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On the Square Distance Function from a Manifold with Boundary

Giovanni Bellettini
Dip. di Ingegneria dell'Informazione e Scienze Matematiche, Università di Siena, Italy; and: International Centre for Theoretical Physics, Mathematics Section, Trieste, Italy
bellettini@diism.unisi.it

Alaa Elshorbagy
Area of Mathematical Analysis, Modelling and Applications, SISSA, Trieste, Italy; and: International Centre for Theoretical Physics, Mathematics Section, Trieste, Italy
alaa.elshorbagy@sissa.it

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We characterize arbitrary codimensional smooth manifolds $\mathcal{M}$ with boundary embedded in $\mathbb{R}^n$ using the square distance function from $\mathcal{M}$ and from its boundary. The results are localized in an open set.

Keywords: Square distance function, smooth manifolds with boundary, signed distance function.

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1. Introduction

It is well known that the smoothness of the boundary of a bounded open subset of $\mathbb{R}^n$ can be characterized using the signed distance function (see for instance [9, 11, 12, 13]). This characterization is useful for several purposes, in particular it is related to the Hamilton-Jacobi equations [6] and it can be used to face the mean curvature flow of a one-codimensional family of smooth embedded hypersurfaces without boundary [10].

For a compact smooth embedded manifold without boundary of arbitrary codimension, it turns out that the meaningful function to be considered is the square distance function: in [7] De Giorgi conjectured[1] that if E is a connected subset of an open set $\Omega \subseteq \mathbb{R}^n$ such that $E \cap \Omega = \overline{E} \cap \Omega$ and the $\frac{1}{2}$-square distance function from E,

$$\eta_E(x) := \frac{1}{2} \inf_{y \in E} |x-y|^2, \qquad x \in \mathbb{R}^n,$$

is smooth in a neighborhood of E, then E is an embedded smooth manifold[2] without boundary in Ω of codimension equal to rank $(\nabla^2 \eta_E)$. Such a conjecture has been proven in [3, 4] (see also [15] and [9]) and can be considered as one of the motivations of this paper.

[1] If $\mathcal{M}$ is a compact smooth embedded manifold without boundary then the square distance function $\eta_{\mathcal{M}}$ is smooth in a suitable tubular neighborhood of $\mathcal{M}$, see Theorem 2.2.

[2] See Theorem 2.3.

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Investigations on arbitrary codimensional mean curvature flow lead De Giorgi [8] to further express the motion using the Laplacian of the gradient of the square distance function from the evolving manifolds, and also to describe the flow passing to a level set formulation: we refer to [3] for more details, and to [5, 16, 2] for further applications.

In this paper we want to characterize a smooth arbitrary codimensional manifold *with boundary* embedded in $\mathbb{R}^n$, using the distance functions. We start our discussion on the smoothness of the distance function from a manifold with boundary with a simple observation. Let E be a smooth compact curve in $\mathbb{R}^n$ with two end points (like the ones in Fig. 3.1 for $n = 2$ or in Fig. 4.1 for $n = 3$): then η_E turns out to be smooth in a sufficiently small neighborhood of E, excluding portions of a smooth hypersurface orthogonal to the boundary of E (the two dashed segments in Fig. 3.1, and the two disks in Fig. 4.1)[3]. This suggests that we have to exclude the boundary and possibly some portions of a hypersurface containing it, if we are hoping to get some sort of regularity for the square distance function from a manifold $\mathcal{M}$ with boundary. In fact, in Propositions 4.2(2) and 4.4(2) we show that, in general, $\eta_{\mathcal{M}}$ is smooth in a neighborhood of $\mathcal{M}$ out of a suitable hypersurface containing the boundary.

Supposing $\overline{\mathcal{M}} = \mathcal{M}$ is a smooth manifold with boundary, roughly speaking $\mathcal{M}$ is the union of two sets: the relative interior $\mathcal{M}^\circ$ (a relatively open subset of $\mathcal{M}$) and the boundary $\partial\mathcal{M}$ (a smooth submanifold of $\mathcal{M}$ of codimension one so that $\mathcal{M}$ lies locally "on one side" of $\partial\mathcal{M}$), joined smoothly; in particular $\mathcal{M}$ is contained in the relative interior of a larger smooth manifold of the same dimension.

We want to mimic the above properties for a pair of subsets of $\mathbb{R}^n$, making use only of the distance functions and their regularity properties. Therefore, let E and L be two subsets of $\mathbb{R}^n$ and $\Omega \subseteq \mathbb{R}^n$ be an open set. We want to isolate a set of necessary and sufficient conditions to be satisfied by the signed distance function and the square distance function from E and from L so that $E \cup L$ is a smooth manifold with boundary L in Ω. Our main Definition 3.2 reformulate the above properties as follows. We say that $E \cup L$ is a smooth manifold with boundary *in the sense of distance functions*, and we write $(E, L) \in D_hB\mathcal{C}^k(\Omega)$ (where h stands for the dimension of E and k for its smoothness degree), if:

- $\overline{L} \cap \Omega = L \cap \Omega$ and η_L is smooth in a neighborhood of L in Ω: this guarantees the smoothness of L;
- $\overline{E} \cap (\Omega \setminus L) = E \cap (\Omega \setminus L)$ and η_E is smooth in a neighborhood of E in $\Omega \setminus L$: this guarantees the smoothness of E in $\Omega \setminus L$;
- all points of L are limits of sequences of points of $E \setminus L$;
- there is a neighborhood B of $E \setminus L$ in Ω, $B \cap L = \emptyset$, such that the signed distance function d_B from B (negative in B) is smooth in a neighborhood A of L: this guarantees the smoothness of the boundary of B in A. Such a boundary is, roughly, represented by the two dashed segments in Fig. 3.1 and the two disks in Fig. 4.1. Hence the set $E \setminus L$ must lie on one side of L. In particular points of L do not belong to the relative interior of $E \cup L$, see Fig. 3.3;

[3] We can even consider the case $n = 1$: take a bounded closed interval $E = [a, b] \subset R$. Then $\eta_E \in \mathcal{C}^{1,1}$ but not $\mathcal{C}^2$ in any neighborhood of E in $\mathbb{R}$; however, η_E is smooth in $\mathbb{R} \setminus \{a, b\}$.

- there exists a smooth extension $\widehat{\eta}$ of η_E in an open neighborhood of $\overline{B} \cap A$ such that $\mathrm{rank}(\nabla^2 \widehat{\eta}(x)) = n - h$ on points $x \in L$: in particular this ensures that E and L join smoothly.

The main results of this paper are Theorems 4.1 and 5.1, where we show that Definition 3.2 is equivalent to the classical definition of smooth manifold embedded in $\mathbb{R}^n$ with boundary. The results are valid in any codimension and localized in an open set. We remark that localization of Definition 2.4 (on which Definition 3.2 is based) in an open set is necessary: for instance, even in the simplest case $\Omega = \mathbb{R}^n$ in the above list, the regularity on η_E is required only in $\mathbb{R}^n \setminus L$, which is an open set.

The content of the paper is the following. In Section 2 we introduce the class $D_h\mathcal{C}^k(\Omega)$, of h-dimensional embedded $\mathcal{C}^k$-manifolds without boundary in Ω in the sense of distance functions (Definition 2.4). After quoting some known results, we recall the correspondence between the classical definition of manifolds without boundary and sets in $D_h\mathcal{C}^k(\Omega)$ (Remark 2.5).

In Definition 3.2 we introduce the class $D_h\, B\, \mathcal{C}^k(\Omega)$; in Section 3 we illustrate the motivations behind this definition through several observations (Remark 3.3) and examples.

In Section 4 we prove our first main result (Theorem 4.1) showing that h-dimensional embedded $\mathcal{C}^k$-manifolds with boundary in Ω are elements of $D_hB\,\mathcal{C}^k(\Omega)$.

In Section 5 we prove our second main result[4] (Theorem 5.1), showing that sets in $D_hB\,\mathcal{C}^k(\Omega)$ are h-dimensional embedded $\mathcal{C}^{k-1}$-manifolds with boundary.

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2. Manifolds without boundary and distance functions

In this section we recall the notion of smooth (resp. analytic) manifold without boundary of arbitrary codimension, using the distance function, and the relation with the classical definition of smooth manifold. In what follows $\mathbb{N}$ stands for the set of positive natural numbers, and $n \in \mathbb{N}$.

Given $E \subseteq \mathbb{R}^n$, we set

$$d_E(x) := \mathrm{dist}(x, E) - \mathrm{dist}(x, \mathbb{R}^n \setminus E), \qquad x \in \mathbb{R}^n,$$

and

$$\eta_E(x) := \frac{1}{2}(\mathrm{dist}(x, E))^2, \qquad x \in \mathbb{R}^n,$$

where $\mathrm{dist}(x, E) := \inf_{y \in E} |x - y|$ and by convention $\inf \emptyset := +\infty$. Note that $\mathrm{dist}(\cdot, E) = \mathrm{dist}(\cdot, \overline{E})$, where $\overline{E}$ denotes the closure of E in $\mathbb{R}^n$, and d_E is a one-Lipschitz function. If E has empty interior then $d_E(\cdot) = \mathrm{dist}(\cdot, E)$.

[4] In the C^∞ or analytic case, this is the converse of Theorem 4.1.

If $\rho > 0$, we set $E^+_\rho := \{\xi \in \mathbb{R}^n : \mathrm{dist}(\xi, E) < \rho\}$, and if $\rho : E \to (0, +\infty]$ is a function, we set

$$E^+_{\rho(\cdot)} := \bigcup_{x\in E} \{\xi \in \mathbb{R}^n : |\xi - x| < \rho(x)\}.$$

We denote by $\mathcal{C}^\omega(\Omega)$ the class of real analytic functions in the open set $\Omega \subseteq \mathbb{R}^n$ and by $B_\rho(x)$ (resp. $B^h_\rho(x)$) the open ball of $\mathbb{R}^n$ (resp. of $\mathbb{R}^h, h < n$) centered at x of radius $\rho > 0$.

Let us recall the definition of the class of h-dimensional embedded $\mathcal{C}^k$-manifolds[5] without boundary in a nonempty open set $\Omega \subset \mathbb{R}^n$ (see for instance [17, 7]).

