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Boundedness of Critical Points in the Symmetric Mountain Pass Lemma

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We study the boundedness of critical points obtained by the symmetric mountain pass lemma. We construct three types of functionals, which have an unbounded sequence of critical values. For the first one, the set of all critical points is bounded although the set of critical values is unbounded. The second one has both an unbounded sequence and a bounded sequence of critical points. For the last one, the set of critical points is countably infinite and unbounded.

Keywords: Symmetric mountain pass lemma, critical point, variational method.

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1. Introduction

We study the boundedness or unboundedness of critical points obtained by the symmetric mountain pass lemma. This lemma gives a divergent sequence of critical values for a Lagrangian functional and is applicable to superlinear elliptic equations. Let X be an infinite dimensional real Banach space and let $I : X \to \mathbb{R}$ be a C^1 functional in the sense of Fréchet derivative. A point $u \in X$ is called a *critical point* of I if $I'(u) = 0$. A number $c \in \mathbb{R}$ is called a *critical value* if there exists a $u \in X$ such that $I'(u) = 0$ and $I(u) = c$. We state the symmetric mountain pass lemma due to Ambrosetti and Rabinowitz [1].

Assumption 1.1. Let X be an infinite dimensional real Banach space. Suppose that $X = V \oplus W$, where V is finite dimensional and $V \oplus W$ denotes a direct sum. Assume that $I : X \to \mathbb{R}$ is a C^1 functional which satisfies the following conditions.

(I_1) I is even (i.e., $I(-u) = I(u)$ for $u \in X$), $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS),

(PS) any sequence $\{u_k\}$ in X such that $\{I(u_k)\}$ is bounded and $I'(u_k) \to 0$ in X^* as $k \to \infty$ has a convergent subsequence.

(I_2) There exist constants $\alpha, \rho > 0$ such that

$$I(u) \geq \alpha \quad \text{if } \|u\| = \rho \text{ and } u \in W.$$

(I_3) There exist two sequences $\{X_k\}_{k=1}^\infty$ and $\{R_k\}_{k=1}^\infty$ such that X_k is a closed linear subspace of X, $\dim X_k = k$, $X_k \subset X_{k+1}$, $R_k > 0$ for $k \geq 1$ and

$$I(u) \leq 0 \quad \text{if } u \in X_k \text{ and } \|u\| \geq R_k.$$

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Remark 1.2. In [1], [8] or [9], instead of (I_3), they assumed the next assumption.

$(I_3)'$ For any finite dimensional subspace Y of X, there exists an $R > 0$ such that $I(u) \leq 0$ if $u \in Y$ and $\|u\| \geq R$.

The assumption (I_3) is weaker than $(I_3)'$. In [1], it is mentioned that (I_3) is enough for the symmetric mountain pass lemma. Indeed, the proof of the theorem is still valid for (I_3).

We here note that (I_3) is more useful than $(I_3)'$ for the application to elliptic equations. Indeed, some equations satisfy (I_3) but do not satisfy $(I_3)'$. We give an example of such an equation. Consider the semilinear elliptic equation,

$$-\Delta u = a(x)|u|^{p-1}u \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $a(x)$ is a continuous function on $\overline{\Omega}$, $1 < p < \infty$ when $N = 1, 2$ and $1 < p < (N+2)/(N-2)$ when $N \geq 3$. Denote a ball in $\mathbb{R}^N$ centered at x with radius $r > 0$ by $B(x, r)$. We assume that there exist two balls $B(x_i, \varepsilon)$ with $i = 0, 1$ in Ω such that they are disjoint and $a(x) \equiv 0$ in $B(x_0, \varepsilon)$ and $a(x) \geq a_0$ in $B(x_1, \varepsilon)$ with a constant $a_0 > 0$. Define I by

$$I(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}a(x)|u|^{p+1} \right) dx \quad \text{for } u \in H_0^1(\Omega).$$

Then I satisfies (I_3) but does not satisfy $(I_3)'$. We shall show this claim. Let Y be a finite dimensional linear subspace of $X = H_0^1(\Omega)$ whose each element has its support in $B(x_0, \varepsilon)$. For example, we consider the eigenvalue problem,

$$-\Delta u = \lambda u \quad \text{in } B(x_0, \varepsilon), \qquad u = 0 \quad \text{on } \partial B(x_0, \varepsilon). \tag{1}$$

Let λ_k be the k-th eigenvalue and u_k be the corresponding eigenfunction. Extend $u_k(x)$ by putting $u_k(x) := 0$ for $x \in \Omega \setminus B(x_0, \varepsilon)$. Put $Y := \mathrm{span}\{u_1, u_2, \dots, u_k\}$. Here span stands for the linear span. Since $a(x) \equiv 0$ in $B(x_0, \varepsilon)$, we have

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx > 0 \quad \text{for } u \in Y \setminus \{0\}.$$

Hence I does not satisfy $(I_3)'$. Let v_k be the k-th eigenfunction of (1) with $B(x_0, \varepsilon)$ replaced by $B(x_1, \varepsilon)$. Extend v_k by $v_k(x) := 0$ for $x \in \Omega \setminus B(x_1, \varepsilon)$. Define $X_k := \mathrm{span}\{v_1, v_2, \dots, v_k\}$. Since $a(x) \geq a_0$ in $B(x_1, \varepsilon)$ and each $u \in X_k$ has its support in $B(x_1, \varepsilon)$, we have

$$I(u) \leq \int_{B(x_1,\varepsilon)} \left(\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}a_0|u|^{p+1} \right) dx \leq 0,$$

when $u \in X_k$ and $\|u\|_{H_0^1} \geq R_k$ with a large R_k. Thus I satisfies (I_3). Therefore it is important to assume (I_3). □

To state the symmetric mountain pass lemma, we define the *Krasnoselskii genus*.

