

Parabolic Quasi-Variational Inequalities. I: Semimonotone Operator Approach

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Variational inequalities, formulated on unknown-dependent convex sets, are called quasi-variational inequalities (QVI). This paper is concerned with an abstract approach to a class of parabolic QVIs arising in many biochemical/mechanical problems. The approach is based on a compactness theorem for parabolic variational inequalities shown previously by the authors [*A new compactness theorem for variational inequalities of parabolic type*, Houston J. Math. 44 (2018) 319–350]. The prototype of our model for QVIs of parabolic type is formulated in a reflexive Banach space as the sum of the time-derivative operator under unknown convex constraints and a semimonotone operator, including a feedback system which selects a convex constraint. The main objective of this work is to specify a class of unknown-state dependent convex constraints and to give a precise formulation of QVIs.

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1. Introduction

The theory of quasi-variational inequalities (QVI) has been attractive since its concept was formulated in [3]. Its development can be found in [15, 21, 25, 29] for the stationary case and in [17, 23, 28] for the evolution case. But there are still many questions in the solvability of quasi-variational evolution inequalities, for instance, the establishment of compactness method from both of theoretical and numerical points of view. In this context, many mathematical models, arising in biochemical/mechanical problems (cf. [1, 2, 8, 11, 13, 14, 16, 19, 26, 27]), have been discussed.

Our motivation for considering an abstract quasi-variational problem is as follows. Let u_0 be some initial state: e.g., an initial magnetic field and/or electric current,

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or the velocity of a flowing liquid, evolving in time to some unknown $u(t)$. As a function of space, u_0 and $u(t)$ belong to some Hilbert space H . The evolution is governed by some physical phenomenon that we describe by a system of equations, like the Maxwell or the Navier-Stokes systems. Here, we consider two general points.

1. **Feedback.** Within the described physical phenomenon, the evolution may additionally depend on some parameter, that we call θ : e.g., the initial temperature, which is evolving according to its own physical law (e.g., the heat equation), but may be coupled to our unknown u (the temperature is strongly coupled to the magnetic state or to the concentration of the substrate in the flow).
2. **Constraint.** The very nature of the physical phenomena that our variable is undergoing may depend on space, on time, but also directly on the unknown itself. In our examples, the magnetic material passes to the superconductive state (zero or very low resistance) when the magnetic field, the electric current and the temperature are low enough (actually, when some convex combination of these is low enough [6]); the flow velocity may depend on obstacles appearing due to accumulation of some substrate that the flow is transporting – a phenomenon well known to cardiologists and their patients. In the mathematical description, this means that the physical equations governing the evolution are valid only in some subset of the space H , which is in general closed and convex. This subset imposes a *constraint* on the initial datum and on the unknown. Let us call this constraint set $\overline{K(\theta; t)}$.

We thus have a feedback operator for θ , which is taken from some metric space Θ :

$$\theta = \Lambda_{u_0} u, \quad (1)$$

a family of convex sets $K(\theta; t)$ in H , and we look for $u(t) \in K(\theta; t)$, which satisfies some evolution equation. We formulate this last in an abstract way as

$$L_{u_0}(\theta; u) + A(u; u) \ni f. \quad (2)$$

Here, L_{u_0} is a time derivative operator including the constraint $u(t) \in K(\theta; t)$, and A is a coercive semimonotone mapping, possibly multivalued.

Finally, the problem writes: given $u_0 \in H$, find $u(t) \in H$ such that

$$L_{u_0}(\theta; u) + A(u; u) \ni f, \quad \theta = \Lambda_{u_0} u. \quad (3)$$

We give the functional framework for these abstract equations and define precisely the convex sets in Section 2, we define the operators L and A in Sections 3 and 4, and we give two examples of these in Section 6. In particular, in Section 6.2 we give the concrete setting for the superconductivity type II problem, in a description inspired by [1, 2, 19, 27]. As for the description of the flow problem, see our previous work [11, 12] and references therein.

The main objective of this paper is to use the time-derivative operator $L_{u_0}(\theta; \cdot)$, by a systematic application of the semimonotone operator theory [22], to solve a class of parabolic quasi-variational inequalities of the form (3).

In the authors' knowledge, there are roughly two approaches to the solvability of parabolic (QVI).

- (i) One is to consider (QVI) in a Hilbert or Banach lattice.

The construction of a solution of (QVI) is made by using the order property. This method has been applied by many authors. Most of results on (QVI) treated so far are based on this approach; see e.g. [23, 28, 29]. However, for instance in (QVI) with gradient constraints, the constraint sets have no longer lattice structure, so we need some other approach.

- (ii) Another one is to make use of the compactness result, see e.g. [12, 13, 16, 27].

Conditions $(\Lambda 1)$, $(\Lambda 2)$, imposed on the feedback system Λ_{u_0} in Section 5, are quite usual in this context, as they grant strong continuity of Λ_{u_0} from $L^p(0, T; H)$ into Θ .

Let us note that the compactness method for the parabolic case and for the elliptic case are quite different. In fact, in the elliptic case, the feedback system Λ is given as a mapping from V into Θ . Therefore, if Λ is weakly continuous from V into Θ , then by the compactness of the embedding $V \subset H$, it is automatically strongly continuous from H into Θ . However, in the parabolic case, the situation is quite different, since $L^p(0, T; H)$ is not compactly embedded in $L^p(0, T; V)$.

The paper is organized as follows. Section 2 introduces mathematical notation that we are going to use. In Section 2.2, we give the precise definition on the convex sets $K(\theta; t)$ and the assumptions on how they are related between them: we want some continuity of these with respect to t and θ . The continuity that we need for the sequel is given in Lemma 2.5, but we also give geometric conditions for this to occur: these are assumptions (A1)–(A3). In Section 3, we give the definition of the operator $L_{u_0}(\theta; \cdot)$, which is our "time derivative with constraint" operator. Fundamental properties of $L_{u_0}(\theta; \cdot)$ are discussed. In particular, geometric conditions (A1)–(A3) are shown to assure continuity of its graph (Theorem 3.8). Section 4 is devoted to the operator A and to a parabolic variational inclusion, intermediate for (2):

$$L_{u_0}(\theta; u) + A(v; u) \ni f.$$

We prove existence and uniqueness of its solution, as well as its continuous dependence upon the parameters θ and v . These are Theorems 4.1 and 4.2. In Section 5, the quasi-variational inequality (3) is formulated in detail and existence of solution is proved. Theorem 5.4, stating existence of solution to (3), is the main theorem of this paper. We show immediately below that uniqueness cannot be expected. In Section 6 some applications, quasi-variational ordinary or partial differential inequalities, are given, showing that our main assumptions are not too much restrictive nor complicated to check.

2. Basic tools

2.1. Duality, subdifferential

In general, for a (real) Banach space X , we denote by X^* the dual space of X , by $|\cdot|_X$ and $|\cdot|_{X^*}$ their norms and by $\langle \cdot, \cdot \rangle_{X^*, X}$ the duality between X^* and X . In the case when X is a Hilbert space, the inner product may be denoted by $(\cdot, \cdot)_X$.

For a proper, lower semicontinuous and convex (l.s.c.) function φ on X , its effective domain $D(\varphi)$ and subdifferential $\partial_{X,X^*}\varphi : X \rightarrow X^*$ are respectively defined by

$$D(\varphi) := \{z \in X \mid \varphi(z) < \infty\},$$

and $\partial_{X,X^*}\varphi(z) := \{z^* \in X^* \mid \langle z^*, w - z \rangle_{X^*,X} \leq \varphi(w) - \varphi(z), \forall w \in X\}, \forall z \in X$.

In the case when X is a Hilbert space $\partial_{X,X^*}\varphi$ may be denoted simply by $\partial\varphi$.

Throughout this paper, let H be a (real) Hilbert space and V be a (real) reflexive Banach space such that V is dense and compactly embedded in H . Then, identifying H with the dual space H^* of H , we have

$$V \subset H \subset V^* \text{ with dense and compact embeddings.}$$

We suppose that V and V^* are uniformly convex; hence the duality mapping F is singlevalued, continuous and strictly monotone from V onto V^* . Also, we write $\langle \cdot, \cdot \rangle$ for $\langle \cdot, \cdot \rangle_{V^*,V}$. The positive numbers p , p' and T are fixed so that

$$2 \leq p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 0 < T < \infty.$$

In this paper, we use the duality mapping $F : V \rightarrow V^*$ associated with the gauge function $r \rightarrow r^{p-1}$, $r \geq 0$, namely

$$Fz \in V^*, \quad |Fz|_{V^*} = |z|_V^{p-1}, \quad \langle Fz, z \rangle = |z|_V^p, \quad \forall z \in V.$$

2.2. Family of convex sets, weak and strong parameters

We are given here a complete metric space Θ , with metric $d_\Theta(\cdot, \cdot)$, consisting of parameters θ , and a family $\{K(\theta; t)\}_{\theta \in \Theta, t \in [0, T]}$ of closed and convex subsets of V (see the previous section).

Definition 2.1. With the above assumptions, let

$$\psi^t(\theta; z) = \begin{cases} I_{K(\theta; t)}(z) + \frac{1}{p}|z|_V^p, & \text{if } z \in V, \\ \infty, & \text{if } z \in H - V, \end{cases}$$

for all $t \in [0, T]$ and for all $\theta \in \Theta$. This is proper, l.s.c. and convex on H and on V . By the definition of subdifferential $\partial\psi^t(\theta; z)$ in H , for any $z^* \in \partial\psi^t(\theta; z)$ we see that $\langle z^*, w - z \rangle_H \leq \langle Fz, w - z \rangle$, $\forall w \in K(\theta; t)$. $\square$

We aim at the precise formulation of the time-derivative $L_{u_0}(\theta; \cdot)$, $\theta \in \Theta$. In this aim, we first introduce a subset Θ_S of Θ , which is the class of "strong parameters".

Definition 2.2. Θ_S consists of all parameters $\theta \in \Theta$ such that $\{K(\theta; t)\}_{0 \leq t \leq T}$ satisfies:

(Θ_S) for each positive number r there are real-valued functions $a_{\theta, r} \in W^{1,2}(0, T)$ and $b_{\theta, r} \in W^{1,1}(0, T)$ having the following property that for any $s, t \in [0, T]$ and any $z \in K(\theta; s)$ with $|z|_H \leq r$ there is $\tilde{z} \in K(\theta; t)$ such that

$$|\tilde{z} - z|_H \leq |a_{\theta, r}(t) - a_{\theta, r}(s)|(1 + |z|_V^{\frac{p}{2}}),$$

$$\text{and} \quad |\tilde{z}|_V^p - |z|_V^p \leq |b_{\theta, r}(t) - b_{\theta, r}(s)|(1 + |z|_V^p).$$

Remark 2.3. It is well-known (cf. [18, 30]) that for a fixed $\theta \in \Theta_S$ condition (Θ_S) is sufficient for the Cauchy problem

$$u'(t) + \partial\psi^t(\theta; u(t)) \ni f(t) \text{ in } H, \quad t \in [0, T], \quad u(0) = u_0, \quad (4)$$

to have a unique solution u such that $u \in C([0, T]; H) \cap L^p(0, T; V)$ satisfying $\sqrt{t}u' \in L^2(0, T; H)$ and $t \rightarrow |u(t)|_V^p$ is absolutely continuous on any compact interval of $(0, T]$ if $u_0 \in \overline{K(\theta; 0)}$, and $f \in L^2(0, T; H)$; here $\partial\psi^t(\theta; \cdot)$ is the subdifferential of $\psi^t(\theta; \cdot)$ in H and the closure is taken in H . In particular, if $u_0 \in K(\theta; 0)$, then the solution u of (4) belongs to $W^{1,2}(0, T; H)$ and $t \rightarrow |u(t)|_V^p$ is absolutely continuous on $[0, T]$. This shows that the set

$$K_0(\theta) := \{\eta \in L^p(0, T; V) \mid \eta' \in L^{p'}(0, T; V^*), \eta(t) \in K(\theta; t), \text{ a.e. } t \in (0, T)\}; \quad (5)$$

which will be our class of test functions, is non-empty. $\square$

In the rest of this paper, we assume that the strong class Θ_S is non-empty in Θ .

For a wider application of our theory we introduce the weak class Θ_W :

Definition 2.4. Θ_W , the weak class of parameters, is the closure of Θ_S in Θ . $\square$

We now specify the dependence of $K(\theta; t)$ on $\theta \in \Theta_W$ for each $t \in [0, T]$ by a family

$$\{R_{\theta\bar{\theta}}(t) \in B(H) \cap B(V) \mid \theta, \bar{\theta} \in \Theta_W, t \in [0, T]\}$$

of bounded linear invertible operators in H and V , where $B(H)$ (resp. $B(V)$) stands for the space of all bounded linear operators in H (resp. V), and by a family $\{\sigma'_{\theta\bar{\theta}, \varepsilon} \in C([0, T]; V) \mid \sigma'_{\theta\bar{\theta}, \varepsilon} \in L^{p'}(0, T; V^*), \theta, \bar{\theta} \in \Theta_W, 0 < \varepsilon \leq 1\}$ as follows:

(A1) There is a positive constant R_0 such that for all $\theta, \bar{\theta} \in \Theta_W$

$$|R_{\theta\bar{\theta}} - I|_{C([0, T]; B(H))} + |R_{\theta\bar{\theta}} - I|_{C([0, T]; B(V))} + |R'_{\theta\bar{\theta}}|_{L^{p'}(0, T; B(H))} \leq R_0 d_\Theta(\theta, \bar{\theta}), \quad (6)$$

where $R'_{\theta\bar{\theta}}(t) = \frac{d}{dt} R_{\theta\bar{\theta}}(t)$ in $B(H)$. Moreover, for any $\theta, \bar{\theta} \in \Theta_W$ we have

$$R_{\theta\bar{\theta}}^*(t) = R_{\theta\bar{\theta}}^{-1}(t) = R_{\bar{\theta}\theta}(t), \quad \forall \theta, \bar{\theta} \in \Theta_W, \quad \forall t \in [0, T], \quad (7)$$

namely the adjoint $R_{\theta\bar{\theta}}^*(t)$ of $R_{\theta\bar{\theta}}(t)$ coincides with the inverse of $R_{\theta\bar{\theta}}(t)$ and it is $R_{\bar{\theta}\theta}(t)$ in $B(H)$.