Definition 2.1. (Smooth embedded manifold without boundary) Let us take some $k \in \mathbb{N} \cup \{\infty, \omega\}$ and $h \in \{1, \dots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. We say that $\Gamma \subset \mathbb{R}^n$ is an *h-dimensional embedded $\mathcal{C}^k$-manifold without boundary in Ω* if

$$\Gamma \cap \Omega = \overline{\Gamma} \cap \Omega, \tag{1}$$

and for all $x \in \Gamma \cap \Omega$ there exist an open set $R \subset \mathbb{R}^n$, an open set $G \subset \mathbb{R}^h$, and maps $\phi \in \mathcal{C}^k(G; \mathbb{R}^n)$, $\psi \in \mathcal{C}^k(R; \mathbb{R}^h)$ such that

$$x \in R,\ \psi(\phi(y)) = y \quad \forall y \in G, \quad \text{and} \quad \Gamma \cap R = \{\phi(y) : y \in G\}. \tag{2}$$

Theorem 2.2. *Let $k \in \mathbb{N}$, $k \geq 2$, or $k \in \{\infty, \omega\}$ and $h \in \{1, \dots, n\}$. Let $\Gamma \subset \mathbb{R}^n$ be a compact h-dimensional embedded $\mathcal{C}^k$-manifold without boundary in $\mathbb{R}^n$. Then η_Γ is $\mathcal{C}^k$ in a tubular neighbourhood Γ^+_ρ of Γ and $\eta_\Gamma(x + p) = \frac{1}{2}|p|^2$ for any $x \in \Gamma$ and any p in the normal space to Γ at x, with $x + p \in \Gamma^+_\rho$. In particular the matrix $\nabla^2\eta_\Gamma(x)$ represents the orthogonal projection on the normal space to Γ at x.*

Proof. See [15, Section 1], [1, Theorem 2] and [9]. □

Theorem 2.2 is still valid if Γ is a h-dimensional embedded $\mathcal{C}^k$-manifold without boundary (not necessarily compact) in some open set Ω, provided that Γ^+_ρ is replaced by a neighborhood $\Gamma^+_{\rho(\cdot)} \subset \Omega$ of $\Gamma \cap \Omega$. Indeed, arguing as in the proof of Theorem 2.2 in [1], and using also [15, Section 1], it follows that that for any $x \in \Gamma \cap \Omega$ there exists $\rho(x) > 0$ such that $B_{\rho(x)}(x) \subset \Omega$, $\eta_\Gamma \in \mathcal{C}^k(B_{\rho(x)}(x))$, $\eta(x + p) = \frac{1}{2}|p|^2$ for any $p \in N_x\Gamma$ such that $x + p \in B_{\rho(x)}(x)$, and $\mathrm{rank}(\nabla^2\eta_\Gamma(x)) = n - h$. Defining $\Gamma^+_{\rho(\cdot)} := \cup_{x\in\Gamma} B_{\rho(x)}(x)$ we get the assertion.

Theorem 2.3. *Let $k \in \mathbb{N}$, $k \geq 2$ or $k \in \{\infty, \omega\}$. Let $A \subseteq \mathbb{R}^n$ be an open set, $E \subset \mathbb{R}^n$ a closed subset and suppose that $\eta_E \in \mathcal{C}^k(A)$. Then any connected component of $E \cap A$ is an embedded $\mathcal{C}^k$-manifold without boundary in A.*

Proof. See [15, Section 3], [4, Theorem 2.4]. □

Note that if $\mathrm{rank}(\nabla^2\eta_E(x)) = n - h$ for any x in a connected component of $E \cap A$, then such a connected component must have dimension h. Furthermore it is sufficient to have E closed in A to get the thesis of Theorem 2.3. Indeed it is enough to apply Theorem 2.3 to $\overline{E}$, hence any connected component of $\overline{E} \cap A$ $(= E \cap A)$ is a manifold without boundary.

[5] k stands for a positive natural number. We also consider the cases $k = +\infty$ or $k = \omega$ (analytic manifolds), in these cases $k - 1 = +\infty$ (resp. ω).

The above results suggest to introduce the following class of sets (which we shall consider as "the class of h-dimensional embedded $\mathcal{C}^k$-manifolds without boundary in Ω in the sense of distance functions"):

Definition 2.4. (The class $D_h\mathcal{C}^k(\Omega)$) Assume $k \in \mathbb{N}$, $k \geq 2$, or $k \in \{\infty, \omega\}$ and $h \in \{0, \dots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set and $E \subset \mathbb{R}^n$. We write $E \in D_h\mathcal{C}^k(\Omega)$ if

(i) $E \cap \Omega = \{x \in \Omega : \eta_E(x) = 0\}$;

(ii) there exists an open set $A \subseteq \Omega$ with $E \cap \Omega \subseteq A$ such that $\eta_E \in \mathcal{C}^k(A)$;

(iii) $\operatorname{rank}(\nabla^2 \eta_E(x)) = n - h$ for any $x \in E \cap \Omega$.

Remark 2.5. (I) If $E \in D_h\mathcal{C}^k(\Omega)$, $k \geq 2$ or $k \in \{\infty, \omega\}$ then E is closed in Ω and $E \cap \Omega$ is a $h-$dimensional embedded manifold of class $\mathcal{C}^k(\Omega)$ without boundary in Ω. Conversely, if $k \geq 2$ or $k \in \{\infty, \omega\}$ and Γ is a $h-$dimensional embedded manifold of class $\mathcal{C}^k(\Omega)$ without boundary in Ω then $\Gamma \in D_h\mathcal{C}^k(\Omega)$.

(II) $E = \emptyset \in D_h\mathcal{C}^k(\Omega)$ for any h, k and any open set Ω.

3. Manifolds with boundary and distance functions

We start this section by defining what we mean by an embedded h-dimensional $\mathcal{C}^k$-manifold in an open set with boundary in the sense of distance functions. But first we recall the classical definition (see for instance [17, 7]).

Definition 3.1. (Smooth embedded manifold with boundary) Let $k \in \mathbb{N} \cup \{\infty, \omega\}$ and $h \in \{1, \dots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. We say that $\mathcal{M} \subseteq \mathbb{R}^n$ is an *h-dimensional embedded manifold of class $\mathcal{C}^k$ with boundary of class $\mathcal{C}^k$ in* Ω (a h-dimensional $\mathcal{C}^k$-manifold in Ω with boundary, for short) if

$$\mathcal{M} \cap \Omega = \overline{\mathcal{M}} \cap \Omega$$

and for all $x \in \mathcal{M} \cap \Omega$ there exist an open set $R \subseteq \mathbb{R}^n$, an open set $G \subseteq \mathbb{R}^h$, maps $\phi \in \mathcal{C}^k(G; \mathbb{R}^n)$, $\psi \in \mathcal{C}^k(R; \mathbb{R}^h)$ and a point $z \in \mathbb{R}^h$ such that

$$x \in R, \quad \psi(\phi(y)) = y \quad \forall y \in G, \quad \mathcal{M} \cap R = \{\phi(y) : y \in G, \langle y, z \rangle \geq 0\}. \tag{3}$$

The boundary of $\mathcal{M}$ in Ω, denoted by $\partial_\Omega \mathcal{M}$ ($\partial \mathcal{M}$ when $\Omega = \mathbb{R}^n$), is the set of all points $\overline{x} \in \mathcal{M} \cap \Omega$ such that

$$\overline{x} = \phi(\overline{y}), \qquad \overline{y} \in G, \;\; \langle \overline{y}, z \rangle = 0, \;\; z \neq 0.$$

We denote by $\mathcal{M}^\circ = \mathcal{M} \setminus \partial_\Omega \mathcal{M}$ the relative interior of $\mathcal{M}$ and by $T_x\mathcal{M}$ (resp. $N_x\mathcal{M}$) the tangent (resp. normal) space to $\mathcal{M}$ at $x \in \mathcal{M}$.

Our main definition of smooth manifold with boundary using the distance functions reads as follows.

Definition 3.2. (The class $D_hB\mathcal{C}^k(\Omega)$) Let $k \in \mathbb{N}$, $k \geq 2$ or $k \in \{\infty, \omega\}$ and $h \in \{1, \dots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set, and $E, L \subseteq \mathbb{R}^n$. We write $(E, L) \in D_hB\mathcal{C}^k(\Omega)$ if:

(i) $L \in D_{h-1}\mathcal{C}^k(\Omega)$ and $E \in D_h\mathcal{C}^k(\Omega \setminus L)$;

(ii) $\operatorname{dist}(x, (E \setminus L) \cap \Omega) \leq \operatorname{dist}(x, L \cap \Omega)$ for any $x \in \mathbb{R}^n$;

(iii) if we define

$$B := \{x \in \Omega : \mathrm{dist}(x, (E \setminus L) \cap \Omega) < \mathrm{dist}(x, L \cap \Omega)\}, \tag{4}$$

then there exists an open set $A \subseteq \Omega$ with $L \cap \Omega \subset A$ such that $d_B \in \mathcal{C}^{k-1}(A)$;

(iv) there exist an open set $\widehat{C} \supset \{x \in A : d_B(x) \leq 0\}$ and a function $\widehat{\eta} \in \mathcal{C}^k(\widehat{C})$ such that $\widehat{\eta} = \eta_E$ on $\{x \in A : d_B(x) \leq 0\}$, and $\mathrm{rank}(\nabla^2 \widehat{\eta}) = n - h$ on $E \cup L$; in particular $\eta_E \in \mathcal{C}^k\big(\{x \in A : d_B(x) \leq 0\}\big)$.

Since Definition 3.2 is crucial, some comments are in order. Informally the set $L \cap \Omega$ should be considered as the "boundary" of $E \cup L$ in Ω, and by condition (i) it must satisfy Definition 2.4, with $h-1$ in place of h, while E must satisfy Definition 2.4 *not in the whole of* Ω, *but only in the open set* $\Omega \setminus L$ (remember that L is closed in Ω by condition (i) in Definition 2.4), see Fig. 3.1 for an elementary example.

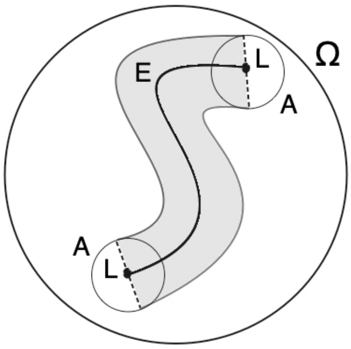

Figure 3.1: $\Omega = B_1(0)$ is an open disk in $\mathbb{R}^2$, $k = \infty$, $h = 1$, E is the bold curve (including the two endpoints), L consists of the two endpoints of E; $\mathrm{dist}(\cdot, (E \setminus L) \cap \Omega) = \mathrm{dist}(\cdot, E)$, B contains the shaded region, A is the union of the two small disks containing L. In Section 4 it will also be useful to consider $H := A \cap \partial B$, which in this example consists of two dashed segments containing L and normal to E. Finally, $\{x \in A : d_B(x) \leq 0\}$ consists of the grey areas inside the two disks, including the dashed segments.

To understand condition (iii), which is a regularity requirement on ∂B, we refer to Examples 3.5 and 3.8. We notice here that, if $h = 1$, then we can require $d_B \in \mathcal{C}^\infty(A)$.