Definition 1.3. We call a subset A of X *symmetric* if $u \in A$ implies $-u \in A$. Let $\mathcal{E}$ denote the family of sets A such that A is a closed symmetric subset of X and does not contain the origin. For $A \in \mathcal{E}$, we define a *genus* $\gamma(A)$ of A by the smallest integer n such that there exists an odd continuous mapping from A to $\mathbb{R}^n \setminus \{0\}$. If there does not exist such a finite n, we define $\gamma(A) = \infty$. Furthermore, we put $\gamma(\emptyset) = 0$. □

For the properties of the genus, we refer the readers to [8] or [9]. Following [8] and [9], we define a sequence of critical values under Assumption 1.1. Let X_k and R_k be given by (I_3). We define

$$D_k := \{u \in X_k,\ \|u\| \le R_k\}, \quad \partial D_k := \{u \in X_k,\ \|u\| = R_k\}$$

$$G_k := \{g \in C(D_k, X) : g \text{ is odd},\ g(u) = u \text{ on } \partial D_k\},$$

where $C(D_k, X)$ denotes the set of continuous mappings from D_k to X. We define

$$H_k := \{g(\overline{D_m \setminus A}) : g \in G_m,\ m \ge k,\ A \in \mathcal{E},\ \gamma(A) \le m-k\},$$

where $\mathcal{E}$ and $\gamma(A)$ are given by Definition 1.3. We define two sequences

$$a_k := \inf_{g \in G_k} \max_{u \in D_k} I(g(u)), \quad b_k := \inf_{A \in H_k} \max_{u \in A} I(u). \tag{2}$$

Theorem 1.4. ([1] Symmetric mountain pass lemma) *Impose Assumption* 1.1. *Then, for $k > \dim V$, it holds that $a_k \ge b_k \ge \alpha > 0$, a_k and b_k are critical values of I and $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = \infty$. Here α is given by (I_2). If $b_k = b_{k+1} = \cdots = b_{k+p} \equiv b$ and $k > \dim V$, then $\gamma(K_b) \ge p+1$. Here K_b is defined by*

$$K_b := \{u \in X :\ I'(u) = 0,\ I(u) = b\}. \tag{3}$$

Therefore I has a sequence of critical values which diverges to infinity.

Proof. In [8, Propositions 9.29, 9.30, 9.33], it is proved that when $k > \dim V$, $b_k \ge \alpha$, b_k is a critical value diverging to infinity as $k \to \infty$ and $\gamma(K_b) \ge p+1$. The same method as above is valid for a_k also (see [9, Theorem 6.5]).

For any $g \in G_k$, we take $A = \emptyset$ in the definition of H_k. Then $g(D_k) \in H_k$. Therefore

$$b_k \le \max_{v \in g(D_k)} I(v) \quad \text{for } g \in G_k,$$

which shows that $b_k \le a_k$. The proof is complete. □

We consider two conditions.

(I_4) For any $C > 0$, $\sup_{\|u\| \le C} I(u) < \infty$.

$(I_4)'$ If $u_k \in X$ and $I(u_k) \to \infty$ as $k \to \infty$, then $\|u_k\| \to \infty$.

It is easy to verify that the above two conditions are equivalent. When X is finite dimensional, the condition (I_4) (hence $(I_4)'$ also) holds. Indeed, the set of points u satisfying $\|u\| \le C$ is compact if $\dim X < \infty$. However, when X is infinite, (I_4) (and $(I_4)'$) does not necessarily hold. For a critical value c of I, we say that u is a critical point corresponding to c if $I'(u) = 0$ and $I(u) = c$. If (I_4) (hence $(I_4)'$) holds, then any critical point corresponding to a_k or b_k must diverge to infinity in X as $k \to \infty$. If (I_4) does not hold, the above result may not be necessarily valid.

Problem 1.5. *We suppose Assumption* 1.1 *without* (I_4) *and consider two problems.*

(i) *Does any critical point corresponding to a_k or b_k diverge to infinity as $k \to \infty$?*

(ii) *Does there exist a functional I which satisfies Assumption* 1.1 *but the set of all the critical points for I is bounded in X?*

The purpose of the present paper is to solve the problem above. We shall explain the reason why we consider this problem. To this end, we introduce another type of the symmetric mountain pass lemma.

Assumption 1.6. Let X be an infinite dimensional real Banach space and assume that $I \in C^1(X, \mathbb{R})$ satisfies (A1) and (A2) below.

(A1) $I(u)$ is even, bounded from below, $I(0) = 0$ and $I(u)$ satisfies the Palais-Smale condition (PS).

(A2) For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} I(u) < 0$, where

$$\Gamma_k := \{A \in \mathcal{E} : \gamma(A) \geq k\}.$$

Under Assumption 1.6, we define c_k by

$$c_k := \inf_{A \in \Gamma_k} \sup_{u \in A} I(u). \tag{4}$$

We state the symmetric mountain pass lemma due to Clark [2].

Theorem 1.7. ([2] symmetric mountain pass lemma) *Suppose that Assumption 1.6 holds. Then each c_k is a critical value of $I(u)$, $c_k \leq c_{k+1} < 0$ for $k \in \mathbb{N}$ and* $\lim_{k\to\infty} c_k = 0$. *Moreover, if $c_k = c_{k+1} = \cdots = c_{k+p} \equiv c$, then $\gamma(K_c) \geq p + 1$. Here K_c is defined by* (3).

For the proof of the theorem above, we refer the readers to [1], [2], [8] or [9]. The convergence of c_k to zero is proved in [5]. Theorem 1.7 gives us a sequence of critical values, which converges to 0. We give another type of the symmetric mountain pass lemma, which was proved in our paper [5].

Theorem 1.8. ([5]) *Under Assumption* 1.6, *either* (i) *or* (ii) *below holds.*

(i) *There exists a sequence $\{u_k\}$ such that $I'(u_k) = 0$, $I(u_k) < 0$ and $\{u_k\}$ converges to zero.*

(ii) *There exist two sequences $\{u_k\}$ and $\{v_k\}$ such that $I'(u_k) = 0$, $I(u_k) = 0$, $u_k \neq 0$,* $\lim_{k\to\infty} u_k = 0$, *$I'(v_k) = 0$, $I(v_k) < 0$,* $\lim_{k\to\infty} I(v_k) = 0$, *and $\{v_k\}$ converges to a non-zero limit.*

In any case (i) *or* (ii)*, there exists a sequence $\{u_k\}$ of critical points such that $I'(u_k) = 0$, $I(u_k) \leq 0$, $u_k \neq 0$ and* $\lim_{k\to\infty} u_k = 0$.

A similar result to Theorem 1.8 was proved in [4] and [7] also. It seems that u_k given by Theorem 1.8 can be obtained as a critical point corresponding to c_k defined by (4). We consider this problem in the following remark.

