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Hyperplane Separation Type Theorems in Hadamard Manifolds

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We prove a hyperplane separation theorem for disjoint and convex sets in two-dimensional Hadamard manifolds. Furthermore, in higher dimensions we prove such a theorem if additionally the two involved sets are closed. This extends a recent generalization of J. Shenawy [*Horosphere slab separation theorems in manifolds without conjugate points*, J. Egyptian Math. Soc. 27 (2019)] and R. Bergmann et al. [*Fenchel duality and a separation theorem on Hadamard manifolds*, SIAM J. Optimization (2022)] of the hyperplane separation theorem in Euclidean space to Hadamard manifolds at which one of the disjoint sets is assumed to be convex and compact and the other convex and closed; the proof relies on results of these references. Note that generalizations of the hyperplane separation theorem from Euclidean space to topological vector spaces are the classical and well-known Hahn-Banach separation theorems.

Keywords: Hadamard manifolds, separation theorem.

2010 Mathematics Subject Classification: 52A20.

1. Introduction

Hyperplane separation theorems are well-known in Euclidean spaces and have also classical generalizations to topological vector spaces, see Appendix A. Proving analogous theorems in curved spaces is a different but natural direction of generalization. Aiming for generalizations in curved spaces (not specifically of hyperplane separation theorems but of several other properties) can also be found in a large well-established program of studying various phenomena in curved spaces, see for example the monograph by Ballmann, Gromov, and Schroeder [1] about manifolds with nonpositive curvature. The goal of this paper is to extend the hyperplane separation theorem in Euclidean space to the case of convex and disjoint sets in n-dimensional Hadamard manifolds where we assume that the two sets are in addition closed if $n \geq 3$. This generalizes recent work about separation theorems in Hadamard manifolds by Shenawy [11] and Bergmann et al. [3]. The theorem says that there exists a hyperplane which separates both sets from each other, see Theorem 3.1 and Theorem 3.2 below for details. For it we rely on results of [3, 11]. The aforementioned works arose in a framework of a growing interest in studying concepts of mathematical optimization on curved spaces, see, e.g., [4] and references therein. In that reference a primal-dual algorithm on Riemannian manifolds is con-

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sidered and a Chambolle-Pock type algorithm for these manifolds is established. In the context of imaging, recently a parallel Douglas-Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifold has been considered, see Bergmann, Persch, and Steidl [5]. Very closely related to optimization is the property of convexity for domains (and functionals). For a quite recent book about convex analysis and optimization in Hadamard spaces we refer to Bačák [2], and for some characterization of geodesic convexity on Hadamard manifolds and applications, to Zhou, Xiao, and Huang [12]. Ferreira and Oliveira [7, Thm 3.1] (see Proposition 2.4 below) focus on the mathematical optimization perspective and prove a version of the hyperplane separation theorem in Hadamard manifolds where one of the disjoint sets is closed and convex and the other consists only of one point which is contained in the boundary of the first set (also called supporting hyperplane theorem). In Shenawy [11] there appear three versions of the hyperplane separation theorem. Two of them are not directly related to our setting. The third of them holds in Hadamard manifolds for two disjoint and convex sets where one set is closed and the other set is compact. The concluded separation is then strict. Bergmann, Herzog and Louzeiro, [3] gave independently from Shenawy a proof of the aforementioned result from [11] and conjecture that a version of the hyperplane separation theorem also holds when the compactness assumption for one of the two sets is replaced by the assumption that both sets are (only) closed, see [3, Sec. 5]. The latter is achieved in our first main result, Theorem 3.1, through reduction the possibly unbounded case to the compact case by intersecting the involved sets with an exhausting sequence of balls. Then a suitable limit consideration gives the claim. In a second main result, Theorem 3.2, we remove the closedness assumption in the two dimensional case by a careful consideration of several cases for the size of the intersection of the closures of the two involved sets. Note that from an algorithmically optimization perspective the compactness assumption is probably not very restrictive and allows for most of the relevant cases.

Our paper is organized as follows. In Section 2 we recall some concepts from the geometry of Hadamard manifolds, introduce our notation and state known separation results in Hadamard manifolds from the literature. In Section 3 we prove our two versions of the hyperplane separation theorem in Hadamard manifolds. In Appendix A we prove an auxiliary lemma and include for convenience and comparison purposes a well-known geometric version of the Hahn-Banach theorem as well as one in $\mathbb{R}^n$.