Condition (iv) implies that $E \cap \Omega$ is smooth up to $L \cap \Omega$. Note carefully that, in general, η_E is not of class $\mathcal{C}^k$ in an open neighborhood of $E \cup L$. For instance, if $n = 1 = h$, $E = [-1, 1] \subset \mathbb{R}$, $L = \{\pm 1\}$ then η_E is of class $\mathcal{C}^{1,1}$ but not of class $\mathcal{C}^2$ in a neighborhood of L.

Note that $(E, \emptyset) \in D_h B\,\mathcal{C}^k(\Omega)$ if and only if $E \in D_h \mathcal{C}^k(\Omega)$. Moreover if $\overline{E} = E$, $\overline{L} = L$, and $E \cup L \subseteq \Omega$ then $(E, L) \in D_h B\,\mathcal{C}^k(\Omega)$ implies $(E, L) \in D_h B\,\mathcal{C}^k(\Omega')$ for every open set $\Omega' \supseteq \Omega$.

Remark 3.3. Suppose $k \in \mathbb{N}$, $k \geq 2$ or $k \in \{\infty, \omega\}$ and $(E, L) \in D_h B\,\mathcal{C}^k(\Omega)$.

(I) By Definition 3.2 (i) we have $L \in D_{h-1}\mathcal{C}^k(\Omega)$ hence, recalling Remark 2.5, we have that L is an embedded $(h-1)$-dimensional $\mathcal{C}^k$-manifold without boundary in Ω. Also, since $E \in D_h \mathcal{C}^k(\Omega \setminus L)$, E is an embedded h-dimensional $\mathcal{C}^k$-manifold without boundary in $\Omega \setminus L$.

(II) In Definition 3.2, we do not specify whether or not points of L belong to E. However, condition (ii) says that, if L is nonempty, all points of $L \cap \Omega$ are limits of sequences of points of $(E \setminus L) \cap \Omega$. Indeed, if $x \in L \cap \Omega$ then $\mathrm{dist}(x, L \cap \Omega) = 0$, hence (ii) implies

$$\mathrm{dist}(x, (E \setminus L) \cap \Omega) \leq 0, \quad \text{and so} \quad x \in \overline{(E \setminus L) \cap \Omega}.$$

(III) We have $\overline{(E \cup L)} \cap \Omega = (E \cup L) \cap \Omega$. Indeed, by Definition 3.2(i) we have

$$\overline{E \cup L} \cap \Omega = (\overline{E} \cup \overline{L}) \cap \Omega = (\overline{E} \cup (\Omega \setminus L)) \cup (\overline{L} \cap \Omega) = (E \cup L) \cap \Omega.$$

(IV) By (II) and (III) it follows $(E \cup L) \cap \Omega = \{x \in \Omega : \eta_E(x) = 0\}$, i.e.,

$$\overline{E} \cap \Omega = (E \cup L) \cap \Omega. \tag{5}$$

(V) Recalling (4), we have $\qquad (E \setminus L) \cap \Omega \subseteq B. \qquad (6)$

Indeed, let $x \in (E \setminus L) \cap \Omega$ so that $\mathrm{dist}(x, (E \setminus L) \cap \Omega) = 0$. Since L is closed in Ω we have $\mathrm{dist}(x, L \cap \Omega) > 0$ and therefore $\mathrm{dist}(x, (E \setminus L) \cap \Omega) < \mathrm{dist}(x, L \cap \Omega)$.

(VI) We have $\qquad L \cap \Omega \subset$ topological boundary of B. $\qquad (7)$

Indeed, from (II) and (III) it follows $L \cap \Omega \subseteq \overline{B}$, and $L \cap \Omega \cap B = \emptyset$.

(VII) In a neighborhood of $L \cap \Omega$, the topological boundary of B is an embedded hypersurface of class $\mathcal{C}^{k-1}$. Indeed since B is an open set and there exists an open set $A \supset L \cap \Omega$ such that $d_B \in \mathcal{C}^{k-1}(A)$, it follows from [15, Section 3], [9] that in A the topological boundary of B is a $\mathcal{C}^{k-1}$ hypersurface. Consistently with our notation in Definition 3.1, we indicate by $\partial_A B$ the boundary of B in A.

(VIII) For $h = n$ we have $\qquad B = (E \setminus L) \cap \Omega. \qquad (8)$

Indeed the inclusion $(E \setminus L) \cap \Omega \subseteq B$ is in (6). To show the converse inclusion we argue by contradiction. Assume that $B \not\subset (E \setminus L) \cap \Omega$. From (I) we know that $(E \setminus L) \cap \Omega$ is an open set and $L \cap \Omega$ is a hypersurface, moreover $L \cap \Omega$ is the topological boundary of $(E \setminus L) \cap \Omega$ in Ω from (II) and (III). Hence the intersection of B with the topological boundary of $\overline{(E \setminus L) \cap \Omega}$ is not empty; it follows $L \cap B \neq \emptyset$ which contradicts (7).

Example 3.4. We start from the simplest nontrivial case (Fig. 3.2, left): we take $n = 2$, $h = 1$, $k \in \{\infty, \omega\}$, $\Omega = \mathbb{R}^2$, $L = \{(\pm 1, 0)\}$, and $E = (-1, 1) \times \{0\}$ ($E = (-1, 1] \times \{0\}$ or $E = [-1, 1) \times \{0\}$ or $E = [-1, 1] \times \{0\}$ would not affect the discussion). In this case it is immediate to verify that the set B in condition (iii) equals $B = (-1, 1) \times \mathbb{R}$; the largest A fulfilling condition (iii) is $A = \mathbb{R}^2 \setminus \{x_1 = 0\}$, and $\{(x_1, x_2) \in A : d_B((x_1, x_2)) \leq 0\} = ([-1, 1] \times \mathbb{R}) \setminus \{x_1 = 0\}$. Finally, in order to fulfill (iv), it is sufficient to take $\widehat{\eta} = \eta_{\widehat{E}}$, where $\widehat{E} = (-1 - \delta, 1 + \delta) \times \{0\}$, for any $\delta > 0$ so that $\widehat{\eta} \in \mathcal{C}^k([-1, 1] \times \mathbb{R})$. Note that η_E is not of class $\mathcal{C}^2$ on $\{x = \pm 1\}$.

If we choose L to be only one point of the two points $\{(\pm 1, 0)\}$, say $L = \{(1, 0)\}$, then $E = (-1, 1) \times \{0\}$ is no longer closed in $\Omega \setminus L$ hence it does not belong to $D_1\mathcal{C}^k(\Omega \setminus L)$. On the other hand $E = [-1, 1) \times \{0\}$ is closed in $\Omega \setminus L$ but condition (ii) of Definition 2.4 (with Ω replaced by $\Omega \setminus L$) is not satisfied, hence E does not belong to $D_1\mathcal{C}^k(\Omega \setminus L)$.

Example 3.5. Take $n = 2$, $h = 1$, $k \in \{\infty, \omega\}$, $E = \{(\cos\theta, \sin\theta), \theta \in (\frac{5\pi}{4}, \frac{7\pi}{4})\}$, $\Omega = \mathbb{R}^2$, and $L = \{(\frac{\pm 1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})\}$, see Fig. 3.2. We have

$$B \cap (\mathbb{R} \times (-\infty, 0)) = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| < |x_2|, x_2 < 0\}.$$

A can be taken to be any open subset of $\mathbb{R} \times (-\infty, 0)$ containing L that does not contain the origin. Finally, taking $\widehat{\eta} := \eta_{\widehat{E}}$ where

$$\widehat{E} = \{(\cos\theta, \sin\theta),\ \theta \in (\frac{5\pi}{4} - \delta, \frac{7\pi}{4} + \delta)\},\ 0 < \delta < \frac{\pi}{4},$$

condition (iv) is fulfilled. Note that ∂B is smooth close to L, but not necessarily far from L (for instance at the origin).

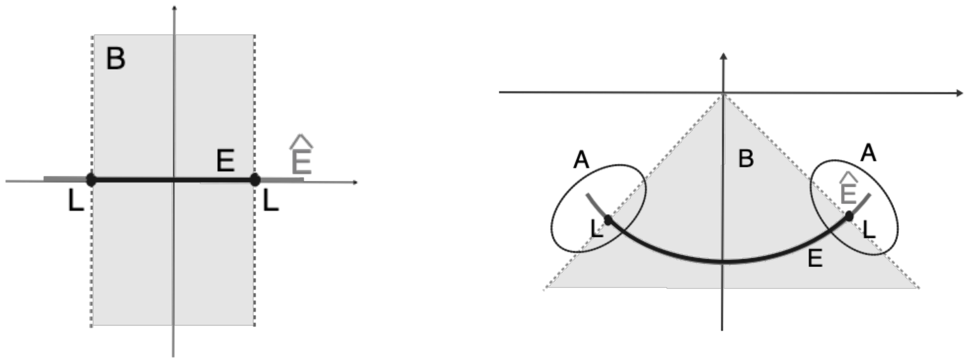

Figure 3.2: Left: E is a segment in $\mathbb{R}^2$. Right: E is an arc of a circle in $\mathbb{R}^2$ and L its two end points.

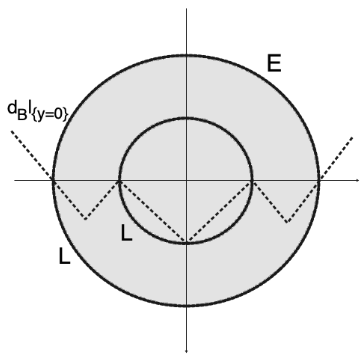

Figure 3.3: Example 3.8: L is the union of the two bold circles, one being included in the larger open disk, B is the grey region. Dashed segments: graph of the signed distance function d_B along $\{y = 0\}$.

Example 3.6. Take $n = h \geq 1$, $k \in \{\infty, \omega\}$, $E = \overline{B_1(0)}$, $\Omega = \mathbb{R}^n$, and $L = \partial B_1(0)$. Note that $\eta_E \in \mathcal{C}^1(B_{1+\epsilon}(0)) \setminus \mathcal{C}^2(B_{1+\epsilon}(0))$ for any $\epsilon > 0$.

We have $\overline{B_1(0)} \in D_n\mathcal{C}^k(\mathbb{R}^n \setminus \partial B_1(0))$: indeed E is closed in $\mathbb{R}^n \setminus \partial B_1(0)$, $\eta_E|_E = 0$ thus $\eta_E \in \mathcal{C}^k(E \setminus L)$ and $\mathrm{rank}(\nabla^2 \eta_E(x)) = 0$ for all $x \in E \setminus L$. Moreover $L \in D_{n-1}\mathcal{C}^k(\mathbb{R}^n)$ from Remark 2.5 (I). Hence condition (i) is fulfilled.

Condition (ii) is immediate and we also have $B = B_1(0)$ and $A = \mathbb{R}^n \setminus \{0\}$. Finally, $\widehat{\eta} = 0$ in $\mathbb{R}^n$ allows to check condition (iv).