Remark 1.9. We shall explain the difference between Theorems 1.7 and 1.8. Theorem 1.7 guarantees the existence of a sequence of critical values converging to zero in $\mathbb{R}$. On the other hand, Theorem 1.8 ensures the existence of a sequence of critical points converging to zero in X. There is a question whether a critical point corresponding to c_k converges to zero in X. This is true under the condition:

$$\text{if } \ I(u) = 0 \ \text{ and } \ I'(u) = 0, \ \text{ then } \ u = 0. \tag{5}$$

Indeed, the condition above with the Palais-Smale condition (PS) ensures that any critical point corresponding to c_k must converge to zero in X. However, if (5) does not hold, then a critical point corresponding to c_k does not necessarily converge to zero. Indeed, we constructed an example of a functional I and a Hilbert space X in [5] such that I is defined on X and it satisfies Assumption 1.6, but $\|u\| \geq c > 0$ whenever $I(u) = c_k$ and $I'(u) = 0$, where c is independent of k. Therefore no critical point corresponding to c_k converges to zero. The similar example was obtained in [4] also. □

Theorem 1.4 is applicable to superlinear elliptic equations and Theorems 1.7 and 1.8 are applicable to sublinear elliptic equations. We compare Assumptions 1.1 with 1.6. The former gives us a sequence of critical values diverging to infinity and the latter provides a sequence of critical values converging to zero. If (5) is assumed, then any critical point corresponding to c_k converges to zero. Even if (5) is not assumed, Theorem 1.8 ensures that there exists a sequence of critical points converging to zero. Therefore, we guess that Assumption 1.1 only (without (I_4)) guarantees the existence of a sequence of critical points diverging to infinity. This is a reason why we consider Problem 1.5. We give a complete answer to Problem 1.5 (i) and provide an almost complete answer to (ii). In fact, we shall construct a functional I which satisfies Assumption 1.1 without (PS) but the set of all critical points for I is bounded. We shall state this result in Theorem 2.1.

Moreover, we shall construct a functional J, which satisfies Assumption 1.1 and the condition below: The set of critical values for J consists only of nonnegative integers and there exists two sequences $\{u_n\}$ and $\{v_n\}$ such that

$$J'(u_n) = J'(v_n) = 0, \quad J(u_n) = J(v_n) = n, \quad \lim_{n\to\infty} \|u_n\| = \infty, \quad \sup_{n\in\mathbb{N}} \|v_n\| < \infty.$$

Thus a sequence $\{n\}$ of critical values diverges to infinity, however a sequence $\{u_n\}$ of critical points is unbounded and another sequence $\{v_n\}$ is bounded. We shall state this result in Theorem 2.2.

Furthermore, we shall construct a functional K which satisfies Assumption 1.1 such that the set of critical points for K is countably infinite and unbounded. This result will be stated in Theorem 2.3.

The present paper is organized in five sections. In Section 2, we state the main theorems. In Section 3, 4 and 5, we prove Theorems 2.3, 2.1 and 2.2, respectively.

2. Main results

In this section, we state the main theorems.

Theorem 2.1. *Let X be a separable infinite dimensional real Hilbert space. Then there exists a functional $I : X \to \mathbb{R}$ such that I satisfies Assumption* 1.1 *without* (PS)*, the set of all critical values for I is unbounded and the set of all critical points for I is bounded., i.e.,*

$$\sup\{I(u) : I'(u) = 0\} = \infty, \quad \sup\{\|u\| : I'(u) = 0\} < \infty.$$

Since the functional I in Theorem 2.1 does not satisfy (PS) (see Remark 4.3), we do not know whether a_k and b_k given by (2) are critical values and diverge to infinity as $k \to \infty$. Thus our theorem is not a complete answer to Problem 1.5 (ii), however it ensures the existence of a functional I such that the set of critical values for I is unbounded and the set of critical points is bounded. In the next theorem, we give a different functional from I in Theorem 2.1.

Theorem 2.2. *Let X be a separable infinite dimensional real Hilbert space. Then there exists a functional $J : X \to \mathbb{R}$ such that J satisfies Assumption* 1.1 *and the following condition: The set of critical values for J consists only of nonnegative integers and there exist two sequences $\{u_n\}$ and $\{v_n\}$ such that*

$$J'(u_n) = J'(v_n) = 0,\ J(u_n) = J(v_n) = n,\ \lim_{n\to\infty} \|u_n\| = \infty,\ \sup_{n\in\mathbb{N}} \|v_n\| < \infty. \tag{6}$$

Since the functional J in Theorem 2.2 satisfies Assumption 1.1, a_k and b_k are well defined by (2) with I replaced by J and they are critical values of J. Since we have $a_k \geq b_k \geq \alpha > 0$, a_k and b_k are positive integers by Theorem 2.2. Let u_n and v_n be defined by Theorem 2.2. Substitute $n = a_k$ in u_n and v_n and put $U_k := u_{a_k}$ and $V_k := v_{a_k}$. Substituting $n = a_k$ in (6), we have

$$J'(U_k) = J'(V_k) = 0, \quad J(U_k) = J(V_k) = a_k.$$

Therefore there exist two sequences of critical points corresponding to a_k such that one diverges to infinity and another is bounded in X. The result above is valid for b_k also.

In the next theorem, we give an example of a functional such that the set of critical points is unbounded.

Theorem 2.3. *Let X be a separable infinite dimensional real Hilbert space. Then there exists a functional $K : X \to \mathbb{R}$ such that K satisfies Assumption 1.1 and the following condition: The set of all critical points for K is a countably infinite set $\{u_k\}_{k=1}^{\infty}$ and $\lim_{k\to\infty} \|u_k\| = \infty$. In particular, any sequence of critical points corresponding to a_k (or b_k) must diverge to infinity, i.e., If $K(v_k) = a_k$ (or b_k) and $K'(v_k) = 0$, then $\lim_{k\to\infty} \|v_k\| = \infty$.*

Theorems 2.2 and 2.3 give an answer to Problem 1.5 (i).

3. Proof of Theorem 2.3

The proof of Theorem 2.3 is easiest in all the proofs of the theorems. Therefore we shall prove it in this section. We first show that it is enough to prove Theorem 2.1 (2.2 or 2.3) for a certain Hilbert space.

Lemma 3.1. *Let X and Y be any separable infinite dimensional real Hilbert spaces. If Theorem* 2.1 (2.2 *or* 2.3) *is valid for X, then it holds for Y also.*

Proof. Let X and Y be separable infinite dimensional real Hilbert spaces. Then there exists a linear isometric isomorphism $T : Y \to X$, i.e., T is bijective, linear and

$$\|Tu\|_X = \|u\|_Y, \quad (Tu, Tv)_X = (u, v)_Y \quad \text{for } u, v \in Y, \tag{7}$$

where $\|\cdot\|_Z$ and $(\cdot,\cdot)_Z$ are norms and inner products for $Z = X, Y$.