2. Related results from the literature and notations

A Hadamard manifold is defined as follows, see e.g. Ballmann, Gromov, and Schroeder [1, 2.2. Def.].

Definition 2.1. A connected, simply connected complete Riemannian manifold of nonpositive curvature is called a *Hadamard-manifold* (or a *Cartan-Hadamard-manifold*). □

By the Hopf-Rinow theorem (see Petersen [9, Thm. 7.1]) any two points of a Hadamard manifold are connected by a unique geodesic segment with length equal to the distance of the points. In the following let M be a n-dimensional Hadamard manifold with tangent bundle TM.

Given $(p, \xi) \in TM$ we denote the hyperplane through p with normal ξ as

$$h = h(p, \xi) := \{q \in M \mid \langle \xi, \log_p q \rangle = 0\} \tag{1}$$

where $\langle \cdot, \cdot \rangle$ denotes the induced scalar product in T_pM and $\log_p$ is the inverse of the exponential map $\exp_p : T_pM \to M$ in p. We recall from e.g. [3] the following definition of separation applied to our notion of hyperplane. (In that reference the definition (1) with 0 replaced by some other number is contrary to here also considered as hyperplane.)

Definition 2.2. The hyperplane $h(p, \xi)$ separates two disjoint subsets A and B of M, if

$$\langle \xi, \log_p q \rangle \leq 0 \text{ for all } q \in A \text{ and } \langle \xi, \log_p q \rangle \geq 0 \text{ for all } q \in B, \tag{2}$$

or vice versa. If such a h exists we shortly say that A and B are separated and that A lies on one side of h. If the inequalities in (2) hold both strictly for one hyperplane h then A and B are said to be strictly separated. If the first inequality holds for one hyperplane h strictly we say that h separates A strictly and that A lies strictly on one side of h. □

For p and q in $\mathbb{R}^n$ we denote the straight segment from p to q by $\overline{pq}$. For p and q in M, $p \neq q$, we denote the geodesic segment connecting p with q by $g(p, q)$. We stipulate that p and q belong always to that geodesic segment. And $\hat{g}$ denotes the not extendable geodesic which contains a given geodesic segment g. Furthermore, if $p \in M$ and $\xi \in T_pM$ we let $g(p, \xi)$ denote the closed half-geodesic starting in p in direction ξ (and escaping to infinity). Correspondingly, $\hat{g}(p, \xi)$ stands for the not extendable geodesic containing $g(p, \xi)$. Especially in two dimensions, we sometimes parameterize directions in the tangent space in terms of an angle variable. More generally, we therefore agree upon the following. If a parameter θ ranges in some parameter set for which we have a bijective correspondence to a subset of T_pM we will use the notation $g(p, \theta)$ for $g(p, \tilde{\theta})$ where $\tilde{\theta} \in T_pM$ is the image of θ under that bijective correspondence. Note that in the case $n = 2$ we have the identity $h(p, \xi) = \hat{g}(p, \xi^\perp)$ for $p \in M$, $\xi \in T_pM$ and $\xi^\perp \in T_pM$ perpendicular to ξ with respect to the induced scalar product in T_pM. For $p \in M$ and $R > 0$ we denote the closed geodesic ball with center in p and radius R by $K_R(p)$. For any set S we denote the number of its elements by $|S|$, if S is finite, and set $|S| = \infty$, otherwise.

We recall from [1] the definition of convexity for sets and functions and first elementary properties. A subset W of M is geodesically convex (or short convex), if for p and q in W the unique shortest geodesic from p to q in M is contained in W. A function f in M is called convex, if for every nontrivial geodesic $\gamma : [0, 1] \to M$ the function $f \circ \gamma$ is convex. If $W \subset M$ is convex and f is a convex function on W then for any $a \in \mathbb{R}$ the sub level $\{f \leq a\}$ is convex. The induced distance function $d : W \times W \to \mathbb{R}$ is convex for every convex subset $W \subset M$. Hence the distance function from a point $p \in M$ is convex. This implies that geodesic balls $K_R(p)$ are convex. We denote the Euclidean distance function in the n-dimensional Euclidean space by d_e.

The following hyperplane separation result in Hadamard manifolds is (independently) due to [11, Cor. 2] and [3, Thm. 4.4], see Figure 2.1 for an illustration of a possible scenario.