(A2) There is a positive constant σ_0 such that

$$|\sigma_{\theta\bar{\theta}, \varepsilon}|_{C([0, T]; V)} + |\sigma'_{\theta\bar{\theta}, \varepsilon}|_{L^{p'}(0, T; V^*)} \leq \sigma_0 (d_\Theta(\theta, \bar{\theta}) + \varepsilon), \quad \forall \theta, \bar{\theta} \in \Theta_W, \quad \forall \varepsilon \in (0, 1].$$

(A3) There is a continuous function $c_0(\cdot)$ on $[0, 1]$ with $c_0(0) = 0$ satisfying the following property:

$$\left\{ \begin{array}{l} \forall \varepsilon \in (0, 1], \exists \delta_\varepsilon \in (0, \varepsilon) \text{ such that} \\ \bar{z} = \mathcal{F}_{\theta\bar{\theta}, \varepsilon}(t)z := (1 + c_0(\varepsilon))R_{\theta\bar{\theta}}(t)z + \sigma_{\theta\bar{\theta}, \varepsilon}(t) \in K(\bar{\theta}; t), \\ \forall z \in K(\theta; t), \quad \forall \theta, \bar{\theta} \in \Theta_W \text{ with } d_\Theta(\theta, \bar{\theta}) \leq \delta_\varepsilon, \quad \forall t \in [0, T]. \end{array} \right. \quad (8)$$

For $t \in [0, T]$, $\theta, \bar{\theta} \in \Theta_W$, (A3) says that $K(\theta; t)$ is mapped into $K(\bar{\theta}; t)$ by the operator $\bar{z} = \mathcal{F}_{\theta\bar{\theta}, \varepsilon}(t)z$ given by (8), which is a composition of rotation $R_{\theta\bar{\theta}}(t)$, contraction/expansion with modulus $(1 + c_0(\varepsilon))$ close to 1 and parallel transformation $\sigma_{\theta\bar{\theta}, \varepsilon}(t)$.

This type of transformation (8) was earlier introduced in [7] and [20].

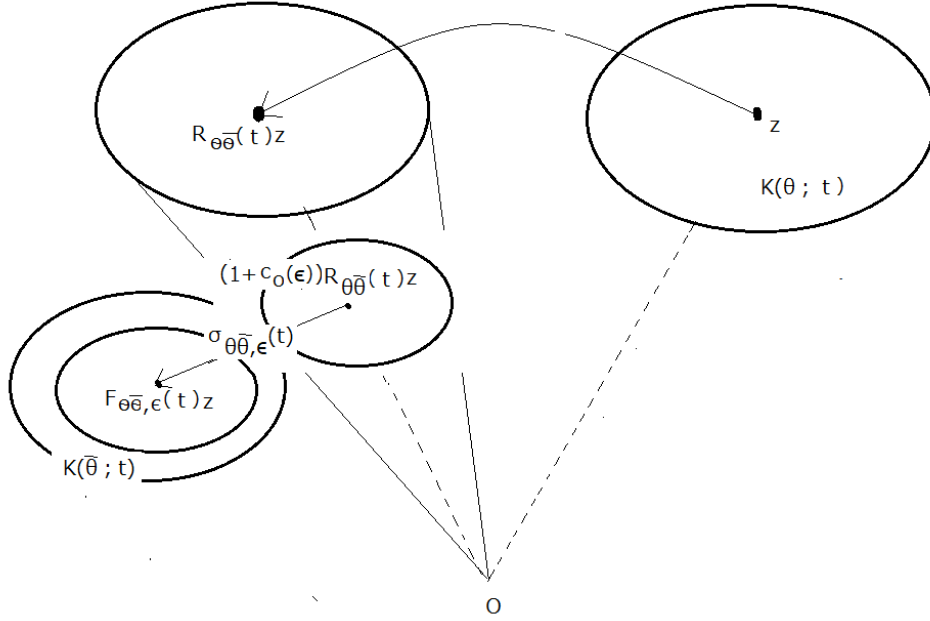


Figure 2.1: An illustration for conditions (A1)–(A3): how $K(\theta; t)$ may depend on $\theta \in \Theta_W$.

2.3. Test functions for weak parameters

As the class of test functions corresponding to parameter $\theta \in \Theta_W$, we employ

$$\mathcal{K}(\theta) := \{\eta \in L^p(0, T; V) \mid \eta(t) \in K(\theta; t) \text{ a.e. } t \in (0, T)\} \quad (9)$$

The more narrow class $\mathcal{K}_0(\theta)$ of test functions corresponding to parameter $\theta \in \Theta_S$ is defined by (5). We see that $\mathcal{K}_0(\theta)$ is continuously embedded in $C([0, T]; H)$.

The following lemma states that for the weak parameters, test functions from the above set subject to our transformation $\mathcal{F}_{\theta\bar{\theta}, \epsilon}(t)$, converge to the original function when ϵ tends to zero and $\bar{\theta}$ tends to θ . This is a condition for a kind of Mosco convergence [24].

Lemma 2.5. *Let $\theta, \bar{\theta} \in \Theta_W$ and ϵ be any small positive number. For each $\eta \in \mathcal{K}(\theta)$, put $\eta_{\theta\bar{\theta}, \epsilon}(t) := \mathcal{F}_{\theta\bar{\theta}, \epsilon}(t)\eta(t)$ for a.e. $t \in (0, T)$. Then, $\eta_{\theta\bar{\theta}, \epsilon} \in \mathcal{K}(\bar{\theta})$ and $\eta_{\theta\bar{\theta}, \epsilon} \rightarrow \eta$ in $L^p(0, T; V)$ as $\bar{\theta} \rightarrow \theta$ in Θ and $\epsilon \downarrow 0$, and moreover, if $\eta \in \mathcal{K}_0(\theta)$, then*

$$\eta_{\theta\bar{\theta}, \epsilon} \rightarrow \eta \text{ in } C([0, T]; H), \quad \eta'_{\theta\bar{\theta}, \epsilon} \rightarrow \eta' \text{ in } L^{p'}(0, T; V^*), \quad (10)$$

as $\bar{\theta} \rightarrow \theta$ in Θ and $\epsilon \downarrow 0$.

Proof. Let $\eta \in \mathcal{K}(\theta)$. We see from (6)–(7) of (A1) and (A2) that $\eta_{\theta\bar{\theta}, \epsilon}(t) \in K(\bar{\theta}; t)$ for a.e. $t \in (0, T)$ and

$$\begin{aligned} |\eta_{\theta\bar{\theta}, \epsilon}(t) - \eta(t)|_V &= |(R_{\theta\bar{\theta}}(t) - I)\eta(t) + c_0(\epsilon)R_{\theta\bar{\theta}}(t)\eta(t) + \sigma_{\theta\bar{\theta}, \epsilon}(t)|_V \\ &\leq (R_0 d_\Theta(\theta, \bar{\theta}) + |c_0(\epsilon)| \|R_{\theta\bar{\theta}}\|_{C([0, T]; B(V))}) |\eta(t)|_V + \sigma_0(d_\Theta(\theta, \bar{\theta}) + \epsilon) \end{aligned} \quad (11)$$

whence $\eta_{\theta\bar{\theta}, \epsilon} \rightarrow \eta$ in $L^p(0, T; V)$ as $\bar{\theta} \rightarrow \theta$ in Θ and $\epsilon \downarrow 0$.

Next, let $\eta \in \mathcal{K}_0(\theta)$. Then $\eta_{\theta\bar{\theta},\varepsilon}$, $\eta \in C([0, T]; H)$ and in a similar way to (11)

$$|\eta_{\theta\bar{\theta},\varepsilon}(t) - \eta(t)|_H \leq (R_0 d_\Theta(\theta, \bar{\theta}) + |c_0(\varepsilon)| |R_{\theta\bar{\theta}}|_{C([0,T];B(H))}) |\eta(t)|_H + \sigma_0(d_\Theta(\theta, \bar{\theta}) + \varepsilon)$$

which implies that $\eta_{\theta\bar{\theta},\varepsilon}$ converges to η in H uniformly on $[0, T]$, namely in $C([0, T]; H)$ as $\bar{\theta} \rightarrow \theta$ in Θ and $\varepsilon \rightarrow 0$. As to the time derivative of $\eta_{\theta\bar{\theta},\varepsilon}$, for every $z \in V$ we observe that

$$\begin{aligned} \frac{d}{dt} \langle R_{\theta\bar{\theta}}(t) \eta(t), z \rangle &= \frac{d}{dt} (R_{\theta\bar{\theta}}(t) \eta(t), z)_H = \frac{d}{dt} (\eta(t), R_{\theta\bar{\theta}}(t) z)_H \\ &= \langle \eta'(t), R_{\theta\bar{\theta}}(t) z \rangle + (\eta(t), R'_{\theta\bar{\theta}}(t) z)_H \end{aligned} \quad (12)$$

for a.e. $t \in (0, T)$. This shows that $R_{\theta\bar{\theta}}(t) \eta(t)$ is (strongly) differentiable in t a.e. on $(0, T)$, since the last term of (12) is the uniform limit of

$$\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \{ \langle \eta(t + \Delta t) - \eta(t), R_{\theta\bar{\theta}}(t) z \rangle + (\eta(t), (R_{\theta\bar{\theta}}(t + \Delta t) - R_{\theta\bar{\theta}}(t)) z)_H \}$$

with respect to z with $|z|_V \leq 1$ at a.e. $t \in (0, T)$. Also, it follows from (12) that if $|z|_V \leq 1$, then

$$\begin{aligned} \left| \frac{d}{dt} R_{\theta\bar{\theta}}(t) \eta(t) \right|_{V^*} &\leq \sup_{|z|_V \leq 1} \{ |\eta'(t)|_{V^*} |R_{\theta\bar{\theta}}(t)|_{B(V)} |z|_V + |\eta(t)|_H |R'_{\theta\bar{\theta}}(t)|_{B(H)} \cdot c_V |z|_V \} \\ &\leq |\eta'(t)|_{V^*} |R_{\theta\bar{\theta}}(t)|_{B(V)} + c_V |\eta(t)|_H |R'_{\theta\bar{\theta}}(t)|_{B(H)} \\ &\leq |\eta'(t)|_{V^*} |R_{\theta\bar{\theta}}|_{C([0,T];B(V))} + c_V |\eta|_{C([0,T];H)} |R'_{\theta\bar{\theta}}(t)|_{B(H)}, \end{aligned}$$

where c_V is a positive constant satisfying that $|w|_H \leq c_V |w|_V$ for all $w \in V$. This shows that $\frac{d}{dt} R_{\theta\bar{\theta}} \eta \in L^{p'}(0, T; V^*)$ and hence $\eta'_{\theta\bar{\theta},\varepsilon} = (1 + c_0(\varepsilon)) \frac{d}{dt} R_{\theta\bar{\theta}} \eta + \sigma'_{\theta\bar{\theta},\varepsilon} \in L^{p'}(0, T; V^*)$. Besides, we see from (12) that

$$\left\langle \frac{d}{dt} R_{\theta\bar{\theta}}(t) \eta(t) - \eta'(t), z \right\rangle = \langle \eta'(t), (R_{\theta\bar{\theta}}(t) - I) z \rangle + (\eta(t), R'_{\theta\bar{\theta}}(t) z)_H,$$

which shows that

$$\left| \frac{d}{dt} R_{\theta\bar{\theta}}(t) \eta(t) - \eta'(t) \right|_{V^*} \leq |\eta'(t)|_{V^*} |R_{\theta\bar{\theta}} - I|_{C([0,T];B(V))} + c_V |\eta|_{C([0,T];H)} |R'_{\theta\bar{\theta}}(t)|_{B(H)}. \quad (13)$$

Since $\eta'_{\theta\bar{\theta},\varepsilon}(t) - \eta'(t) = \frac{d}{dt} R_{\theta\bar{\theta}}(t) \eta(t) - \eta'(t) + c_0(\varepsilon) \frac{d}{dt} R_{\theta\bar{\theta}}(t) \eta(t) + \sigma'_{\theta\bar{\theta},\varepsilon}(t)$,

it results from (13) that $\eta'_{\theta\bar{\theta},\varepsilon} \rightarrow \eta'$ in $L^{p'}(0, T; V^*)$ as $\bar{\theta} \rightarrow \theta$ in Θ and $\varepsilon \downarrow 0$.

Thus (10) is obtained. $\square$

3. Time-derivative operator $L_{u_0}(\theta; \cdot)$

We are now ready to define our “time-derivative with constraint” operator. This concept has been already used in our previous work [10], where it was used so as to set formally a parabolic variational inequalities corresponding to a reaction-diffusion system generated by semimonotone mappings, solutions of which were subject to a varying in time constraint, and the solvability is based on a new compactness

method. However, related quasi-variational problems have not been treated. Our constraint $K(\theta; t)$ is typically a bound on the solution (its value or its gradient). The parametrized time-derivative operator $L_{u_0}(\theta; u)$ is a weak variational formulation of $\frac{du}{dt} + \partial I_{K(\theta; t)}(u)$, which is given in the following definition. This concept enables us to discuss systematically a wide class of parabolic quasi-variational inequalities.

Definition 3.1. Let $\theta \in \Theta_W$ and $u_0 \in \overline{K(\theta; 0)}$. We define the operator $L_{u_0}(\theta; \cdot)$ by: $g \in L_{u_0}(\theta; u)$ if and only if

$$u \in \mathcal{K}(\theta), \quad g \in L^{p'}(0, T; V^*),$$

$$\int_0^T \langle \eta' - g, u - \eta \rangle dt \leq \frac{1}{2} |u_0 - \eta(0)|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta), \quad (14)$$

where $\mathcal{K}(\theta)$ and $\mathcal{K}_0(\theta)$ are the classes of test functions given by (9) and (5), respectively. $\square$

The fundamental properties of $L_{u_0}(\theta; \cdot)$ as well as its characterization are derived from the proposition stated below.

Proposition 3.2. Suppose that (Θ_S) and (A1)–(A3) are fulfilled. Let ε be any small positive number and $\theta_i \in \Theta_S$, $i = 1, 2$, such that $d_\Theta(\theta_1, \theta_2) < \varepsilon$. Also, let M be any positive number, $u_{0i} \in \overline{K(\theta_i; 0)}$, and $f_i \in L^2(0, T; H)$, $i = 1, 2$, such that

$$\sum_{i=1}^2 \left\{ |f_i|_{L^{p'}(0, T; V^*)} + |u_{0i}|_H \right\} \leq M. \quad (15)$$

Then, for the strong solutions u_i of

$$u_i'(t) + \partial \psi^t(\theta_i; u_i(t)) \ni f_i(t) \quad \text{in } H, \quad \text{a.e. } t \in (0, T), \quad u_i(0) = u_{0i}, \quad i = 1, 2, \quad (16)$$

we have that for all $s, t \in [0, T]$ with $s \leq t$

$$\begin{aligned} & \frac{1}{2} |u_1(t) - u_2(t)|_H^2 + \int_0^t \langle F u_1(\tau) - F u_2(\tau), u_1(\tau) - u_2(\tau) \rangle d\tau \\ & \leq \frac{1}{2} |u_1(s) - u_2(s)|_H^2 + \int_s^t (f_1(\tau) - f_2(\tau), u_1(\tau) - u_2(\tau))_H d\tau + C_0^*(M) C_1^*(\varepsilon), \end{aligned} \quad (17)$$

where $C_0^*(M)$ is a positive constant depending only on $M > 0$ and $C_1^*(\varepsilon)$ is a positive continuous function of ε satisfying $C_1^*(\varepsilon) \rightarrow 0$ as $\varepsilon \downarrow 0$.