In what follows, we write $\|\cdot\|$ instead $\|\cdot\|_X$ and $\|\cdot\|_Y$. Let $I : X \to \mathbb{R}$ satisfy Theorem 2.1. Define $J := I \circ T$. Then J is a real valued functional defined on Y. We shall show that J satisfies Theorem 2.1 with X replaced by Y. It is well known that T is of C^∞ class. Indeed, differentiating the relation

$$T(x+ty) = Tx + tTy \quad \text{for } t \in \mathbb{R},$$

with respect to t at $t = 0$, we have $T'(x)y = Ty$. Hence $T''(x)[y,z] = 0$. Thus T is of C^∞ class. Since $I \in C^1(X,\mathbb{R})$, J belongs to $C^1(Y,\mathbb{R})$. We shall show that J satisfies Assumption 1.1. Since I is even and T is linear, J is even. It is clear that $J(0) = I(T(0)) = 0$. We shall confirm (PS) for J. Since $J'(u)v = I'(Tu)Tv$, it holds that $\|J'(u)\| = \|I'(Tu)T\|$. We shall show that the right hand side is equal to $\|I'(Tu)\|$. Indeed, for $v \in Y$, we put $w = Tv$. Using (7), we obtain

$$\|I'(Tu)T\| = \sup_{\|v\|=1} \|I'(Tu)Tv\| = \sup_{\|w\|=1} \|I'(Tu)w\| = \|I'(Tu)\|.$$

Accordingly,
$$\|J'(u)\| = \|I'(Tu)\|. \tag{8}$$

Suppose that a sequence $\{u_n\} \subset Y$ satisfies

$$\lim_{n\to\infty} \|J'(u_n)\| = 0, \quad \sup_n |J(u_n)| < \infty.$$

By (8), we have $\lim_{n\to\infty} \|I'(Tu_n)\| = 0, \quad \sup_n |I(Tu_n)| < \infty.$

Put $v_n := Tu_n$. By (PS) for I, $\{v_n\}$ has a convergent subsequence (denoted by $\{v_n\}$ again) and we denote the limit by v_∞. Then $u_n = T^{-1}v_n \to T^{-1}v_\infty$. Therefore J satisfies (PS).

Since $X = V \oplus W$, it holds that $Y = T^{-1}X = T^{-1}V \oplus T^{-1}W$. Then J satisfies (I_2) with W replaced by $T^{-1}W$. Similarly, J satisfies (I_3) with X_k replaced by $T^{-1}X_k$.

Let $\{u_k\}$ be a sequence in Y. Put $v_k := Tu_k$. Then $J(u_k) = I(v_k)$. It follows from (8) that u_k is a critical point of J if and only if v_k is a critical point of I. Since $\|u_k\| = \|v_k\|$, we find that $\sup_{k\in\mathbb{N}} \|u_k\| < \infty$ if and only if $\sup_{k\in\mathbb{N}} \|v_k\| < \infty$; $\lim_{k\to\infty} \|u_k\| = \infty$ if and only if $\lim_{k\to\infty} \|v_k\| = \infty$. These assertions prove that Theorem 2.1 (2.2 or 2.3) for X implies that for Y. The proof is complete. □

We shall show Theorem 2.3. Define $X := H_0^1(0,1)$, which is a usual Sobolev space. Because of the Poincaré inequality, we can define the $H_0^1(0,1)$-norm by

$$\|u\|_{H_0^1} := \left(\int_0^1 u'(x)^2 dx \right)^{1/2}.$$

We consider the one-dimensional Emden-Fowler equation,

$$u'' + |u|^{p-1}u = 0 \quad \text{in } (0,1), \quad u(0) = u(1) = 0, \tag{9}$$

where $1 < p < \infty$. We extract the first equation from (9) and consider

$$u'' + |u|^{p-1}u = 0. \tag{10}$$

The next lemma is well known (for example, see [3] or [6, Lemma 3.1]).

Lemma 3.2. *Let u be a nontrivial solution of* (10)*. Then $u(x)$ is a periodic solution having zeros. For any $\lambda > 0$, $\lambda^{2/(p-1)}u(\lambda x)$ is also a solution of* (10).

Using the lemma above, we obtain the next one.

Lemma 3.3. *For each positive integer n,* (9) *has a unique solution u which has exactly $n-1$ zeros in* $(0,1)$ *and* $u'(0)>0$*. Denote it by $u_n(x)$. Then the set of all solutions for* (9) *consists only of $\pm u_n(x)$ with $n\in\mathbb{N}$ and the zero solution. Moreover,* $\lim_{n\to\infty}\|u_n\|_{H_0^1}=\infty$.

Proof. Denote a unique solution of (10) satisfying $u(0)=0$ and $u'(0)=1$ by $\phi(x)$. Let $T>0$ be the first zero of $\phi(x)$ in $(0,\infty)$. Then $\phi(nT)=0$ for any $n\in\mathbb{N}$. Define $u_n(x):=(nT)^{2/(p-1)}\phi(nTx)$. By Lemma 3.2, it is a solution of (9) which has exactly $n-1$ zeros in $(0,1)$ and satisfies $u_n'(0)>0$. If $v(x)$ is a solution of (9) which has exactly $n-1$ zeros in $(0,1)$ and $v'(0)>0$, then $v(x)=\lambda^{(p-1)/2}\phi(\lambda x)$ with some $\lambda>0$. Indeed, put $w(x):=\lambda^{(p-1)/2}\phi(\lambda x)$. Here we choose λ which satisfies $v'(0)=w'(0)$. Since $v(0)=w(0)=0$ and $v'(0)=w'(0)$, $v(x)$ is identically equal to $w(x)$. Hence $v(x)=\lambda^{(p-1)/2}\phi(\lambda x)$. Since $v(1)=0$ and $v(x)$ has exactly $n-1$ zeros in $(0,1)$, $\lambda=nT$ and so $v(x)=u_n(x)$. Therefore the set $\pm u_n(x)$ with $n\in\mathbb{N}$ covers all the nontrivial solutions of (9).

Using the change of variables $nTx=t$, we compute

$$\|u_n\|_{H_0^1}^2=(nT)^{2(p+1)/(p-1)}\int_0^1\phi'(nTx)^2dx=n^{2(p+1)/(p-1)}T^{(p+3)/(p-1)}\int_0^T\phi'(t)^2dt.$$

Therefore the $H_0^1(0,1)$ norm of u_n diverges to infinity as $n\to\infty$. □

We define a functional

$$K(u):=\int_0^1\left(\frac{1}{2}u'(x)^2-\frac{1}{p+1}|u(x)|^{p+1}\right)dx\quad\text{for } u\in X=H_0^1(0,1).$$

Then u is a critical point of K if and only if it satisfies (9). Therefore

$$\{u\in X: K'(u)=0\}=\{\pm u_n(x): n\in\mathbb{N}\}\cup\{0\},\tag{11}$$

where u_n is given by Lemma 3.3.