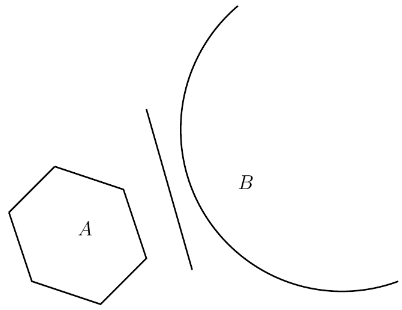

Figure 2.1: Two disjoint, convex sets, one is compact, the other closed

Proposition 2.3. *Let M be an n-dimensional Hadamard manifold, $n \geq 2$, and A and B two non-empty, disjoint, and convex subsets of M. Then A and B are strictly separated if one of them is compact and the other one is closed.*

A supporting hyperplane version of the separation hyperplane theorem in Hadamard manifolds is due to [7, Thm 3.1] as follows.

Proposition 2.4. *Let C be a nonempty, closed, and convex subset of a Hadamard manifold M and $x \in \partial C$. Then, there exists a hyperplane which separates C and $\{x\}$.*

Note that a separating hyperplane h for C and $\{x\}$ in Proposition 2.4 necessarily satisfies (2) with equality for the set $\{x\}$. In this special case h is also called a supporting hyperplane of C in x.

3. Hyperplane separation type theorems

Our first main result is as follows.

Theorem 3.1. *Let A and B be non-empty, closed, convex and disjoint subsets of the n-dimensional Hadamard manifold M, $n \geq 2$. Then there exists a hyperplane h in M which separates A and B.*

Proof. Choose $p \in A$ and $q \in B$. Via parallel transport along the geodesic segment $g(p,q)$ we may identify T_rM with T_pM for all $r \in g(p,q)$. Since the length of a tangent vector is invariant under parallel transport we may identify all

$$S_r^n = \{\xi \in T_rM \mid |\xi| = 1\}, \quad r \in g(p,q), \tag{3}$$

with S_p^n via parallel transport. Hereby we denote the induced norm in T_rM by $|\cdot|$. Let $\gamma = \gamma(t)$, $\gamma : [0,1] \to M$, be a parameterization of $g(p,q)$ proportional to arc length, so that $\gamma(0) = p$ and $\gamma(1) = q$. We observe that any hyperplane h which separates A and B necessarily intersects $g(p,q)$ which can be seen by using the definition for h in (1) and the intermediate value theorem for continuous functions in one variable. For $R > 0$ we define $A_R = A \cap K_R(p)$ and $B_R = B \cap K_R(p)$. Denote by h_R a separating hyperplane for A_R and B_R which exists in view of Proposition 2.3, cf. also Figure 3.1.

Let $m_R \in g(p,q) \cap h_R$ and $\xi_R \in T_pM$ be a normal direction of h_R (in m_R) so that $h_R = h(m_R, \xi_R)$. Hereby, we choose the orientation of ξ_R so that the corresponding separation inequality, see (2), gives for elements in A_R always the ≤ 0 inequality. Then $(m_R, \xi_R) \in [0,1] \times S^n_p$ for all $R > 0$ via the above identification. Taking a suitable sequence $R_k \to \infty$ we may assume that $m_{R_k} \to m \in [0,1]$ and $\xi_{R_k} \to \xi \in S^n_p$.

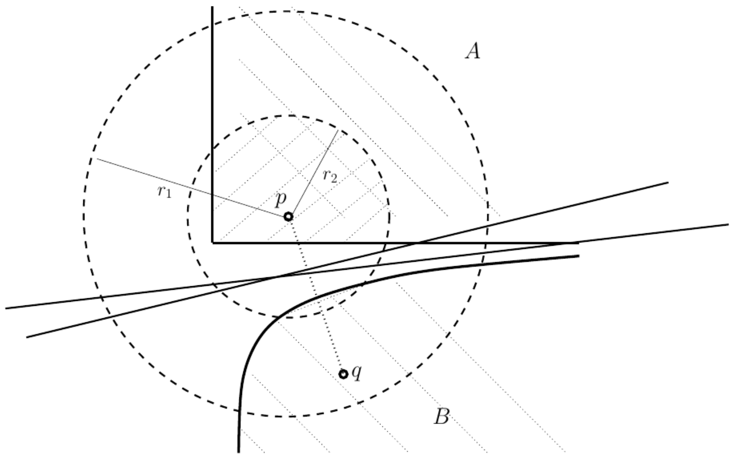

Figure 3.1: Two disjoint, convex and closed sets

We claim that the hyperplane $h(m, \xi)$ separates A and B. We divide the statement into two claims, each of which may be visualized best by the reader in the Euclidean picture. But here we proof the more general curved scenario.