Under the same assumptions and same notation as in Proposition 3.2, as was noted in Remark 2.3, problem (4) admits a unique solution $u_i \in C([0, T]; H)$ satisfying $\sqrt{t} u_i' \in L^2(0, T; H)$. Put $u_i^*(\tau) := f_i(\tau) - u_i'(\tau) \in \partial \psi^\tau(\theta_i; u_i(\tau))$ for a.e. $\tau \in (0, T)$. Since $\psi^\tau(\theta_i; \cdot)$ is coercive, namely $\psi^\tau(\theta_i; z) \geq \frac{1}{p} |z|_V^p$ for all $z \in V$, we note from the usual energy estimate for problem (4) that

$$|u_i|_{C([0, T]; H)} + |u_i|_{L^p(0, T; V)} \leq M_i := M_i(|f_i|_{L^{p'}(0, T; V^*)}, |u_{0i}|_H), \quad i = 1, 2, \quad (18)$$

where $M_i(\cdot, \cdot)$ is a non-negative and non-decreasing function on $\mathbb{R}^2$.

We take the inner product between the both sides of $u'_1(\tau) + u_1^*(\tau) = f_1(\tau)$ and $u_1(\tau) - \mathcal{F}_{\theta_2\theta_1,\varepsilon}(\tau)u_2(\tau)$ to obtain by the definition of the subdifferential of $\partial\psi^\tau(\theta_1; \cdot)$

$$\begin{aligned} & (u'_1(\tau), u_1(\tau) - \mathcal{F}_{\theta_2\theta_1,\varepsilon}(\tau)u_2(\tau))_H + \langle Fu_1(\tau), u_1(\tau) - \mathcal{F}_{\theta_2\theta_1,\varepsilon}(\tau)u_2(\tau) \rangle \\ & \leq (f_1(\tau), u_1(\tau) - \mathcal{F}_{\theta_2\theta_1,\varepsilon}(\tau)u_2(\tau))_H, \quad \text{a.e. } \tau \in (0, T). \end{aligned}$$

Here we observe that

$$\begin{aligned} & (u'_1(\tau), u_1(\tau) - u_2(\tau))_H + \langle Fu_1(\tau), u_1(\tau) - u_2(\tau) \rangle \\ & \leq (f_1(\tau), u_1(\tau) - u_2(\tau))_H + \Gamma_{\theta_1\theta_2,\varepsilon}(\tau) + \tilde{\Gamma}_{\theta_1\theta_2,\varepsilon}(\tau) \end{aligned} \quad (19)$$

with

$$\begin{aligned} \Gamma_{\theta_1\theta_2,\varepsilon}(\tau) &:= (u'_1(\tau), (R_{\theta_2\theta_1}(\tau) - I)u_2(\tau))_H + c_0(\varepsilon)(u'_1(\tau), R_{\theta_2\theta_1}(\tau)u_2(\tau))_H \\ &\quad + \frac{d}{d\tau}(u_1(\tau), \sigma_{\theta_2\theta_1,\varepsilon}(\tau))_H \\ \tilde{\Gamma}_{\theta_1\theta_2,\varepsilon}(\tau) &:= (f_1(\tau), (I - R_{\theta_2\theta_1}(\tau))u_2(\tau) - c_0(\varepsilon)R_{\theta_2\theta_1}(\tau)u_2(\tau) - \sigma_{\theta_2\theta_1,\varepsilon}(\tau))_H \\ &\quad - \langle \sigma'_{\theta_2\theta_1,\varepsilon}(\tau), u_1(\tau) \rangle - \langle Fu_1(\tau), u_2(\tau) - \mathcal{F}_{\theta_2\theta_1,\varepsilon}(\tau)u_2(\tau) \rangle. \end{aligned} \quad (20)$$

Similarly, by exchanging parameters θ_1 and θ_2 we have

$$\begin{aligned} & (u'_2(\tau), u_2(\tau) - u_1(\tau))_H + \langle Fu_2(\tau), u_2(\tau) - u_1(\tau) \rangle \\ & \leq (f_2(\tau), u_2(\tau) - u_1(\tau))_H + \Gamma_{\theta_2\theta_1,\varepsilon}(\tau) + \tilde{\Gamma}_{\theta_2\theta_1,\varepsilon}(\tau), \end{aligned} \quad (21)$$

Adding (19) and (21) yields

$$\begin{aligned} & \frac{1}{2}|u_1(t) - u_2(t)|_H^2 + \int_0^t \langle Fu_1 - Fu_2, u_1 - u_2 \rangle d\tau \\ & \leq \frac{1}{2}|u_{01} - u_{02}|_H^2 + \int_0^t (f_1 - f_2, u_1 - u_2)_H d\tau \\ & \quad + \int_0^t (\Gamma_{\theta_2\theta_1,\varepsilon} + \Gamma_{\theta_1\theta_2,\varepsilon} + \tilde{\Gamma}_{\theta_2\theta_1,\varepsilon} + \tilde{\Gamma}_{\theta_1\theta_2,\varepsilon}) d\tau, \quad \forall t \in [0, T]. \end{aligned}$$

Now, recall the rearrangement of $\Gamma_{\theta_2\theta_1,\varepsilon}$, $\tilde{\Gamma}_{\theta_2\theta_1,\varepsilon}$ and $\Gamma_{\theta_1\theta_2,\varepsilon}$, $\tilde{\Gamma}_{\theta_1\theta_2,\varepsilon}$, which can be derived from the assumptions (A1), (A2) and (A3) on $R_{\theta\bar{\theta}}(t)$ and $\sigma_{\theta\bar{\theta},\varepsilon}$ together with (20).

Lemma 3.3. ([20; Lemma 3.2]) *We have for a.e. $\tau \in (0, T)$,*

$$\begin{aligned} & \Gamma_{\theta_1\theta_2,\varepsilon}(\tau) + \Gamma_{\theta_2\theta_1,\varepsilon}(\tau) \\ &= \frac{d}{d\tau}(u_1(\tau), (R_{\theta_2\theta_1}(\tau) - I)u_2(\tau))_H + c_0(\varepsilon)\frac{d}{d\tau}(u_1(\tau), R_{\theta_2\theta_1}(\tau)u_2(\tau))_H \\ &\quad + \frac{d}{d\tau}\{(u_1(\tau), \sigma_{\theta_2\theta_1,\varepsilon}(\tau))_H + (u_2(\tau), \sigma_{\theta_1\theta_2,\varepsilon}(\tau))_H\} \\ &\quad - (1 + c_0(\varepsilon))(u_1(\tau), R'_{\theta_2\theta_1}(\tau)u_2(\tau))_H, \end{aligned}$$

$$\begin{aligned}
& |\tilde{\Gamma}_{\theta_1\theta_2,\varepsilon}(\tau)| + |\tilde{\Gamma}_{\theta_2\theta_1,\varepsilon}(\tau)| \\
& \leq |f_1(\tau)|_{V^*} \{ |I - R_{\theta_2\theta_1}(\tau)|_{B(V)} |u_2(\tau)|_V + |c_0(\varepsilon)| |u_2(\tau)|_V + |\sigma_{\theta_2\theta_1,\varepsilon}(\tau)|_V \} \\
& \quad + |u_1(\tau)|_V |\sigma'_{\theta_2\theta_1,\varepsilon}(\tau)|_{V^*} + |u_1(\tau)|_V^{p-1} |\mathcal{F}_{\theta_2\theta_1,\varepsilon}(\tau) u_2(\tau) - u_2(\tau)|_V \\
& \quad + |f_2(\tau)|_{V^*} \{ |I - R_{\theta_1\theta_2}(\tau)|_{B(V)} |u_1(\tau)|_V + |c_0(\varepsilon)| |u_1(\tau)|_V + |\sigma_{\theta_1\theta_2,\varepsilon}(\tau)|_V \} \\
& \quad + |u_2(\tau)|_V |\sigma'_{\theta_1\theta_2,\varepsilon}(\tau)|_{V^*} + |u_2(\tau)|_V^{p-1} |\mathcal{F}_{\theta_1\theta_2,\varepsilon}(\tau) u_1(\tau) - u_1(\tau)|_V, \quad \text{and} \\
& |u_1(\tau)|_V^{p-1} |\mathcal{F}_{\theta_2\theta_1,\varepsilon} u_2(\tau) - u_2(\tau)|_V + |u_2(\tau)|_V^{p-1} |\mathcal{F}_{\theta_1\theta_2,\varepsilon} u_1(\tau) - u_1(\tau)|_V \\
& \leq |u_1(\tau)|_V^{p-1} (|I - R_{\theta_2\theta_1}(\tau)|_{B(V)} |u_2(\tau)|_V + |c_0(\varepsilon)| |R_{\theta_2\theta_1}(\tau)|_{B(V)} |u_2(\tau)|_V \\
& \quad + |\sigma_{\theta_2\theta_1,\varepsilon}(\tau)|_V) + |u_2(\tau)|_V^{p-1} (|I - R_{\theta_1\theta_2}(\tau)|_{B(V)} |u_1(\tau)|_V \\
& \quad + |c_0(\varepsilon)| |R_{\theta_1\theta_2}(\tau)|_{B(V)} |u_1(\tau)|_V + |\sigma_{\theta_1\theta_2,\varepsilon}(\tau)|_V).
\end{aligned}$$

Proof of Proposition 3.2. Let M and M_i be the same constants as in (15) and (18), and let ε be any small positive number. Then, by using (A1)–(A3), we have on account of Lemma 3.3 that for all $s, t \in [0, T]$, $s \leq t$.

$$\begin{aligned}
& \left| \int_s^t (\Gamma_{\theta_2\theta_1,\varepsilon} + \Gamma_{\theta_1\theta_2,\varepsilon}) d\tau \right| \\
& \leq 2M_1 M_2 |I - R_{\theta_2\theta_1}|_{C([0,T];B(H))} + 2|c_0(\varepsilon)| M_1 M_2 + 2M_1 |\sigma_{\theta_2\theta_1,\varepsilon}|_{C([0,T];H)} \\
& \quad + 2M_2 |\sigma_{\theta_1\theta_2,\varepsilon}|_{C([0,T];H)} + (1 + |c_0(\varepsilon)|) M_1 M_2 T^{\frac{1}{p}} |R'_{\theta_2\theta_1}|_{L^{p'}(0,T;B(H))} \\
& \leq 2d_\Theta(\theta_1, \theta_2) + 2|c_0(\varepsilon)| M_1 M_2 + 2(M_1 + M_2) c_V \sigma_0(d_\Theta(\theta_1, \theta_2) + \varepsilon) \\
& \quad + 2M_1 M_2 T^{\frac{1}{p}} R_0 d_\Theta(\theta_1, \theta_2);
\end{aligned}$$

we used above $|R_{\theta_i\theta_j}(\tau)|_{B(H)} = 1$ for all τ and the inequality $|z|_H \leq c_V |z|_V$ for all $z \in V$ with an embedding constant $c_V > 0$ to obtain $|\sigma_{\theta_2\theta_1,\varepsilon}|_{C([0,T];H)} \leq c_V |\sigma_{\theta_2\theta_1,\varepsilon}|_{C([0,T];V)}$. Similarly,

$$\begin{aligned}
& \int_0^T \{ |\tilde{\Gamma}_{\theta_2\theta_1,\varepsilon}| + |\tilde{\Gamma}_{\theta_1\theta_2,\varepsilon}| \} d\tau \\
& \leq M(M_1 + M_2) (|I - R_{\theta_2\theta_1}|_{C([0,T];B(V))} + |I - R_{\theta_1\theta_2}|_{C([0,T];B(V))}) \\
& \quad + |c_0(\varepsilon)| T^{\frac{1}{p'}} (M_1 + M_2) + M (|\sigma_{\theta_2\theta_1,\varepsilon}|_{C([0,T];V)} + |\sigma_{\theta_1\theta_2,\varepsilon}|_{C([0,T];V)}) \\
& \quad + M_1 |\sigma'_{\theta_2\theta_1,\varepsilon}|_{L^{p'}(0,T;V^*)} + M_2 |\sigma'_{\theta_1\theta_2,\varepsilon}|_{L^{p'}(0,T;V^*)} \\
& \quad + (M_1^{\frac{p}{p'}} + M_2^{\frac{p}{p'}}) (|I - R_{\theta_2\theta_1}|_{C([0,T];B(V))} + |I - R_{\theta_1\theta_2}|_{C([0,T];B(V))} + 2|c_0(\varepsilon)|) \\
& \quad + M_1^{\frac{p}{p'}} |\sigma_{\theta_1\theta_2,\varepsilon}|_{L^p(0,T;V)} + M_2^{\frac{p}{p'}} |\sigma_{\theta_2\theta_1,\varepsilon}|_{L^p(0,T;V)} \\
& \leq 2M(M_1 + M_2) R_0 d_\Theta(\theta_1, \theta_2) + T^{\frac{1}{p'}} (M_1 + M_2) |c_0(\varepsilon)| + 2M \sigma_0(d_\Theta(\theta_1, \theta_2) + \varepsilon) \\
& \quad + (M_1 + M_2) \sigma_0(d_\Theta(\theta_1, \theta_2) + \varepsilon) + (M_1^{\frac{p}{p'}} + M_2^{\frac{p}{p'}}) (2R_0 d_\Theta(\theta_1, \theta_2) + |c_0(\varepsilon)|) \\
& \quad + (M_1^{\frac{p}{p'}} + M_2^{\frac{p}{p'}}) T^{\frac{1}{p}} \sigma_0(d_\Theta(\theta_1, \theta_2) + \varepsilon).
\end{aligned}$$

By these estimates we can find some constant $C_0^*(M)$ and function $C_1^*(\varepsilon)$ of $\varepsilon \in (0, 1]$ so that (17) holds, for instance,

$$\begin{aligned} C_0^*(M) &:= (M_1 + M_2)(4c_V\sigma_0 + 2MR_0 + T^{\frac{1}{p'}} + 2\sigma_0 + T^{\frac{1}{p'}}) + 2R_0(M_1^p + M_2^p) \\ &\quad + 4(M_1^{\frac{p}{p'}} + M_2^{\frac{p}{p'}})(2R_0 + T^{\frac{1}{p}}\sigma_0) + 2M_1M_2(1 + T^{\frac{1}{p}}R_0) + 2 \end{aligned}$$

and $C_1^*(\varepsilon) := \varepsilon + |c_0(\varepsilon)|$. □

Now, we investigate some fundamental properties of $L_{u_0}(\theta; \cdot)$ which are derived from Proposition 3.2.