Proof of Theorem 2.3. K defined above satisfies Assumption 1.1 (see [8] or [9]). By (11), the set of critical points for K is countably infinite. By Lemma 3.3, $\|u_n\|_{H_0^1}\to\infty$ as $n\to\infty$. The proof is complete. □

4. Proof of Theorem 2.1

In this section, we shall prove Theorem 2.1. We shall construct a functional I in $X:=\ell^2$, which is defined by

$$\ell^2:=\{x=(x_1,x_2,x_3,\dots):\ x_i\in\mathbb{R},\ \sum_{i=1}^\infty x_i^2<\infty\}.$$

This is a Hilbert space equipped with the inner product and the norm,

$$(x,y)_2:=\sum_{i=1}^\infty x_iy_i,\quad \|x\|_2:=\left(\sum_{i=1}^\infty x_i^2\right)^{1/2},\tag{12}$$

for $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$.

We choose a C^∞ function $h(t)$ such that $0 \le h(t) \le 1$ in $[0,\infty)$ and

$$h(t) = 0 \quad \text{in } [0,1/4], \quad h(t) = 1 \quad \text{in } [1/2,\infty), \quad h'(t) \ge 0 \quad \text{in } [0,\infty). \tag{13}$$

For a positive integer k, we define $g_k(t) := (k-1)th(t) + t$. Then $g_k(t)$ satisfies

$$g_k(t) = t \quad \text{for } t \in [0,1/4], \qquad g_k(t) = kt \quad \text{for } t \ge 1/2, \tag{14}$$

$$t \le g_k(t) \le kt, \quad 1 \le g_k'(t) \le Ck \quad \text{for } t \ge 0, \tag{15}$$

with some $C > 0$ independent of k. Put $X := \ell^2$ and define

$$I(x) := \left(\sum_{k=1}^{\infty} g_k(x_k^2) \right) \left(2 - \|x\|_2^2\right) \quad \text{for } x = (x_1, x_2, \ldots) \in X, \tag{16}$$

where $\|x\|_2$ is defined by (12). We shall show that I is well defined in $X = \ell^2$. In what follows, for a point x in X, we denote its k-th coordinate by x_k. Let $x \in X$ be any point. Since $\lim_{k\to\infty} x_k = 0$, we have $x_k^2 < 1/4$ for $k \ge k_0$ with some k_0. By (14), $\sum_{k=k_0}^{\infty} g_k(x_k^2) = \sum_{k=k_0}^{\infty} x_k^2 < \infty$. Thus $I(x)$ is well defined.

Lemma 4.1. *I defined by* (16) *is a C^1 functional. Its Fréchet derivative is given by*

$$I'(x)y = 2\left(\sum_{k=1}^{\infty} g_k'(x_k^2) x_k y_k \right)(2 - \|x\|_2^2) - 2\left(\sum_{k=1}^{\infty} g_k(x_k^2) \right) \left(\sum_{n=1}^{\infty} x_n y_n \right). \tag{17}$$

Proof. We put

$$G(x) := \sum_{k=1}^{\infty} g_k(x_k^2), \quad H(x) := 2 - \|x\|_2^2. \tag{18}$$

Then $I(x) = G(x)H(x)$. We shall compute the derivatives of G and H. It is well known that

$$H'(x)y = -2(x,y)_2 = -2\sum_{n=1}^{\infty} x_n y_n,$$

where $(x,y)_2$ is the inner product defined by (12). We shall show that G is a C^1 functional and

$$G'(x)y = 2\sum_{k=1}^{\infty} g_k'(x_k^2) x_k y_k. \tag{19}$$

Once this relation would be proved, (17) follows. For any $x \in X$, there exists a positive integer k_0 such that $x_k^2 < 1/4$ for $k \ge k_0$. By (14), $g_k'(x_k^2) = 1$ for $k \ge k_0$, and hence the right hand side of (19) converges.

We compute the Gâteaux derivative of G. Fix $x, y \in X$ arbitrarily.

For $t \in \mathbb{R}$, we define $\phi(t) := G(x+ty)$, $\phi_k(t) := g_k((x_k + ty_k)^2)$.

Then $\phi(t) = \sum_{k=1}^{\infty} \phi_k(t)$. Choose $k_0 \in \mathbb{R}$ such that $x_k^2 + y_k^2 < 1/8$ for $k \ge k_0$.

For $|t| \le 1$, we have

$$(x_k + ty_k)^2 \le 2(x_k^2 + y_k^2) < 1/4 \quad \text{for } k \ge k_0,$$

and so, $\phi_k(t) = (x_k + ty_k)^2$ for $k \ge k_0$. Since $0 \le \phi_k(t) \le 2(x_k^2 + y_k^2)$ and $x, y \in \ell^2$, $\sum_{k=1}^{\infty} \phi_k(t)$ uniformly converges on $[-1,1]$.

Since $g_k'(t) = 1$ for $|t| \le 1/4$, we have

$$\phi_k'(t) = 2g_k'((x_k + ty_k)^2)(x_k + ty_k)y_k = 2(x_k + ty_k)y_k \quad \text{for } k \ge k_0.$$

Thus $\sum_{k=1}^{\infty} \phi_k'(t)$ uniformly converges on $[-1, 1]$. Therefore we have

$$\frac{d}{dt}G(x + ty) = \phi'(t) = 2\sum_{k=1}^{\infty} g_k'((x_k + ty_k)^2)(x_k + ty_k)y_k.$$

Substituting $t = 0$, we obtain $\displaystyle \frac{d}{dt}G(x + ty)|_{t=0} = 2\sum_{k=1}^{\infty} g_k'(x_k^2)x_k y_k.$

This is a Gâteaux derivative of G. Denote the right hand side by $F(y)$, i.e.,

$$F(y) := 2\sum_{k=1}^{\infty} g_k'(x_k^2)x_k y_k.$$

We shall show that F is a bounded linear functional on X. We fix x and vary y. Choose k_1 such that $x_k^2 < 1/4$ for $k \ge k_1$. Since $g_k'(x_k^2) = 1$ for $k \ge k_1$ and $1 \le g_k'(t) \le Ck$ by (15), we have

$$|F(y)| \le 2\sum_{k=1}^{k_1-1} Ck|x_k||y_k| + 2\sum_{k=k_1}^{\infty} |x_k||y_k| \le C'\|x\|_2\|y\|_2.$$