Claim 1: The points p and q are on different sides of h.

Proof of Claim 1: The metrics of T_rM, $r \in g(p,q)$, here expressed in local coordinates $g_{ij}(r)$, are all equivalent and the positive constants to estimate these metrics against each other can be chosen uniformly in r. Hence the preimages of p and q under $\exp_r$ are uniformly bounded for all $r \in g(p,q)$ since

$$\max_{r \in g(p,q)} \operatorname{dist}(r, \{p,q\}) < \infty. \tag{4}$$

This means that we find $c_1 > 0$ such that $\exp_r$ for $r \in g(p,q)$ maps a ball around the origin of radius c_1 in T_rM onto a set which contains p and q. We conclude that $\exp^{-1}_{m_{R_k}}(p)$ and $\exp^{-1}_{m_{R_k}}(q)$ converge to $\exp^{-1}_m(p)$ and $\exp^{-1}_m(q)$. Note, that via parallel transport the first two sequences may be considered in T_mM. Therein they are bounded and hence, one can extract a subsequence converging to some limit, e.g. $\exp^{-1}_{m_{R_k}}(p) \to \xi_p \in T_mM$. In view of the local Lipschitz continuity of the geodesic flow ξ_p is also a preimage of p under $\exp_m$. Since the exponential function is a diffeomorphism we have $\exp^{-1}_m(p) = \xi_p$. Then we deduce that

$$0 \geq \langle \xi_{R_k}, \exp^{-1}_{m_{R_k}}(p) \rangle \to \langle \xi, \exp^{-1}_m(p) \rangle \tag{5}$$

and

$$0 \leq \langle \xi_{R_k}, \exp^{-1}_{m_{R_k}}(q) \rangle \to \langle \xi, \exp^{-1}_m(q) \rangle \tag{6}$$

from which we conclude that h separates $\{p\}$ and $\{q\}$.

Claim 2: A (resp. B) lies on the same side of h as p (resp. q).

Proof of Claim 2: The proof is analogous to the proof of Claim 1. We test for an arbitrary but fixed element $x \in A$ (and $x \in B$) on which side of h it is. For large R_k, the element x has already been taken into account for the choice of h_{R_k}, so that x lies on the correct side of h_{R_k}. Inserted in the corresponding expressions of (2) for h_{R_k} we get a sign which is preserved by the approximation of h by h_{R_k}. $\square$

In two dimensions we obtain a hyperplane separation theorem in Hadamard manifolds in the full natural generality, see Theorem A.3 for the Euclidean case, which does not require an assumption like closedness.

Theorem 3.2. *Let A and B be non-empty, convex, and disjoint subsets of the two-dimensional Hadamard manifold M. Then there exists a hyperplane h in M which separates A and B.*

Proof. If even the closures $\bar{A}$ and $\bar{B}$ are disjoint we can apply Theorem 3.1. Note for it that the distance function is continuous and the geodesic segment between two different points unique. This implies that the closure of a convex set is convex, i.e. $\bar{A}$ and $\bar{B}$ are convex. Hence we assume that $\bar{A} \cap \bar{B} \neq \emptyset$, see Figure 3.2.

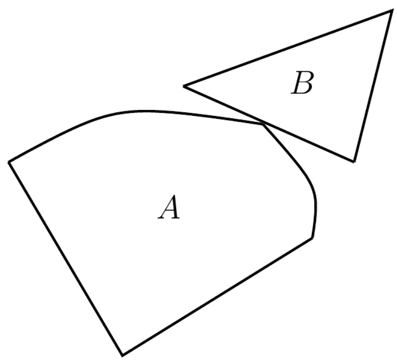

Figure 3.2: Two disjoint, convex sets with common point in their closure

If $|A| = 1$ or $|B| = 1$ we can apply Proposition 2.4, since then the single element of A lies in the boundary ∂B (or the same with interchanged roles of A and B). Hence we may assume that A and B contain both at least two points (and by convexity then also a segment of a geodesic).