Lemma 3.4. *Let $\theta \in \Theta_W$, $u_0 \in \overline{K(\theta; 0)}$ and $f \in L^{p'}(0, T; V^*)$, and let $\{\theta_n\} \subset \Theta_S$, $\{u_{0n}\} \subset H$ and $\{f_n\} \subset L^2(0, T; H)$ such that $u_{0n} \in \overline{K(\theta_n; 0)}$,*

$$\theta_n \rightarrow \theta \text{ in } \Theta, \quad u_{0n} \rightarrow u_0 \text{ in } H, \quad f_n \rightarrow f \text{ in } L^{p'}(0, T; V^*) \quad (\text{as } n \rightarrow \infty). \quad (22)$$

Then the strong solution u_n of

$$u_n'(t) + \partial\psi^t(\theta_n; u_n(t)) \ni f_n(t) \text{ in } H, \text{ a.e. } t \in (0, T), \quad u_n(0) = u_{0n}, \quad (23)$$

converges in $C([0, T]; H) \cap L^p(0, T; V)$ to a solution u of

$$L_{u_0}(\theta; u) + Fu \ni f \text{ in } L^{p'}(0, T; V^*). \quad (24)$$

Proof. Given any small $\varepsilon > 0$ and a constant $M > 0$ satisfying

$$|f|_{L^{p'}(0, T; V^*)} + |u_0|_H + 1 \leq \frac{M}{2},$$

choose a positive integer N_ε such that $d_\Theta(\theta_n, \theta) \leq \frac{\varepsilon}{2}$ and $|f_n|_{L^{p'}(0, T; V^*)} + |u_{0n}|_H \leq \frac{M}{2}$ for all $n \geq N_\varepsilon$. Then it follows from Proposition 3.2 with (22) that

$$\begin{aligned} &\frac{1}{2}|u_n(t) - u_m(t)|_H^2 + \int_0^t \langle Fu_n - Fu_m, u_n - u_m \rangle d\tau \\ &\leq \frac{1}{2}|u_{0n} - u_{0m}|_H^2 + \int_0^t (f_n - f_m, u_n - u_m)_H d\tau + C_0^*(M)C_1^*(\varepsilon), \end{aligned} \quad (25)$$

for all $t \in [0, T]$, for all $n, m \geq N_\varepsilon$, which implies that $u_n \rightarrow u$ in $C([0, T]; H)$ as $n \rightarrow \infty$ and $\varepsilon \rightarrow 0$ for a certain function $u \in C([0, T]; H) \cap L^p(0, T; V)$ and

$$\lim_{n, m \rightarrow \infty} \int_0^T \langle Fu_n - Fu_m, u_n - u_m \rangle d\tau = 0. \quad (26)$$

By the uniform convexity of V , it follows from (26) that $u_n \rightarrow u$ in $L^p(0, T; V)$. Since for large n , small $\varepsilon > 0$ and any $\eta \in K_0(\theta)$, with notation $\eta_{n, \varepsilon}(t) := \mathcal{F}_{\theta\theta_n, \varepsilon}(t)\eta(t)$ we see from (23) that for any interval $[s, t] \subset [0, T]$

$$\int_s^t (u_n', u_n - \eta_{n, \varepsilon})_H d\tau + \int_s^t \langle Fu_n, u_n - \eta_{n, \varepsilon} \rangle d\tau \leq \int_s^t \langle f_n, u_n - \eta_{n, \varepsilon} \rangle d\tau$$

from which we obtain by integration by parts

$$\begin{aligned} &\int_s^t \langle \eta_{n, \varepsilon}', u_n - \eta_{n, \varepsilon} \rangle d\tau + \frac{1}{2}|u_n(t) - \eta_{n, \varepsilon}(t)|_H^2 + \int_0^t \langle Fu_n, u_n - \eta_{n, \varepsilon} \rangle d\tau \\ &\leq \frac{1}{2}|u_n(s) - \eta_{n, \varepsilon}(s)|_H^2 + \int_s^t \langle f_n, u_n - \eta_{n, \varepsilon} \rangle d\tau. \end{aligned}$$

Now, passing to the limit in this inequality as $\theta_n \rightarrow \theta$ in Θ and $\varepsilon \downarrow 0$ by using Lemma 2.5, we conclude that

$$\int_s^t \langle \eta' - f + Fu, u - \eta \rangle d\tau + \frac{1}{2} |u(t) - \eta(t)|_H^2 \leq \frac{1}{2} |u(s) - \eta(s)|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta). \quad (27)$$

This implies $f - Fu \in L_{u_0}(\theta; u)$. Thus (24) is obtained. $\square$

Corollary 3.5. *Let $\theta_i \in \Theta_W$, $u_{0i} \in \overline{K(\theta_i; 0)}$ and $f_i \in L^{p'}(0, T; V^*)$, $i = 1, 2$, and let u_i be a solution of $L_{u_{0i}}(\theta_i; u_i) + Fu_i \ni f_i$ in $L^{p'}(0, T; V^*)$, $i = 1, 2$. Let $M > 0$ and ε be any positive numbers such that*

$$\sum_{i=1}^2 \left\{ |f_i|_{L^{p'}(0, T; V^*)} + |u_{0i}|_H \right\} \leq M, \quad d_\Theta(\theta_1, \theta_2) \leq \varepsilon.$$

Then, for any $s, t \in [0, T]$, $s \leq t$,

$$\begin{aligned} & \frac{1}{2} |u_1(t) - u_2(t)|_H^2 + \int_s^t \langle Fu_1 - Fu_2, u_1 - u_2 \rangle d\tau \\ & \leq \frac{1}{2} |u_1(s) - u_2(s)|_H^2 + \int_s^t \langle f_1 - f_2, u_1 - u_2 \rangle d\tau + C_0^*(M)C_1^*(\varepsilon), \end{aligned} \quad (28)$$

where $C_0^*(M)$ and $C_1^*(\varepsilon)$ are the same ones as in Proposition 3.2.

Proof. Just as in the proof of Lemma 3.4 choose approximate sequences $\{\theta_{i,n}\} \subset \Theta_S$, $\{f_{i,n}\} \subset L^2(0, T; H)$ and $\{u_{0i,n}\} \subset H$ for $i = 1, 2$, such that $u_{0i,n} \in \overline{K(\theta_{i,n}; 0)}$,

$\theta_{i,n} \rightarrow \theta_i$ in Θ , $f_{i,n} \rightarrow f_i$ in $L^{p'}(0, T; V^*)$, $u_{0i,n} \rightarrow u_{0i}$ in H , $i = 1, 2$, (as $n \rightarrow \infty$).

Denote by $u_{i,n}$ the strong solution of

$$u'_{i,n} + \partial\psi^t(\theta_{i,n}; u_{i,n}) \ni f_i, \quad u_{i,n}(0) = u_{0i,n}.$$

Then, for any large n and small $\varepsilon > 0$, it follows from (25) in the proof of Lemma 3.4

$$\begin{aligned} & \frac{1}{2} |u_{1,n}(t) - u_{2,n}(t)|_H^2 + \int_s^t \langle Fu_{1,n} - Fu_{2,n}, u_{1,n} - u_{2,n} \rangle d\tau \\ & \leq \frac{1}{2} |u_{1,n}(s) - u_{2,n}(s)|_H^2 + \int_s^t \langle f_{1,n} - f_{2,n}, u_{1,n} - u_{2,n} \rangle_H d\tau + C_0^*(M)C_1^*(\varepsilon), \end{aligned}$$

for all $s, t \in [0, T]$, $s \leq t$, for all $n, m \geq N_\varepsilon$. By Lemma 3.4, $\{u_{i,n}\}$, $i = 1, 2$, converges in $C([0, T]; H) \cap L^p(0, T; V)$ to solutions u_i of $L_{u_{0i}}(\theta_i; u_i) + Fu_i \ni f_i$ as $n \rightarrow \infty$. Hence, letting $n \rightarrow \infty$ in the above inequality, we see that (28) holds. $\square$

The following corollary is immediately obtained by letting $\varepsilon \downarrow 0$ in (28) with $f_1 = f_2$ and $u_{10} = u_{20}$.

Corollary 3.6. *For every $\theta \in \Theta_W$, $u_0 \in \overline{K(\theta; 0)}$ and $f \in L^{p'}(0, T; V^*)$, the solution u of $L_{u_0}(\theta; u) + Fu \ni f$ is unique.*

Theorem 3.7. Assume (Θ_S) and $(A1)–(A3)$. Then we have:

- (a) Let $\theta \in \Theta_W$. Then for any $u_0 \in \overline{K(\theta; 0)}$, $L_{u_0}(\theta; \cdot)$ is a maximal monotone operator from $D(L_{u_0}(\theta; \cdot)) \subset L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$ and

$$D(L_{u_0}(\theta; \cdot)) \subset \left\{ w \in C([0, T]; H) \cap L^p(0, T; V) \left| \begin{array}{l} w(0) = u_0, \quad w(t) \in K(\theta; t) \\ \text{for a.e. } t \in (0, T) \end{array} \right. \right\}.$$

- (b) Let $\theta \in \Theta_W$, $f, \bar{f} \in L^{p'}(0, T; V^*)$, $u_0, \bar{u}_0 \in \overline{K(\theta; 0)}$, $f \in L_{u_0}(\theta; u)$, $\bar{f} \in L_{\bar{u}_0}(\theta; \bar{u})$. Then, for any $s, t \in [0, T]$, $s \leq t$,

$$\frac{1}{2}|u(t) - \bar{u}(t)|_H^2 \leq \frac{1}{2}|u(s) - \bar{u}(s)|_H^2 + \int_s^t \langle f(\tau) - \bar{f}(\tau), u(\tau) - \bar{u}(\tau) \rangle d\tau. \quad (29)$$

- (c) Let $u_0 \in \overline{K(\theta; 0)}$ and $f \in L_{u_0}(\theta; u)$. Then, for any $s, t \in [0, T]$, $s \leq t$,

$$\int_s^t \langle \eta' - f, u - \eta \rangle d\tau + \frac{1}{2}|u(t) - \eta(t)|_H^2 \leq \frac{1}{2}|u(s) - \eta(s)|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta). \quad (30)$$

Proof. First we prove (a) and (b). Let $\theta \in \Theta_W$, $u_0 \in \overline{K(\theta; 0)}$, $f, \bar{f} \in L^{p'}(0, T; V^*)$. Assume that $f \in L_{u_0}(\theta; u)$ and $\bar{f} \in L_{u_0}(\theta; \bar{u})$. Then, since $f + Fu \in L_{u_0}(\theta; u) + Fu$ and $\bar{f} + F\bar{u} \in L_{u_0}(\theta; \bar{u}) + F\bar{u}$, it follows from (28) by letting $\varepsilon \rightarrow 0$ that

$$\begin{aligned} & \frac{1}{2}|u(t) - \bar{u}(t)|_H^2 + \int_s^t \langle Fu(\tau) - F\bar{u}(\tau), u(\tau) - \bar{u}(\tau) \rangle d\tau \\ & \leq \frac{1}{2}|u(s) - \bar{u}(s)|_H^2 + \int_s^t \langle f + Fu(\tau) - \bar{f} - F\bar{u}(\tau), u(\tau) - \bar{u}(\tau) \rangle d\tau, \end{aligned}$$

which is just (29). By (29) with $s = 0$,

$$\frac{1}{2}|u(t) - \bar{u}(t)|_H^2 \leq \int_0^t \langle f - \bar{f}, u - \bar{u} \rangle d\tau, \quad \forall t \in [0, T].$$

This implies that $L_{u_0}(\theta; \cdot)$ is strictly monotone from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$. Moreover, by Lemma 3.4, the range of $L_{u_0}(\theta; \cdot) + F$ is the whole of $L^{p'}(0, T; V^*)$, so that $L_{u_0}(\theta; \cdot)$ is maximal monotone from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$ and

$$D(L_{u_0}(\theta; \cdot)) \subset \left\{ w \in L^p(0, T; V) \cap C([0, T]; H) \left| \begin{array}{l} w(0) = u_0, \quad w(t) \in K(\theta; t) \\ \text{for a.e. } t \in (0, T) \end{array} \right. \right\}.$$

Thus (a) and (b) are obtained.

Next we show (c). Assume that $f \in L_{u_0}(\theta; u)$. Putting $g := f + Fu$, we observe that $g \in L_{u_0}(\theta; u) + Fu$. By (27) in the proof of Lemma 3.4, we have

$$\int_s^t \langle \eta' - g + Fu, u - \eta \rangle d\tau + \frac{1}{2}|u(t) - \eta(t)|_H^2 \leq \frac{1}{2}|u(s) - \eta(s)|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta).$$

Since $g - Fu = f$, we obtain (30). □

Another important property of $L_{u_0}(\theta; \cdot)$ is stated in the following theorem.

Theorem 3.8. *Assume (Θ_S) and (A1)–(A3). Let $\{\theta_n\}$ be a sequence in Θ_W such that $\theta_n \rightarrow \theta$ in Θ and let $\{u_{0n}\}$ be a sequence in H and $u_0 \in \overline{K(\theta; 0)}$ such that $u_{0n} \in \overline{K(\theta_n; 0)}$ for all n and $u_{0n} \rightarrow u_0$ in H (as $n \rightarrow \infty$). Then $\{L_{u_{0n}}(\theta_n; \cdot)\}$ converges to $L_{u_0}(\theta; \cdot)$ in the graph sense; namely, if $g \in L_{u_0}(\theta; u)$, then there are sequences $\{g_n\}$ and $\{u_n\}$ such that*

$$g_n \in L_{u_{0n}}(\theta_n; u_n), \quad \forall n, \quad g_n \rightarrow g \text{ in } L^{p'}(0, T; V^*), \quad u_n \rightarrow u \text{ in } L^p(0, T; V). \quad (31)$$

Proof. Assume that $g \in L_{u_0}(\theta; u)$. We see that $\tilde{g} := g + Fu \in L_{u_0}(\theta; u) + Fu$. Take a sequence $\{\tilde{g}_n\}$ in $L^2(0, T; H)$ so that $\tilde{g}_n \rightarrow \tilde{g} + Fu$ in $L^{p'}(0, T; V^*)$, and consider the sequence $\{u_n\}$ of strong solutions to

$$u'_n(t) + \partial\psi^t(\theta_n; u_n(t)) \ni \tilde{g}_n(t) \text{ in } H, \text{ a.e. } t \in (0, T), \quad u_n(0) = u_{0n},$$

or equivalently $\tilde{g}_n \in L_{u_{0n}}(\theta_n; u_n) + Fu_n$ in $L^{p'}(0, T; V^*)$.

By virtue of Lemma 3.4 and its proof, we observe that u_n converges to the solution u of $L_{u_0}(\theta; u) + Fu \ni \tilde{g}$ in $L^{p'}(0, T; V^*)$ in the sense that $u_n \rightarrow u$ in the space $C([0, T]; H) \cap L^p(0, T; V)$. Since $\tilde{g}_n - Fu_n \rightarrow \tilde{g} - Fu = g$ in $L^{p'}(0, T; V^*)$ and the sequence $\{u_n, g_n\}$ with $g_n := \tilde{g}_n - Fu_n$ satisfies (31). $\square$

4. Parabolic variational inclusions

We begin with the precise assumptions on (multivalued) semimonotone operator $A = A(v; w)$. Let $\mathcal{V}_A$ be a closed convex subset of $L^p(0, T; V)$ such that

$$\mathcal{K}(\Theta_W) := \bigcup_{\theta \in \Theta_W} \mathcal{K}(\theta) \subset \mathcal{V}_A. \quad (32)$$

Let $A := A(v; u)$ be a mapping from $\mathcal{V}_A \times L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$, satisfying

(B0) if $v_1, v_2 \in \mathcal{V}_A$ and $w_1, w_2 \in L^p(0, T; V)$ such that $v_1 = v_2$ in V and $w_1 = w_2$ in V a.e. on $(0, t)$, $t \in [0, T]$, then $[A(v_1; w_1)](\tau) = [A(v_2; w_2)](\tau)$ in V^* for a.e. $\tau \in (0, t)$.