Thus F is bounded linear. Since the Gâteaux derivative is a bounded linear functional, it coincides with the Fréchet derivative and we have

$$G'(x)y = 2\sum_{k=1}^{\infty} g_k'(x_k^2)x_k y_k.$$

We shall show the continuity of $G'(x)$. We denote the operator norm of $G'(x)$ by $\|G'(x)\|$. For $x, y \in X$, we have

$$\|G'(x) - G'(y)\|^2 = 4\sum_{k=1}^{\infty} \left(g_k'(x_k^2)x_k - g_k'(y_k^2)y_k\right)^2.$$

Fix $y \in X$. We shall show that $\|G'(x) - G'(y)\|^2 \to 0$ as $x \to y$ in X. Choose a positive integer k_0 such that $|y_k| < 1/4$ for $k \ge k_0$. Let $\|x - y\|_2 < 1/4$. Hence $|x_k - y_k| < 1/4$ for all k. In particular, $|x_k| < 1/2$ and $x_k^2 < 1/4$ for $k \ge k_0$. Therefore $g_k'(x_k^2) = g_k'(y_k^2) = 1$ for $k \ge k_0$, and so

$$\|G'(x) - G'(y)\|^2 = 4\sum_{k=1}^{k_0-1} \left(g_k'(x_k^2)x_k - g_k'(y_k^2)y_k\right)^2 + 4\sum_{k=k_0}^{\infty} (x_k - y_k)^2 \to 0,$$

as $\|x - y\|_2 \to 0$. Consequently, G is a C^1 functional and so is I. □

Observing (17), we define $z_k(x)$ and $z(x)$ by

$$z_k(x) := 2x_k\{g_k'(x_k^2)(2 - \|x\|_2^2) - G(x)\}, \quad z(x) := (z_1(x), z_2(x), \ldots) \in \ell^2. \tag{20}$$

Then (17) can be rewritten as

$$I'(x)y = (z(x), y)_2 \quad \text{for } x, y \in \ell^2. \tag{21}$$

By the Riesz representation theorem, any bounded linear functional is written as an inner product. The relation (21) is exactly this representation.

We shall show that I satisfies (I_2) and (I_3) with $V = \{0\}$ and $W = X$.

Lemma 4.2. *I satisfies (I_2) with $\alpha = 1$ and $W = X$ and fulfills (I_3).*

Proof. Since $I(u) \le 0$ for $\|x\|_2 \ge \sqrt{2}$, (I_3) holds. Let $G(x)$ be given by (18). Since $G(x) \ge \|x\|_2^2$ by (15), we have

$$I(x) = G(x)(2 - \|x\|_2^2) \ge \|x\|_2^2(2 - \|x\|_2^2) = 1,$$

provided that $\|x\|_2 = 1$. Thus (I_2) holds with $\rho = \alpha = 1$. The proof is complete. □

Remark 4.3. I does not satisfy (PS). Indeed, we define

$$x_n := (1/2, 1/2, 1/2, 0, \dots, 0, 1/2, 0, 0, \dots),$$

where the first three and the n-th coordinates of x_n are $1/2$ and the others are 0. Then $\|x_n\|_2 = 1$, $G(x_n) = 1$, $I(x_n) = 1$ and $I'(x_n) = 0$, however $\{x_n\}$ has no subsequence converging.

Proof of Theorem 2.1. It is clear that I is even, $I(0) = 0$ and I fulfills (I_2) and (I_3) by Lemma 4.2.

Let e_n be a point in X whose n-th coordinate is 1 and the others are 0, i,e.,

$$e_n = (0, \dots, 0, 1, 0, 0, \dots) \in X. \tag{22}$$

Since $g_n'(1) = n$ and $G(e_n) = g_n(1) = n$, it holds that $z_k(e_n) = 0$ for all k and n.

Therefore $I'(e_n) = 0$. Moreover $I(e_n) = n$, $I'(0) = 0$ and $I(0) = 0$. Consequently, all nonnegative integers are critical values and hence the set of critical values is unbounded.

We shall show that the set of all critical points for $I(x)$ is bounded. Let x be any point satisfying $\|x\|_2 \ge \sqrt{2}$. Suppose that $I'(x) = 0$. Let $z_k(x)$ be given by (20). Then $z_k(x) = 0$ for all k. Since $x \ne 0$, $x_k \ne 0$ at some k. Then

$$g_k'(x_k^2)(2 - \|x\|_2^2) - G(x) = 0.$$

However the left hand side is negative because $G(x) > 0$ and $g'(x_k^2) \ge 1$. A contradiction occurs. Therefore $I'(x) \ne 0$. Consequently, we have

$$\{x : I'(x) = 0\} \subset \{x : \|x\|_2 < \sqrt{2}\}.$$

The proof is complete. □

5. Proof of Theorem 2.2

In this section, we prove Theorem 2.2 for $X := \ell^2$. Let $h(t)$ satisfy (13). For each positive integer k, we define

$$f_k(t) := kt^2(2-t^2)h(t) + t^2(1-h(t)) \quad \text{for } t \geq 0.$$

By (13), we have

$$f_k(t) = t^2 \quad \text{in } [0,1/4], \quad f_k(t) = kt^2(2-t^2) \quad \text{in } [1/2,\infty). \tag{23}$$

We shall show that

$$f_k'(t) > 0 \quad \text{in } (0,1), \quad f_k'(t) < 0 \quad \text{in } (1,\infty), \quad f_k'(0) = f_k'(1) = 0. \tag{24}$$

By (23), $f_k'(0) = f_k'(1) = 0$ and

$$f_k'(t) > 0 \quad \text{in } (0,1/4] \cup [1/2,1), \quad f_k'(t) < 0 \quad \text{in } (1,\infty).$$

Differentiating $f_k(t)$, we have

$$f_k'(t) = 4kt(1-t^2)h(t) + [k(2-t^2)-1]t^2h'(t) + 2t(1-h(t)) > 0,$$

for $t \in [1/4,1/2]$. Thus (24) holds. We extend the function $f_k(t)$ as an even function by putting $f_k(-t) := f_k(t)$. We define

$$J(x) := \sum_{k=1}^{\infty} f_k(x_k) \quad \text{for } x = (x_1,x_2,x_3,\ldots) \in X = \ell^2. \tag{25}$$

Let $x \in X$. Then $|x_k| < 1/4$ for $k \geq k_0$ with some k_0. Therefore

$$\sum_{k=k_0}^{\infty} f_k(x_k) = \sum_{k=k_0}^{\infty} x_k^2 < \infty$$

and J is well-defined. We shall verify (I_2) and (I_3).