Case 1: $\bar{A} \cap \bar{B}$ contains exactly one element.

Let $p \in \bar{A} \cap \bar{B}$. We parameterize the unit sphere S_p^1 in T_pM with the angle parameter $\theta \in (0, 2\pi]$. Therefore we identify via local coordinates T_pM with $\mathbb{R}^2$ equipped with the induced metric of T_pM which we write in these coordinates as $g_{ij}(p)$. In $\mathbb{R}^2$ the set S_p^1 is an ellipse, especially strictly convex. Hence θ is well-defined if we identify $r \in S_p^1$ with $\theta \in (0, 2\pi]$ so that the ray $\{\lambda(\cos\theta, \sin\theta) \mid \lambda \geq 0\}$ meets S_p^1 in r. We then extend the definition of the angle θ as parameter by 2π-periodicity to all real numbers.

Since $\bar{A}$ contains a point $q \neq p$, we may shift the definition of θ by a constant so that $q \in g(p, 2\pi)$ using our notation from Section 2. Furthermore, we may w.l.o.g. assume that there is $\tilde{q} \in \bar{B}$ with $\tilde{q} \neq p$ and $\tilde{\theta} \in (0, 2\pi]$ such that $\tilde{q} \in g(p, \tilde{\theta})$. Note that indeed $\tilde{\theta} \in (0, 2\pi)$, otherwise, q and $\tilde{q}$ lie both on $g(p, 0)$. Then the geodesic segment from $p \in \bar{A}$ to $q \in \bar{A}$ is by convexity in $\bar{A}$ and contains $\tilde{q} \in \bar{B}$, or, the geodesic segment from $p \in \bar{B}$ to $\tilde{q} \in \bar{B}$ is contained in $\bar{B}$ and contains $q \in \bar{A}$. This implies $q \in \bar{A} \cap \bar{B}$ or $\tilde{q} \in \bar{A} \cap \bar{B}$, which is in both cases a contradiction to the fact that this intersection is assumed to contain only one element.

We define angles $\theta_l^A, \theta_r^A, \theta_l^B, \theta_r^B \in [0, 2\pi]$ as follows,

$$\theta_l^B = \sup\{\theta \in [\tilde{\theta}, 2\pi] \mid \forall \bar{\theta} \in [\tilde{\theta}, \theta) : \ |g(p, \bar{\theta}) \cap \bar{B}| > 1\}, \tag{7}$$

$$\theta_r^B = \inf\{\theta \in [0, \tilde{\theta}] \mid \forall \bar{\theta} \in (\theta, \tilde{\theta}] : \ |g(p, \bar{\theta}) \cap \bar{B}| > 1\}, \tag{8}$$

$$\theta_l^A = \inf\{\theta \in [\tilde{\theta}, 2\pi] \mid \forall \bar{\theta} \in (\theta, 2\pi] : \ |g(p, \bar{\theta}) \cap \bar{A}| > 1\}, \tag{9}$$

$$\theta_r^A = \sup\{\theta \in [0, \tilde{\theta}] \mid \forall \bar{\theta} \in [0, \theta) : \ |g(p, \bar{\theta}) \cap \bar{A}| > 1\}. \tag{10}$$

First note that all four directions θ_l^A, θ_r^A, θ_l^B, and θ_r^B are well-defined and lie in $[0, 2\pi]$. In a simplified picture one might visualize that 2π is the downward direction, $\tilde{\theta}$ is the upward direction and the subscripts l and r refer to phenomena happening left and right, respectively. We have the inequalities

$$0 \overset{(i)}{\leq} \theta_l^B - \theta_r^B \overset{(ii)}{\leq} \pi, \quad \theta_r^A \overset{(iii)}{\leq} \theta_r^B, \quad \theta_l^B \overset{(iv)}{\leq} \theta_l^A, \quad \pi \overset{(v)}{\leq} \theta_l^A - \theta_r^A \overset{(vi)}{\leq} 2\pi. \tag{11}$$

Inequalities (ii), (iii), (iv) and (v) are the non-trivial ones. Inequality (iv) is of the same type as (iii) and inequality (v) is of the same type as inequality (ii). So, we say that all inequalities are clear if we explain (only) inequalities (ii) and (iv) in the following.

Claim 1a: Inequality (ii) in (11) holds.