Example 3.7. Take $n = 2$, $h = 1$, $k \in \{\infty, \omega\}$, $E = \mathbb{S}^1$ the unit circle centered at the origin, $\Omega = \mathbb{R}^2$, and $L = \emptyset$. Then condition (i) is immediate. Notice that $\mathrm{dist}(\cdot, L) \equiv +\infty$ hence $B = \mathbb{R}^2$, and $A = \emptyset$ so that also condition (iv) is trivially satisfied.

Example 3.8. Take $n = h = 2$, $E = \overline{B_2(0)}$, $\Omega = \mathbb{R}^2$, and $L = \partial B_1(0) \cup \partial B_2(0)$. Then (i) and (ii) of Definition 3.2 are fulfilled. We have $B = B_2(0) \setminus L$, moreover there is no $A \supset \partial B_1(0)$ such that $d_B \in \mathcal{C}^1(A)$ hence (iii) is not satisfied (note that $\eta_E = 0$ in $\overline{B_2(0)}$, *i.e.,* fulfilling (iv) also depends on the existence of A), see Fig. 3.3.

4. Smooth manifolds with boundary are in $D_h B\mathcal{C}^k(\Omega)$

In this section we show that smooth manifolds with boundary in the classical sense (Definition 3.1) are smooth manifolds with boundary in the sense of distance functions (Definition 3.2), more precisely:

Theorem 4.1. *Let $k \in \mathbb{N}$, $k \geq 2$ or $k \in \{\infty, \omega\}$, and $h \in \{1, \dots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set and $\mathcal{M} \subset \mathbb{R}^n$ be an embedded $\mathcal{C}^k$-manifold of dimension h with nonempty boundary in Ω. Then $(\mathcal{M}, \partial_\Omega \mathcal{M}) \in D_h B\mathcal{C}^k(\Omega)$.*

First we need the following result.

Proposition 4.2. *Let $k \in \mathbb{N}$, $k \geq 2$, or $k \in \{\infty, \omega\}$, and $h \in \{1, \dots, n\}$. Assume that $\mathcal{M} \subset \mathbb{R}^n$ is a compact embedded $\mathcal{C}^k$-manifold of dimension h with nonempty boundary in $\mathbb{R}^n$. Then there exists $\epsilon > 0$ such that, setting*

$$H_\epsilon := \bigcup_{x \in \partial\mathcal{M}} B_\epsilon(x) \cap N_x\mathcal{M}, \tag{9}$$

the following properties hold:

(1) *H_ϵ is an embedded $\mathcal{C}^{k-1}$-hypersurface without boundary in $\mathcal{M}_\epsilon^+$, and $N_x H_\epsilon \subseteq T_x\mathcal{M}$ for any $x \in \partial\mathcal{M}$;*

(2) *$\eta_\mathcal{M} \in \mathcal{C}^k(\mathcal{M}_\epsilon^+ \setminus H_\epsilon)$.*

Proof. Since we can work separately on each connected component of $\mathcal{M}$, from now on we suppose that $\mathcal{M}$ is connected. Suppose first $h = n$. In this case the interior of $\mathcal{M}$ is a nonempty open set with $\mathcal{C}^k$-boundary, and [9, 12, 13] if $\epsilon > 0$ is sufficiently small, $d_\mathcal{M}$ is of class $\mathcal{C}^k$ in the tubular neighborhood $(\partial\mathcal{M})_\epsilon^+$ of $\partial\mathcal{M}$. From (9) we have $H_\epsilon = \partial\mathcal{M}$, since $N_x\mathcal{M} = \{x\}$. Then (1) holds because $T_x\mathcal{M} = \{x\} \times \mathbb{R}^n$ for any $x \in \partial\mathcal{M}$. Moreover $\mathrm{dist}(\cdot, \mathcal{M}) \in \mathcal{C}^k(\mathcal{M}_\epsilon^+ \setminus H_\epsilon)$, since $\mathrm{dist}(\cdot, \mathcal{M}) = 0$ in the interior of $\mathcal{M}$ and $\mathrm{dist}(\cdot, \mathcal{M}) = d_\mathcal{M}(\cdot)$ in $\mathcal{M}_\epsilon^+ \setminus \mathcal{M}$, hence also (2) follows.

Now suppose $h \in \{1, \dots, n-1\}$. We divide the proof into 3 steps.

Step 1. There exists $\epsilon_1 > 0$ such that

$$\eta_\mathcal{M} \in \mathcal{C}^k(V_{\epsilon_1}), \tag{10}$$

where V_{ϵ_1} is the neighborhood of the relative interior $\mathcal{M}^\circ$ of $\mathcal{M}$ defined as

$$V_{\epsilon_1} := \bigcup_{x \in \mathcal{M}^\circ} B_{\epsilon_1}(x) \cap N_x\mathcal{M}^\circ. \tag{11}$$

The initial part of the proof of this step is rather standard, see for instance [1].

Take any $x_\circ \in \mathcal{M}^\circ$. Since $\mathcal{M}$ is a smooth manifold embedded in $\mathbb{R}^n$, there exists $\rho = \rho(x_\circ) > 0$ such that

$$B_\rho(x_\circ) \cap \mathcal{M} = B_\rho(x_\circ) \cap \mathcal{M}^\circ, \tag{12}$$

and there are smooth orthonormal vector fields $\nu^1(x), \ldots, \nu^{n-h}(x)$ spanning $N_x\mathcal{M}^\circ$ for any $x \in B_\rho(x_\circ) \cap \mathcal{M}^\circ$. Consider the function

$$\widetilde{\Phi} = \widetilde{\Phi}_{\mathcal{M}^\circ} : (B_\rho(x_\circ) \cap \mathcal{M}^\circ) \times \mathbb{R}^{n-h} \longrightarrow \mathbb{R}^n, \quad \widetilde{\Phi}(x, \alpha) := x + \sum_{i=1}^{n-h} \alpha_i \nu^i(x), \tag{13}$$

where $\alpha = (\alpha_1, \ldots, \alpha_{n-h}) \in \mathbb{R}^{n-h}$. Let $G \subset \mathbb{R}^h$ be an open set and assume that $f : G \to B_\rho(x_\circ) \cap \mathcal{M}^\circ$ is a local parametrization of $\mathcal{M}^\circ$ with $f(y_\circ) = x_\circ$, $y_\circ \in G$. Then $\widetilde{\Phi}$ in local coordinates becomes

$$\Phi : G \times \mathbb{R}^{n-h} \to \mathbb{R}^n, \quad \Phi(y, \alpha) := \widetilde{\Phi}(f(y), \alpha) = f(y) + \sum_{i=1}^{n-h} \alpha_i \nu^i(f(y)).$$

Clearly Φ is $\mathcal{C}^{k-1}$ and therefore $d\Phi_{(y_\circ,0)}$ is represented by a matrix with columns

$$f_{y_1}(y_\circ),\ f_{y_2}(y_\circ),\ \ldots,\ f_{y_h}(y_\circ),\ \nu^1(f(y_\circ)),\ \nu^2(f(y_\circ)),\ \ldots,\ \nu^{n-h}(f(y_\circ)),$$

where $y = (y_1, \ldots, y_h)$ and $f_{y_i} = \frac{\partial}{\partial y_i}$. Since $\operatorname{span}\{f_{y_1}(y_\circ), \ldots, f_{y_h}(y_\circ)\} = T_{x_\circ}\mathcal{M}$, the columns of $d\Phi_{(y_\circ,0)}$ are linearly independent. Hence, by the implicit function theorem, Φ is locally invertible with inverse of class $\mathcal{C}^{k-1}$. Let

$$O := (B_{r_\circ}(x_\circ) \cap \mathcal{M}^\circ) \times B^{n-h}_{r_\circ}(0),$$

where $0 < r_\circ = r(x_\circ) \leq \rho$ is so that the implicit function theorem holds, and let

$$\Psi : \widetilde{\Phi}(O) \subset \mathbb{R}^n \to O, \quad \Psi(\xi) = (x(\xi), \alpha(\xi)),$$

be the local inverse of $\widetilde{\Phi}$. Take $\delta_\circ \in (0, r_\circ/2)$ and $\xi \in B_{\delta_\circ}(x_\circ) \subset \widetilde{\Phi}(O)$, and let $x \in \mathcal{M}$ be so that $\operatorname{dist}(\xi, \mathcal{M}) = |x - \xi|$ (recall that $\mathcal{M}$ is closed by Definition 3.1).

Since $|x - \xi| \leq |x_\circ - \xi| < \delta_\circ$ it follows that $x \in B_{r_\circ}(x_\circ) \cap \mathcal{M}^\circ$ (recall (12) and $r_\circ \leq \rho$), hence $x = x(\xi)$[6] and $\operatorname{dist}(\xi, \mathcal{M}) = |\alpha(\xi)|$. Thus,

$$\eta_{\mathcal{M}}(\xi) = \frac{1}{2}|\alpha(\xi)|^2 = \frac{1}{2}\sum_{i=1}^{n-h}(\alpha_i(\xi))^2 \qquad \forall \xi \in B_{\delta_\circ}(x_\circ),$$

where $\alpha(\xi) = (\alpha_1(\xi) \ldots \alpha_{n-h}(\xi))$. Therefore $\eta_{\mathcal{M}} \in \mathcal{C}^{k-1}(B_{\delta_\circ}(x_\circ))$. Arguing as in [15, Section 1], it turns out that $\nabla \eta_{\mathcal{M}} \in (\mathcal{C}^{k-1}(B_{\delta_\circ}(x_\circ)))^n$, and hence $\eta_{\mathcal{M}} \in \mathcal{C}^k(B_{\delta_\circ}(x_\circ))$.

[6] Indeed ξ is normal to $\mathcal{M}^\circ$ at x, *i.e.*, for any tangent vector ω to $\mathcal{M}$ at x we have $< x - \xi, \omega >= 0$. To prove that, consider a local chart f around x such that $f(p) = x$ and $df_p(\tau) = \omega$. Since p is a minimum point for the function $|\xi - f(p + \sigma\tau)|^2$ where $|\sigma|$ is small enough, we then have $0 = \frac{d}{d\sigma}|\xi - f(p + \sigma\tau)|^2|_{\sigma=0} =< \xi - x, \omega >$.

