \lambda$ and let $z = z_n$. Choose now $z^* \in T(z)$ and let

$$M_0 = \|z^*\|(2+2d+M)/\gamma, \quad \text{where } d = \operatorname{diam} K_1. \tag{3}$$

Since $z \in \overline{D_S} \setminus D_S$ it is known that

$$\lim_{n\to\infty} \inf\{\|s^*\| : s^* \in S(s), \ s \in B(z,1/n)\} = \infty$$

(see for example [14, Fact 2.16]). Choose $0 < \varepsilon < \lambda$ such that

$$\|s^*\| \geq \frac{3Md - \gamma M_0 - \gamma d\|x^*\|}{(1-\gamma)\rho} \quad \text{for any} \quad s \in B(z,\varepsilon) \cap D_S \quad \text{and} \quad s^* \in S(s). \tag{4}$$

Clearly either $2z \neq x_0$ or $3z \neq x_0$. Assume $2z \neq x_0$, the other case being similar. Choose $0 < \beta < \max\{1/3, \lambda, \varepsilon/\|2z - x_0\|\}$ and let $\alpha = 1 - 2\beta$. Let

$$v = \alpha z + \beta x_0 = \alpha z + \beta x_0 + \beta 0 \ \ (\text{of course } v \in \operatorname{int} D_S). \tag{5}$$

Then

$$z - v = (1-\alpha)z - \beta x_0 = 2\beta z - \beta x_0 = \beta(2z - x_0) \ \ \text{and} \ \ v \in B(z,\varepsilon). \tag{6}$$

Our aim is to show that there exist $u \in D_T \cap D_S \cap B(z,\varepsilon)$ and $u^* \in T(u)$ such that

$$\langle u^*, u - x\rangle \le M_0. \tag{7}$$

Notice that for any $z_1, z_2 \in K_1$ we have

$$\|z_1 - z_2\| \le d = \operatorname{diam} K_1. \tag{8}$$

We consider now two cases.

Case 1: $z^* \in T(v)$. Set $u = v$ and $u^* = z^*$. We have

$$\langle u^*, u - x\rangle = \langle z^*, v - x\rangle \le \|z^*\|\,\|v - x\| \le d\|z^*\| \le M_0$$

proving (7) in this case.

Case 2: $z^* \notin T(v)$. Choose $0 < \delta < \beta$ such that $B(v,\delta) \subset B(z,\varepsilon) \cap D_S$.
Since $0, z, x_0 \in \overline{D_T}$, which is convex it follows that we have $v \in \overline{D_T}$ and therefore $B(v,\delta) \cap D_T \neq \emptyset$. Since T is FPV and $z^* \notin T(v)$, there exist $u \in D_T \cap B(v,\delta)$ and $u^* \in T(u)$ such that

$$\langle u^* - z^*, v - u\rangle > 0 \ \text{ or } \ \langle u^*, u - v\rangle < \langle z^*, u - v\rangle.$$

Since $v = \alpha z + \beta x_0$ and $x_0 = \gamma x$ we have

$$\begin{aligned}
\langle u^*, u - x\rangle &= \langle u^*, u - v\rangle + \langle u^*, v - x\rangle \\
&\le \langle z^*, u - v\rangle + \langle u^*, \alpha z + \beta\gamma x - x\rangle \\
&= \langle z^*, u - v\rangle + \langle u^*, \alpha z - \alpha u\rangle + \langle u^*, \alpha u\rangle + \langle u^*, (\beta\gamma - 1)x\rangle \\
&= \langle z^*, u - v\rangle + \langle u^*, \alpha z - \alpha u\rangle + (\alpha + \beta\gamma - 1)\langle u^*, u\rangle + (1 - \beta\gamma)\langle u^*, u - x\rangle \\
&= \langle z^*, u - v\rangle + \alpha\langle u^*, z - u\rangle + \beta(\gamma - 2)\langle u^*, u\rangle + (1 - \beta\gamma)\langle u^*, u - x\rangle
\end{aligned}$$

and thus

$$\begin{aligned}
\beta\gamma\langle u^*, u - x\rangle &\le \langle z^*, u - v\rangle + \alpha\langle u^*, z - u\rangle + \beta(\gamma - 2)\langle u^*, u\rangle \quad (\text{use } \langle u^*, u\rangle \ge 0) \\
&\le \|z^*\|\|u - v\| + \alpha\langle z^*, u - z\rangle + 0 \\
&= \|z^*\|\|u - v\| + \alpha(\langle z^*, u - v\rangle + \langle z^*, v - z\rangle) \\
&\le \|z^*\|\,\|u - v\| + \alpha\|z^*\|(\|u - v\| + \|v - z\|) \\
&\qquad\qquad (\text{use } u \in B(v,\delta) \text{ and (6) and next } \delta \le \beta) \\
&\le \delta\|z^*\| + \alpha\|z^*\|(\delta + \beta\|2z - x_0\|) \\
&\le \beta\|z^*\| + \alpha\|z^*\|(\beta + 2d\beta) \le \beta\|z^*\|((2 + 2d)
\end{aligned}$$

It follows that

$$\langle u^*, u - x\rangle \le \|z^*\|((2 + 2d)/\gamma \le M_0$$

and thus (7) is true.

To prove the theorem (by contradiction) it is enough to show that

$$\langle s^* + u^* - x^*, x - u\rangle > 0 \ \text{ for some } s^* \in S^*(u).$$

First, from the choices of $z = z_n$, ε, δ, v and u, we have

$$\|\gamma x - u\| \le \|\gamma x - z\| + \|z - v\| + \|v - u\| \le \lambda + \varepsilon + \delta \le 3\lambda \le \rho(1-\gamma)/2.$$

Therefore, for $s^* \in S(u)$, we have (by Lemma 3.13)

$$\begin{aligned}\langle s^*, x - u\rangle &= \frac{(1-\gamma)\langle s^*, u\rangle}{\gamma} + \frac{\langle s^*, \gamma x - u\rangle}{\gamma}\\ &\ge \frac{(1-\gamma)(\rho\|s^*\| - M(\|u\| + \rho))}{\gamma} - \frac{\|s^*\|\rho(1-\gamma)}{2\gamma}\\ &= \frac{(1-\gamma)}{2\gamma}\big(\rho\|s^*\| - 4Md\big).\end{aligned}$$

Also, by (7),

$$\langle u^* - x^*, x - u\rangle \ge -M_0 - d\|x^*\|.$$

By combining the last two results and using the choice of ε we obtain

$$\langle s^* + u^* - x^*, x - u\rangle \ge \frac{(1-\gamma)(\rho\|s^*\| - 4Md)}{\gamma} - M_0 - d\|x^*\| > 0$$

thus proving the theorem. □

Remark 4.2. The choice of u and u^* in the proof above was inspired by [14]. However, our computations and other arguments are considerably shorter.

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