(B1) (Boundedness) There are positive constants a_1, a_2 such that

$$\sup_{\alpha^* \in A(v, w)} |\alpha^*|_{L^{p'}(0, T; V^*)}^{p-1} \leq a_1 |w|_{L^p(0, T; V)}^{p-1} + a_2, \quad \forall v \in \mathcal{V}_A, \quad \forall w \in L^p(0, T; V).$$

(B2) (Coerciveness) There are positive constants a_3, a_4 such that

$$\int_0^T \langle \alpha^*, w \rangle dt \geq a_3 |w|_{L^p(0, T; V)}^p - a_4, \quad \forall v \in \mathcal{V}_A, \quad \forall \alpha^* \in A(v, w).$$

(B3) (Semimonotonicity) For each $v \in \mathcal{V}_A$, $w \rightarrow A(v; w)$ is a (multivalued) maximal monotone mapping from $D(A(v; \cdot)) = L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$. Moreover, for any sequence $\{v_n\} \subset \mathcal{V}_A$ with $v_n \rightarrow v$ in $L^p(0, T; H)$ and weakly in $L^p(0, T; V)$ (as $n \rightarrow \infty$), the maximal monotone mapping $A(v_n; \cdot)$ converges to $A(v; \cdot)$ in the graph sense, namely for any $w \in L^p(0, T; V)$ and any $\alpha^* \in A(v; w)$ there exist sequences $\{w_n\} \subset L^p(0, T; V)$ and $\{\alpha_n^*\}$ with $\alpha_n^* \in A(v_n; w_n)$ such that

$$w_n \rightarrow w \text{ in } L^p(0, T; V), \quad \alpha_n^* \rightarrow \alpha^* \text{ in } L^{p'}(0, T; V^*).$$

For simplicity we use the following notation:

$$\langle\langle g, w \rangle\rangle = \int_0^T \langle g(t), w(t) \rangle dt, \quad \forall w \in L^p(0, T; V), \quad \forall g \in L^{p'}(0, T; V^*);$$

namely $\langle\langle \cdot, \cdot \rangle\rangle$ means the duality between $L^{p'}(0, T; V^*)$ and $L^p(0, T; V)$.

For each $\theta \in \Theta_W$, $v \in \mathcal{V}_A$, $f \in L^{p'}(0, T; V^*)$ and $u_0 \in \overline{K(\theta; 0)}$ we consider a nonlinear inclusion of the form:

$$L_{u_0}(\theta; u) + A(v; u) \ni f \quad \text{in } L^{p'}(0, T; V^*), \quad (33)$$

more precisely, there are $u \in D(L_{u_0}(\theta; \cdot))$, $\ell^* \in L_{u_0}(\theta; u)$ and $\alpha^* \in A(v; u)$ such that

$$\ell^*(t) + \alpha^*(t) = f(t) \quad \text{in } V^*, \quad \text{a.e. } t \in (0, T),$$

which is written as

$$u \in \mathcal{K}(\theta), \quad \alpha^* \in A(v; u),$$

$$\langle\langle \eta' - f + \alpha^*, u - \eta \rangle\rangle \leq \frac{1}{2} \|u_0 - \eta(0)\|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta).$$

We now prove:

Theorem 4.1. *Assume (Θ_S) , (A1)–(A3) and (B0)–(B3). Then, for each $v \in \mathcal{V}_A$, $\theta \in \Theta_W$, $f \in L^{p'}(0, T; V^*)$ and $u_0 \in \overline{K(\theta; 0)}$, there is a unique solution u of (33).*

Proof. By (a) of Theorem 3.7, $L_{u_0}(\theta; \cdot)$ is a maximal monotone operator from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$. Also, $A(v; \cdot)$ is everywhere defined on $L^p(0, T; V)$, coercive and maximal monotone from $L^p(0, T; V)$ into $L^{p'}(0, T; V^*)$ by condition (B0)–(B3). Therefore, it follows from the general theory on monotone operators (cf. [5]) that the range of the sum $L_{u_0}(\theta; \cdot) + A(v; \cdot)$ is the whole of $L^{p'}(0, T; V^*)$; in other words, for any $f \in L^{p'}(0, T; V^*)$ problem (33) has a solution u , which is unique by the strict monotonicity of $L_{u_0}(\theta; \cdot)$. $\square$

As to the continuous dependence of the solution u of (33) upon the parameters θ and v , we have:

Theorem 4.2. *Suppose that (Θ_S) , (A1)–(A3) and (B0)–(B3) are fulfilled. Let $f \in L^{p'}(0, T; V^*)$, $\theta \in \Theta_W$, $v \in \mathcal{V}_A$ and $u_0 \in \overline{K(\theta; 0)}$. Assume that $\{\theta_n\} \subset \Theta_W$, $\{v_n\} \subset \mathcal{V}_A$ and $\{u_{0n}\} \subset H$ such that $u_{0n} \in \overline{K(\theta_n; 0)}$ for all n , $\{v_n\}$ is bounded in $L^p(0, T; V)$ and*

$$u_{0n} \rightarrow u_0 \text{ in } H, \quad \theta_n \rightarrow \theta \text{ in } \Theta, \quad v_n \rightarrow v \text{ in } L^p(0, T; H) \quad (\text{as } n \rightarrow \infty).$$

Then, the sequence $\{u_n\}$ of solutions of (33) with $\theta = \theta_n$, $v = v_n$ and $u_0 = u_{0n}$ converges to the solution u of (33) in $C([0, T]; H)$ and weakly in $L^p(0, T; V)$.

By virtue of Theorem 4.1, for each n there is a unique solution u_n of

$$L_{u_{0n}}(\theta_n; u_n) + A(v_n; u_n) \ni f \quad \text{in } L^{p'}(0, T; V^*), \quad (34)$$

namely

$$\ell_n^* \in L_{u_{0n}}(\theta_n; u_n), \quad \alpha_n^* \in A(v_n; u_n), \quad \ell_n^*(t) + \alpha_n^*(t) = f(t) \quad \text{in } V^*, \quad \text{a.e. } t \in (0, T).$$

Lemma 4.3. *The sequence $\{u_n\}$ of solutions to (34) is bounded in $L^p(0, T; V)$ and $C([0, T]; H)$; in fact we have:*

$$|u_n|_{L^p(0, T; V)}^p + |u_n|_{C([0, T]; H)}^2 \leq N_0 \left\{ |f|_{L^{p'}(0, T; V^*)}^{p'} + |u_0|_H^2 + 1 \right\}, \quad \forall n, \quad (35)$$

where N_0 is a positive constant independent of n .

Proof. By Lemma 2.5 there is a sequence $\{\eta_n\}$ with $\eta_n \in \mathcal{K}_0(\theta_n)$ with a positive constant N'_0 such that

$$|\eta_n|_{L^p(0, T; V)} + |\eta'_n|_{L^{p'}(0, T; V^*)} + |\eta_n|_{C([0, T]; H)} \leq N'_0.$$

For each n we have by (c) of Theorem 3.7

$$\begin{aligned} & \int_0^t \langle \eta'_n, u_n - \eta_n \rangle d\tau + \int_0^t \langle \alpha_n^*, u_n - \eta_n \rangle d\tau + \frac{1}{2} |u_n(t) - \eta_n(t)|_H^2 \\ & \leq \int_0^t \langle f, u_n - \eta_n \rangle d\tau + \frac{1}{2} |u_{0n} - \eta_n(0)|_H^2, \quad \forall t \in [0, T]. \end{aligned}$$

From the above inequality with (B1) and (B2) we obtain the following estimate:

$$\frac{a_3}{2} \int_0^t |u_n|_V^p d\tau + \frac{1}{4} |u_n(t)|_H^2 \leq N''_0 \left\{ \int_0^t |f|_{V^*}^{p'} d\tau + |u_0|_H^2 + 1 \right\}, \quad \forall t \in [0, T],$$

where a_3 is the same constant as in (B2), N''_0 is a positive constant independent of $t \in [0, T]$, η_n and f ; actually it depends only on N'_0 . Hence we have (35). $\square$

On account of Lemma 4.3, we can find a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\begin{aligned} u_{n_k} & \rightarrow u \text{ weakly in } L^p(0, T; V), \quad \alpha_{n_k}^* \rightarrow \alpha^* \text{ weakly in } L^{p'}(0, T; V^*), \\ \ell_{n_k}^* = f - \alpha_{n_k}^* & \rightarrow f - \alpha^* =: \ell^* \text{ weakly in } L^{p'}(0, T; V^*), \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Lemma 4.4. $\liminf_{k \rightarrow \infty} \langle \alpha_{n_k}^*, u_{n_k} - u \rangle \geq 0$ and $\limsup_{k \rightarrow \infty} \langle \ell_{n_k}^*, u_{n_k} - u \rangle \leq 0$.

Proof. We first observe from (34) that

$$\langle \langle f, u_{n_k} - w \rangle \rangle = \langle \langle \ell_{n_k}^*, u_{n_k} - w \rangle \rangle + \langle \langle \alpha_{n_k}^*, u_{n_k} - w \rangle \rangle, \quad \forall w \in L^p(0, T; V). \quad (36)$$

By assumption (B3), for any $\tilde{\alpha}^* \in A(v; u)$ there are sequences $\{w_k\}$ in $L^p(0, T; V)$ and $\{\tilde{\alpha}_k^*\}$ with $\tilde{\alpha}_k^* \in A(v_{n_k}; w_k)$ such that

$$w_k \rightarrow u \text{ in } L^p(0, T; V), \quad \tilde{\alpha}_k^* \rightarrow \tilde{\alpha}^* \text{ in } L^{p'}(0, T; V^*).$$

Now, taking w_k as w in (36) and passing to the limit as $k \rightarrow \infty$, we have

$$0 = \lim_{k \rightarrow \infty} \langle \langle f, u_{n_k} - w_k \rangle \rangle \geq \limsup_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, u_{n_k} - w_k \rangle \rangle + \liminf_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} - w_k \rangle \rangle. \quad (37)$$

Here, since $\alpha_{n_k}^* \in A(v_{n_k}; u_{n_k})$ and $\tilde{\alpha}_k^* \in A(v_{n_k}; w_k)$, we note from the monotonicity of $A(v_{n_k}; \cdot)$ that

$$\begin{aligned} \langle \langle \alpha_{n_k}^*, u_{n_k} - w_k \rangle \rangle & = \langle \langle \alpha_{n_k}^* - \tilde{\alpha}_k^*, u_{n_k} - w_k \rangle \rangle + \langle \langle \tilde{\alpha}_k^*, u_{n_k} - w_k \rangle \rangle \\ & \geq \langle \langle \tilde{\alpha}_k^*, u_{n_k} - w_k \rangle \rangle. \end{aligned}$$

Therefore,

$$\liminf_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} - w_k \rangle \rangle \geq \lim_{k \rightarrow \infty} \langle \langle \tilde{\alpha}_k^*, u_{n_k} - w_k \rangle \rangle = \langle \langle \tilde{\alpha}^*, u - u \rangle \rangle = 0,$$

so that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} - u \rangle \rangle &= \liminf_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} - w_k + w_k - u \rangle \rangle \\ &= \liminf_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} - w_k \rangle \rangle + \lim_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, w_k - u \rangle \rangle \\ &= \liminf_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} - w_k \rangle \rangle \geq 0 \end{aligned}$$

and another inequality of the lemma follows similarly from (37). $\square$

Proof of Theorem 4.2: First we shall show that

$$\lim_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} \rangle \rangle = \langle \langle \alpha^*, u \rangle \rangle \quad \text{and} \quad \alpha^* \in A(v; u). \quad (38)$$

Let η be any function in $\mathcal{K}_0(\theta)$ and put $\eta_{n,\varepsilon}(t) := \mathcal{F}_{\theta\theta_{n,\varepsilon}}(t)\eta(t)$; note from Lemma 2.5 that $\eta_{n,\varepsilon} \rightarrow \eta$ in $L^p(0, T; V)$, $\eta'_{n,\varepsilon} \rightarrow \eta'$ in $L^{p'}(0, T; V^*)$ and $\eta_{n,\varepsilon} \rightarrow \eta$ in $C([0, T]; H)$ as well, when $n \rightarrow \infty$ and $\varepsilon \downarrow 0$. Since $f - \alpha_n^* \in L_{u_{0n}}(\theta_n; u_n)$, it follows that

$$\langle \langle \eta'_{n,\varepsilon} - f + \alpha_n^*, u_n - \eta_{n,\varepsilon} \rangle \rangle \leq \frac{1}{2} |u_{0n} - \eta_{n,\varepsilon}(0)|_H^2, \quad (39)$$

Now, let $n = n_k \rightarrow \infty$ and $\varepsilon \downarrow 0$ in (39) and note $\liminf_{n \rightarrow \infty} \langle \langle \alpha_n^*, u_n \rangle \rangle \geq \langle \langle \alpha^*, u \rangle \rangle$ by Lemma 4.4, to see that

$$\langle \langle \eta' - f + \alpha^*, u - \eta \rangle \rangle \leq \frac{1}{2} |u_0 - \eta(0)|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta),$$

which implies by definition that $f - \alpha^* \in L_{u_0}(\theta; u)$.

Next we show that $\alpha^* \in A(v; u)$. In fact, since $\ell^* := f - \alpha^* \in L_{u_0}(\theta; u)$, it follows from Theorem 3.8 that there exists a sequence $\{\tilde{u}_n, \tilde{\ell}_n^*\}$ such that

$$\tilde{\ell}_n^* \in L_{u_{0n}}(\theta_n; \tilde{u}_n), \quad \tilde{u}_n \rightarrow u \text{ in } L^p(0, T; V), \quad \tilde{\ell}_n^* \rightarrow \ell^* = f - \alpha^* \text{ in } L^{p'}(0, T; V^*). \quad (40)$$

Using this sequence, we see that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, u_{n_k} - u \rangle \rangle &= \liminf_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, u_{n_k} - \tilde{u}_{n_k} + \tilde{u}_{n_k} - u \rangle \rangle \\ &\geq \liminf_{k \rightarrow \infty} \langle \langle \tilde{\ell}_{n_k}^*, u_{n_k} - \tilde{u}_{n_k} \rangle \rangle + \liminf_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, \tilde{u}_{n_k} - u \rangle \rangle \\ &\geq \lim_{k \rightarrow \infty} \langle \langle \tilde{\ell}_{n_k}^*, u_{n_k} - \tilde{u}_{n_k} \rangle \rangle + \lim_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, \tilde{u}_{n_k} - u \rangle \rangle = 0. \end{aligned}$$

Therefore, together with $\limsup_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, u_{n_k} - u \rangle \rangle \leq 0$ in Lemma 4.4, we have that

$$\lim_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, u_{n_k} - u \rangle \rangle = 0, \quad \text{i.e.} \quad \lim_{k \rightarrow \infty} \langle \langle \ell_{n_k}^*, u_{n_k} \rangle \rangle = \langle \langle \ell^*, u \rangle \rangle.$$

Since $\ell_{n_k}^* = f - \alpha_{n_k}^*$ and $\alpha_{n_k}^* \rightarrow \alpha^*$ weakly in $L^{p'}(0, T; V^*)$, it results from the above equality that

$$\lim_{k \rightarrow \infty} \langle \langle \alpha_{n_k}^*, u_{n_k} \rangle \rangle = \langle \langle \alpha^*, u \rangle \rangle, \quad (41)$$

namely (38) holds.