Now we deal with points on $\partial\mathcal{M}$. Since $\mathcal{M}$ is an embedded $\mathcal{C}^k$-manifold with boundary, it can be extended[7] to a connected h-dimensional $\mathcal{C}^k$-manifold $\mathcal{N}$ with boundary such that $\partial\mathcal{M} \subset \mathcal{N}^\circ$. Let $\bar{x} \in \partial\mathcal{M} \subset \mathcal{N}$ and repeat the argument at the beginning of this step with $\mathcal{M}$ replaced by $\mathcal{N}$, to conclude that $\eta_{\mathcal{N}}$ is $\mathcal{C}^k$ in $B_\delta(\bar{x}) \subset \widetilde{\Phi}_{\mathcal{N}^\circ}((B_r(\bar{x}) \cap \mathcal{N}^\circ) \times B_r^{n-h}(0))$ for $r > 0$ sufficient small and $\delta \in (0, \frac{r}{2})$. Consider the open set

$$\mathcal{W}(\bar{x}) := B_\delta(\bar{x}) \cap \left(\widetilde{\Phi}_{\mathcal{N}^\circ}((B_r(\bar{x}) \cap \mathcal{M}^\circ) \times B_r^{n-h}(0))\right).$$

We claim that
$$\eta_{\mathcal{M}} = \eta_{\mathcal{N}} \quad \text{in} \quad \mathcal{W}(\bar{x}), \tag{14}$$

and hence $\eta_{\mathcal{M}} \in \mathcal{C}^k(\mathcal{W}(\bar{x}))$. Indeed, take $\xi \in \mathcal{W}(\bar{x})$; then $\xi \in B_\delta(\bar{x})$ implies the existence of a unique $x(\xi) \in \mathcal{N}$ such that $\mathrm{dist}(\xi, \mathcal{N}) = |x(\xi) - \xi|$ and $\xi \in N_{x(\xi)}\mathcal{N}$. Moreover $\xi \in \widetilde{\Phi}_{\mathcal{N}^\circ}\big((B_r(\bar{x}) \cap \mathcal{M}^\circ) \times B_r^{n-h}(0)\big)$ implies $x(\xi) \in \mathcal{M}^\circ$ by the definition of $\widetilde{\Phi}_{\mathcal{N}^\circ}$, and claim (14) follows. In particular, any point of $\mathcal{W}(\bar{x})$ has a unique point of minimal distance to $\mathcal{N}$ on $B_r(\bar{x}) \cap \mathcal{M}^\circ$.

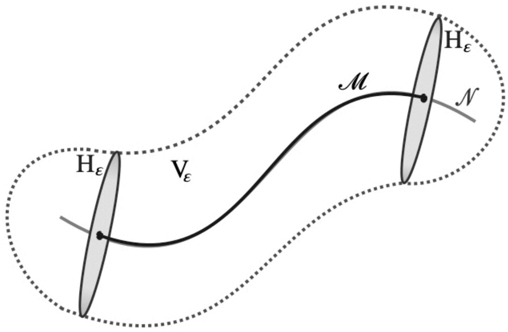

Figure 4.1: $\mathcal{M}$ is a curve (smooth up to the boundary) embedded in $\mathbb{R}^3$, $\mathcal{N}$ is a smooth extension of $\mathcal{M}$, and H_ϵ consists of two open disks normal to $\mathcal{M}$ at the endpoints (the boundary of $\mathcal{M}$).

By the compactness of $\mathcal{M}$, we can select $\epsilon_1 > 0$ such that:

- (10) holds;
- for any $\xi \in V_{\epsilon_1}$ there exists a unique $x(\xi) \in \mathcal{M}^\circ$ such that we have $\mathrm{dist}(\xi, \mathcal{M}) = |\xi - x(\xi)|$, in particular
$$\mathrm{dist}(\cdot, \mathcal{M}) < \mathrm{dist}(\cdot, \partial\mathcal{M}) \qquad \text{in} \quad V_{\epsilon_1}; \tag{15}$$
- by construction $V_{\epsilon_1} \subset \mathcal{M}^+_{\epsilon_1}$ and the topological boundary of V_{ϵ_1} is $K \cup H_{\epsilon_1}$, where $K \subset \partial(\mathcal{M}^+_{\epsilon_1})$ and H_{ϵ_1} is defined in (9) with ϵ replaced by ϵ_1. Hence the closure of V_{ϵ_1} in $\mathcal{M}^+_{\epsilon_1}$ is H_{ϵ_1} (see Fig. 4.1);
- $\eta_{\mathcal{N}} \in \mathcal{C}^k(\mathcal{M}^+_{\epsilon_1})$ and by (14)
$$\eta_{\mathcal{N}} = \eta_{\mathcal{M}} \qquad \text{in} \quad V_{\epsilon_1} \cup H_{\epsilon_1}. \tag{16}$$

Step 2. For $\epsilon_2 > 0$ small enough, H_{ϵ_2} is a $\mathcal{C}^{k-1}$ embedded hypersurface without boundary in $\mathcal{M}^+_{\epsilon_2}$.

[7] This directly follows from Definition 3.1.

The statement follows from the smooth embeddedness of $\partial\mathcal{M}$ in $\mathbb{R}^n$ and its compactness; we write the details for convenience of the reader. Let $\bar{x} \in \partial\mathcal{M}$, $\rho > 0$ and $g : G'_{\bar{x}} \subset \mathbb{R}^{h-1} \longrightarrow B_\rho(\bar{x}) \cap \partial\mathcal{M}$ be a local chart on $\partial\mathcal{M}$. Define

$$X(y', \alpha) = g(y') + \sum_{i=1}^{n-h} \alpha_i \nu^i(g(y')), \qquad y' \in G'_{\bar{x}},\ \alpha \in B^{n-h}_{\epsilon(\bar{x})}(0), \tag{17}$$

where $\{\nu^i(g(y'))\}_{i=1,\dots,n-h}$ are orthonormal vector fields of class $\mathcal{C}^{k-1}$ spanning the normal space to $\mathcal{M}$ at $g(y') \in \partial\mathcal{M}$ and $\epsilon(\bar{x}) > 0$ is sufficiently small. Clearly X is $\mathcal{C}^{k-1}$, $dX_{(y',0)}$ is non singular and X is a local homeomorphism onto its image. Now, we use the compactness of $\partial\mathcal{M}$ to get a finite subcovering

$$\bigcup_{i=1}^{l} g_i(G'_{\bar{x}_i}) = \bigcup_{i=1}^{l} B_{\rho_i}(\bar{x}_i) \cap \partial\mathcal{M} = \partial\mathcal{M}, \bar{x}_i \in \partial\mathcal{M},$$

and define

$$\epsilon_2 := \min\{\epsilon(\bar{x}_i) : i = 1, \dots, l\} > 0, \text{ so that } H_{\epsilon_2} = \bigcup_{i=1}^{l} X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0)). \tag{18}$$

It remains to prove that, possibly reducing the value of ϵ_2, any $\zeta \in H_{\epsilon_2}$ has an open neighborhood $\mathcal{V}$ such that $\mathcal{V} \cap H_{\epsilon_2}$ is exactly the image of one of the charts $X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0))$ (that is, H_{ϵ_2} has no self-intersections).

Assume that $\epsilon_2 > 0$ is small enough so that $H_{\epsilon_2} \subset (\partial\mathcal{M})^+_{\epsilon_2}$, and

- for every $\xi \in (\partial\mathcal{M})^+_{\epsilon_2}$ there exists unique $x_\xi \in \partial\mathcal{M}$ such that $\xi \in N_{x_\xi}\partial\mathcal{M}$ and $\mathrm{dist}(\xi, \partial\mathcal{M}) = |\xi - x_\xi|$;
- the projection map $P : (\partial\mathcal{M})^+_{\epsilon_2} \to \partial\mathcal{M}$, $P(\xi) := x_\xi$, is $\mathcal{C}^{k-1}$, see [14].

Now let $\zeta \in H_{\epsilon_2}$ and $i \in \{1, \dots, l\}$ be such that $x_\zeta \in B_{\rho_i}(\bar{x}_i) \cap \partial\mathcal{M}$. Define $\mathcal{V} := P^{-1}(B_{\rho_i}(\bar{x}_i) \cap \partial\mathcal{M})$ which is an open neighborhood of ζ in $\mathbb{R}^n$. We have

$$\mathcal{V} \cap H_{\epsilon_2} = X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0)).$$

Indeed, if $\xi \in \mathcal{V} \cap H_{\epsilon_2}$ then $x_\xi \in B_{\rho_i}(\bar{x}_i) \cap \partial\mathcal{M}$, $|\xi - x_\xi| < \epsilon_2$ and $\xi \in N_{x_\xi}\mathcal{M}$, *i.e.*, $\xi \in X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0))$ by the definition of X in (17). On the other hand the inclusion $\mathcal{V} \cap H_{\epsilon_2} \supseteq X(G'_{\bar{x}_i} \times B^{n-h}_{\epsilon_2}(0))$ is immediate from (18).

Now assertion (1) follows from (9), with ϵ replaced by ϵ_2.

Step 3. There exists $\epsilon \in (0, \min(\epsilon_1, \epsilon_2))$ such that

$$\eta_{\mathcal{M}} \in \mathcal{C}^k(\mathcal{M}^+_\epsilon \setminus H_\epsilon). \tag{19}$$

From Theorem 2.2, we may assume that $\eta_{\partial\mathcal{M}}$ is $\mathcal{C}^k$ in a tubular neighborhood $(\partial\mathcal{M})^+_{\epsilon_3}$ of $\partial\mathcal{M}$ of radius $\epsilon_3 > 0$. Define

$$\epsilon := \min\{\epsilon_1, \epsilon_2, \epsilon_3\}. \tag{20}$$

If V_ϵ is as in Step 1 (with ϵ replaced by ϵ_1), we have $V_\epsilon \subset \mathcal{M}^+_\epsilon$ and, from Step 1, $\eta_{\mathcal{M}}$ is $\mathcal{C}^k$ in V_ϵ.

We claim that
$$\eta_{\mathcal{M}} = \eta_{\partial\mathcal{M}} \quad \text{in } \ \mathcal{M}^+_\epsilon \setminus V_\epsilon. \tag{21}$$

Since $\partial\mathcal{M} \subset \mathcal{M}$, $\mathrm{dist}(\cdot, \mathcal{M}) \leq \mathrm{dist}(\cdot, \partial\mathcal{M})$ hence $\eta_{\mathcal{M}} \leq \eta_{\partial\mathcal{M}}$. Assume by contradiction that there exists $\xi \in \mathcal{M}^+_\epsilon \setminus V_\epsilon$ such that $\mathrm{dist}(\xi, \mathcal{M}) < \mathrm{dist}(\xi, \partial\mathcal{M})$. Then there exists $x \in \mathcal{M} \setminus \partial\mathcal{M}$ such that $|\xi - x| = \mathrm{dist}(\xi, \mathcal{M}) < \epsilon$ which, by the definition of V_ϵ, implies $\xi \in V_\epsilon$, a contradiction.

Since the closure of V_ϵ in $\mathcal{M}^+_\epsilon$ is H_ϵ (see the end of Step 1) it follows directly that $\mathcal{M}^+_\epsilon \setminus (V_\epsilon \cup H_\epsilon)$ is an open subset of $(\partial\mathcal{M})^+_{\epsilon_3}$ in which $\eta_{\mathcal{M}}$ is $\mathcal{C}^k$. Hence assertion (2) is proven. $\square$

Now, we prove Theorem 4.1 when $\Omega = \mathbb{R}^n$ and supposing that the manifold is compact.