Proof of Claim 1a: Assume that $\theta_l^B - \theta_r^B > \pi$. Then by possibly increasing θ_r^B and decreasing of θ_l^B such that the inequality is preserved and denoting the resulting values again with the same symbols, we may assume that there are points $r \in g(p, \theta_r^B) \cap \bar{B}$, $s \in g(p, \theta_l^B) \cap \bar{B}$ both different from p such that $g(r, s)$ intersects $g(p, q)$ in a point p' different from p. Given such p' it belongs to $\bar{A} \cap \bar{B}$ which is a contradiction. It remains to argue that such intersection point p' indeed appears. This would be immediate clear in the Euclidean case. Here it can be seen from considering a small neighborhood around p in which M can be considered as being flat in the sense of a first order approximation. The details can be found in Lemma A.1.

Claim 1b: Inequality (iv) in (11) holds.

Proof of Claim 1b: Let us assume the contrary, i.e. $\theta_l^A < \theta_l^B$. Necessarily, $\pi \leq \theta_l^A < 2\pi$, $\theta_l^B > \tilde{\theta}$. Choose $\theta_l^A < \theta_1 < 2\pi$ close to θ_l^A and $\tilde{\theta} < \theta_2 < \theta_l^B$ close to θ_l^B so that $\tilde{\theta} < \theta_1 < \theta_2 < 2\pi$. Choose $r \in (g(p, \theta_1) \setminus \{p\}) \cap \bar{A}$, $s \in (g(p, \theta_2) \setminus \{p\}) \cap \bar{B}$. Then $g(r, p), g(q, p) \subset \bar{A}$ and $g(s, p) \subset \bar{B}$. Roughly spoken, the first two named geodesic segments in $\bar{A}$ meet in p with an angle $< \pi$ and the third one in $\bar{B}$ arrives in between. Again the Euclidean picture immediately implies that a point $\neq p$ is also in the intersection of $\bar{A}$ with $\bar{B}$. For the curved case we again apply Lemma A.1. Hence we have a contradiction to the assumption that $|\bar{A} \cap \bar{B}| = 1$.

Case 2: $\bar{A} \cap \bar{B}$ contains at least two elements.

Let p and q be in $\bar{A} \cap \bar{B}$, $p \neq q$. Consider the geodesic $\hat{g}(p, q)$. If the latter does not separate $\bar{A}$ from $\bar{B}$ we may w.l.o.g. choose $r \in \bar{A}$ and $s \in \bar{B}$ which lie on the same side of $\hat{g}(p, q)$ but not on that geodesic. If $r = s$ then $\bar{A}$ and $\bar{B}$ have a common interior, w.l.o.g. a small geodesic ball $K_R(z)$, $z \in \bar{A} \cap \bar{B}$, $R > 0$, compare for it also the second proof of Lemma A.1. By varying z slightly and replacing R by $R/2$ (not renamed), w.l.o.g. $z \in A$. Let $r \in A \cap K_R(z) \setminus \{z\}$ and $s \in g(z, r) \subset A$, $s \neq z, r$. A small geodesic ball $K_\varepsilon(s)$, $\varepsilon > 0$, is separated by $g(z, r)$ into two open half-balls B_1 and B_2 in the Euclidean picture (here we use the identification via $\exp_s$). Since $K_\varepsilon(s)$ is convex, for two points $w_i \in B_i$, $i = 1, 2$, the geodesic segment $g(w_1, w_2)$ meets $g(z, r)$. Since B_i are open, $i = 1, 2$, we can vary the w_i slightly, so that everything holds as before and in addition $w_i \in B$, $i = 1, 2$. Now the intersection point of $g(w_1, w_2)$ and $g(z, r)$ is contained in both, A and B, a contradiction.

Hence we may assume that $r \neq s$. If r and s are in the same half-geodesic starting from p, then r or s belongs to $\bar{A} \cap \bar{B}$, and we are in the situation from before.

So let us assume that they lie on different half-geodesics starting in p. But then via Lemma A.1 we can construct a point in $\bar{A} \cap \bar{B}$ which differs from p as well as q. Hence $\bar{A} \cap \bar{B}$ has an interior, a contradiction. □

Corollary 3.3. *If in the situation of Theorem 3.2 A is open then there exists a hyperplane h which separates A and B such that the separation with respect to A is strict.*

Proof. In view of Theorem 3.2 a separating hyperplane h exists for A and B. Necessarily we have $h \cap A = \emptyset$. □

Remark 3.4. We think that a higher-dimensional version of Corollary 3.3 probably holds and that the basic idea from the two dimensional case above could be useful for it. We leave this open here.