We are now in a position to show $\alpha^* \in A(v; u)$. Let w be any element in V and α_w^* be any element of $A(v; w)$. Using (B3), choose a sequence $\{\alpha_{w,k}^*, w_k\}$ so that $\alpha_{w,k}^* \in A(v_{n_k}; w_k)$, $w_k \rightarrow w$ in $L^p(0, T; V)$ and $\alpha_{w,k}^* \rightarrow \alpha_w^*$ in $L^{p'}(0, T; V^*)$ as $k \rightarrow \infty$. Then, by the monotonicity of $A(v_{n_k}; \cdot)$, we have that

$$\langle \langle \alpha_{n_k}^* - \alpha_{w,k}^*, u_{n_k} - w_k \rangle \rangle \geq 0.$$

Passing to the limit in $k \rightarrow \infty$, we obtain by (41)

$$\langle \langle \alpha^* - \alpha_w^*, u - w \rangle \rangle \geq 0, \quad \forall \alpha_w^* \in A(v; w), \quad \forall w \in V,$$

which implies that $\alpha^* \in A(v; u)$ by the maximal monotonicity of $A(v; \cdot) : V \rightarrow V^*$.

Finally we show that $u_{n_k} \rightarrow u$ in $C([0, T]; H)$ as $k \rightarrow \infty$. From (b) of Theorem 3.7 it follows that for all $t \in [0, T]$

$$\frac{1}{2} |u_{n_k}(t) - u(t)|_H^2 + \int_0^t \langle \alpha_{n_k}^* - \alpha^*, u_{n_k} - u \rangle d\tau \leq \frac{1}{2} |u_{0n_k} - u_0|_H^2 + C_0^*(M)C_1^*(\varepsilon), \quad (42)$$

for all large k and any small $\varepsilon > 0$.

Here we show by the same idea as getting (41) under (40) that

$$\lim_{k \rightarrow \infty} \int_0^t \langle \alpha_{n_k}^* - \alpha^*, u_{n_k} - u \rangle d\tau = 0. \quad (43)$$

Indeed, take sequences $\{\tilde{u}_n\}$ and $\{\tilde{\alpha}_n^*\}$ so that $\tilde{u}_n \rightarrow u$ in $L^p(0, T; V)$, $\tilde{\alpha}_n^* \rightarrow \alpha^*$ in $L^{p'}(0, T; V^*)$ and $\tilde{\alpha}_n^* \in A(v_n; \tilde{u}_n)$ for all n . Then, by putting

$$\bar{u}_{n_k} := \begin{cases} \tilde{u}_{n_k} & \text{on } [0, t), \\ u_{n_k} & \text{on } [t, T], \end{cases} \quad \bar{\alpha}_{n_k}^*(t) := \begin{cases} \tilde{\alpha}_{n_k}^* & \text{on } [0, t), \\ \alpha_{n_k}^* & \text{on } [t, T], \end{cases}$$

we see from condition (B0) that $\bar{\alpha}_{n_k}^* \in A(v_{n_k}; \bar{u}_{n_k})$, and clearly $\bar{u}_{n_k} \rightarrow u$ in $L^p(0, T; V)$ and $\bar{\alpha}_{n_k}^* \rightarrow \alpha^*$ in $L^{p'}(0, T; V^*)$ as $k \rightarrow \infty$. Hence we can obtain

$$\begin{aligned} & \liminf_{k \rightarrow \infty} \int_0^t \langle \alpha_{n_k}^* - \alpha^*, u_{n_k} - u \rangle d\tau \\ &= \liminf_{k \rightarrow \infty} \int_0^t \langle \alpha_{n_k}^* - \bar{\alpha}_{n_k}^* + \bar{\alpha}_{n_k}^* - \alpha^*, u_{n_k} - \tilde{u}_{n_k} + \tilde{u}_{n_k} - u \rangle d\tau \\ &= \liminf_{k \rightarrow \infty} \int_0^t \langle \alpha_{n_k}^* - \bar{\alpha}_{n_k}^*, u_{n_k} - \tilde{u}_{n_k} \rangle d\tau \\ &= \liminf_{k \rightarrow \infty} \int_0^T \langle \alpha_{n_k}^* - \bar{\alpha}_{n_k}^*, u_{n_k} - \bar{u}_{n_k} \rangle d\tau \geq 0; \end{aligned}$$

the last equality follows from (B3). As $k \rightarrow \infty$ and $\varepsilon \downarrow 0$ in (42), (43) is obtained and $u_{n_k} \rightarrow u$ in $C([0, T]; H)$. $\square$

5. Parabolic quasi-variational inequalities

In this section we give the formulation of a class of parabolic quasi-variational inequalities and an abstract existence result.

Given an initial datum $u_0 \in H$, we introduce a feedback system Λ_{u_0} , which is an operator from $\mathcal{V}_A$ (cf. (32)) into Θ_W satisfying the following conditions:

- (A1) Λ_{u_0} maps $\mathcal{V}_A$ into $\Theta_W(u_0) := \{\theta \in \Theta_W \mid u_0 \in \overline{K(\theta; 0)}\}$.
- (A2) If $\{w_n\} \subset \mathcal{V}_A$ and is bounded in $L^p(0, T; V)$ and $w_n \rightarrow w$ in $L^p(0, T; H)$, then $\Lambda_{u_0} w_n \rightarrow \Lambda_{u_0} w$ in Θ as $n \rightarrow \infty$.

Definition 5.1. Given $u_0 \in H$ and $f \in L^{p'}(0, T; V^*)$, we denote by $QVI(\Lambda_{u_0}; f, u_0)$ the problem to find a pair $\{\theta, u\} \in \Theta_W(u_0) \times L^p(0, T; V)$ such that

$$\begin{cases} L_{u_0}(\theta; u) + A(u; u) \ni f \text{ in } L^{p'}(0, T; V^*), \\ \theta = \Lambda_{u_0} u \text{ in } \Theta_W. \end{cases} \quad (44)$$

We need an additional technical set-up in order to establish the solvability of $QVI(\Lambda_{u_0}; f, u_0)$. Let W be a reflexive and *separable* Banach space which is densely and continuously embedded in V ; in this case, since $V^* \subset W^*$,

$$V \subset H \subset W^* \text{ with compact embeddings.} \quad (45)$$

Moreover, we suppose that there is a positive number δ_0 such that

$$\delta_0 B_W(0) \subset K(\theta; t), \quad \forall \theta \in \Theta_W(u_0), \quad \forall t \in [0, T], \quad (46)$$

where $B_W(0)$ is the unit closed ball around the origin in W .

Under the additional conditions (45), (46) and a given constant $M^* > 0$, we consider the set $Z(\delta_0, M^*, u_0)$ in $L^p(0, T; V) \cap L^\infty(0, T; H)$ given as:

$$\left\{ u \left| \begin{array}{l} |u|_{L^p(0, T; V)} \leq M^*, \quad |u|_{L^\infty(0, T; H)} \leq M^*, \\ \exists g \in L^{p'}(0, T; V^*) \text{ such that} \\ \langle \langle g, u \rangle \rangle \leq M^*, \quad |g|_{L^1(0, T; W^*)} \leq M^*, \\ \langle \langle \eta' - g, u - \eta \rangle \rangle \leq \frac{1}{2} |u_0 - \eta(0)|_H^2, \quad \forall \eta \in L^p(0, T; V) \\ \text{with } \eta' \in L^{p'}(0, T; V^*) \text{ and } \eta(t) \in \delta_0 B_W(0), \quad \forall t \in [0, T] \end{array} \right. \right\} \quad (47)$$

The compactness lemma stated below is one of important mathematical tools for the solvability of $QVI(\Lambda_{u_0}; f, u_0)$.

Lemma 5.2. ([10; Theorem 4.1]). *For any $\delta_0 > 0$, $M^* > 0$ and $u_0 \in H$ the set $Z(\delta_0, M^*, u_0)$ is relatively compact in the space $L^p(0, T; H)$ and its convex closure $\overline{\text{conv}}(Z(\delta_0, M^*, u_0))$ (in $L^p(0, T; V)$) is compact in $L^p(0, T; H)$.*

Next, let $A(v; u)$ be the same semimonotone operator as in the last section. Then we note that there is a positive constant N_0 such that

$$|u|_{C([0, T]; H)}^2 + |u|_{L^p(0, T; V)}^p \leq N_0(|f|_{L^{p'}(0, T; V^*)}^{p'} + |u_0|_H^2) =: N_1, \quad (48)$$

for all solutions u of $L_{u_0}(\theta; u) + A(v; u) \ni f$ as long as $\theta \in \Theta_W(u_0)$ and $v \in \mathcal{V}_A$.

In fact, by (c) of Theorem 3.7 we have

$$\int_0^t \langle \eta' - f + \alpha^*, u - \eta \rangle d\tau + \frac{1}{2} |u(t) - \eta(t)|_H^2 \leq \frac{1}{2} |u_0 - \eta(0)|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta), \quad \forall t \in [0, T],$$

where $\alpha^* \in A(v; u)$. By (46), we take 0 as $\eta \in \mathcal{K}_0(\theta_n)$ to get

$$\frac{1}{2} |u(t)|_H^2 + \int_0^t \langle \alpha^*, u \rangle d\tau \leq \int_0^t \langle f, u \rangle d\tau + \frac{1}{2} |u_0|_H^2, \quad \forall t \in [0, T].$$

By using conditions (B1) and (B2), it is easy to derive (48) for some constant $N_1 > 0$ from this inequality.

Corollary 5.3. *Let $\{\theta_n\} \subset \Theta_W(u_0)$ and $\{v_n\} \subset \mathcal{V}_A$ be sequences such that $\{v_n\}$ is bounded in $L^p(0, T; V)$ and*

$$\theta_n \rightarrow \theta \text{ in } \Theta, \quad v_n \rightarrow v \text{ in } L^p(0, T; H) \quad (\text{as } n \rightarrow \infty).$$

Let $\{u_n\}$ be the sequence of solutions u_n of $L_{u_0}(\theta_n; u_n) + A(v_n; u_n) \ni f$. Then $\{u_n\}$ is bounded in $C([0, T]; H) \cap L^p(0, T; V)$ and is relatively compact in $L^p(0, T; H)$.

We see by (c) of Theorem 3.7, together with (45)–(48), and Theorem 4.2 that $u_n \in Z(\delta_0, M^*, u_0)$ for a certain constant $M^* > 0$. Hence this corollary is a direct consequence of Lemma 5.2.

Now we formulate an existence result for $QVI(\Lambda_{u_0}; f, u_0)$.

Theorem 5.4. *Assume that (Θ_S) , (A1)–(A3), (B0)–(B3) and (45)–(46) are fulfilled. Let $u_0 \in H$ be a given initial datum and Λ_{u_0} be a feedback system from $\mathcal{V}_A$ into $\Theta_W(u_0) \neq \emptyset$, satisfying $(\Lambda 1)$ and $(\Lambda 2)$. Then, for each $f \in L^{p'}(0, T; V^*)$, the problem $QVI(\Lambda_{u_0}; f, u_0)$ from Definition 5.1 admits at least one solution $\{\theta, u\}$.*

Proof. By estimate (48), for a large constant M^* we see that any solution u of $L_{u_0}(\theta; u) + A(v; u) \ni f$ belongs to $Z(\delta_0, M^*, u_0)$, as long as $\theta \in \Theta_W(u_0)$ and $v \in \mathcal{V}_A$.

We put $\mathcal{X}(u_0) := \overline{\text{conv}}(Z(\delta_0; M^*, u_0))$, which is compact and convex in $L^p(0, T; H)$ by Lemma 5.2. Now, for each $u \in \mathcal{X}(u_0) \cap \mathcal{V}_A$, we denote by $\bar{u}$ a unique solution of

$$\theta = \Lambda_{u_0} u, \quad L_{u_0}(\theta; \bar{u}) + A(u; \bar{u}) \ni f \text{ in } L^{p'}(0, T; V^*),$$

and define a mapping $S: \mathcal{X}(u_0) \cap \mathcal{V}_A \rightarrow \mathcal{X}(u_0) \cap \mathcal{V}_A$ by

$$\bar{u} = Su \text{ (} \in Z(\delta_0, M^*, u_0) \subset \mathcal{X}(u_0) \cap \mathcal{V}_A \text{)}.$$

We are going to prove that S is continuous in $\mathcal{X}(u_0) \cap \mathcal{V}_A$ with respect to the topology of $L^p(0, T; H)$. Let $\{u_n\}$ be any sequence in $\mathcal{X}(u_0) \cap \mathcal{V}_A$ such that $u_n \rightarrow u$ in $L^p(0, T; H)$. Then, by $(\Lambda 2)$, $\theta_n := \Lambda_{u_0} u_n \rightarrow \Lambda_{u_0} u =: \theta$ in Θ . Therefore, by Corollary 5.3, the solution $\bar{u}_n$ of $L_{u_0}(\theta_n; \bar{u}_n) + A(u_n; \bar{u}_n) \ni f$ converges to the solution $\bar{u}$ of $L_{u_0}(\theta; \bar{u}) + A(u; \bar{u}) \ni f$ in the sense that $\bar{u}_n \rightarrow \bar{u}$ in $C([0, T]; H)$ and weakly in $L^p(0, T; V)$. This shows that $\bar{u}_n = Su_n \rightarrow Su = \bar{u}$ in $L^p(0, T; H)$.

Now we are in a position to apply the Schauder's fixed-point theorem for S in $\mathcal{X}(u_0) \cap \mathcal{V}_A$ in order to find a function $u \in \mathcal{X}(u_0) \cap \mathcal{V}_A$ such that $u = Su$. In this case, the pair $\{\theta, u\}$ with $\theta = \Lambda_{u_0} u$ satisfies (44) and u is a solution of $QVI(\Lambda_{u_0}; f, u_0)$. $\square$

In general, the quasi-variational inequality (44) has multiple solutions as the following simple example shows.