Theorem 4.3. *Let $k \in \mathbb{N}$, $k \geq 2$ or $k \in \{\infty, \omega\}$, and $h \in \{1, \ldots, n\}$. Let $\mathcal{M} \subset \mathbb{R}^n$ be an embedded compact $\mathcal{C}^k$-manifold of dimension h with nonempty boundary $\partial\mathcal{M}$ in $\mathbb{R}^n$. Then*
$$(\mathcal{M}, \partial\mathcal{M}) \in D_h B\mathcal{C}^k(\mathbb{R}^n).$$

Proof. We have to check conditions (i)–(iv) of Definition 3.2.

Suppose $h = n$. By Remark 2.5 (I) it follows $\partial\mathcal{M} \in D_{n-1}\mathcal{C}^k(\mathbb{R}^n)$. One also immediately checks that $\mathcal{M} \in D_n\mathcal{C}^k(\mathbb{R}^n \setminus \partial\mathcal{M})$. Moreover $\mathrm{dist}(\cdot, \mathcal{M}) \leq \mathrm{dist}(\cdot, \partial\mathcal{M})$ and $B = \mathcal{M} \setminus \partial\mathcal{M}$, hence [13, 12, 1] the function d_B is $\mathcal{C}^k$ in a tubular neighborhood of $\partial\mathcal{M}$, which shows condition (iii). Clearly $\{x \in \mathbb{R}^n : d_B(x) \leq 0\} = \mathcal{M}$; thus $\widehat{\eta} \equiv 0$ on $\mathbb{R}^n \setminus \mathcal{M}$ is a $\mathcal{C}^k(\mathbb{R}^n)$ extension of $\eta_{\mathcal{M}}$, so that condition (iv) is fulfilled.

Now suppose $h < n$. From Remark 2.5 (I) we have $\partial\mathcal{M} \in D_{h-1}\mathcal{C}^k(\mathbb{R}^n)$.

Moreover, since $\mathcal{M}$ is a $\mathcal{C}^k$-manifold without boundary in $\mathbb{R}^n \setminus \partial\mathcal{M}$ then, again by Remark 2.5(I),
$$\mathcal{M} \in D_h\mathcal{C}^k(\mathbb{R}^n \setminus \partial\mathcal{M}),$$
which shows (i). We also have $\mathrm{dist}(\cdot, \mathcal{M}) \leq \mathrm{dist}(\cdot, \partial\mathcal{M})$, which shows (ii).

Let B be defined as in (4) with $\Omega = \mathbb{R}^n$, and $(E, L) := (\mathcal{M}, \partial\mathcal{M})$. Then by (15) and the last comments in Step 1 in Proposition 4.2 we have
$$B \cap \mathcal{M}^+_\epsilon = V_\epsilon \qquad \text{and} \qquad \mathcal{M}^+_\epsilon \cap \partial B = H_\epsilon,$$
with ϵ, V_ϵ and H_ϵ as in (20), (11) and (9), respectively.

Since by Proposition 4.2 (1) H_ϵ is an embedded hypersurface (without boundary) of class $\mathcal{C}^{k-1}$ in $\mathcal{M}^+_\epsilon$, following the same argument in the comment after Theorem 2.2 and using [1, 12, 13], there exists an open neighborhood $A \subset \mathcal{M}^+_\epsilon$ of H_ϵ such that $d_B \in \mathcal{C}^{k-1}(A)$. Hence condition (iii) is satisfied.

Finally, since $\{x \in A : d_B(x) \leq 0\} \subset V_\epsilon \cup H_\epsilon$, condition (iv) follows from (16). $\square$

Now we generalize Proposition 4.2: if we drop the compactness of $\mathcal{M}$ we can not take tubular neighborhoods of constant width.

Proposition 4.4. *Let $k \in \mathbb{N}$, $k \geq 2$ or $k \in \{\infty, \omega\}$, and $h \in \{1, \ldots, n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set. Let $\mathcal{M} \subset \mathbb{R}^n$ be an embedded $\mathcal{C}^k$-manifold of dimension h*

with nonempty boundary in Ω. *Then there exists a function* $\epsilon : \mathcal{M} \cap \Omega \to (0, +\infty]$ *such that, setting*

$$H_{\overline{\epsilon}(\cdot)} := \bigcup_{\bar{x} \in \partial_\Omega \mathcal{M}} B_{\overline{\epsilon}(\bar{x})}(\bar{x}) \cap N_{\bar{x}}\mathcal{M} \tag{22}$$

where $\overline{\epsilon}(\bar{x}) := \sup\{\rho > 0 : B_\rho(\bar{x}) \cap N_{\bar{x}}\mathcal{M} \subset (\mathcal{M} \cap \Omega)^+_{\epsilon(\cdot)}\}$, *we have*

$$\partial_\Omega \mathcal{M} \subseteq H_{\overline{\epsilon}(\cdot)} \subseteq \bigcup_{\bar{x} \in \partial_\Omega \mathcal{M}} N_{\bar{x}}\mathcal{M}$$

and the following properties hold:

(1) $H_{\overline{\epsilon}(\cdot)}$ *is an embedded* $\mathcal{C}^{k-1}$*-hypersurface without boundary in* $(\mathcal{M} \cap \Omega)^+_{\epsilon(\cdot)}$, *and* $N_{\bar{x}} H_{\overline{\epsilon}(\cdot)} \subseteq T_{\bar{x}}\mathcal{M}$ *for any* $\bar{x} \in \partial_\Omega \mathcal{M}$;

(2) $\eta_{\mathcal{M}} \in \mathcal{C}^k((\mathcal{M} \cap \Omega)^+_{\epsilon(\cdot)} \setminus H_{\overline{\epsilon}(\cdot)})$.

Proof. The proof is similar to the proof of Proposition 4.2 with slight modifications. We suppose that $\mathcal{M} \cap \Omega$ is connected. Let us write for simplicity $\epsilon^x = \epsilon(x)$, $\overline{\epsilon}^x = \overline{\epsilon}(x)$ and so on.

Assume first $h = n$; then the interior of $\mathcal{M} \cap \Omega$ is a nonempty open set with $\mathcal{C}^k$-boundary in Ω, and for every $\bar{x} \in \partial_\Omega \mathcal{M}$ there exists $\epsilon^{\bar{x}} > 0$ such that $B_{\epsilon^{\bar{x}}}(\bar{x}) \subset \Omega$ and $d_{\mathcal{M}} \in \mathcal{C}^k(B_{\epsilon^{\bar{x}}}(\bar{x}))$ [13, 12, 9], hence $d_{\mathcal{M}}$ is of class $\mathcal{C}^k$ in $(\partial_\Omega \mathcal{M})^+_{\epsilon(\cdot)} \subset \Omega$.

Recall that from (22) we have $H_{\overline{\epsilon}(\cdot)} = \partial_\Omega \mathcal{M}$. Then the assertions (1)–(2) follow as in the proof of Proposition 4.2.

Now suppose $h \in \{1, \dots, n-1\}$. As in Step 1 in the proof of Proposition 4.2 it follows that:

- for every $x_\circ \in (\mathcal{M} \cap \Omega)^\circ$ there exists $\epsilon^1_{x_\circ} > 0$ such that $B_{\epsilon^1_{x_\circ}}(x_\circ) \subset \Omega$, $B_{\epsilon^1_{x_\circ}}(x_\circ) \cap \partial_\Omega \mathcal{M} = \emptyset$ and $\eta_{\mathcal{M}} \in \mathcal{C}^k(B_{\epsilon^1_{x_\circ}}(x_\circ))$;
- if $\mathcal{N} \subset \Omega$ is a connected embedded $\mathcal{C}^k$-manifold of dimension h containing $\mathcal{M}$ and so that $\partial_\Omega \mathcal{M} \subset \mathcal{N}^o$ then, for every $\bar{x} \in \partial_\Omega \mathcal{M}$, there exists $\epsilon^{\bar{x}}_1 > 0$ such that $B_{\epsilon^{\bar{x}}_1}(\bar{x}) \subset \Omega$ and

$$\begin{aligned} \mathcal{W}_{\epsilon^{\bar{x}}_1}(\bar{x}) : &= \{\xi \in B_{\epsilon^{\bar{x}}_1}(\bar{x}) : \operatorname{dist}(\xi, \mathcal{N}) = |\xi - x_\xi|,\ x_\xi \in (\mathcal{M} \cap \Omega)^\circ\} \\ &= \bigcup_{y \in (\mathcal{M} \cap \Omega)^\circ} B_{\epsilon^{\bar{x}}_1}(\bar{x}) \cap N_y \mathcal{M} \end{aligned}$$

  is an open subset of Ω, and

$$\overline{\mathcal{W}_{\epsilon^{\bar{x}}_1}(\bar{x})} \cap B_{\epsilon^{\bar{x}}_1}(\bar{x}) = \mathcal{W}_{\epsilon^{\bar{x}}_1}(\bar{x}) \cup \Big(\bigcup_{y \in \partial_\Omega \mathcal{M}} (B_{\epsilon^{\bar{x}}_1}(\bar{x}) \cap N_y \mathcal{M}) \Big);$$

- $\eta_{\mathcal{M}} = \eta_{\mathcal{N}}$ in $\overline{\mathcal{W}_{\epsilon^{\bar{x}}_1}(\bar{x})}$, hence $\eta_{\mathcal{M}} \in \mathcal{C}^k\big(\overline{\mathcal{W}_{\epsilon^{\bar{x}}_1}(\bar{x})} \cap B_{\epsilon^{\bar{x}}_1}(\bar{x})\big)$.

Define $$\widetilde{V}_{\epsilon_1(\cdot)} := \Big(\bigcup_{x_\circ \in (\mathcal{M} \cap \Omega)^\circ} B_{\epsilon^1_{x_\circ}}(x_\circ) \Big) \cup \Big(\bigcup_{\bar{x} \in \partial_\Omega \mathcal{M}} \mathcal{W}_{\epsilon^{\bar{x}}_1}(\bar{x}) \Big),$$

where $\epsilon_1(x) = \epsilon^x$ for any $x \in \mathcal{M} \cap \Omega$. By construction $\widetilde{V}_{\epsilon_1(\cdot)} \subset (\mathcal{M} \cap \Omega)^+_{\epsilon_1(\cdot)}$ and the topological boundary of $\widetilde{V}_{\epsilon_1(\cdot)}$ splits as $K \cup H_{\overline{\epsilon}_1(\cdot)}$, where K is a subset of the topological boundary of $((\mathcal{M}\cap\Omega)^+_{\epsilon_1(\cdot)})$ and $H_{\bar\epsilon_1(\cdot)}$ is as in (22) with $\bar\epsilon$ replaced by $\bar\epsilon_1$. Hence the topological boundary of $\widetilde V_{\epsilon_1(\cdot)}$ in $(\mathcal{M}\cap\Omega)^+_{\epsilon_1(\cdot)}$ is $H_{\bar\epsilon_1(\cdot)}$. Moreover we have $\eta_{\mathcal{N}} \in \mathcal{C}^k((\mathcal{M}\cap\Omega)^+_{\epsilon_1(\cdot)})$ and $\eta_{\mathcal{N}} = \eta_{\mathcal{M}}$ in $\widetilde V_{\epsilon_1(\cdot)} \cup H_{\bar\epsilon_1(\cdot)}$. We conclude that $\eta_{\mathcal{M}} \in \mathcal{C}^k(\widetilde V_{\epsilon_1(\cdot)})$.