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A. Appendix

We prove here a lemma which states that a specific property which is clear in two-dimensional Euclidean space $\mathbb{R}^2$ holds also in a Hadamard manifold. This lemma is used several times in the proofs. The property is illustrated in Figure A.1 and roughly says the following: Given three pairwise different straight segments s_i, $i = 1, 2, 3$, meeting in a point $p \in \mathbb{R}^2$ such that s_1 and s_3 form an angle strictly less than π which is divided by the third segment s_2. Then a certain straight segment (in the picture from r to s) which connects s_1 with s_3 without containing p meets s_2.

Lemma A.1. *Assume that three geodesic segments arrive in $p \in M$ from different directions. Furthermore, assume that all three segments lie on one side of a (auxiliary) geodesic through p and that at most one of the segments is contained in the latter. Then one can choose two of these three segments* (*clear from picture, see Figure* A.1) *and two points $r, s \neq p$ on them such that the geodesic segment connecting r with s intersects the third segment in a point $\neq p$.*

Proof. We present two proofs, the first is a quantitative comparison of lengths with the Euclidean setting. The second proof argues more topologically.

First proof: Let $R > 0$ be small, it suffices to make all considerations in $K_R(p)$, see Figure A.1. W.l.o.g. we choose normal coordinates around p in $K_R(p)$, see [8, Prop. 5.11] for a definition. These establish especially a first order approximation of the curved metric by an Euclidean metric in the whole of $K_R(0)$ which is exact in p. Geodesics through p are straight lines through the origin in the normal coordinate system. Any shortest geodesic between points in $K_R(p)$ lies entirely in $K_R(p)$ since $K_R(p)$ is convex. Now we choose two of the three given segments (see Figure A.1 how we make this choice), denote them by s_1 and s_3, as well as points $r, s \neq p$ on both of the selected segments which also lie in $K_R(p)$. The geodesic segment from r to s meets the third segment, denoted by s_2, (we refer to is as case (i)) or the following piece of its geodesic extension,

$$(\hat{s}_2 \cap K_R(p)) \setminus s_2 \qquad \text{(case (ii))}. \tag{12}$$

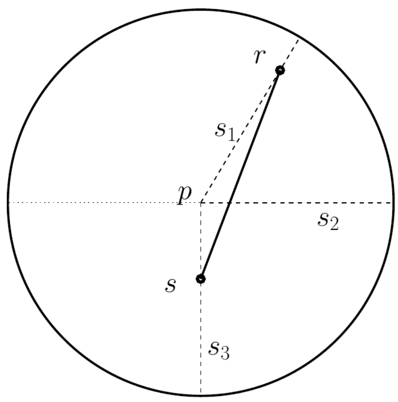

Figure A.1: In a small geodesic ball $K_R(0)$ around p the geodesics through p are straight lines through 0 in T_pM. Geodesic segments between arbitrary points in $K_R(p)$ are contained in $K_R(p)$. The geodesic segment from r to s necessary meets s_2 or its backward geodesic extension. The last named case can be excluded since the length of the geodesic segment would then be too long.

We study now both cases in the purely Euclidean setting and make then an approximation error deliberation to translate this back to the curved setting. Given the angles at which the s_i, $i = 1, 2, 3$, meet in p the Euclidean setting shows (which we assume here for a moment) that a lower bound for the length of a smooth curve from r to s in case (ii) is the length of $\overline{rps} := \overline{rp} \cup \overline{ps}$ given by

$$d_e(r, p) + d_e(p, s) \geq (1 + \tilde{\mu}) d_e(r, s), \tag{13}$$

with some $\tilde{\mu} > 0$. Note that d_e denotes the Euclidean distance and that the above inequality is scaling invariant with respect to scaling from p. By scaling down R, r and s all with the same factor, the last two along geodesics from p, we can arrange that the Euclidean length of any curve in $K_R(p)$ is estimated from below and above by the $(1 - \mu)$ multiple and $(1 + \mu)$ multiple, respectively, of its length in the curved metric. Note that $\mu > 0$ can be chosen arbitrarily small if we scale down sufficiently, which can be quantitatively expressed by saying that $R > 0$ is small after that scaling. Since $\tilde{\mu}$ from above is invariant under this type of scaling, we can achieve that $0 < \mu \ll \tilde{\mu}$. This shows that in the situation after scaling the lower bound for the length of the geodesic segment from r to s (obtained from the Euclidean value and the error factor estimate) in case (ii) is in the curved case greater than the curved distance between r and s. For the curved length l and the corresponding Euclidean length l_e of a curve from r to s in case (ii) we summarize the statements of this paragraph in the following inequalities