Lemma 5.1. *J given by* (25) *satisfies* (I_2) *and* (I_3).

Proof. Let $0 < \rho < 1/4$ and let $\|x\|_2 = \rho$. Since $|x_k| \leq \|x\|_2 < 1/4$, $f_k(x_k) = x_k^2$ for all k and we have $J(x) = \|x\|_2^2 = \rho^2$. This proves (I_2).

Let n be a positive integer. We define

$$X_n := \operatorname{span}\{e_1,e_2,\ldots,e_n\} = \{(x_1,x_2,\ldots,x_n,0,0,\ldots) : \ x_i \in \mathbb{R}\},$$

where e_n is defined by (22). Let $R \in (\sqrt{n},\infty)$ be so large that

$$\frac{n(n+1)}{2} + t^2(2-t^2) < 0 \quad \text{for } |t| \geq R/\sqrt{n}. \tag{26}$$

We shall show that J satisfies (I_3). Let $x \in X_n$ and $\|x\|_2 \geq R$. Then $\sum_{k=1}^n x_k^2 \geq R^2$ and hence $|x_i| \geq R/\sqrt{n}$ with a certain i. Since $|x_i| \geq R/\sqrt{n} \geq 1$, it holds that $h(x_i) = 1$ and $f_i(x_i) = ix_i^2(2-x_i^2)$. By (24), $f_k(t)$ attains its maximum at $t = 1$. Accordingly, $f_k(t) \leq f_k(1) = k$ for all $t \in \mathbb{R}$. Using (26), we obtain

$$J(x) = \sum_{k\neq i, 1\leq k\leq n} f_k(x_k) + f_i(x_i) \leq \frac{n(n+1)}{2} + ix_i^2(2-x_i^2) < 0.$$

Consequently, J satisfies (I_3). The proof is complete. □

In the same way as in the proof of Lemma 4.1, we can prove the next lemma.

Lemma 5.2. *J is a C^1 functional and it holds that*

$$J'(x)y = \sum_{k=1}^{\infty} f_k'(x_k)y_k, \quad \|J'(x)\|^2 = \sum_{k=1}^{\infty} f_k'(x_k)^2 \quad \text{for } x, y \in X. \tag{27}$$

Lemma 5.3. *J satisfies the Palais-Smale condition.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in X satisfying

$$\lim_{n\to\infty} \|J'(x_n)\| = 0, \quad \sup_{n\in\mathbb{N}} |J(x_n)| \le C, \tag{28}$$

with some $C > 0$. We use the notation $x_n = (x_{n,1}, x_{n,2}, x_{n,3}, \ldots)$. Choose a positive integer n_0 satisfying

$$\|J'(x_n)\| \le 1 \quad \text{for } n \ge n_0. \tag{29}$$

By (27), we have
$$|f_k'(x_{n,k})| \le \|J'(x_n)\| \quad \text{for } n, k \in \mathbb{N}. \tag{30}$$

We fix a positive integer k_0 grater than $2C$, where C is given by (28). We show that

$$\lim_{n\to\infty} x_{n,k} = 0 \quad \text{for } k \ge k_0. \tag{31}$$

We choose $\delta \in (0, 1/4)$ so small that

$$t^2(2-t^2) > \frac{1}{2} \quad \text{when } |t-1| < \delta \text{ or } |t+1| < \delta,$$

which implies that

$$f_k(t) > \frac{k}{2} \quad \text{when } |t-1| < \delta \text{ or } |t+1| < \delta. \tag{32}$$

Give $\varepsilon \in (0, \delta)$ arbitrarily. We denote an ε-neighborhood of $t = 0, 1, -1$ by $A(\varepsilon)$ and its complement by $B(\varepsilon)$;

$$A(\varepsilon) := (-\varepsilon, \varepsilon) \cup (1-\varepsilon, 1+\varepsilon) \cup (-1-\varepsilon, -1+\varepsilon), \quad B(\varepsilon) := \mathbb{R} \setminus A(\varepsilon).$$

We put $c_0(\varepsilon) := \inf\{|f_k'(t)| : t \in B(\varepsilon),\ k \in \mathbb{N}\}$. Then $c_0(\varepsilon) > 0$.
Choose $n_1(\varepsilon) \in \mathbb{N}$ so large that $\|J'(x_n)\| < c_0(\varepsilon)$ for $n \ge n_1(\varepsilon)$. By (30), we have

$$|f_k'(x_{n,k})| \le \|J'(x_n)\| < c_0(\varepsilon) \quad \text{for } n \ge n_1(\varepsilon).$$

The inequality above with the definition of $c_0(\varepsilon)$ implies that $x_{n,k}$ belongs to $A(\varepsilon)$ for $n \ge n_1(\varepsilon)$ and $k \in \mathbb{N}$. That is, for $n \ge n_1(\varepsilon)$ and $k \in \mathbb{N}$, $x_{n,k}$ satisfies one of the following inequalities:

$$|x_{n,k}| < \varepsilon, \quad |x_{n,k} - 1| < \varepsilon, \quad |x_{n,k} + 1| < \varepsilon. \tag{33}$$

These inequalities imply that

$$f_k(x_{n,k}) \ge 0 \quad \text{for } n \ge n_1(\varepsilon),\ k \in \mathbb{N}. \tag{34}$$

We shall show that

$$|x_{n,k} - 1| \geq \varepsilon \quad \text{for } n \geq n_1(\varepsilon),\ k \geq k_0, \tag{35}$$

where k_0 has already been defined before (31). Let $n \geq n_1(\varepsilon)$ and $k \geq k_0$. Suppose to the contrary that $|x_{n,k} - 1| < \varepsilon$. By (28), (34) and (32), we have

$$C \geq J(x_n) = \sum_{j=1}^{\infty} f_j(x_{n,j}) \geq f_k(x_{n,k}) > \frac{k}{2} \geq \frac{k_0}{2}.$$

This contradicts the choice of k_0. Therefore (35) holds. In the same way, we have

$$|x_{n,k} + 1| \geq \varepsilon \quad \text{for } n \geq n_1(\varepsilon),\ k \geq k_0. \tag{36}$$

By (33), (35) and (36), we conclude that

$$|x_{n,k}| < \varepsilon \quad \text{for } n \geq n_1(\varepsilon),\ k \geq k_0. \tag{37}$$

This shows (31).