As in Steps 1, 2 in the proof of Proposition 4.2, for each $\bar x \in \partial_\Omega\mathcal{M}$ there exist $\epsilon_2^{\bar x} > 0$, an open set $G'_{\bar x} \subset \mathbb{R}^{h-1}$, an open set $B_{\bar x} \subset \mathbb{R}^{n-h+1}$ and a $\mathcal{C}^{k-1}$-diffeomorphism

$$\overline{X} : G'_{\bar x} \times B_{\bar x} \to B_{\epsilon_2^{\bar x}}(\bar x) \subset \mathbb{R}^n, \quad \overline{X}(y',\overline{\alpha}) := g(y') + \sum_{i=1}^{n-h+1} \overline{\alpha}_i \nu^i(g(y')),$$

where the map $g : G'_{\bar x} \subset \mathbb{R}^{h-1} \longrightarrow B_{\epsilon_2^{\bar x}}(\bar x) \cap \partial_\Omega\mathcal{M}$ is a local chart on $\partial_\Omega\mathcal{M}$ and $\{\nu^i(g(y'))\}_{i=1,\dots,n-h+1}$ with $\nu^{n-h+1}(g(y'))$ tangent to $\mathcal{M}$ at $g(y')$, are orthonormal vector fields of class $\mathcal{C}^{k-1}$ spanning the normal space to $\partial_\Omega\mathcal{M}$ at $g(y') \in \partial_\Omega\mathcal{M}$.

Thus
$$(\partial_\Omega\mathcal{M})^+_{\epsilon_2(\cdot)} = \bigcup_{\bar x \in \partial_\Omega\mathcal{M}} \overline{X}(G'_{\bar x} \times B_{\bar x}).$$

Let
$$X : G'_{\bar x} \times B^{n-h}_{\bar x} \to B_{\epsilon_2^{\bar x}}(\bar x), \quad X(y',\alpha) := g(y') + \sum_{i=1}^{n-h} \alpha_i \nu^i(g(y')),$$

where $B^{n-h}_{\bar x} := \{(\overline{\alpha}_1, \cdots, \overline{\alpha}_{n-h+1}) \in B_{\bar x} : \overline{\alpha}_{n-h+1} = 0\} \subset \mathbb{R}^{n-h}$ (note that X equals the restriction of $\overline{X}$ in $G'_{\bar x} \times B^{n-h}_{\bar x}$, hence it is a $\mathcal{C}^{k-1}$-diffeomorphism). Setting $H_{\bar\epsilon_2(\cdot)}$ as in (22) with $\bar\epsilon$ replaced by $\bar\epsilon_2$, we have

$$H_{\bar\epsilon_2(\cdot)} = \bigcup_{\bar x \in \partial_\Omega\mathcal{M}} X(G'_{\bar x} \times B^{n-h}_{\bar x}).$$

Thus for each $\zeta \in H_{\bar\epsilon_2(\cdot)}$ there exists $\bar x \in \partial_\Omega\mathcal{M}$ such that $\zeta \in X(G'_{\bar x} \times B^{n-h}_{\bar x})$. Letting $\mathcal{V} := \overline{X}(G'_{\bar x} \times B_{\bar x})$, following the argument in Step 2 of Proposition 4.2, we may show that the maps $X(G'_{\bar x} \times B^{n-h}_{\bar x})$ are local charts covering $H_{\bar\epsilon_2(\cdot)}$. This completes the proof of assertion (1).

To prove (2) we assume that $\eta_{\partial\mathcal{M}}$ is $\mathcal{C}^k$ in a neighborhood $(\partial_\Omega\mathcal{M})^+_{\epsilon_3(\cdot)}$ of $\partial_\Omega\mathcal{M}$, see the comment after Theorem 2.2, and we define $\epsilon(x) := \min\{\epsilon_1^x, \epsilon_2^x, \epsilon_3^x\}$. Again following Step 3 in the proof of Proposition 4.2 we have $\eta_{\mathcal{M}} \in \mathcal{C}^k((\mathcal{M}\cap\Omega)^+_{\epsilon(\cdot)} \setminus H_{\bar\epsilon(\cdot)})$. □

Conclusion of the proof of Theorem 4.1. It follows from Proposition 4.4 the same way the proof of Theorem 4.3 follows from Proposition 4.2. □

5. Sets in $D_hB\,\mathcal{C}^k(\Omega)$ are smooth manifolds with boundary

The goal of this section is to prove a sort of converse[8] of Theorem 4.1. That is, we want to show the following:

Theorem 5.1. *Let $k \in \mathbb{N}$, $k \geq 3$ or $k \in \{\infty, \omega\}$ and $h \in \{1,\dots,n\}$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty open set, and let $E, L \subset \mathbb{R}^n$ be such that $(E, L) \in D_hB\,\mathcal{C}^k(\Omega)$. Then $(E \cup L) \cap \Omega$ is a h-dimensional $\mathcal{C}^{k-1}$-manifold in Ω with boundary $L \cap \Omega$.*

[8] In the $\mathcal{C}^\infty$ or analytic case, it is the converse.

Proof. Let us assume first $\Omega = \mathbb{R}^n$ (which includes the converse of Theorem 4.3). We can suppose $L \neq \emptyset$, since if $L = \emptyset$ the result follows from Remark 2.5 (I) and the comment above Remark 3.3.

Recall from Remark 3.3 (I) that L is an embedded $\mathcal{C}^k$-manifold in $\mathbb{R}^n$ without boundary of dimension $h-1$, and E is an embedded $\mathcal{C}^k$-manifold without boundary in $\mathbb{R}^n \setminus L$ of dimension h. Let A, B,

$$C := \{x \in A : d_B(x) \leq 0\}, \tag{23}$$

$\widehat{C}$ and $\widehat{\eta}$ be as in Definition 3.2 (iii), (iv). We divide the proof of the theorem into four steps.

Step 1. We have
$$\overline{E} \cap C \subseteq \{x \in \widehat{C} : \nabla\widehat{\eta}(x) = 0\}. \tag{24}$$
From [4, Lemma 2.1], η_E is differentiable in $\overline{E}$ and $\overline{E} = \{x \in \mathbb{R}^n : \nabla\eta_E(x) = 0\}$. Hence, using (5),

$$\overline{E} \cap C = (E \cup L) \cap C = \{x \in C : \nabla\eta_E(x) = 0\}. \tag{25}$$

Let $x \in (E \setminus L) \cap C$; then from (6) and (23) it follows

$$x \in B \cap C = \{x \in A : d_B(x) < 0\}. \tag{26}$$

Hence x is an interior point of C, and since $\widehat{\eta} = \eta_E$ on C, from (25) we deduce $\nabla\widehat{\eta}(x) = 0$. Now, let $x \in L \cap C$; recall from Remark 3.3 (VII) that the topological boundary of B is of class $\mathcal{C}^{k-1}$ in a neighborhood of x.
Then $x \in \partial_A B = \{x \in A : d_B(x) = 0\}$ from (7). We shall show that

$$\nabla\widehat{\eta}(x)\nu = 0 \qquad \forall \nu \in \mathbb{R}^n \setminus \{0\}. \tag{27}$$

Let $n \geq 2$ (the case $n = 1$ being trivial); if $\nu \in T_x\partial_A B$ then there exist $\epsilon > 0$ and $\alpha : (-\epsilon, \epsilon) \to \partial_A B$ of class $\mathcal{C}^1$ such that $\alpha(0) = x$, $\alpha'(0) = \nu$. Hence, using also that $\widehat{\eta} = \eta_E$ on C,

$$\nabla\widehat{\eta}(x)\nu = \frac{d}{dt}\widehat{\eta}(\alpha(t))|_{t=0} = \frac{d}{dt}\eta_E(\alpha(t))|_{t=0} = 0.$$

If $\nu \in N_x\partial_A B$ then we can select $\beta : (-\epsilon, \epsilon) \to \mathbb{R}^n$ of class $\mathcal{C}^1$ such that $\beta(0) = x$, $\beta'(0) = \nu$ and $\beta((-\epsilon, 0))$ is contained in the interior of C. Hence, denoting by $\frac{d}{dt_-}$ the left derivative, we have

$$\nabla\widehat{\eta}(x)\nu = \frac{d}{dt}\widehat{\eta}(\beta(t))|_{t=0} = \frac{d}{dt_-}\widehat{\eta}(\beta(t))|_{t=0} = \frac{d}{dt_-}\eta_E(\beta(t))|_{t=0} = \frac{d}{dt}\eta_E(\beta(t))|_{t=0} = 0.$$

This concludes the proof of (27), and then (24) follows.

Step 2. There exists an embedded h-dimensional manifold $\mathcal{N}$ of class $\mathcal{C}^{k-1}$, without boundary in a sufficiently small neighborhood of $E \cup L$, such that

$$E \cup L \subset \mathcal{N}.$$

For $h = n$, it is sufficient to take $\mathcal{N} = \mathbb{R}^n$. Hence, suppose $h < n$. Take $\overline{x} \in L$ and, recalling that $\operatorname{rank}(\nabla^2\widehat{\eta}) = n - h$ on $(E \cup L) \cap C$, let $\{\overline{\nu}^1, \overline{\nu}^2, \ldots, \overline{\nu}^{n-h}\}$ be an orthonormal basis of $\operatorname{Im}\big(\nabla^2\widehat{\eta}(\overline{x})\big)$.