$$l \geq l_e(1-\mu) \geq (1-\mu)\left(d_e(r,p) + d_e(p,s)\right) \geq (1-\mu)(1+\tilde{\mu})d_e(r,s) \tag{14}$$

$$\geq (1-\mu)^2(1+\tilde{\mu})d(r,s). \tag{15}$$

Since for $0 < \mu \ll \tilde{\mu}$ the factor $(1-\mu)^2(1+\tilde{\mu})$ is larger than 1 we conclude that l is bounded from below by a number slightly larger that $d(r,s)$. Hence only case (i) can occur.

Second proof: We denote the geodesic segments as in the picture by s_i, $i = 1,2,3$ and choose normal coordinates in p. Let g_1 and g_3 be half-geodesics starting from p and containing s_1 and s_3, respectively. Then g_1 and g_3 specify a closed and unbounded sector S of opening angle strictly less than π. Any geodesic segment which begins in a point p_1 from $g_1 \setminus \{p\}$ and ends in a point p_3 from $g_3 \setminus \{p\}$ necessarily runs entirely through S without meeting p. For it we remark that if the segment $g(p_1,p_3)$, written now as $\{\gamma(t) : 0 \leq t \leq 1\}$, $\gamma : [0,1] \to M$, $\gamma(0) = p_1$, $\gamma(1) = p_3$, is outside S at some point $t' \in (0,1)$, i.e. $\gamma(t') \notin S$, then there are $0 \leq t_1 < t' < t_2 \leq 1$ such that $\gamma(t_1)$ and $\gamma(t_2)$ lie both on $\hat{g}(p,p_1)$ or both on $\hat{g}(p,p_2)$. Then locally, more precisely, between t_1 and t_2 the route along γ would not be of optimal length, a contradiction.

Finally, $p \notin g(p_1,p_3)$ since $p_1 \notin \hat{g}(p,p_3)$. For any $R > 0$ the ball $K_R(p)$ is convex and can be chosen so small that each s_i, $i = 1,2,3$ has an endpoint outside that ball. Then, if p_1 and p_3 as above are chosen in $K_R(p)$, also $g(p_1,p_3) \subset (K_R(p) \cap S) \setminus \{p\}$ in view of the convexity of the ball and the considerations above about non-leaving of S. Then, clearly, g meets s_2 in a point different from p and the choice $r = p_1$ and $s = p_3$ is as desired. □

We recall from Rudin [10, p.105] the following generalization of the hyperplane separation theorem to topological vector spaces (which is only one of several classical versions).

Theorem A.2. (Hahn-Banach) *Suppose that A and B are disjoint, convex, non-empty sets in a topological vector space X.*

(i) *If A is open there exists a $\Lambda \in X^*$ and $\gamma \in \mathbb{R}$ such that for all $(x,y) \in A \times B$*

$$\operatorname{Re}\Lambda x < \gamma \leq \operatorname{Re}\Lambda y \tag{16}$$

(ii) *If A is compact, B is closed, X is locally convex, then there exists $\Lambda \in X^*$ and $(\gamma_1,\gamma_2) \in \mathbb{R}^2$ such that for all $(x,y) \in A \times B$*

$$\operatorname{Re}\Lambda x \leq \gamma_1 < \gamma_2 \leq \operatorname{Re}\Lambda y \tag{17}$$

Moreover, we recall from Boyd and Vandenberghe [6, p. 46].

Theorem A.3. (Hahn-Banach in finite dimensional spaces) *Suppose A and B are nonempty, disjoint, and convex subsets of $\mathbb{R}^n$, $n \geq 2$. Then there exist $0 \neq a \in \mathbb{R}^n$ and $b \in \mathbb{R}$ such that $a \cdot x \leq b$ for all $x \in A$ and $a \cdot x \geq b$ for all $x \in B$ (where the dot $\cdot$ between two vectors denotes their Euclidean inner product).*

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