Example 5.5. (Non-uniqueness) We consider the case $H = V = \mathbf{R}$. For a fixed positive constant c_0 , put

$$X_0 := \{z \in W^{1,2}(0, T) \mid 0 \leq z' \leq c_0 \text{ a.e. on } (0, T), z(0) = 1\},$$

which is compact and convex in $L^2(0, T)$. Clearly

$$\{2 - e^{-ct} \mid 0 \leq c \leq c_0\} \subset X_0. \quad (49)$$

As the space Θ of parameters we take X_0 with metric $d_\Theta(z_1, z_2) = |z_1 - z_2|_{C(\overline{Q})}$, and define for every $z \in \Theta$

$$K(z; t) := \{r \in \mathbf{R} \mid r \geq z(t) - 2\}, \quad \forall t \in [0, T].$$

It is easy to check the conditions (Θ_S) and (A1)–(A3). By the general results in section 3, the time-derivative $L_1(z; \cdot)$ is defined as well corresponding to the initial value 1 and the constraint set $K(z; t)$. Next we formulate Λ_1 , with initial value 1, as a feedback system from $L^2(0, T)$ into Θ as follows: for each $v \in L^2(0, T)$, define $z := \Lambda_1 v \in \Theta$ by

$$|z - v|_{L^2(0, T)}^2 = \min_{\zeta \in \Theta} |\zeta - v|_{L^2(0, T)}^2;$$

we note from (49) that

$$\Lambda_1(2 - e^{-ct}) = 2 - e^{-ct}, \quad 0 \leq \forall c \leq c_0. \quad (50)$$

Now, consider the quasi-variational inequality (44) with $A(v; u) \equiv 0$ and $f = 0$:

$$L_1(z; u) \ni 0 \quad \text{in } L^2(0, T), \quad z = \Lambda_1 u,$$

which can be written as

$$\int_0^T \eta'(u - \eta) dt \leq \frac{1}{2} |1 - \eta(0)|^2, \quad \forall \eta \in \mathcal{K}_0(z), \quad z = \Lambda_1 u, \quad (51)$$

where $\mathcal{K}_0(z)$ is the class of smooth test functions, namely

$$\mathcal{K}_0(z) = \{\eta \in W^{1,2}(0, T) \mid \eta(t) \in K(z; t), \quad \forall t \in [0, T]\}.$$

For instance, pay attention to the function $u(t) := 2 - e^{-ct}$ for any constant c with $0 \leq c \leq c_0$. Then we see that $u(0) = 1$ and for any $\eta \in \mathcal{K}_0(u)$

$$\int_0^T \eta'(u - \eta) dt \leq \frac{1}{2} |1 - \eta(0)|^2.$$

In fact, since $u'(t) = ce^{-ct} \geq 0$ and $u(t) - \eta(t) \leq 0$, it follows that

$$\begin{aligned} \int_0^T \eta'(u - \eta) dt &= \int_0^T u'(u - \eta) dt - \frac{1}{2} |u(T) - \eta(T)|^2 + \frac{1}{2} |1 - \eta(0)|^2 \\ &\leq \frac{1}{2} |1 - \eta(0)|^2. \end{aligned}$$

Besides, we have by (50) that $\Lambda_1 u = 2 - e^{-ct} = u$. Thus $u := 2 - e^{-ct}$ satisfies (51) with $z = u$ for all $c \in [0, c_0]$, which shows that problem (51) possesses infinitely many solutions. $\square$

Remark 5.6. In Theorem 5.4 an existence result of $QVI(\Lambda_{u_0}; f, u_0)$ was established, based on a compactness property of $L_{u_0}(\theta; \cdot)$ (cf. Lemma 5.2). Of course, there are a variety of existence results for $QVI(\Lambda_{u_0}; f, u_0)$ without such a compactness property, depending on the choice of feedback system Λ_{u_0} . For instance, see Application 6.1 of the next section.

6. Applications

6.1. Quasi-variational ordinary differential inequality

In the first application we are going to treat a sweeping process with quasi-variational structure in two dimensional space $\mathbf{R}^2$.

Let us consider the case of $H = V = W = \mathbf{R}^2$ and

$$\Theta := \left\{ \theta := [\mathbf{a}, \gamma, \boldsymbol{\zeta}] \left| \begin{array}{l} \mathbf{a} \in W^{1,p}(0, T; \mathbf{R}^2), \quad |\mathbf{a}(t)| = 1, \quad \forall t \in [0, T], \\ \gamma \in C_b(\mathbf{R}^2), \quad \gamma_* \leq \gamma \leq \gamma^* \text{ on } \mathbf{R}, \\ \boldsymbol{\zeta} \in C([0, T]; \mathbf{R}^2) \end{array} \right. \right\},$$

where $2 \leq p < \infty$ and γ_*, γ^* are positive constants with $\gamma_* < \gamma^*$; $C_b(\mathbf{R}^2)$ is the space of all functions γ in $C(\mathbf{R}^2)$ such that $\lim_{|\mathbf{r}| \rightarrow \infty} \gamma(\mathbf{r})$ exists. Here the space Θ is a complete metric space with metric

$$d_\Theta(\theta, \bar{\theta}) := |\mathbf{a} - \bar{\mathbf{a}}|_{W^{1,p}(0, T; \mathbf{R}^2)} + |\gamma - \bar{\gamma}|_{C_b(\mathbf{R}^2)} + |\boldsymbol{\zeta} - \bar{\boldsymbol{\zeta}}|_{C([0, T]; \mathbf{R}^2)},$$

for $\theta := [\mathbf{a}, \gamma, \boldsymbol{\zeta}]$, $\bar{\theta} := [\bar{\mathbf{a}}, \bar{\gamma}, \bar{\boldsymbol{\zeta}}] \in \Theta$. Also, given $\theta := [\mathbf{a}, \gamma, \boldsymbol{\zeta}]$ and $\bar{\theta} := [\bar{\mathbf{a}}, \bar{\gamma}, \bar{\boldsymbol{\zeta}}]$, we define the rotation $R_{\theta\bar{\theta}}(t)$ by

$$R_{\theta\bar{\theta}}(t) := \begin{pmatrix} \cos \alpha(t) & -\sin \alpha(t) \\ \sin \alpha(t) & \cos \alpha(t) \end{pmatrix}, \quad (52)$$

with the angle $\alpha(t)$ between the vectors

$$\mathbf{a}(t) := (a^{(1)}(t), a^{(2)}(t)) \quad \text{and} \quad \bar{\mathbf{a}}(t) := (\bar{a}^{(1)}(t), \bar{a}^{(2)}(t)).$$

It is easy to see from (52) that for any vector $\mathbf{z} := (z^{(1)}, z^{(2)}) \in \mathbf{R}^2$

$$R_{\theta\bar{\theta}}(t)\mathbf{z} = (z^{(1)} \cos \alpha(t) - z^{(2)} \sin \alpha(t), z^{(1)} \sin \alpha(t) + z^{(2)} \cos \alpha(t)), \quad (53)$$

with

$$\sin \alpha(t) = a^{(1)}(t)\bar{a}^{(2)}(t) - a^{(2)}(t)\bar{a}^{(1)}(t), \quad \cos \alpha(t) = a^{(1)}(t)\bar{a}^{(1)}(t) + a^{(2)}(t)\bar{a}^{(2)}(t). \quad (54)$$

Also, as function $\sigma_{\theta\bar{\theta}, \varepsilon}(t)$ we take

$$\sigma_{\theta\bar{\theta}, \varepsilon}(t) := \varepsilon \bar{\mathbf{a}}(t), \quad \forall \varepsilon \in (0, 1], \quad \forall t \in [0, T]. \quad (55)$$

Now, for each $\theta := [\mathbf{a}, \gamma, \boldsymbol{\zeta}] \in \Theta$, we put

$$K(\theta; t) := \{\mathbf{z} \in \mathbf{R}^2 \mid \mathbf{a}(t) \cdot (\mathbf{z} - \mathbf{a}(t)) = 0, \quad |\mathbf{z} - \mathbf{a}(t)| \leq \gamma(\boldsymbol{\zeta}(t))\}, \quad \forall t \in [0, T]. \quad (56)$$

(Verification of (Θ_S)) Denote by Θ_1 the set of all parameters $\theta := [\mathbf{a}, \gamma, \boldsymbol{\zeta}] \in \Theta$ of C^2 -class. Let $\theta := [\mathbf{a}, \gamma, \boldsymbol{\zeta}] \in \Theta_1$, and $0 = T_0 < T_1 < T_2 < \dots < T_N := T$ be a partition of $[0, T]$ such that

$$|\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))| < \gamma_*, \quad \forall s, t \in [T_{k-1}, T_k], \quad k = 1, 2, \dots, N.$$

Given $\mathbf{z} \in K(\theta; s)$, $s, t \in [T_{k-1}, T_k]$, $s \leq t$, we put

$$\tilde{\mathbf{z}} := \left(1 - \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))|\right) R(s, t)(\mathbf{z} - \mathbf{a}(s)) + \mathbf{a}(t),$$

where $R(s, t)$ is the rotation operator with the angle between $\mathbf{a}(s)$ and $\mathbf{a}(t)$. Then

$$\begin{aligned} |\tilde{\mathbf{z}} - \mathbf{a}(t)| &= \left(1 - \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))|\right) |\mathbf{z} - \mathbf{a}(s)| \\ &\leq \left(1 - \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))|\right) \gamma(\boldsymbol{\zeta}(s)) \\ &= \gamma(\boldsymbol{\zeta}(s)) - \frac{\gamma(\boldsymbol{\zeta}(s))}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))| \leq \gamma(\boldsymbol{\zeta}(t)), \end{aligned}$$

and

$$\begin{aligned} (\tilde{\mathbf{z}} - \mathbf{a}(t)) \cdot \mathbf{a}(t) &= \left(1 - \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))|\right) R(s, t)(\mathbf{z} - \mathbf{a}(s)) \cdot \mathbf{a}(t) \\ &= \left(1 - \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))|\right) R(s, t)(\mathbf{z} - \mathbf{a}(s)) \cdot R(s, t)^{-1} \mathbf{a}(s) \\ &= \left(1 - \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))|\right) (\mathbf{z} - \mathbf{a}(s)) \cdot \mathbf{a}(s) = 0. \end{aligned}$$

Hence $\tilde{\mathbf{z}} \in K(\theta; t)$. Besides, since $|R(s, t) - I|_{B(\mathbf{R}^2)} \leq C_a |\mathbf{a}(t) - \mathbf{a}(s)|$ for all $s, t \in [0, T]$ and for some constant $C_a > 0$ depending only on $\mathbf{a}$ (cf. (52)–(54)), we see that

$$\begin{aligned} |\tilde{\mathbf{z}} - \mathbf{z}| &\leq \left| \left(1 - \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))|\right) R(s, t)(\mathbf{z} - \mathbf{a}(s)) + \mathbf{a}(t) - \mathbf{z} \right| \\ &\leq |R(s, t)(\mathbf{z} - \mathbf{a}(s)) - (\mathbf{z} - \mathbf{a}(s))| + |\mathbf{a}(t) - \mathbf{a}(s)| + \frac{1}{\gamma_*} |\gamma(\boldsymbol{\zeta}(s)) - \gamma(\boldsymbol{\zeta}(t))| |\mathbf{z} - \mathbf{a}(s)| \\ &\leq \text{const.} \int_s^t \left\{ |\mathbf{a}'(\tau)| + \left| \frac{d}{d\tau} \gamma(\boldsymbol{\zeta}(\tau)) \right| \right\} d\tau. \end{aligned}$$

Similarly, for any $p \geq 2$,

$$\frac{1}{p} |\tilde{\mathbf{z}}|^p - \frac{1}{p} |\mathbf{z}|^p \leq \text{const.} |\tilde{\mathbf{z}} - \mathbf{z}| \leq \text{const.} \int_s^t \left\{ |\mathbf{a}'(\tau)| + \left| \frac{d}{d\tau} \gamma(\boldsymbol{\zeta}(\tau)) \right| \right\} d\tau.$$

For any $s, t \in [0, T]$ we get the same type of estimates as above by repeating the above procedure at most N -times. Thus we can see that $\Theta_1 \subset \Theta_S$, since $\gamma(\boldsymbol{\zeta})$ and $\mathbf{a}$ are of C^2 . Hence $\Theta_1 \subset \Theta_W \subset \Theta$. Since Θ_1 is dense in Θ , we conclude that $\Theta_W = \Theta$ and (Θ_S) holds.

(Verification of (A1)–(A3)) For any parameters $\theta := [\mathbf{a}, \gamma, \boldsymbol{\zeta}]$, $\bar{\theta} := [\bar{\mathbf{a}}, \bar{\gamma}, \bar{\boldsymbol{\zeta}}] \in \Theta$, we have by our assumption that $\mathbf{a}, \bar{\mathbf{a}} \in W^{1,p}(0, T; \mathbf{R}^2)$, so that expression (52)–(54) of $R_{\theta\bar{\theta}}$ shows that $R_{\theta\bar{\theta}} \in W^{1,p}(0, T; B(\mathbf{R}^2))$ and (A1) holds. Since $\bar{\mathbf{a}}(t) = R_{\theta\bar{\theta}}(t)\mathbf{a}(t)$, $\bar{\mathbf{z}} := (1 - \varepsilon)R_{\theta\bar{\theta}, \varepsilon}\mathbf{z} + \varepsilon\bar{\mathbf{a}}(t)$, $0 < \varepsilon < 1$, we can write

$$\bar{\mathbf{z}} - \bar{\mathbf{a}}(t) = (1 - \varepsilon)R_{\theta\bar{\theta}}(\mathbf{z} - \mathbf{a}(t)).$$

Hence,

$$(\bar{z} - \bar{a}(t)) \cdot \bar{a}(t) = (1 - \varepsilon) R_{\theta\bar{\theta}}(t)(z - a(t)) \cdot R_{\theta\bar{\theta}}(t)z = (1 - \varepsilon)(z - a(t)) \cdot z = 0.$$

and if $d_{\Theta}(\theta, \bar{\theta}) < \varepsilon\gamma_*$, then

$$|\bar{z} - \bar{a}(t)| \leq (1 - \varepsilon)\gamma(\zeta(t)) \leq \gamma(\zeta(t)) - \varepsilon\gamma_* \leq \bar{\gamma}(\bar{\zeta}(t)).$$

and $\bar{z} \in K(\bar{\theta}; t)$. By (55) and (56), we see that (A2) and (A3) with $\delta_{\varepsilon} = \varepsilon\gamma_*$ are fulfilled.

By virtue of Theorem 3.7, the time-derivative operator $L_{u_0}(\theta; \cdot)$ is defined as a maximal monotone mapping from $L^p(0, T; \mathbf{R}^2)$ into $L^{p'}(0, T; \mathbf{R}^2)$ associated with $\{K(\theta; t)\}$ given by (56) and an initial datum $u_0 \in \bar{K}(\theta; 0)$. By our assumption, $\cup_{\theta \in \Theta, t \in [0, T]} K(\theta; t)$ is bounded in $\mathbf{R}^2$, namely there is a closed ball B_{k_0} around the origin of $\mathbf{R}^2$ and with radius $k_0 > 0$ such that

$$\bigcup_{\theta \in \Theta, t \in [0, T]} K(\theta; t) \subset B_{k_0}.$$

(Feedback system Λ_{u_0}) We define a feedback system Λ_{u_0} in the following way: Let $\mathbf{G} := \mathbf{G}(t, \mathbf{w}, \zeta)$ be a globally bounded and continuous vector field from the set $[0, T] \times \mathbf{R}^2 \times \mathbf{R}^2$ into $\mathbf{R}^2$ such that

$$|\mathbf{G}(t, \mathbf{w}, \zeta) - \mathbf{G}(t, \bar{\mathbf{w}}, \bar{\zeta})| \leq C_{\mathbf{G}}(|\mathbf{w} - \bar{\mathbf{w}}| + |\zeta - \bar{\zeta}|), \quad \forall \mathbf{w}, \bar{\mathbf{w}}, \zeta, \bar{\zeta} \in \mathbf{R}^2, \quad \forall t \in [0, T],$$

where $C_{\mathbf{G}}$ is a positive constant. Let $\mathbf{Y}$ be a closed convex set in $\mathbf{R}^2$ such that $0 \notin \mathbf{Y}$. Now, given $\zeta_0 \in \mathbf{Y}$ and $\mathbf{u} \in \mathcal{U} := \{\mathbf{w} \in L^p(0, T; \mathbf{R}^2) \mid \mathbf{w}(t) \in B_{k_0}, \text{ a.e. } t \in (0, T)\}$, consider the evolution inclusion

$$\zeta'(t) + \partial I_{\mathbf{Y}}(\zeta(t)) \ni \mathbf{G}(t, \int_0^t \mathbf{u}(\tau) d\tau, \zeta(t)), \quad \text{a.e. } t \in (0, T), \quad \zeta(0) = \zeta_0. \quad (57)$$

By the general theory of nonlinear evolution equations (cf. [4]) (57) has a unique solution in $W^{1,2}(0, T; \mathbf{R}^2)$ and for simplicity this solution is denoted by $\zeta(\mathbf{u})$ or $\zeta(\mathbf{u}; t)$.