Define $$F_i(x) := \langle \nabla\widehat{\eta}(x), \overline{\nu}^i\rangle, \qquad i = 1,\dots,n-h, \quad x \in \widehat{C}, \tag{28}$$

and set $$F : \widehat{C} \longrightarrow \mathbb{R}^{n-h}, \qquad F := (F_1, F_2, \dots, F_{n-h}). \tag{29}$$

From (5) and (25) we have

$$(E \cup L) \cap C = \overline{E} \cap C \subseteq \{x \in \widehat{C} : \nabla\widehat{\eta}(x) = 0\} \subseteq \{x \in \widehat{C} : F(x) = 0\}. \tag{30}$$

Moreover, $F \in \mathcal{C}^{k-1}(\widehat{C}; \mathbb{R}^{n-h})$ and if we denote by $JF(\overline{x})$ the Jacobian of F at $\overline{x}$, then
$$JF(\overline{x}) = Q^T \nabla^2\widehat{\eta}(\overline{x}), \tag{31}$$
where Q^T is the transposed of the $n \times (n-h)$ matrix Q having as columns the vectors $(\overline{\nu}^i)_{i=1,\dots,n-h}$. By construction $JF(\overline{x})$ has rank $n-h$. Since the rank is lower semicontinuous and $JF(\overline{x})$ has top rank, we can choose $\sigma = \sigma(\overline{x}) > 0$ so that $\mathrm{rank}(JF) = n-h$ on $B_\sigma(\overline{x})$. Let

$$\Gamma_{\overline{x}} := B_\sigma(\overline{x}) \cap \{x \in \widehat{C} : F(x) = 0\}.$$

The implicit function theorem ensures that $\Gamma_{\overline{x}}$ is an embedded $h-$dimensional manifold without boundary in $B_\sigma(\overline{x})$, of class $\mathcal{C}^{k-1}$.

Note that $B_\sigma(\overline{x}) \cap ((E \setminus L) \cap C)$, nonempty by (6) and (23), is a manifold without boundary in $B_\sigma(\overline{x}) \setminus L$ of dimension h (Remark 3.3 (I)) and it is contained in $\Gamma_{\overline{x}}$ by (30). Hence $\Gamma_{\overline{x}}$ is an extension of $(E \setminus L) \cap C$ in $B_\sigma(\overline{x})$. It is then sufficient to define

$$\mathcal{N} := E \cup \bigcup_{\overline{x} \in L} \Gamma_{\overline{x}}.$$

Step 3. $E \cup L$ is an embedded h-dimensional $\mathcal{C}^{k-1}$-manifold in $\mathbb{R}^n$ with boundary.

We need to check that Definition 3.1 is satisfied. Recall from Remark 3.3 (III) that $\overline{E \cup L} = E \cup L$. If $x \in E \setminus L$, the assertion follows, since $E \setminus L$ is an embedded manifold without boundary in $\mathbb{R}^n \setminus L$ of dimension h (Remark 3.3 (I)).

Now, let $\bar{x} \in L$. Since L is a $\mathcal{C}^{k-1}$ embedded submanifold of $\mathcal{N}$ of codimension 1 (Step 1), there exist an open neighborhood $R \subset \mathbb{R}^n$ of $\bar{x}$ and a $\mathcal{C}^{k-1}$ local parametrization

$$\phi : G := B_1^h(0) \to U := R \cap \mathcal{N} \subset \mathbb{R}^n \tag{32}$$

such that $$R \cap L = \{\phi(y) : y = (y_1, \dots, y_h) \in G, y_h = 0\}. \tag{33}$$

Hence $U \cap L$ divides U into two relatively open connected components U^+ and U^- defined as
$$U^\pm := \{\phi(y) : y \in G,\ \langle y, \pm e_h\rangle > 0\}, \tag{34}$$
where $e_h := (0, \dots, 0, 1) \in \mathbb{R}^h$ (note that $(E \setminus L) \cap (U \setminus L) \neq \emptyset$). Clearly

$$U^+ \cap L = U^- \cap L = \emptyset. \tag{35}$$

Let us show:

$$U^\pm \cap (E \setminus L) \neq \emptyset \qquad \Longrightarrow \qquad U^\pm \cap (E \setminus L) = U^\pm. \tag{36}$$

Assume $U^+ \cap (E \setminus L) \neq \emptyset$ and suppose by contradiction that $U^+ \setminus \big(U^+ \cap (E \setminus L)\big)$ is nonempty (the case for U^- being similar). Recalling that U^+ is connected and that both sets $E \setminus L$ and U^+ are relatively open in $\mathcal{N}$, we have

$$U^+ \cap (E \setminus L) \subsetneqq \overline{U^+ \cap (E \setminus L)} \cap U^+. \tag{37}$$

Thus

$$\overline{U^+ \cap (E \setminus L)} \cap U^+ \subseteq U^+ \cap \overline{(E \setminus L)} \subseteq (U^+ \cap (E \setminus L)) \cup (L \cap U^+), \tag{38}$$

since $\overline{E \setminus L} \subseteq \overline{E \cup L} = E \cup L$. From (37) and (38) we deduce

$$U^+ \cap (E \setminus L) \subsetneqq (U^+ \cap (E \setminus L)) \cup (L \cap U^+).$$

Hence $U^+ \cap L \neq \emptyset$, which contradicts (35).

We now conclude to verify Definition 3.1.

Case 1. $U^- \cap (E \setminus L) = \emptyset$. Then from (36) it follows $U \cap (E \setminus L) = U^+$, and (3) (with $\mathcal{M}$ replaced by $E \cup L$) is a consequence of (33) and (34).

Case 2. $U^+ \cap (E \setminus L) = \emptyset$. Similar to Case 1.

Case 3. $U^+ \cap (E \setminus L) \neq \emptyset$ and $U^- \cap (E \setminus L) \neq \emptyset$. Then from (36) it follows $U \cap (E \setminus L) = U \setminus L$, and (2) follows from (33) and (34).

In the next final step, we shall show that only one among the two sets $U^+ \cap (E \setminus L)$ and $U^- \cap (E \setminus L)$ is not empty, in particular, Case 3 above is impossible. This will allow us to conclude that L is the boundary of $E \cup L$.

Step 4. We have $\partial(E \cup L) = L$.

Since E is a $\mathcal{C}^{k-1}$-manifold without boundary in $\mathbb{R}^n \setminus L$ of dimension h (Remark 3.3 (I)), we have

$$\partial(E \cup L) \subseteq L.$$

To prove the converse inclusion, recalling also the proof of Step 3 (see (32), (33) and (36)), it is sufficient to show that for any $\overline{x} \in L$ there is no relatively open neighborhood U of $\overline{x}$ in $\mathcal{N}$ such that $U \cap (E \setminus L) = U \setminus L$. Once we prove this we are either in case 1 or case 2 of the previous step; if, without loss of generality, we assume to be in case 1 then we conclude

$$(E \cup L) \cap U = \{\phi(y) : y \in G,\ \langle y, e_h \rangle \geq 0\},$$

and

$$L \cap U = \{\phi(y) : y \in G,\ \langle y, e_h \rangle = 0\}.$$

Let $\overline{x} \in L$, and recall once more the definition of B in (4), and that $L \subset \partial_A B$ (see (7) and Remark 3.3 (VII)). From condition (iii) of Definition 3.2 we know that d_B is of class $\mathcal{C}^k$ in a neighborhood of $\overline{x}$. Hence there exist a neighborhhood $R \subset \mathbb{R}^n$ of $\overline{x}$, $\delta > 0$, and a map $\psi \in \mathcal{C}^k(R; \mathbb{R}^n)$ such that

$\psi(R) = B_\delta(0),\ \psi(R \cap B) = B_\delta(0) \cap \{x_n > 0\}$ and $\psi(R \cap \partial B) = B_\delta(0) \cap \{x_n = 0\}$

(in particular B locally lies on one side of ∂B).

If $h = n$, our assertion follows from the fact that $B = E \setminus L$ by (8). Assume now $h < n$. Suppose by contradiction that there exist $\overline{x} \in L$ and a neighborhood U of $\overline{x}$ in $\mathcal{N}$ such that $U \cap (E \setminus L) = U \setminus L$. Since B is locally on one side of ∂B and $\overline{x} \in L \subset \partial_A B$, recalling also (6), we have

$$U \setminus L = U \cap (E \setminus L) \subset E \setminus L \subset B.$$

Moreover since U is relatively open in $\mathcal{N}$ and $L \subset \partial_A B$, we get

$$T_{\overline{x}}\mathcal{N} = T_{\overline{x}}U \subsetneq T_{\overline{x}}\partial_A B.$$

We are then allowed to take a non-zero $\xi \in B \setminus \mathcal{N}$ in the normal space to $\mathcal{N}$ at $\overline{x}$ such that $\mathrm{dist}(\xi, \mathcal{N}) = |\xi - \overline{x}|$. By Remark 3.3 (II) and $\bar{x} \in L$, we have

$$\mathrm{dist}(\xi, L) = \mathrm{dist}(\xi, E \setminus L). \tag{39}$$

Then (39) contradicts the inclusion $\xi \in B = \{z \in \mathbb{R}^n : \mathrm{dist}(z, E \setminus L) < \mathrm{dist}(z, L)\}$.

This concludes the proof when $\Omega = \mathbb{R}^n$. The proof when $\Omega \subset \mathbb{R}^n$ is a nonempty open set follows by replacing E with $E \cap \Omega$, L with $L \cap \Omega$, and $\mathbb{R}^n$ with Ω in the above arguments. □

Remark 5.2. Inspecting the proof of Theorem 5.1, we observe that we do not obtain the $\mathcal{C}^k$-regularity of $(E \cup L) \cap \Omega$ because the map F defined in (28), (29) is just $\mathcal{C}^{k-1}$ and not $\mathcal{C}^k$, and this is due to the $\mathcal{C}^k$-regularity of the function $\widehat{\eta}$. If we knew that $\widehat{\eta}$ were half of the square distance function from some $\mathcal{C}^k$-manifold then, using the results of [15, Section 3] we could get the $\mathcal{C}^k$-regularity of $(E \cup L) \cap \Omega$. In this direction, one could modify Definition 3.2 into a more rigid one, by assuming $\widehat{\eta}$ in Definition 3.2 (iv) to be half of the square distance function from an h-dimensional embedded manifold Γ of class $\mathcal{C}^k$, without boundary in $\widehat{C}$. Adopting this modified definition:

(1) Theorem 4.1 holds by taking $\Gamma = \mathcal{N}$ and $\widehat{\eta} = \eta_{\mathcal{N}}$, where $\mathcal{N}$ is the $\mathcal{C}^k$- manifold defined in Step 1 of the proof of Proposition 4.2 and $\eta_{\mathcal{N}}(\cdot) = \frac{1}{2}\mathrm{dist}(\cdot, \mathcal{N})^2$;

(2) Theorem 5.1 holds and one obtains an improved regularity, namely if we have $(E, L) \in D_h B\,\mathcal{C}^k(\Omega)$ then $(E \cup L) \cap \Omega$ is an h-dimensional $\mathcal{C}^k$-manifold in Ω with boundary $L \cap \Omega$. Indeed Step 1 in the proof of Theorem 5.1 together with [4, Lemma 2.1] and [15, Section 3], imply Step 2 in the proof of Theorem 5.1 with $\mathcal{N} = \Gamma$ of class $\mathcal{C}^k(\Omega)$, hence following the same arguments in Steps 3–4, we get that $(E \cup L) \cap \Omega$ is a $\mathcal{C}^k$-manifold with boundary $L \cap \Omega$.

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