By (29) and (30), we have $|f_k'(x_{n,k})| \leq 1$ for $k \geq 1$ and $n \geq n_0$. If $|x_{n,k}| \geq 2$ with some k and $n \geq n_0$, then $|f_k'(x_{n,k})| = 4k|x_{n,k}(1 - x_{n,k}^2)| \geq 24$ and a contradiction occurs. Therefore

$$|x_{n,k}| \leq 2 \quad \text{for } k \geq 1 \text{ and } n \geq n_0.$$

Since $x_{n,k}$ is bounded, we choose a subsequence (denoted by $\{x_n\}$ again) of $\{x_n\}$ such that for each $1 \leq k \leq k_0 - 1$, $x_{n,k}$ converges to a limit $x_{\infty,k}$ as $n \to \infty$. Put $x_{\infty,k} := 0$ for $k \geq k_0$ and define $x_\infty := (x_{\infty,1}, x_{\infty,2}, x_{\infty,3}, \ldots)$. Then x_∞ belongs to X. It follows from (31) that

$$\lim_{n\to\infty} x_{n,k} = x_{\infty,k} \quad \text{for } k \in \mathbb{N}. \tag{38}$$

Since $\varepsilon < \delta < 1/4$, it follows from (37) that $f_k'(x_{n,k}) = 2x_{n,k}$ for $n \geq n_1(\varepsilon)$ and $k \geq k_0$. Then we have

$$4\sum_{k=k_0}^{\infty} x_{n,k}^2 = \sum_{k=k_0}^{\infty} f_k'(x_{n,k})^2 \leq \|J'(x_n)\|^2 \to 0 \quad \text{as } n \to \infty. \tag{39}$$

Since $x_{\infty,k} = 0$ for $k \geq k_0$, (38) and (39) show that

$$\|x_n - x_\infty\|_2^2 = \sum_{k=1}^{k_0-1} (x_{n,k} - x_{\infty,k})^2 + \sum_{k=k_0}^{\infty} x_{n,k}^2 \to 0 \quad \text{as } n \to \infty.$$

Thus J satisfies (PS). The proof is complete. □

To prove Theorem 2.2, we investigate the critical points and the critical values of J. We denote the set of all critical points for J by CP and the set of all critical values for J by CV, i.e.,

$$CP := \{u : J'(u) = 0\}, \quad CV := \{J(u) : J'(u) = 0\}.$$

Lemma 5.4. *We have*

$$CP = \{x = (x_1, x_2, \ldots) \in X : \textit{each} \ \ x_i = 0, 1, \ \ \textit{or} \ \ -1\}, \tag{40}$$

$$CV = \mathbb{N} \cup \{0\}. \tag{41}$$

By (40), we mean that any coordinate of x in CP is equal to 0, 1 or -1. This says that $x_k = 0$ for $k \geq k_0$ with some k_0 because $x \in \ell^2$.

Proof of Lemma 5.4. $J'(x) = 0$ if and only if $f_k'(x_k) = 0$ for all k. Since $f_k'(t)$ vanishes at $t = 0, \pm 1$ only, we find that $J'(x) = 0$ is equivalent to $x_k = 0, 1$ or -1 for all k. This proves (40).

Since e_n is a critical point by (40), $J(e_n) = n$ is a critical value. Moreover $f(0) = 0$ is also a critical value. Therefore we have $\mathbb{N} \cup \{0\} \subset CV$. Let x be in CP. By (40), we have

$$x = (x_1, x_2, \ldots, x_{k_0 - 1}, 0, 0, \ldots)$$

with some k_0 and each $x_i = 0, 1$ or -1. Since $f_k(\pm 1) = k$ and $f_k(0) = 0$, we conclude that $J(x) = \sum_{k=1}^{k_0 - 1} f_k(x_k)$ is a nonnegative integer. Hence $CV \subset \mathbb{N} \cup \{0\}$ and (41) follows. The proof is complete. □

Theorem 2.2 ensures that there exists an unbounded sequence of critical points. To obtain such a sequence, we prove the following lemma.

Lemma 5.5. *For any integer $n \geq 3$, there exists a point $u_n \in X$ such that*

$$J'(u_n) = 0, \quad J(u_n) = n, \quad \|u_n\|_2 \geq (n/2)^{1/4}. \tag{42}$$

Proof. Let $n \geq 3$ be any integer. We choose the largest integer $m \in \mathbb{N}$ satisfying $m(m+1)/2 \leq n$. We divide the proof into two cases.

Case 1. $n = m(m+1)/2$. Then we define

$$u := e_1 + e_2 + \cdots + e_m = (1, 1, \ldots, 1, 0, 0, \ldots),$$

where e_m is defined by (22). We find that $J'(u) = 0$ by (40) and

$$J(u) = \sum_{k=1}^{m} f_k(1) = 1 + 2 + \cdots + m = \frac{m(m+1)}{2} = n.$$

Moreover, $\|u\|_2 = (1 + 1 + \cdots + 1)^{1/2} = \sqrt{m} \geq n^{1/4}$, because $n = m(m+1)/2 \leq m^2$. Therefore u satisfies (42).

Case 2. $n > m(m+1)/2$. We choose the largest positive integer $p \in \mathbb{N}$ satisfying

$$n > \frac{p(p+1)}{2} + p. \tag{43}$$

Such an integer p exists because $n \geq 3$. Put $q := n - p(p+1)/2$. Note that $p < q$.
We define $u := e_1 + e_2 + \cdots + e_p + e_q = (1, 1, \ldots, 1, 0, \ldots, 0, 1, 0, \ldots)$.
Then $J'(u) = 0$ and $J(u) = 1 + 2 + \cdots + p + q = n$.

Since p is the largest integer satisfying (43), we have

$$\frac{(p+1)(p+2)}{2} + p + 1 \geq n,$$

which shows that $4(p+1)^2 \geq (p+1)(p+4) \geq 2n$. Using this inequality, we compute

$$\|u\|_2 = (1 + 1 + \cdots + 1)^{1/2} = (p+1)^{1/2} \geq (n/2)^{1/4}.$$

The proof is complete. □

We conclude this paper by proving Theorem 2.2.

Proof of Theorem 2.2. By Lemmas 5.1, 5.2 and 5.3, J defined by (25) satisfies Assumption 1.1. By Lemma 5.4, the set of critical values for J consists only of nonnegative integers. We define $v_n := e_n$, which is given by (22). Put $u_1 := e_1$ and $u_2 := e_2$. When $n \geq 3$, we define u_n by Lemma 5.5. By (42), we have

$$J'(u_n) = J'(v_n) = 0, \quad J(u_n) = J(v_n) = n, \quad \lim_{n\to\infty} \|u_n\|_2 = \infty, \quad \|v_n\|_2 = 1.$$

The proof is complete. □

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