Lemma 6.1. *Let $\{\mathbf{u}_n\}$ be a sequence in $\mathcal{U}$ such that $\mathbf{u}_n \rightarrow \mathbf{u}$ weakly in $L^p(0, T; \mathbf{R}^2)$ (as $n \rightarrow \infty$), then the sequence $\zeta(\mathbf{u}_n)$ converges to the solution $\zeta(\mathbf{u})$ of (57) in the sense that*

$$\zeta_n := \zeta(\mathbf{u}_n) \rightarrow \zeta := \zeta(\mathbf{u}) \text{ in } C([0, T]; \mathbf{R}^2), \quad \zeta'_n \rightarrow \zeta' \text{ in } L^p(0, T; \mathbf{R}^2). \quad (58)$$

Proof. Multiplying the both sides of (57) for ζ_n by ζ'_n , we get

$$|\zeta'_n(t)|^2 + \frac{d}{dt} I_{\mathbf{Y}}(\zeta_n(t)) = \mathbf{G}_n(t) \cdot \zeta'_n(t), \quad \text{a.e. } t \in (0, T),$$

where $\mathbf{G}_n(t) := \mathbf{G}(t, \int_0^t \mathbf{u}_n(\tau) d\tau, \zeta_n(t))$. Since $I_{\mathbf{Y}}(\zeta_n) \equiv 0$ on $[0, T]$, we see that $|\zeta'_n(t)| \leq |\mathbf{G}_n(t)|$ and hence $\{\zeta_n\}$ is bounded in $W^{1,\infty}(0, T; \mathbf{R}^2)$. Therefore there exists a subsequence $\{\zeta_{n_k}\}$ of $\{\zeta_n\}$ such that

$$\zeta_{n_k} \rightarrow \zeta \text{ in } C([0, T]; \mathbf{R}^2), \quad \zeta'_{n_k} \rightarrow \zeta' \text{ weakly in } L^p(0, T; \mathbf{R}^2).$$

for some function $\zeta \in W^{1,\infty}(0, T; \mathbf{R}^2)$.

Since $\mathbf{G}_n \rightarrow \mathbf{G} := \mathbf{G}(t, \int_0^t \mathbf{u} d\tau, \boldsymbol{\zeta})$ in $C([0, T]; \mathbf{R}^2)$, it follows that $\boldsymbol{\zeta}_{n_k}$ converges in $C([0, T]; \mathbf{R}^2)$ to the solution of (57) which is nothing but $\boldsymbol{\zeta}$ and consequently $\boldsymbol{\zeta}_n \rightarrow \boldsymbol{\zeta}$ in $C([0, T]; \mathbf{R}^2)$ without extracting any subsequence from $\{\boldsymbol{\zeta}_n\}$ by the uniqueness of solution to (57). Besides we have for any $s, t \in [0, T], s \leq t$,

$$\limsup_{n \rightarrow \infty} \int_s^t |\boldsymbol{\zeta}'_n|^2 d\tau = \lim_{n \rightarrow \infty} \int_s^t G_n \cdot \boldsymbol{\zeta}'_n d\tau = \int_s^t G \cdot \boldsymbol{\zeta}' d\tau = \int_s^t |\boldsymbol{\zeta}'|^2 d\tau,$$

so that $\boldsymbol{\zeta}'_n \rightarrow \boldsymbol{\zeta}'$ in $L^2(0, T; \mathbf{R}^2)$ (hence in $L^p(0, T; \mathbf{R}^2)$) as well as $|\boldsymbol{\zeta}'(t)| \leq |\mathbf{G}(t)|$ for a.e. $t \in (0, T)$. Thus (58) has been obtained. $\square$

Now, we define a mapping $\mathbf{a}(\cdot)$ by putting

$$\mathbf{a}(\mathbf{u}; t) := \frac{\boldsymbol{\zeta}(\mathbf{u}; t)}{|\boldsymbol{\zeta}(\mathbf{u}; t)|}$$

for any $\mathbf{u}$ in $\mathcal{U}$. It is easy to see that

$$\mathbf{a}(\mathbf{u}) \in W^{1,p}(0, T; \mathbf{R}^2), \quad |\mathbf{a}(\mathbf{u}; t)| = 1, \quad \forall t \in [0, T], \quad \mathbf{u} \in \mathcal{U}.$$

Now our feedback system $\Lambda_{\mathbf{u}_0} : \mathcal{U} \rightarrow \Theta$ is given by

$$\Lambda_{\mathbf{u}_0} \mathbf{u} = [\mathbf{a}(\mathbf{u}), \gamma, \boldsymbol{\zeta}(\mathbf{u})], \quad \forall \mathbf{u} \in \mathcal{U}.$$

We observe from Lemma 6.1 that $\Lambda_{\mathbf{u}_0}$ is a compact mapping from $\mathcal{U}$ into Θ ; in fact, if $\mathbf{u}_n \in \mathcal{U}$ and $\mathbf{u}_n \rightarrow \mathbf{u}$ weakly in $L^p(0, T; \mathbf{R}^2)$ (as $n \rightarrow \infty$), then $\boldsymbol{\zeta}(\mathbf{u}_n) \rightarrow \boldsymbol{\zeta}(\mathbf{u})$ in $W^{1,p}(0, T; \mathbf{R}^2)$ and $\theta_n := [\mathbf{a}(\mathbf{u}_n), \gamma, \boldsymbol{\zeta}(\mathbf{u}_n)] \rightarrow \theta := [\mathbf{a}(\mathbf{u}), \gamma, \boldsymbol{\zeta}(\mathbf{u})]$ in Θ .

In the above set-up, given an initial datum $\mathbf{u}_0$, satisfying

$$\mathbf{a}_0 := \frac{\boldsymbol{\zeta}_0}{|\boldsymbol{\zeta}_0|}, \quad \mathbf{a}_0 \cdot (\mathbf{u}_0 - \mathbf{a}_0) = 0, \quad |\mathbf{u}_0 - \mathbf{a}_0| \leq \gamma(\boldsymbol{\zeta}_0), \quad \text{and } \mathbf{f} \in L^{p'}(0, T; \mathbf{R}^2),$$

we formulate a quasi-variational ordinary differential inequality:

$$L_{\mathbf{u}_0}(\theta; \mathbf{u}) \ni \mathbf{f} \quad \text{in } L^{p'}(0, T; \mathbf{R}^2), \quad \theta = \Lambda_{\mathbf{u}_0} \mathbf{u} \tag{59}$$

which can be written as

$$\begin{aligned} \boldsymbol{\zeta}'(t) + \partial I_Y(\boldsymbol{\zeta}(t)) \ni \mathbf{G}(t, \int_0^t \mathbf{u}(\tau) d\tau, \boldsymbol{\zeta}(t)), \quad t \in (0, T), \quad \boldsymbol{\zeta}(0) = \boldsymbol{\zeta}_0, \\ \int_0^T (\boldsymbol{\eta}'(t) - \mathbf{f}(t)) \cdot (\mathbf{u}(t) - \boldsymbol{\eta}(t)) dt \leq \frac{1}{2} |\mathbf{u}_0 - \boldsymbol{\eta}|^2, \quad \forall \boldsymbol{\eta} \in \mathcal{K}_0(\theta). \end{aligned}$$

The existence of a solution $\{\theta, \mathbf{u}\}$ of (59) is not covered by Theorem 5.4, because condition (46) is not fulfilled. Hence the compactness result mentioned in Lemma 5.2 is not obtained. However the existence of a solution of (59) is directly proved by the fixed point argument. In fact, let S be the mapping which assigns to each function $\mathbf{u} \in \mathcal{U}$ the solution $\bar{\mathbf{u}}$ of

$$\theta = \Lambda_{\mathbf{u}_0} \mathbf{u}, \quad L_{\mathbf{u}_0}(\theta; \bar{\mathbf{u}}) \ni \mathbf{f}.$$

Then it follows from the above observations that S is compact mapping from $\mathcal{U}$ into itself in the topology of $L^p(0, T; \mathbf{R}^2)$, so that S has at least one fixed point, $\mathbf{u} = S\mathbf{u}$ in $\mathcal{U}$. Clearly, the pair $\{\theta, \mathbf{u}\}$ with $\theta := [\mathbf{a}(\mathbf{u}), \gamma, \boldsymbol{\zeta}(\mathbf{u})]$ is a solution of (59).

Remark 6.2. For simplicity we treated above in $\mathbf{R}^2$. But similar problems are formulated and solved in the 3d space or more generally infinite dimensional spaces, too, although the computation is more technical for the verification of assumptions (A1)–(A3) and (Θ_S) .

6.2. Quasi-variational partial differential inequality

In this application we treat a model arising in superconductivity. Let Ω be a smooth bounded domain in $\mathbf{R}^N$, $1 \leq N < \infty$, and

$$\Gamma := \partial\Omega, \quad \Sigma := \Gamma \times (0, T), \quad Q := \Omega \times (0, T).$$

We put $V := W_0^{1,p}(\Omega)$, $H := L^2(\Omega)$, $W := W_0^{2,q}(\Omega)$ with $\max\{p, N\} < q < \infty$, and hence $W \subset V \subset H \subset V^* \subset W^*$ with dense and compact embeddings. We suppose that

- $a(x, t, v)$ is a function on $Q \times \mathbf{R}$, satisfying the Carathéodory condition, namely for a.e. $(x, t) \in Q$, the function $v \rightarrow a(x, t, v)$ is continuous and for all $v \in \mathbf{R}$ the function $(x, t) \rightarrow a(x, t, v)$ is measurable on Q . We assume that for some positive constants a_* , a^*

$$a_* \leq a(x, t, v) \leq a^*, \quad \text{a.e. } (x, t) \in Q, \quad \forall v \in \mathbf{R}.$$

- $\gamma(\cdot) \in C_b(\mathbf{R})$, namely γ is bounded and continuous on $\mathbf{R}$ such that $\lim_{r \rightarrow \pm\infty} \gamma(r)$ exists. Furthermore, suppose that, for a positive constant ε_0 , $\gamma(\zeta) \geq \varepsilon_0$, $\forall \zeta \in \mathbf{R}$.
- $h(x, t, u)$ is a Lipschitz continuous function on $\overline{Q} \times \mathbf{R}$.

Next, we put $\Theta := \{\theta := [\gamma, \zeta] \mid \gamma \in C_b(\mathbf{R}), \gamma \geq \varepsilon_0 \text{ on } \mathbf{R}, \zeta \in C(\overline{Q})\}$, and define a semimonotone mapping $A(v; u)$ and a family $\{K(\theta; t)\}$ of closed and convex subsets of V by:

$$A(v; u) := -\operatorname{div}(a(x, t, v)|\nabla u|^{p-2}\nabla u) \quad \text{in } L^{p'}(0, T; V^*),$$

$$\forall v \in \mathcal{V}_A := L^p(0, T; V), \quad \forall u \in L^p(0, T; V),$$

$$K(\theta; t) := \{z \in V \mid |\nabla z(x)| \leq \gamma(\zeta(x, t)) \text{ a.e. } x \in \Omega\},$$

$$\forall t \in [0, T], \quad \forall \theta := [\gamma, \zeta] \in \Theta.$$

Furthermore, for a given $u_0 \in H$, a feedback system Λ_{u_0} is defined by $\theta := [\gamma, \zeta] = \Lambda_{u_0} u$ if and only if $u \in L^p(0, T; V)$ and ζ is a unique solution of the following heat equation

$$\begin{cases} \zeta_t - \Delta \zeta = h(x, t, u) & \text{in } Q, \\ \frac{\partial \zeta}{\partial n} + n_0 \zeta = 0 & \text{on } \Sigma, \quad \zeta(\cdot, 0) = \zeta_0 \text{ in } \Omega, \end{cases} \quad (60)$$

where $\zeta_0 \in H^2(\Omega)$, satisfying $\frac{\partial \zeta_0}{\partial n} + n_0 \zeta_0 = 0$ on Γ and

$$u_0 \in K(\zeta_0; 0) := \{z \in V \mid |\nabla z| \leq \gamma(\zeta_0) \text{ a.e. on } \Omega\}.$$

We know that problem (60) admits a unique solution

$$\zeta \in W^{1,2}(0, T; H^1(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$$

(cf. [7; Appendix]), which is compactly embedded in $C(\overline{Q})$. It is verified that (Θ_S) holds under (45)–(46), $\Theta \cap (C^2(\mathbf{R}) \times C^2(\overline{Q})) \subset \Theta_S$, $\Theta_W = \Theta$ and conditions (A1)–(A3) are satisfied by $R_{\theta\bar{\theta}} = I$, $\sigma_{\theta\bar{\theta}, \varepsilon} = 0$ and $\mathcal{F}_{\theta\bar{\theta}, \varepsilon} = (1 + c_0(\varepsilon))I$; see [7, 20] for the verification of these facts. Also, it is easy to check conditions (B0)–(B3) from the definition of $A(v; u)$ as well as Λ_{u_0} satisfies $(\Lambda 1) - (\Lambda 2)$.

Therefore, by virtue of Theorem 5.4, for a given $f \in L^{p'}(0, T; V^*)$, the quasi-variational inequality

$$L_{u_0}(\theta; u) + A(u; u) \ni f \text{ in } L^{p'}(0, T; V^*), \quad \theta = \Lambda_{u_0} u,$$

has at least one solution $\{\theta, u\}$, which gives a solution of the following system:

$$\begin{aligned} \zeta_t - \Delta \zeta &= h(x, t, u) \text{ in } Q, \quad \frac{\partial \zeta}{\partial n} + n_0 \zeta = 0 \text{ on } \Sigma, \quad \zeta(\cdot, 0) = \zeta_0 \text{ in } \Omega, \\ u &\in C([0, T]; H), \quad u(t) \in K(\theta; t), \quad \forall t \in [0, T], \\ \int_0^T \langle \eta', u - \eta \rangle dt + \int_Q a(x, t, u) |\nabla u|^{p-2} \nabla u \cdot \nabla (u - \eta) dx dt \\ &\leq \int_0^T \langle f, u - \eta \rangle dt + \frac{1}{2} |u_0 - \eta(0)|_H^2, \quad \forall \eta \in \mathcal{K}_0(\theta). \end{aligned}$$

Remark 6.3. The above quasi-variational inequality arises from simplified models of Type II superconductivity; we refer to [1, 2, 19, 27] for related works.

Remark 6.4. The similar approach is possible to quasi-variational Navier-Stokes problems. In the general case the obstacle function $\gamma(\zeta)$ is required to have three phases, $\gamma(\zeta) = 0$, $0 < \gamma(\zeta) < \infty$ and $\gamma(\zeta) = \infty$ which are respectively the solid, mussy and liquid parts in the fluid. Therefore, its mathematical treatment would be much more complicated (cf. [9, 11, 12]).

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